mathematics

MDPI

Article

The Extended Log-Logistic Distribution: Inference and Actuarial Applications

Nada M. Alfaer [1], Ahmed M. Gemeay [2], Hassan M. Aljohani [1] and Ahmed Z. Afify [3,*]

[1] Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; nmfaer@tu.edu.sa (N.M.A.); hmjohani@tu.edu.sa (H.M.A.)
[2] Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt; ahmed.gemeay@science.tanta.edu.eg
[3] Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt
* Correspondence: ahmed.afify@fcom.bu.edu.eg

Abstract: Actuaries are interested in modeling actuarial data using loss models that can be adopted to describe risk exposure. This paper introduces a new flexible extension of the log-logistic distribution, called the extended log-logistic (Ex-LL) distribution, to model heavy-tailed insurance losses data. The Ex-LL hazard function exhibits an upside-down bathtub shape, an increasing shape, a J shape, a decreasing shape, and a reversed-J shape. We derived five important risk measures based on the Ex-LL distribution. The Ex-LL parameters were estimated using different estimation methods, and their performances were assessed using simulation results. Finally, the performance of the Ex-LL distribution was explored using two types of real data from the engineering and insurance sciences. The analyzed data illustrated that the Ex-LL distribution provided an adequate fit compared to other competing distributions such as the log-logistic, alpha-power log-logistic, transmuted log-logistic, generalized log-logistic, Marshall–Olkin log-logistic, inverse log-logistic, and Weibull generalized log-logistic distributions.

Keywords: insurance losses data; expected shortfall; log-logistic distribution; parameter estimation; risk measures

check for updates

Citation: Alfaer, N.M.; Gemeay, A.M.; Aljohani, H.M.; Afify, A.Z. The Extended Log-Logistic Distribution: Inference and Actuarial Applications. *Mathematics* **2021**, *9*, 1386. https://doi.org/10.3390/math9121386

Academic Editor: Lev Klebanov

Received: 5 May 2021
Accepted: 8 June 2021
Published: 15 June 2021

1. Introduction

Modeling insurance losses data has received significant interest from actuaries and risk managers who often evaluate and study the unlikely outcomes that the value-at-risk may express by chance. The insurance data are usually unimodal [1], right-skewed [2], positive [3], have a heavy-tailed density [4], and have a unimodal hump shape [5].

There is a clear need to develop and propose more flexible distributions by extending the well-known classical distributions or by introducing a new family to model several insurance datasets such as financial returns, unemployment insurance data, insurance losses data, and risk management data, among others.

The log-logistic (LL) distribution is also known as the Fisk distribution in the income distribution literature [6]. Some authors, such as [7,8], have referred to the Fisk distribution as the LL distribution, whereas Arnold [9] referred to it as the Pareto Type III distribution and included an additional location parameter. Further details about the LL model can be found in [10]. Several authors have studied different generalized forms of the LL distribution to improve its capability and flexibility. Some notable examples are the following: Kumaraswamy-LL [11], beta-LL [12], Marshall–Olkin LL [13], McDonald LL [14], Zografos-Balakrishnan LL [15], and odd Lomax LL distributions [16].

This article suggests a new version of the LL distribution called the extended log-logistic (Ex-LL) distribution, which can provide more flexibility in modeling insurance data than other competing models. Hence, the aim of the paper was three-fold. The first was devoted to proposing the Ex-LL model as a new form of the LL distribution via the

Type-I heavy-tailed G family [17]. The Ex-LL model has some desirable characteristics as follows: It is a more flexible extension with only an extra shape parameter for the LL distribution, and it improves goodness-of-fit to real-life data; it produces negative and positive skewness, and also, it causes the kurtosis to be more flexible as compared to the baseline LL model. The Ex-LL model has a heavier tail than the LL and exponentiated log-logistic (ELL) models (see Section 6). The Ex-LL hazard rate function (HRF) exhibits an increasing shape, an upside-down bathtub shape, a decreasing-J shape, and a reversed-J shape. Its density exhibits a reversed-J shape, a symmetrical shape, an asymmetrical shape (right-skewed or left-skewed), a unimodal shape, and a J shape. Furthermore, it can be adopted to analyze various data in the engineering and actuarial sciences. Two sets of real data from the insurance and engineering fields were fitted using the Ex-LL model, showing its flexibility over other competing distributions.

Second, we estimated the Ex-LL parameters using some estimation approaches: the least-squares estimators (LSEs), the Anderson–Darling estimators (ADEs), the maximum likelihood estimators (MLEs), the weighted least-squares estimators (WLSEs), and the Cramér–von Mises estimators (CVMEs). The proposed estimators were compared through extensive simulations to determine the best estimation approach to estimate the Ex-LL parameters.

Estimating the exposure to market risk, due to changes in the prices of equity exchange rates and interest rates, is considered an important subject in the actuarial sciences. Therefore, our third objective was devoted to deriving some important actuarial or risk measures based on the Ex-LL distribution called the value-at-risk (VaR), tail value-at-risk (TVaR), tail variance premium (TVP), tail variance (TV), and expected shortfall (ES). These measures are very important in portfolio optimization under uncertainty.

The article is outlined as follows. The new Ex-LL distribution is defined in Section 2. In Section 3, its distributional properties were obtained. Five classical estimation methods were adopted to estimate the Ex-LL parameters, as shown in Section 4. We provide, in Section 5, detailed numerical results for the introduced methods. In Section 6, we explored five important risk measures based on the Ex-LL model, and provided numerical simulation results for these risk measures. In Section 7, two real-life data from the engineering and actuarial sciences were analyzed to explore the importance and flexibility of the Ex-LL distribution. Some final concluding remarks were provided in Section 8.

2. The Ex-LL Distribution

The cumulative distribution function (CDF) of the two-parameter LL model is given (for $x > 0$) by $G(x) = \left(1 + \frac{\lambda}{x^\beta}\right)^{-1}$, β, $\lambda > 0$, and its probability density function (PDF) reduces to $g(x) = \beta\,\lambda\,x^{-\beta-1}\left(1 + \frac{\lambda}{x^\beta}\right)^{-2}$.

We define the CDF of the Ex-LL model based on the extended family [17] which is specified (for $x \in \mathbb{R}$) by the CDF

$$F(x) = 1 - \left(\frac{1 - G(x;\,\varphi)}{1 - (1-\alpha)G(x;\,\varphi)}\right)^{\alpha}, \qquad \alpha > 0, \tag{1}$$

and the PDF

$$f(x) = \frac{\alpha^2\, g(x;\,\varphi)\, [1 - G(x;\,\varphi)]^{\alpha-1}}{[1 - (1-\alpha)G(x;\,\varphi)]^{\alpha+1}}, \qquad \alpha > 0, \tag{2}$$

where $G(x;\,\varphi)$ is any baseline CDF that depends on $\varphi \in \mathbb{R}$. The CDF (1) reduces to the baseline CDF for $\alpha = 1$.

The CDF of the Ex-LL distribution follows, by inserting the LL CDF in Equation (1), as

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + \alpha x^{\beta}}\right)^{\alpha}, \qquad \alpha, \beta, \lambda > 0,\ x > 0. \tag{3}$$

The corresponding PDF of the Ex-LL distribution takes the form

$$f(x) = \alpha^2 \beta \lambda^\alpha x^{\beta-1} \left(\frac{1}{\lambda + \alpha x^\beta} \right)^{\alpha+1}, \qquad \alpha, \beta, \lambda > 0, \ x > 0. \tag{4}$$

The LL model follows as a special case from (4) with $\alpha = 1$. The EX-LL HRF reduces to

$$h(x) = \frac{\alpha^2 \beta x^{\beta-1}}{\lambda + \alpha x^\beta}, \qquad \alpha, \beta, \lambda > 0, \ x > 0. \tag{5}$$

Plots of the PDF and HRF of the Ex-LL distribution are depicted respectively in Figures 1 and 2. These plots show that the Ex-LL density exhibits a symmetrical shape, an asymmetrical shape, a J shape, a unimodal shape, and a reversed-J shape. Furthermore, its HRF exhibits an increasing shape, an upside-down bathtub shape, a decreasing-J shape, and a reversed-J shape.

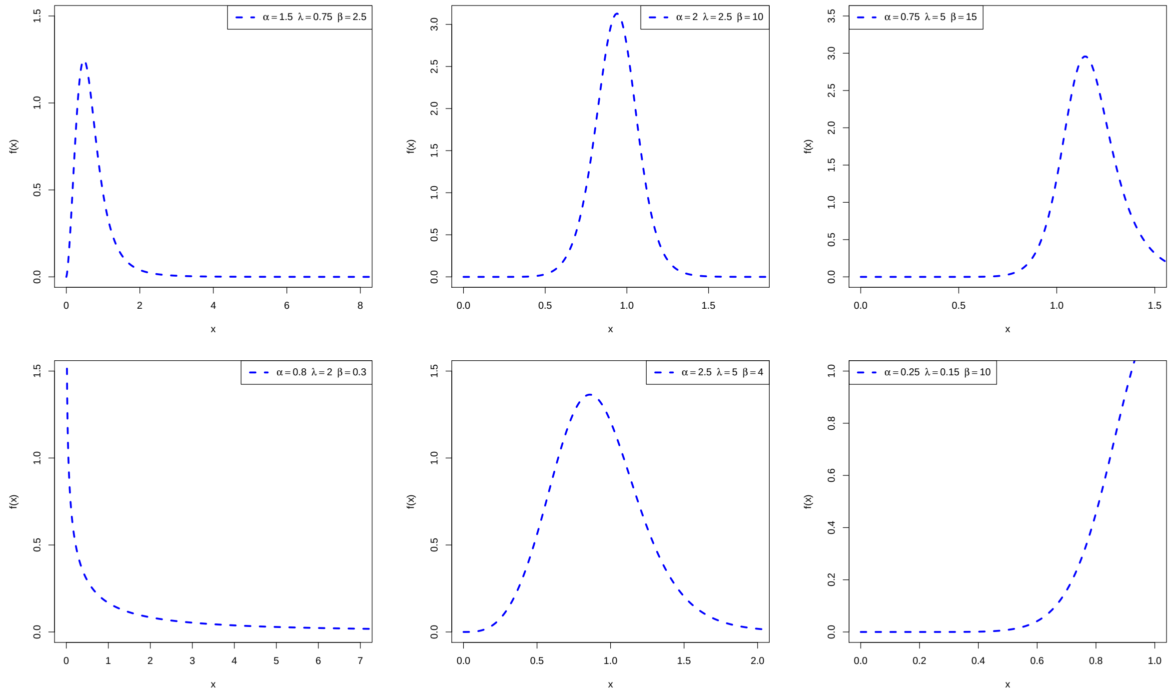

Figure 1. Possible shapes of the Ex-LL density function for various values of α, β and λ.

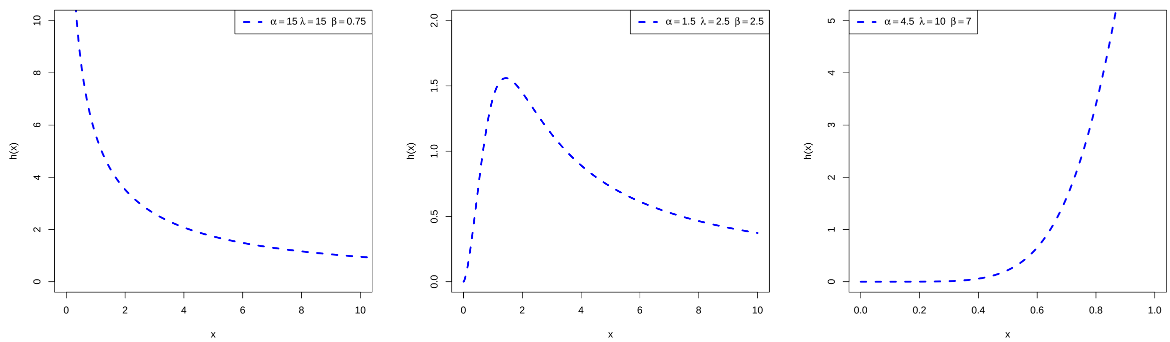

Figure 2. *Cont.*

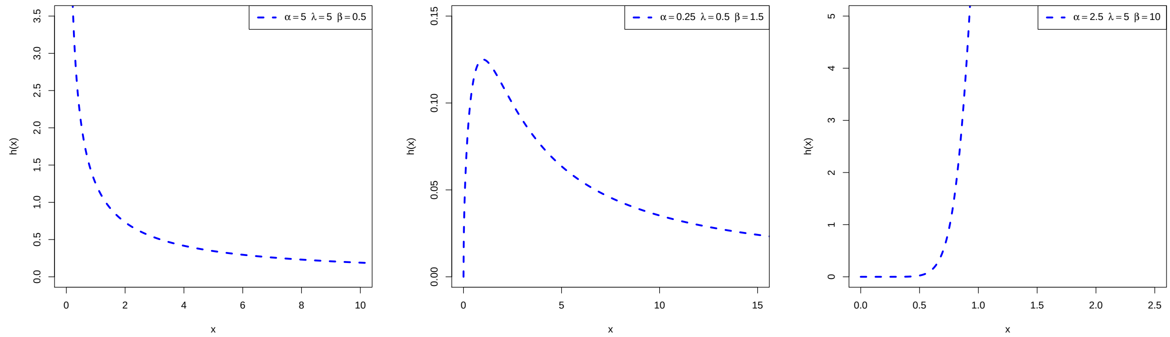

Figure 2. Possible shapes of the Ex-LL hazard function for various values of α, β and λ.

3. Mathematical Properties

3.1. Mode and Quantile Function

On differentiating the logarithm of (4) with respect to x and equating to zero, the unique mode of the Ex-LL distribution follows as

$$x_0 = \left[\frac{(\beta-1)\lambda}{\alpha(\alpha\beta+1)}\right]^{1/\beta}, \ \beta > 1.$$

For $\beta \leq 1$, the Ex-LL distribution has no mode.

The quantile function (QF) of the Ex-LL distribution follows as

$$Q(p) = \left\{\frac{\lambda\left[(1-p)^{-1/\alpha}-1\right]}{\alpha}\right\}^{1/\beta}, \quad 0 < p < 1. \tag{6}$$

Let $p \sim$Uniform$(0,1)$, then the QF of the Ex-LL distribution can be adopted to generate its random data by the formula

$$x_i = \left\{\frac{\lambda\left[(1-p_i)^{-1/\alpha}-1\right]}{\alpha}\right\}^{1/\beta}, \quad i = 1,2,\ldots,n. \tag{7}$$

3.2. Moments and Moment Generating Function

The rth ordinary moment of the Ex-LL distribution is given by

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x)dx = \frac{\Gamma\left(\frac{r+\beta}{\beta}\right)\left(\frac{\alpha}{\lambda}\right)^{-\frac{r}{\beta}}\Gamma\left(\alpha-\frac{r}{\beta}\right)}{\Gamma(\alpha)}, \ \ \frac{r}{\beta} < \alpha.$$

The first four ordinary moments of the Ex-LL distribution can be obtained directly by setting $r = 1,2,3,$ and 4 in the last equation.

The mean (μ'_1), variance ($Var(X)$), skewness ($\gamma_1(X)$), and kurtosis ($\gamma_2(X)$) of the Ex-LL distribution are obtained numerically for some choices of its parameters using the Wolfram Mathematica program version 12.0. These results are reported in Table 1. It is noted that the Ex-LL skewness varies in the interval $(-0.84559, 45.48282)$, whereas the LL skewness varies only within the interval $(0.35216, 1.81999)$. Furthermore, the kurtosis of the Ex-LL distribution spreads much larger in the interval $(4.10572, 2573.493)$, whereas the kurtosis of the LL distribution spreads only in the interval $(4.50847, 14.76564)$ for same parameter values.

Table 1. The numerical values for μ_1', $Var(X)$, $\gamma_1(X)$, and $\gamma_2(X)$ of the Ex-LL distribution for several choices of α, β, and λ.

Parameters	μ_1'	$Var(X)$	$\gamma_1(X)$	$\gamma_2(X)$
$(\alpha = 0.5,\ \beta = 15,\ \lambda = 1.5)$	1.19852	0.05122	2.13181	15.34159
$(\alpha = 0.75,\ \beta = 7.5,\ \lambda = 0.75)$	1.11244	0.11314	2.34462	22.42470
$(\alpha = 1.5,\ \beta = 3,\ \lambda = 3)$	1.17775	0.43811	2.82183	49.06847
$(\alpha = 1.5,\ \beta = 3,\ \lambda = 0.75)$	0.74194	0.17386	2.82183	49.06840
$(\alpha = 1.75,\ \beta = 15,\ \lambda = 2.5)$	0.97440	0.009770	-0.05100	3.75522
$(\alpha = 2,\ \beta = 9,\ \lambda = 3)$	0.94900	0.02378	0.10911	3.67376
$(\alpha = 2,\ \beta = 6,\ \lambda = 3)$	0.93367	0.05104	0.43449	4.10571
$(\alpha = 3,\ \beta = 1.5,\ \lambda = 2)$	0.41012	0.14478	3.58182	63.19551
$(\alpha = 3,\ \beta = 1.5,\ \lambda = 5)$	0.75546	0.49124	3.5818	63.19552
$(\alpha = 3.75,\ \beta = 15,\ \lambda = 2.5)$	0.86925	0.00610	-0.46932	3.66150
$(\alpha = 5,\ \beta = 1.5,\ \lambda = 1.5)$	0.15610	0.01560	1.99387	11.18786
$(\alpha = 10,\ \beta = 10,\ \lambda = 2.5)$	0.82517	0.01221	-0.27530	3.39749
$(\alpha = 15,\ \beta = 0.5,\ \lambda = 0.75)$	1.1×10^{-6}	2.4×10^{-9}	45.48282	2573.493
$(\alpha = 15,\ \beta = 25,\ \lambda = 5)$	0.84142	0.00184	-0.84557	4.31163

The rth incomplete moment of the Ex-LL distribution has the form

$$I_r(t) = \int_0^t x^r f(x)dx = \frac{\alpha^2\beta\lambda^{\alpha-1}t^{\beta+r}\left(\frac{1}{\lambda+\alpha t^\beta}\right)^\alpha {}_2F_1\left(1, \frac{r}{\beta} - \alpha + 1; \frac{r}{\beta} + 2; -\frac{t^\beta\alpha}{\lambda}\right)}{\beta + r},$$

where ${}_2F_1\left(1, \frac{r}{\beta} - \alpha + 1; \frac{r}{\beta} + 2; -\frac{t^\beta\alpha}{\lambda}\right)$ is the hyper geometric function.
The moment generating function of the Ex-LL distribution takes the form

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k\Gamma\left(\frac{k+\beta}{\beta}\right)\left(\frac{\alpha}{\lambda}\right)^{-\frac{k}{\beta}}\Gamma\left(\alpha - \frac{k}{\beta}\right)}{k!\Gamma(\alpha)}, \quad \frac{k}{\beta} < \alpha.$$

3.3. Mean Residual Life, Mean Inactivity Time and Inequality Curves

The mean residual life of the Ex-LL distribution at age t has the form

$$MRL = \frac{1 - I_1(t)}{S(t) - t} = \frac{(\beta+1) - \alpha^2\beta\lambda^{\alpha-1}t^{\beta+1}\left(\frac{1}{\lambda+\alpha t^\beta}\right)^\alpha {}_2F_1\left(1, 1 - \alpha + \frac{1}{\beta}; 2 + \frac{1}{\beta}; -\frac{t^\beta\alpha}{\lambda}\right)}{(\beta+1)\left[\left(\frac{\lambda}{\lambda+\alpha t^\beta}\right)^\alpha - t\right]},$$

where $I_1(t)$ is the first incomplete moments.
The mean inactivity time of the Ex-LL distribution takes the form

$$MIT = t - \frac{I_1(t)}{F(t)} = \frac{\alpha^2\beta\lambda^{\alpha-1}t^{\beta+1}\left(\frac{1}{\lambda+\alpha t^\beta}\right)^\alpha {}_2F_1\left(1, 1 - \alpha + \frac{1}{\beta}; 2 + \frac{1}{\beta}; -\frac{t^\beta\alpha}{\lambda}\right)}{(\beta+1)\left[\left(\frac{\lambda}{\lambda+\alpha t^\beta}\right)^\alpha - 1\right]} + t,$$

The Lorenz, Bonferroni and Zenga curves (see, Lorenz [18] and Bonferroni [19]) are considered the most important inequality curves and their useful applications are common in insurance, reliability, medicine, and economics.

The Lorenz curve is defined for the Ex-LL distribution as follows:

$$L(p) = \frac{I_1(x_p)}{\mu} = \frac{\alpha\beta\lambda^\alpha\Gamma(\alpha)\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{\beta}+1}x_p^{\beta+1}\left(\frac{1}{\lambda+\alpha x_p^\beta}\right)^\alpha {}_2F_1\left(1, 1 - \alpha + \frac{1}{\beta}; 2 + \frac{1}{\beta}; -\frac{\alpha x_p^\beta}{\lambda}\right)}{(\beta+1)\Gamma\left(1 + \frac{1}{\beta}\right)\Gamma\left(\alpha - \frac{1}{\beta}\right)},$$

where $F(x_p) = p$, x_p is the QF, and $I_1(t)$ refers to the first incomplete moments.

The Bonferroni and Zenga inequality curves can be determined, through their relationship with the Lorenz curve, by the following formulae ([20])

$$B(p) = \frac{L(p)}{p} \quad \text{and} \quad Z(p) = \frac{L(p) - p}{p[1 - L(p)]}.$$

3.4. Some Entropies

The entropy of the random variable X has important applications in many applied fields including statistics for testing hypotheses [21] and engineering, physics, and information theory to describe dynamical systems or nonlinear chaotic [22]. Furthermore, Song [23] developed the log-likelihood-based distribution measure based on the Rényi information. Song's measure is exist and can be defined for all distributions. Song's measure provides meaningful comparisons between distributions as compared with traditional kurtosis measure.

The Rényi $P_X(k)$, Tsallis $L_X(k)$, and Shannon $H_X(1)$ entropies of the Ex-LL distribution can be derived for the Ex-LL distribution by the following formulae.

$$\begin{aligned} P_X(k) &= \frac{1}{1-k} \log \int_{x=0}^{\infty} f^k(x)\, dx, \qquad k > 0, k \neq 1 \\ &= \frac{1}{k-1} \left\{ -k \log(\alpha\,\beta) - \frac{(k-1)\log\left(\frac{\alpha}{\lambda}\right)}{\beta} + \log\left[\frac{\beta\,\Gamma(\alpha\,k + k)}{\Gamma\left(\frac{k(\beta-1)+1}{\beta}\right)\Gamma\left(\frac{\alpha\,\beta k + k - 1}{\beta}\right)} \right] \right\} \end{aligned}$$

and

$$\begin{aligned} L_X(k) &= \frac{1}{1-k}\left[\int_{x=0}^{\infty} f^k(x)\, dx - 1 \right] \\ &= \frac{(\alpha\,\beta)^k\,\Gamma\left(\frac{k(\beta-1)+1}{\beta}\right)\left(\frac{\alpha}{\lambda}\right)^{\frac{k-1}{\beta}}\Gamma\left(\frac{\alpha\,\beta k + k - 1}{\beta}\right) - \beta\,\Gamma\left(\alpha\,k + k\right)}{\beta\,(1-k)\,\Gamma\left(\alpha k + k\right)}. \end{aligned}$$

The Shannon entropy, say $H_X(1)$, follows from $P_X(k)$ as $r \longrightarrow 1$. Then, $H_X(1)$ follows for the Ex-LL distribution as

$$\begin{aligned} H_X(1) &= \lim_{r \longrightarrow 1} P_X(k) \\ &= \frac{\frac{\beta}{\alpha} - \beta \log(\alpha\,\beta) + (\beta-1)\Phi(\alpha) - \log(\alpha) + \Upsilon(\beta-1) + \beta + \log(\lambda)}{\beta}, \end{aligned}$$

where $\Phi(z) = \frac{d}{dz}\log[\Gamma(z)]$ and Υ refers to the Euler Mascheroni constant.

3.5. Order Statistics

The PDF of the kth order statistic, $X_{k:n}$, for the Ex-LL distribution is defined by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(n-k)!\,(k-1)!} f(x)\ [F(x)]^{k-1}\ [1 - F(x)]^{n-k} \\ &= \frac{\alpha^2 \beta n! x^{\beta-1} \lambda^{\alpha(n+1-k)} \left(\frac{1}{\lambda + \alpha x^{\beta}}\right)^{\alpha - i\alpha + \alpha n + 1} \left[1 - \left(\frac{\lambda}{\lambda + \alpha x^{\beta}}\right)^{\alpha}\right]^{k-1}}{\Gamma(k)\Gamma(n+1-k)}. \end{aligned}$$

The CDF of $X_{k:n}$ for the Ex-LL distribution takes the form

$$\begin{aligned} F_{k:n}(x) &= \sum_{l=k}^{n} \binom{n}{l} \; [1 - F(x)]^{n-l} \; [F(x)]^{l} \\ &= \binom{n}{k} \left[1 - \left(\frac{\lambda}{\lambda + \alpha x^{\beta}} \right)^{\alpha} \right]^{k} \left(\frac{\lambda}{\lambda + \alpha x^{\beta}} \right)^{\alpha(n-k)} \\ &\times \; {}_2F_1 \left[1,\, k-n;\, k+1; 1 - \left(\frac{\lambda}{\alpha x^{\beta} + \lambda} \right)^{-\alpha} \right], \end{aligned}$$

where ${}_2F_1\left[1, k-n;\, k+1;\, 1 - \left(\frac{\lambda}{\alpha x^{\beta}+\lambda}\right)^{-\alpha}\right]$ denotes the hyper geometric function.

The PDF and CDF of the minimum, say (W_n), and the maximum order statistics, say (Z_n), follows by setting $k = 1$ and $k = n$, respectively. The limiting distributions (Theorem 2.1.5 [24]) of W_n and Z_n are given by

$$\lim_{n \to +\infty} P(W_n < d_n x) = 1 - \exp(-x^{\beta}), \quad d_n = F^{-1}\left(\frac{1}{n}\right)$$

and

$$\lim_{n \to +\infty} P(Z_n < b_n x) = \exp(-x^{-\alpha\beta}), \quad b_n = F^{-1}\left(1 - \frac{1}{n}\right).$$

4. Estimation Methods

Here,the Ex-LL parameters are estimated using some estimation approaches.

4.1. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from the PDF (4), then the log-likelihood function takes the form

$$L = n \log\left(\alpha^2 \beta \lambda^{\alpha}\right) - (\alpha + 1) \sum_{i=1}^{n} \log\left(\alpha x_i^{\beta} + \lambda\right) + (\beta - 1) \sum_{i=1}^{n} \log(x_i). \tag{8}$$

The first derivatives with respect to α, β and λ follows by differentiating Equation (8) and equating them to zero. We obtain

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \frac{n\lambda^{-\alpha}\left[\alpha^2 \beta \lambda^{\alpha} \log(\lambda) + 2\,\alpha\beta\,\lambda^{\alpha}\right]}{\alpha^2\,\beta} - (1+\alpha) \sum_{i=1}^{n} \frac{x_i^{\beta}}{\alpha x_i^{\beta} + \lambda} - \sum_{i=1}^{n} \log\left(\alpha x_i^{\beta} + \lambda\right) = 0, \\ \frac{\partial L}{\partial \lambda} &= \frac{\alpha n}{\lambda} - (\alpha + 1) \sum_{i=1}^{n} \frac{1}{\alpha x_i^{\beta} + \lambda} = 0, \\ \frac{\partial L}{\partial \beta} &= \frac{n}{\beta} - (1 + \alpha) \sum_{i=1}^{n} \frac{\alpha x_i^{\beta} \log(x_i)}{\alpha x_i^{\beta} + \lambda} + \sum_{i=1}^{n} \log(x_i) = 0. \end{aligned}$$

Solving the previous equations, we obtain the MLEs of the Ex-LL parameters α, β and λ. However, these equations cannot be solved analytically, hence statistical software including Maple, R or Mathematica are adopted to solve them numerically using iterative methods such as Newton-Raphson algorithm.

4.2. Least-Squares and Weighted Least-Squares Estimation

Let $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$ be the order statistics of the Ex-LL distribution. Hence, the LSEs of the Ex-LL parameters are provided by minimizing:

$$O = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha} - \frac{i}{n+1}\right]^2,$$

Besides, by solving the following equations we obtain the LSEs of the Ex-LL parameters:

$$\sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha} - \frac{i}{n+1}\right]\Delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3,$$

where

$$\Delta_1(x_{i:n}) = \frac{\partial}{\partial\alpha}F(x_{i:n}) = \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha}\left[-\log\left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right) + \frac{\alpha x_{i:n}^{\beta}}{\alpha x_{i:n}^{\beta} + \lambda}\right], \tag{9}$$

$$\Delta_2(x_{i:n}) = \frac{\partial}{\partial\lambda}F(x_{i:n}) = \alpha\left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha-1}\left[\frac{\lambda}{\left(\alpha x_{i:n}^{\beta} + \lambda\right)^2} - \frac{1}{\alpha x_{i:n}^{\beta} + \lambda}\right], \tag{10}$$

$$\Delta_3(x_{i:n}) = \frac{\partial}{\partial\beta}F(x_{i:n}) = \frac{\alpha^2\,\lambda^{\alpha}\,x_{i:n}^{\beta}\ \log(x_{i:n})\left(\alpha x_{i:n}^{\beta} + \lambda\right)^{1-\alpha}}{\left(\alpha x_{i:n}^{\beta} + \lambda\right)^2}. \tag{11}$$

The WLSEs of the Ex-LL parameters are calculated by minimizing:

$$\begin{aligned} W &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1 - \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha} - \frac{i}{n+1}\right]^2. \end{aligned}$$

Besides, the WLSEs of α, β and λ can be obtained by solving:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]\Delta_s(x_{i:n}) = 0,$$

where $\Delta_s(x_{i:n}), s = 1, 2, 3$ are given in (9)–(11), respectively.

4.3. Anderson—Darling Estimation

The ADEs of the Ex-LL parameters are calculated by minimizing:

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}) + \log S(x_{i:n})].$$

The ADEs are also obtained by solving:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_{i:n})}{F(x_{i:n})} - \frac{\Delta_s(x_{n+1-i:n})}{S(x_{n+1-i:n})}\right] = 0,$$

where $\Delta_s(x_{i:n}), s = 1, 2, 3$ are given in (9)–(11), respectively.

4.4. Cramér—von Mises Estimation

The CVMEs of the Ex-LL parameters are calculated by minimizing:

$$CV = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha} - \frac{2i-1}{2n}\right]^2,$$

These estimators can be also obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\alpha x_{i:n}^{\beta} + \lambda}\right)^{\alpha} - \frac{2i-1}{2n}\right]\Delta_s(x_{i:n}) = 0,$$

where $\Delta_s(x_{i:n}), s = 1,2,3$ are given in (9)–(11), respectively.

5. Numerical Simulations for the Estimation Methods

Now, we provided detailed simulation results to explore the performances of the introduced estimation methods in estimating the parameters of the Ex-LL model. We considered several sample sizes and different values of the parameters, that is, $n = \{20, 60, 100, 200, 400\}$ and $\alpha = \{0.25, 0.50, 0.75, 1.50, 2.0, 2.50\}$, $\beta = \{0.25, 0.50, 0.75, 1.50, 2.0, 2.50, 3.0\}$, and $\lambda = \{0.25, 0.50, 0.75, 1.50, 2.0, 3.0\}$. We generated $N = 5000$ random samples using Equation (7). The behavior of the different estimates is compared with respect to their: average absolute biases ($|BIAS|$), $|BIAS| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\vartheta} - \vartheta|$, mean square errors (MSEs), $MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\vartheta} - \vartheta)^2$, and mean relative errors (MREs), $MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\vartheta} - \vartheta|/\vartheta$.

The Tables 2–9 show the simulation results, average estimates of the parameters (AVEs), $|BIAS|$, MSEs, and MREs, of the Ex-LL parameters using different approaches. These results showed that estimates are very close to their true values and have small biases, MSEs and MREs. The results illustrated that the biases, MSEs, and MREs decrease as n increases, showing that the introduced estimators are consistent. We can conclude that the introduced estimation methods performed very well in estimating the Ex-LL parameters. Generally, based on our study, the ordering performance of these estimators, in terms of their MSEs, is the MLEs, WLSEs, ADEs, CVMEs, and LSEs. Hence, the maximum likelihood (ML) method is the best estimation approach to estimate the Ex-LL parameters and it will be adopted in the subsequent section to estimate the parameters of the Ex-LL model from real datasets.

Table 2. Numerical values of the Ex-LL distribution for the parameters $\alpha = 0.25,\ \beta = 0.75,\ \lambda = 0.5$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	0.45544	0.82322	0.42882	0.20092	0.57817	0.20749	0.05807	0.38638	0.43397	0.40184	0.77090	0.82998
	60	0.49767	0.85403	0.27882	0.13519	0.46744	0.05623	0.02582	0.28113	0.01014	0.27039	0.62326	0.22491
	100	0.50470	0.84475	0.26361	0.10901	0.38399	0.03613	0.01709	0.20715	0.00250	0.21801	0.51198	0.14453
	200	0.50599	0.82369	0.25548	0.08150	0.30228	0.02438	0.00984	0.14093	0.00102	0.16300	0.40304	0.09753
	400	0.50411	0.79550	0.25295	0.06234	0.22406	0.01847	0.00601	0.08300	0.00056	0.12469	0.29875	0.07387
ADEs	20	0.47006	0.85047	0.36550	0.17961	0.55206	0.15321	0.04672	0.36312	0.32044	0.35923	0.73609	0.61282
	60	0.49728	0.85796	0.27238	0.13283	0.45931	0.05222	0.02469	0.27227	0.00536	0.26565	0.61241	0.20888
	100	0.49794	0.83269	0.26512	0.11280	0.39868	0.03958	0.01786	0.22022	0.00306	0.22561	0.53157	0.15832
	200	0.50683	0.83278	0.25593	0.08694	0.32444	0.02628	0.01133	0.15956	0.00116	0.17388	0.43258	0.10510
	400	0.51133	0.81706	0.25103	0.06748	0.24631	0.01921	0.00692	0.09611	0.00059	0.13496	0.32841	0.07682
CVMEs	20	0.43719	0.83250	0.45411	0.20364	0.59180	0.23453	0.05934	0.39988	0.36734	0.40729	0.78907	0.93813
	60	0.47816	0.84229	0.29655	0.14841	0.50060	0.07414	0.03061	0.30716	0.01561	0.29681	0.66747	0.29658
	100	0.49590	0.84021	0.26977	0.12305	0.43532	0.04692	0.02087	0.25272	0.00445	0.24611	0.58043	0.18766
	200	0.49769	0.80971	0.26128	0.10032	0.35700	0.03228	0.01407	0.18118	0.00185	0.20063	0.47601	0.12913
	400	0.50719	0.80858	0.25299	0.07733	0.28006	0.02237	0.00881	0.12201	0.00081	0.15465	0.37341	0.08949

Table 2. *Cont.*

Method	n	AVEs			$\lvert$BIAS$\rvert$			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
LSEs	20	0.45847	0.82578	0.37292	0.20637	0.58905	0.17174	0.05844	0.39551	0.16133	0.41274	0.78540	0.68696
	60	0.48678	0.81618	0.28039	0.15045	0.48006	0.06498	0.03125	0.29138	0.01055	0.30091	0.64008	0.25991
	100	0.50286	0.84874	0.26454	0.12220	0.43261	0.04449	0.02082	0.24967	0.00403	0.24440	0.57681	0.17796
	200	0.50544	0.82622	0.25790	0.09767	0.34795	0.03139	0.01383	0.17468	0.00174	0.19534	0.46394	0.12557
	400	0.50557	0.80299	0.25280	0.07308	0.26556	0.02106	0.00828	0.11398	0.00075	0.14616	0.35407	0.08425
WLSEs	20	0.42128	0.62919	0.37346	0.16618	0.36769	0.16154	0.04217	0.17687	0.20431	0.33236	0.49026	0.64616
	60	0.46021	0.68579	0.28047	0.10123	0.30033	0.05367	0.01668	0.11734	0.00909	0.20246	0.40044	0.21467
	100	0.47383	0.69804	0.26572	0.08736	0.27135	0.03702	0.01195	0.09499	0.00276	0.17472	0.36180	0.14808
	200	0.48150	0.71748	0.25940	0.06735	0.22310	0.02559	0.00691	0.06495	0.00118	0.13469	0.29747	0.10237
	400	0.48988	0.73140	0.25519	0.05457	0.18547	0.01790	0.00428	0.04574	0.00054	0.10915	0.24729	0.07159

Table 3. Numerical values of the Ex-LL distribution for the parameters $\alpha = 1.5,\ \beta = 0.5,\ \lambda = 0.75$.

Method	n	AVEs			$\lvert$BIAS$\rvert$			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	1.41153	0.52791	0.92737	0.52999	0.37135	0.23962	0.37676	0.16123	0.13372	0.35333	0.74271	0.31950
	60	1.48145	0.55751	0.81153	0.40645	0.34424	0.11364	0.21149	0.14125	0.02864	0.27097	0.68848	0.15152
	100	1.50103	0.56398	0.78553	0.35953	0.31680	0.07982	0.16758	0.12507	0.01260	0.23969	0.63360	0.10643
	200	1.50431	0.54874	0.77072	0.29413	0.27135	0.05785	0.11529	0.09963	0.00596	0.19609	0.54270	0.07713
	400	1.54652	0.58012	0.75478	0.24613	0.23991	0.04102	0.08372	0.08260	0.00276	0.16409	0.47981	0.05469
ADEs	20	1.31110	0.47318	0.90698	0.51427	0.37389	0.23058	0.36618	0.16295	0.11341	0.34285	0.74778	0.30744
	60	1.44458	0.54431	0.80616	0.41158	0.34272	0.11176	0.22592	0.14116	0.02706	0.27439	0.68544	0.14901
	100	1.46757	0.54777	0.78539	0.37295	0.32857	0.08549	0.17888	0.13098	0.01374	0.24864	0.65714	0.11399
	200	1.49099	0.54753	0.76732	0.31129	0.28337	0.05973	0.12804	0.10596	0.00605	0.20752	0.56674	0.07964
	400	1.51391	0.55180	0.75876	0.25225	0.23914	0.04359	0.08940	0.08230	0.00308	0.16817	0.47827	0.05812
CVMEs	20	1.27258	0.47091	0.97473	0.56918	0.39746	0.29462	0.44307	0.17760	0.19027	0.37945	0.79492	0.39282
	60	1.44240	0.55826	0.81707	0.44389	0.36743	0.12505	0.25899	0.15689	0.03341	0.29593	0.73487	0.16674
	100	1.45890	0.55626	0.80078	0.41612	0.35277	0.10053	0.22259	0.14669	0.02185	0.27741	0.70554	0.13404
	200	1.47705	0.55032	0.77750	0.36031	0.31680	0.06889	0.16615	0.12463	0.00886	0.24021	0.63360	0.09186
	400	1.49539	0.54816	0.76766	0.30443	0.27381	0.05074	0.12364	0.10077	0.00463	0.20295	0.54763	0.06766
LSEs	20	1.25855	0.48920	0.90873	0.55036	0.40474	0.25849	0.42880	0.18298	0.14353	0.36691	0.80948	0.34465
	60	1.40272	0.54430	0.81088	0.45438	0.37463	0.13135	0.27403	0.16106	0.03822	0.30292	0.74925	0.17513
	100	1.42517	0.53875	0.79324	0.41560	0.35221	0.09948	0.22450	0.14637	0.02011	0.27707	0.70442	0.13265
	200	1.46760	0.55255	0.77457	0.36010	0.31955	0.07163	0.16870	0.12682	0.00979	0.24007	0.63909	0.09550
	400	1.50127	0.55580	0.76257	0.30360	0.27733	0.04843	0.12120	0.10284	0.00414	0.20240	0.55466	0.06457
WLSEs	20	1.28169	0.48653	0.88215	0.54249	0.39246	0.22283	0.40396	0.17376	0.11742	0.36166	0.78492	0.29711
	60	1.37571	0.50967	0.80492	0.43880	0.35596	0.11527	0.25421	0.14886	0.02946	0.29254	0.71192	0.15369
	100	1.46830	0.55319	0.77985	0.37477	0.32993	0.08329	0.18007	0.13256	0.01314	0.24985	0.65986	0.11105
	200	1.49321	0.55431	0.76960	0.31853	0.29148	0.06066	0.13340	0.10988	0.00663	0.21235	0.58295	0.08088
	400	1.51673	0.55744	0.75932	0.26604	0.24956	0.04460	0.09754	0.08778	0.00325	0.17736	0.49912	0.05946

Table 4. Numerical values of the Ex-LL distribution for the parameters $\alpha = 2,\ \beta = 1.5,\ \lambda = 0.25$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
	20	1.85314	1.48999	0.31158	0.73785	1.02471	0.08048	0.74610	1.15826	0.01736	0.36893	0.68314	0.32193
	60	1.90010	1.51124	0.27375	0.57038	0.91181	0.03731	0.43609	0.95168	0.00294	0.28519	0.60787	0.14923
MLEs	100	1.92990	1.52921	0.26407	0.49315	0.84142	0.02720	0.32231	0.82352	0.00139	0.24658	0.56095	0.10879
	200	2.00622	1.61720	0.25649	0.39765	0.72007	0.01727	0.21150	0.64493	0.00056	0.19883	0.48004	0.06909
	400	1.99447	1.56795	0.25460	0.33997	0.63142	0.01338	0.15306	0.51363	0.00030	0.16998	0.42095	0.05354
	20	1.72200	1.38696	0.29740	0.69804	1.02208	0.07323	0.70656	1.16539	0.01284	0.34902	0.68138	0.29294
	60	1.82869	1.44691	0.26858	0.56224	0.90890	0.03673	0.43659	0.94701	0.00280	0.28112	0.60594	0.14693
ADEs	100	1.87063	1.45446	0.26288	0.50596	0.83793	0.02689	0.34733	0.82977	0.00141	0.25298	0.55862	0.10754
	200	1.91945	1.49514	0.25741	0.42940	0.76274	0.01887	0.24265	0.69713	0.00063	0.21470	0.50850	0.07549
	400	1.95698	1.51378	0.25461	0.36280	0.65160	0.01326	0.17309	0.53883	0.00029	0.18140	0.43440	0.05304
	20	1.61667	1.27273	0.33639	0.81359	1.07118	0.10798	0.93812	1.29057	0.02862	0.40679	0.71412	0.43192
	60	1.78705	1.42209	0.28117	0.66140	1.00275	0.04712	0.59267	1.11512	0.00494	0.33070	0.66850	0.18848
CVMEs	100	1.84697	1.46006	0.26819	0.57122	0.92311	0.03330	0.44008	0.96553	0.00237	0.28561	0.61541	0.13318
	200	1.90668	1.51328	0.25988	0.48834	0.83406	0.02229	0.31400	0.81293	0.00094	0.24417	0.55604	0.08916
	400	1.93295	1.50741	0.25568	0.41197	0.73034	0.01572	0.22306	0.64811	0.00042	0.20599	0.48690	0.06286
	20	1.60099	1.32167	0.30946	0.76927	1.05451	0.09292	0.89264	1.25585	0.02188	0.38464	0.70301	0.37167
	60	1.78712	1.44056	0.27176	0.63829	0.99005	0.04311	0.55917	1.08696	0.00395	0.31915	0.66003	0.17243
LSEs	100	1.83845	1.46657	0.26526	0.56044	0.92200	0.03126	0.42632	0.96197	0.00209	0.28022	0.61467	0.12505
	200	1.87112	1.47376	0.26070	0.50198	0.84759	0.02308	0.33393	0.84025	0.00106	0.25099	0.56506	0.09230
	400	1.95600	1.54985	0.25485	0.41293	0.73354	0.01568	0.22228	0.65567	0.00043	0.20647	0.48902	0.06273
	20	1.62026	1.33165	0.30114	0.74460	1.06107	0.08086	0.82260	1.24371	0.01652	0.37230	0.70738	0.32345
	60	1.79192	1.41344	0.26863	0.60238	0.95058	0.03833	0.49514	1.01211	0.00307	0.30119	0.63372	0.15332
WLSEs	100	1.86971	1.47801	0.26166	0.51454	0.86538	0.02850	0.36243	0.87311	0.00161	0.25727	0.57692	0.11401
	200	1.93741	1.54087	0.25636	0.44350	0.78729	0.01905	0.25577	0.73088	0.00065	0.22175	0.52486	0.07619
	400	1.97894	1.56557	0.25381	0.35977	0.65664	0.01394	0.17146	0.55077	0.00033	0.17988	0.43776	0.05575

Table 5. Numerical values of the Ex-LL distribution for the parameters $\alpha = 2.5,\ \beta = 3,\ \lambda = 1.5$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
	20	2.24560	2.64903	1.86198	0.77288	1.47282	0.46844	0.98041	2.89801	0.57068	0.30915	0.49094	0.31230
	60	2.25729	2.63147	1.61863	0.59737	1.36952	0.19191	0.56221	2.38995	0.07931	0.23895	0.45651	0.12794
MLEs	100	2.33810	2.74689	1.57806	0.52603	1.26338	0.14449	0.41949	2.00936	0.04052	0.21041	0.42113	0.09632
	200	2.42485	2.93535	1.53902	0.41483	1.08489	0.09320	0.25331	1.46911	0.01563	0.16593	0.36163	0.06213
	400	2.42592	2.89977	1.52715	0.36439	0.97387	0.07037	0.18736	1.20044	0.00867	0.14576	0.32462	0.04691
	20	1.96643	2.22208	1.81809	0.86504	1.67317	0.46106	1.17772	3.60422	0.53668	0.34601	0.55772	0.30737
	60	2.15967	2.49301	1.63461	0.64438	1.45352	0.22280	0.67819	2.70468	0.10543	0.25775	0.48451	0.14853
ADEs	100	2.23278	2.60175	1.58387	0.56446	1.33984	0.15538	0.51867	2.32278	0.04837	0.22578	0.44661	0.10358
	200	2.31263	2.72185	1.54640	0.47481	1.18999	0.10990	0.35107	1.82351	0.02218	0.18992	0.39666	0.07327
	400	2.36909	2.79513	1.52576	0.39547	1.03967	0.07402	0.23015	1.38118	0.00967	0.15819	0.34656	0.04935
	20	1.91910	2.21165	2.01548	0.93924	1.62648	0.62234	1.44421	3.73631	0.95156	0.37570	0.54216	0.41489
	60	2.05856	2.35779	1.69367	0.72607	1.54565	0.27475	0.88149	3.14012	0.19211	0.29043	0.51522	0.18317
CVMEs	100	2.14048	2.44608	1.62859	0.66368	1.48803	0.20009	0.70178	2.82250	0.08101	0.26547	0.49601	0.13339
	200	2.27589	2.66818	1.56418	0.53037	1.27897	0.12895	0.44065	2.09930	0.03015	0.21215	0.42632	0.08596
	400	2.35226	2.79380	1.53631	0.44741	1.14970	0.08734	0.29189	1.64528	0.01335	0.17896	0.38323	0.05823

Table 5. *Cont.*

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
LSEs	20	1.85172	2.18912	1.86952	0.91448	1.60553	0.56718	1.42335	3.73097	0.78003	0.36579	0.53518	0.37812
	60	2.06210	2.40349	1.64521	0.72344	1.52299	0.26579	0.88558	3.08153	0.15334	0.28938	0.50766	0.17719
	100	2.15266	2.49861	1.59454	0.63051	1.45032	0.18940	0.64817	2.69959	0.08013	0.25221	0.48344	0.12627
	200	2.21830	2.56388	1.56615	0.56182	1.33145	0.13133	0.49994	2.28895	0.03368	0.22473	0.44382	0.08755
	400	2.31187	2.70032	1.53389	0.45719	1.15281	0.08493	0.31498	1.70177	0.01303	0.18288	0.38427	0.05662
WLSEs	20	1.87451	2.17124	1.84122	0.93047	1.74185	0.49880	1.37506	3.87423	0.68065	0.37219	0.58062	0.33253
	60	2.09905	2.38931	1.63869	0.69261	1.52041	0.23253	0.77223	2.96559	0.12164	0.27704	0.50680	0.15502
	100	2.19562	2.52031	1.57188	0.59081	1.38196	0.15574	0.54307	2.40593	0.04481	0.23633	0.46065	0.10383
	200	2.28638	2.65047	1.55164	0.49905	1.24121	0.11369	0.37638	1.94273	0.02405	0.19962	0.41374	0.07579
	400	2.37214	2.79379	1.52799	0.39952	1.05013	0.07544	0.23121	1.39508	0.00997	0.15981	0.35004	0.05030

Table 6. Numerical values of the Ex-LL distribution for the parameters $\alpha = 0.5,\ \beta = 2,\ \lambda = 3$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	0.50243	2.24341	3.28224	0.13748	0.94306	0.67856	0.02881	1.02669	0.58293	0.27496	0.47153	0.22619
	60	0.48696	2.11286	3.21149	0.10820	0.78843	0.50011	0.01652	0.76828	0.36328	0.21640	0.39422	0.16670
	100	0.49448	2.07243	3.14715	0.09661	0.68849	0.41255	0.01330	0.61490	0.26279	0.19322	0.34424	0.13752
	200	0.49851	2.07641	3.07894	0.07651	0.57391	0.30177	0.00849	0.45547	0.14912	0.15302	0.28695	0.10059
	400	0.50212	2.07711	3.04037	0.05995	0.47700	0.21511	0.00547	0.33087	0.07603	0.11990	0.23850	0.07170
ADEs	20	0.44274	2.01193	4.20196	0.16895	1.00339	1.60789	0.04291	1.18999	15.31004	0.33791	0.50169	0.53596
	60	0.48139	2.05736	3.27801	0.11600	0.80984	0.60048	0.02003	0.80657	0.78694	0.23201	0.40492	0.20016
	100	0.49631	2.11680	3.15742	0.09698	0.73466	0.43798	0.01399	0.67913	0.38749	0.19396	0.36733	0.14599
	200	0.49408	2.06002	3.09391	0.07580	0.60346	0.30384	0.00860	0.49172	0.16549	0.15160	0.30173	0.10128
	400	0.50434	2.08353	3.03178	0.06162	0.47980	0.22724	0.00570	0.33226	0.08575	0.12324	0.23990	0.07575
CVMEs	20	0.47191	2.15401	3.34405	0.14059	0.97436	0.73788	0.02910	1.09771	0.66518	0.28118	0.48718	0.24596
	60	0.48488	2.13509	3.24755	0.11422	0.83868	0.56069	0.01828	0.84367	0.42795	0.22844	0.41934	0.18690
	100	0.48909	2.07422	3.16734	0.10394	0.78330	0.44853	0.01536	0.75647	0.30584	0.20788	0.39165	0.14951
	200	0.49244	2.07016	3.11885	0.08338	0.65596	0.34937	0.01004	0.56122	0.19839	0.16676	0.32798	0.11646
	400	0.50562	2.10875	3.03711	0.06724	0.51606	0.25395	0.00668	0.37635	0.10448	0.13447	0.25803	0.08465
LSEs	20	0.48450	2.00972	3.20069	0.15733	1.01402	0.72415	0.03516	1.18199	0.65321	0.31466	0.50701	0.24138
	60	0.48641	2.00471	3.15364	0.12065	0.86235	0.54057	0.02068	0.89473	0.40918	0.24130	0.43118	0.18019
	100	0.49040	2.03666	3.13691	0.10653	0.76666	0.45869	0.01595	0.72757	0.31329	0.21305	0.38333	0.15290
	200	0.49555	2.03812	3.09514	0.08997	0.67809	0.34966	0.01139	0.58659	0.19512	0.17994	0.33905	0.11655
	400	0.50241	2.07492	3.04581	0.07053	0.53769	0.26944	0.00729	0.40341	0.11977	0.14105	0.26884	0.08981
WLSEs	20	0.48996	1.97273	3.14295	0.15384	0.97597	0.70515	0.03462	1.11621	0.62033	0.30768	0.48799	0.23505
	60	0.49369	2.08173	3.14092	0.11013	0.78946	0.49356	0.01750	0.77294	0.35424	0.22026	0.39473	0.16452
	100	0.49501	2.05356	3.09485	0.09771	0.71978	0.41366	0.01356	0.65721	0.25556	0.19543	0.35989	0.13789
	200	0.49602	2.03650	3.07105	0.07612	0.59806	0.29513	0.00837	0.47520	0.14243	0.15224	0.29903	0.09838
	400	0.50016	2.05893	3.04193	0.06049	0.47467	0.22367	0.00550	0.32886	0.08310	0.12098	0.23734	0.07456

Table 7. Numerical values of the Ex-LL distribution for the parameters $\alpha = 0.75,\ \beta = 0.25,\ \lambda = 2$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	0.83492	0.46297	2.24933	0.33119	0.37160	0.49496	0.14717	0.21310	0.36254	0.44159	1.48642	0.24748
	60	0.81841	0.40549	2.08986	0.24451	0.28605	0.32478	0.08961	0.14744	0.16990	0.32602	1.14421	0.16239
	100	0.79325	0.36502	2.09112	0.21510	0.24491	0.28637	0.07033	0.11362	0.13963	0.28680	0.97962	0.14318
	200	0.78636	0.32651	2.02668	0.15142	0.17165	0.18665	0.03876	0.06234	0.05959	0.20189	0.68661	0.09333
	400	0.76520	0.28487	2.02040	0.10656	0.11451	0.13671	0.01870	0.02552	0.03135	0.14208	0.45803	0.06836
ADEs	20	0.79889	0.43582	2.21286	0.31913	0.36183	0.50847	0.13592	0.20087	0.37146	0.42550	1.44731	0.25424
	60	0.79511	0.38007	2.10999	0.24857	0.27758	0.33594	0.09053	0.13852	0.18240	0.33143	1.11032	0.16797
	100	0.79254	0.36612	2.07709	0.21196	0.24546	0.27375	0.06934	0.11501	0.12517	0.28261	0.98184	0.13688
	200	0.78689	0.32595	2.02369	0.15182	0.17020	0.18431	0.03880	0.06073	0.05613	0.20243	0.68080	0.09215
	400	0.76688	0.28961	2.01599	0.11201	0.12121	0.14362	0.02059	0.02891	0.03319	0.14935	0.48485	0.07181
CVMEs	20	0.82472	0.46533	2.26876	0.33654	0.38787	0.54958	0.14941	0.22490	0.42798	0.44872	1.55148	0.27479
	60	0.80202	0.40289	2.15938	0.27591	0.31266	0.38080	0.10554	0.16672	0.23498	0.36788	1.25062	0.19040
	100	0.80891	0.39274	2.09950	0.25005	0.28103	0.31747	0.08981	0.13985	0.16916	0.33340	1.12413	0.15874
	200	0.79265	0.35382	2.05850	0.19409	0.21992	0.24060	0.05996	0.09739	0.09882	0.25879	0.87967	0.12030
	400	0.77411	0.30589	2.02115	0.13685	0.14992	0.16024	0.02992	0.04506	0.04103	0.18246	0.59969	0.08012
LSEs	20	0.79370	0.45012	2.16879	0.33067	0.37739	0.53675	0.14176	0.21563	0.40326	0.44089	1.50957	0.26837
	60	0.78499	0.38649	2.12663	0.27131	0.30138	0.38899	0.10137	0.15467	0.23666	0.36174	1.20554	0.19449
	100	0.80486	0.39098	2.06359	0.24347	0.27741	0.31188	0.08636	0.13776	0.15902	0.32463	1.10964	0.15594
	200	0.78174	0.33450	2.04287	0.18178	0.19988	0.22408	0.05314	0.08086	0.08415	0.24237	0.79951	0.11204
	400	0.78216	0.31606	2.01207	0.13903	0.15462	0.16389	0.03244	0.05026	0.04381	0.18537	0.61848	0.08195
WLSEs	20	0.81442	0.46710	2.16386	0.33118	0.38694	0.51276	0.14430	0.22623	0.37333	0.44158	1.54776	0.25638
	60	0.80606	0.40889	2.08138	0.26430	0.30403	0.35064	0.09941	0.15848	0.20094	0.35240	1.21612	0.17532
	100	0.79000	0.35923	2.06213	0.21368	0.24001	0.28231	0.06998	0.10946	0.13010	0.28491	0.96003	0.14115
	200	0.77623	0.32028	2.03488	0.15935	0.17806	0.20080	0.04068	0.06383	0.06655	0.21247	0.71225	0.10040
	400	0.76726	0.28926	2.01469	0.10864	0.11795	0.13961	0.01983	0.02892	0.03080	0.14485	0.47179	0.06980

Table 8. Numerical values of the Ex-LL distribution for the parameters $\alpha = 0.5,\ \beta = 2.5,\ \lambda = 0.75$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	0.46153	2.71277	0.95455	0.18905	1.33791	0.30181	0.04932	2.08963	0.15880	0.37809	0.53517	0.40241
	60	0.48363	2.64010	0.83440	0.12721	1.09299	0.16535	0.02427	1.49659	0.05432	0.25441	0.43720	0.22047
	100	0.50133	2.68608	0.78893	0.10527	0.94619	0.11357	0.01660	1.16592	0.02334	0.21054	0.37848	0.15143
	200	0.50499	2.70207	0.77399	0.08325	0.76398	0.08087	0.01013	0.82362	0.01201	0.16650	0.30559	0.10782
	400	0.50172	2.61169	0.76254	0.05946	0.57668	0.05462	0.00544	0.51692	0.00494	0.11892	0.23067	0.07282
ADEs	20	0.47205	2.76543	0.89752	0.17182	1.29764	0.26027	0.04250	1.96457	0.11997	0.34365	0.51906	0.34702
	60	0.49022	2.69752	0.81884	0.12708	1.11096	0.15657	0.02315	1.51754	0.04735	0.25416	0.44438	0.20877
	100	0.49542	2.64261	0.79174	0.10748	0.97352	0.11482	0.01683	1.23172	0.02410	0.21496	0.38941	0.15310
	200	0.50214	2.63223	0.76715	0.08471	0.79713	0.08182	0.01075	0.87469	0.01163	0.16943	0.31885	0.10910
	400	0.50391	2.60425	0.75934	0.06225	0.57941	0.05607	0.00585	0.51021	0.00508	0.12451	0.23176	0.07475
CVMEs	20	0.45366	2.78292	0.95677	0.18705	1.35660	0.30773	0.04764	2.11219	0.15895	0.37410	0.54264	0.41030
	60	0.48393	2.79942	0.84624	0.13555	1.14799	0.18161	0.02632	1.61091	0.06553	0.27109	0.45920	0.24214
	100	0.48905	2.67878	0.81234	0.11520	1.01129	0.13716	0.01938	1.30345	0.03667	0.23040	0.40451	0.18289
	200	0.49558	2.64128	0.78536	0.09394	0.85106	0.09981	0.01302	1.00094	0.01855	0.18787	0.34043	0.13308
	400	0.50474	2.63547	0.76320	0.07443	0.68258	0.06820	0.00830	0.68628	0.00778	0.14887	0.27303	0.09093

Table 8. *Cont.*

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
LSEs	20	0.46485	2.48793	0.89969	0.20045	1.40815	0.29410	0.05378	2.29446	0.14425	0.40089	0.56326	0.39213
	60	0.47964	2.60632	0.83485	0.14323	1.17969	0.18224	0.02876	1.67940	0.06566	0.28646	0.47187	0.24298
	100	0.48933	2.61250	0.80603	0.12078	1.04900	0.13704	0.02053	1.37201	0.03665	0.24157	0.41960	0.18272
	200	0.50076	2.62396	0.77556	0.09639	0.85770	0.09788	0.01346	0.97896	0.01749	0.19278	0.34308	0.13050
	400	0.50605	2.63179	0.75787	0.07262	0.68215	0.06507	0.00786	0.68552	0.00676	0.14523	0.27286	0.08676
WLSEs	20	0.48081	2.62103	0.87745	0.19272	1.35298	0.27448	0.05168	2.13699	0.13228	0.38545	0.54119	0.36598
	60	0.48762	2.61203	0.81217	0.12827	1.10342	0.15156	0.02411	1.51895	0.04482	0.25654	0.44137	0.20208
	100	0.50143	2.68694	0.78520	0.10823	0.99737	0.11820	0.01709	1.26456	0.02637	0.21646	0.39895	0.15760
	200	0.50581	2.69735	0.76913	0.08709	0.82057	0.08384	0.01109	0.91942	0.01160	0.17419	0.32823	0.11178
	400	0.50639	2.60810	0.75496	0.06065	0.58635	0.05395	0.00571	0.52327	0.00462	0.12131	0.23454	0.07194

Table 9. Numerical values of the Ex-LL distribution for the parameters $\alpha = 2,\ \beta = 0.75,\ \lambda = 1.5$.

Method	n	AVEs			|BIAS|			MSEs			MREs		
		α	β	λ	α	β	λ	α	β	λ	α	β	λ
MLEs	20	2.02056	0.87123	1.75268	0.75361	0.60040	0.37441	0.72176	0.40677	0.24670	0.37680	0.80053	0.24961
	60	2.04596	0.90606	1.58218	0.59371	0.55020	0.18982	0.45335	0.35824	0.06828	0.29685	0.73360	0.12655
	100	2.03551	0.88457	1.56159	0.53118	0.50936	0.14718	0.36724	0.31873	0.04348	0.26559	0.67914	0.09812
	200	2.03433	0.85937	1.53411	0.45860	0.45067	0.10724	0.27391	0.26548	0.02019	0.22930	0.60089	0.07150
	400	2.05591	0.86550	1.51165	0.36412	0.37504	0.07650	0.18431	0.20053	0.00972	0.18206	0.50006	0.05100
ADEs	20	1.59765	0.53685	1.76036	0.66074	0.43279	0.38211	0.66528	0.23666	0.25494	0.33037	0.57706	0.25474
	60	1.73484	0.61443	1.61213	0.48879	0.36901	0.20326	0.39090	0.17490	0.08294	0.24440	0.49202	0.13550
	100	1.80913	0.65080	1.57623	0.40531	0.33106	0.15308	0.27071	0.14146	0.04761	0.20265	0.44141	0.10205
	200	1.86591	0.67683	1.54618	0.33465	0.29295	0.10695	0.17778	0.11082	0.02107	0.16732	0.39059	0.07130
	400	1.90085	0.69379	1.53028	0.28738	0.26020	0.07660	0.12418	0.08740	0.01073	0.14369	0.34693	0.05107
CVMEs	20	1.60021	0.53982	1.84033	0.71141	0.43441	0.45749	0.76888	0.24432	0.33633	0.35571	0.57922	0.30499
	60	1.70061	0.59676	1.69293	0.56319	0.39183	0.27727	0.51422	0.19922	0.15218	0.28160	0.52243	0.18485
	100	1.76260	0.62855	1.61511	0.47324	0.36525	0.19052	0.36568	0.17017	0.07308	0.23662	0.48700	0.12701
	200	1.79483	0.64084	1.58012	0.41277	0.33641	0.13674	0.27498	0.14544	0.03813	0.20639	0.44855	0.09116
	400	1.87348	0.68178	1.53910	0.32180	0.28266	0.08595	0.16065	0.10293	0.01438	0.16090	0.37688	0.05730
LSEs	20	1.58241	0.58488	1.69564	0.65138	0.41516	0.37637	0.68332	0.22305	0.24430	0.32569	0.55354	0.25091
	60	1.68550	0.60271	1.63653	0.53091	0.38273	0.23533	0.47255	0.18963	0.11102	0.26546	0.51031	0.15689
	100	1.74643	0.62611	1.59427	0.47012	0.36568	0.18290	0.36683	0.16959	0.06656	0.23506	0.48758	0.12193
	200	1.80075	0.64544	1.56000	0.39626	0.32886	0.12646	0.25467	0.13800	0.03148	0.19813	0.43848	0.08431
	400	1.88573	0.69750	1.53286	0.32604	0.28731	0.08698	0.16230	0.10455	0.01368	0.16302	0.38308	0.05799
WLSEs	20	1.61320	0.58453	1.69765	0.64398	0.41260	0.36127	0.66479	0.21989	0.23351	0.32199	0.55013	0.24085
	60	1.73319	0.61795	1.61011	0.48655	0.36708	0.20733	0.39413	0.17357	0.08445	0.24327	0.48944	0.13822
	100	1.75203	0.61316	1.59444	0.44046	0.34836	0.16668	0.31873	0.15808	0.05398	0.22023	0.46448	0.11112
	200	1.86835	0.67969	1.55063	0.34866	0.30253	0.10877	0.18761	0.11574	0.02104	0.17433	0.40338	0.07251
	400	1.92492	0.71359	1.52427	0.27331	0.25280	0.07229	0.11114	0.08122	0.00918	0.13665	0.33707	0.04819

6. Actuarial Measures

In this section, we discuss the mathematical and computational aspects of five important actuarial measures in portfolio optimization under uncertainty namely VaR, TVaR, TV, TVP, and ES for the Ex-LL distribution.

6.1. VaR Measure

The VaR has some other names such as the quantile premium principle or quantile risk measure, and it can be specified with a typical degree of confidence q, where $q = 90\%, 95\%$ or 99%).

The VaR, say VaR_q, of a random variable X is the qth QF of its CDF, i.e., $VaR_q = Q(q)$ (Artzner [25]).

The VaR of the Ex-LL model is derived as

$$VaR_q = \left\{ \frac{\lambda\left[(1-q)^{-1/\alpha} - 1\right]}{\alpha} \right\}^{1/\beta}.$$

6.2. TVaR and TV Measures

The TVaR quantifies the expected value of loss given that an event outside a certain probability level has occurred. The TVaR is defined by the formula

$$TVaR_q = \frac{1}{(1-q)} \int_{VaR_q}^{\infty} x\, f(x)\, dx.$$

The TVaR of the Ex-LL distribution has the form

$$TVaR_q = \frac{\alpha^{1-\alpha}\beta\lambda^{\alpha} VaR_q^{1-\alpha\beta}\, {}_2F_1\left(\alpha+1, \alpha-\frac{1}{\beta}; \alpha-\frac{1}{\beta}+1; -\frac{VaR_q^{-\beta}\lambda}{\alpha}\right)}{(\alpha\beta-1)(1-q)}, \tag{12}$$

where ${}_2F_1\left(\alpha+1, \alpha-\frac{1}{\beta}; \alpha-\frac{1}{\beta}+1; -\frac{VaR_q^{-\beta}\lambda}{\alpha}\right)$ is the hyper geometric function.

The TV measure (Landsman [26]) represents the variance of a loss distribution beyond a particular critical value, and it pays attention to the TV beyond VaR. The TV of the Ex-LL distribution is defined as

$$TV_q(X) = E(X^2|X > x_q) - (TVaR_q)^2 = \frac{1}{(1-q)} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2, \tag{13}$$

where

$$E(X^2|X > x_q) = \frac{\alpha^{1-\alpha}\beta\lambda^{\alpha} VaR_q^{2-\alpha\beta}\, {}_2F_1\left(\alpha+1, \alpha-\frac{2}{\beta}; \alpha-\frac{2}{\beta}+1; -\frac{VaR_q^{-\beta}\lambda}{\alpha}\right)}{(\alpha\beta-2)(1-q)}. \tag{14}$$

Using Equation (12)–(14), we obtain the TV of the Ex-LL distribution.

6.3. TVP and ES Measures

The TVP plays an important role in insurance field. The TVP of the Ex-LL distribution is defined by

$$TVP_q(x) = TVaR_q + \lambda TV_q, \tag{15}$$

where $0 < \lambda < 1$. The TVP of the Ex-LL distribution follows by substituting the expressions (12) and (13) in Equation (15).

Artzner [25] introduced another important measure of financial risk called the ES. The ES of the Ex-LL distribution takes the form

$$ES_q(x) = \frac{1}{q} \int_0^q VaR_t\, dt = \frac{(\frac{\lambda}{\alpha})^{1/\beta}}{q} \int_0^q \left[(1-t)^{-1/\alpha} - 1\right]^{1/\beta} dt. \tag{16}$$

6.4. Numerical Computations for Actuarial Measures

In this section, we presented numerical simulations for the studied risk measures, VaR, TVaR, TV, TVP, and ES, of the Ex-LL, ELL, and LL distributions. We generated a random sample, $n = 100$, from the Ex-LL, ELL, and LL distributions, and the parameters were estimated by the ML method. The results were obtained after 1000 repetitions to calculate the five risk measures. The numerical results of these measures were reported in Tables 10 and 11 for the Ex-LL, ELL, and LL distributions. For visual comparisons, we displayed the results, in Tables 10 and 11, graphically as shown in Figures 3 and 4.

The values and plots in Tables 10 and 11 and Figures 3 and 4 reveal that the introduced Ex-LL model has a heavier tail than the tails of the ELL and LL distributions. Hence, it may be adopted to model heavy-tailed datasets.

Table 10. Simulation results of the five risk measures for the Ex-LL, ELL, and LL distributions.

Distribution	Parameters	Significance Level	VaR	TVaR	TV	TVP	ES
Ex-LL	$\alpha = 0.9$, $\lambda = 0.5$, $\beta = 2.5$	0.60	0.99108	2.01266	17.07036	12.25487	0.58926
		0.65	1.08279	2.15221	19.34957	14.72942	0.62361
		0.70	1.19084	2.32171	22.36834	17.97955	0.66015
		0.75	1.32315	2.53515	26.56123	22.45607	0.69977
		0.80	1.49396	2.81781	32.79004	29.04984	0.74382
		0.85	1.73244	3.22189	43.04476	39.80993	0.79451
		0.90	2.11268	3.88033	63.21386	60.77280	0.85620
		0.95	2.92095	5.30891	122.11669	121.31977	0.94018
LL	$\lambda = 0.5$, $\beta = 2.5$	0.60	0.89125	1.70053	3.51346	3.80861	0.53540
		0.65	0.97023	1.81064	3.91554	4.35574	0.56574
		0.70	1.06244	1.94324	4.44114	5.05204	0.59783
		0.75	1.17421	2.10862	5.15965	5.97836	0.63240
		0.80	1.31676	2.32522	6.20622	7.29019	0.67052
		0.85	1.51287	2.63068	7.88611	9.33387	0.71395
		0.90	1.81955	3.11972	11.07641	13.08849	0.76609
		0.95	2.45266	4.15248	19.87667	23.03531	0.83561
ELL	$\lambda = 0.5$, $\beta = 2.5$, $c = 0.75$	0.60	0.76627	1.49988	4.52012	4.21195	0.43416
		0.65	0.83911	1.59959	5.08234	4.90311	0.46245
		0.70	0.92366	1.71949	5.82319	5.79572	0.49229
		0.75	1.02554	1.86882	6.84576	7.00314	0.52433
		0.80	1.15475	2.06421	8.35301	8.74662	0.55951
		0.85	1.33160	2.33966	10.80959	11.52781	0.59941
		0.90	1.60692	2.78093	15.57473	16.79819	0.64706
		0.95	2.17369	3.71548	29.18379	31.44008	0.71019

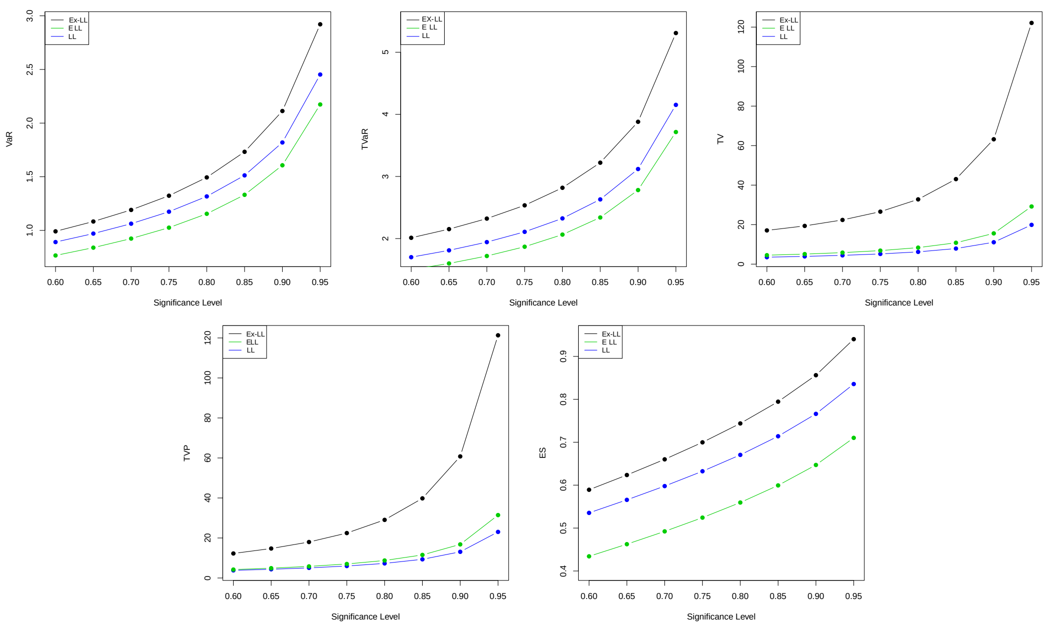

Figure 3. Shapes of the VaR, TVaR, TV, TVP, and ES using the numerical values in Table 10.

Table 11. Simulation results of the five risk measures for the Ex-LL, ELL, and LL distributions.

Distribution	Parameters	Significance Level	VaR	TVaR	TV	TVP	ES
Ex-LL	$\alpha = 0.5,\ \lambda = 1.5,\ \beta = 5$	0.60	1.73116	2.94389	10.75330	9.39587	1.23942
		0.65	1.84031	3.10954	12.06103	10.94921	1.28134
		0.70	1.96950	3.31065	13.77559	12.95357	1.32574
		0.75	2.12859	3.56358	16.12836	15.65985	1.37375
		0.80	2.33518	3.89777	19.57230	19.55561	1.42707
		0.85	2.62516	4.37342	25.13698	25.73985	1.48847
		0.90	3.08890	5.14234	35.80597	37.36772	1.56331
		0.95	4.07034	6.78324	65.71723	69.21461	1.66531
LL	$\lambda = 1.5,\ \beta = 5$	0.60	1.17575	1.56193	0.20470	1.68475	0.89085
		0.65	1.22690	1.61351	0.21234	1.75153	0.91470
		0.70	1.28405	1.67328	0.22233	1.82891	0.93900
		0.75	1.35008	1.74470	0.23566	1.92145	0.96415
		0.80	1.42987	1.83374	0.25417	2.03708	0.99067
		0.85	1.53286	1.95205	0.28165	2.19146	1.01939
		0.90	1.68128	2.12727	0.32773	2.42222	1.05173
		0.95	1.95220	2.45547	0.42938	2.86338	1.09106
ELL	$\lambda = 1.5,\ \beta = 5,\ c = 1.5$	0.60	1.54762	2.55035	3.77461	4.81512	1.05488
		0.65	1.65298	2.68624	4.15888	5.38951	1.09674
		0.70	1.77511	2.84855	4.65763	6.10890	1.14072
		0.75	1.92191	3.04906	5.33340	7.04911	1.18774
		0.80	2.10724	3.30870	6.30683	8.35416	1.23916
		0.85	2.35894	3.66969	7.84720	10.33981	1.29719
		0.90	2.74555	4.23665	10.71635	13.88136	1.36600
		0.95	3.52044	5.39809	18.39376	22.87216	1.45596

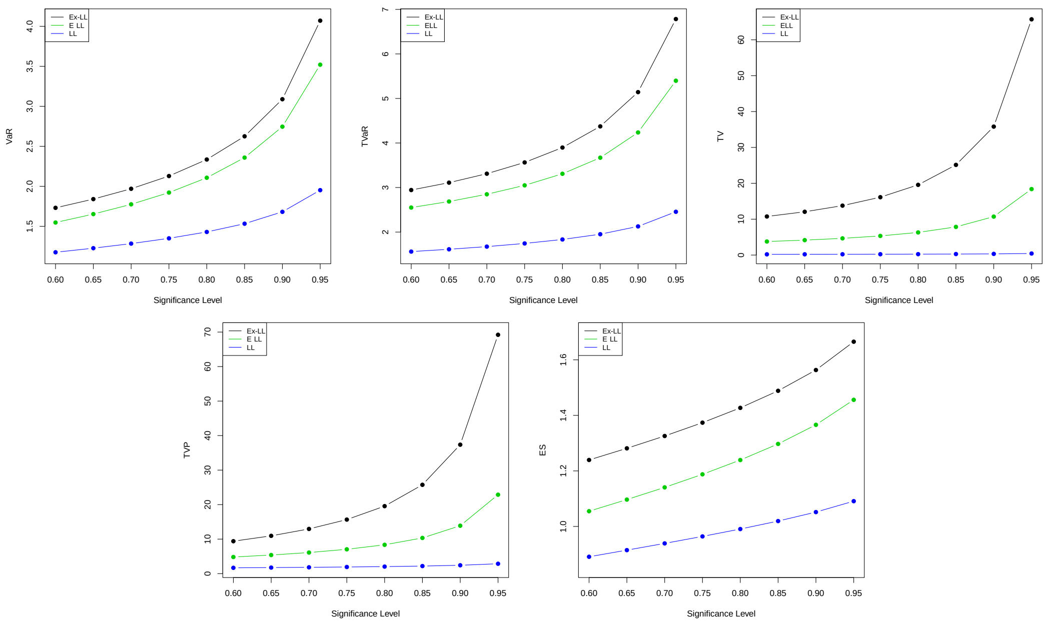

Figure 4. Shapes of the VaR, TVaR, TV, TVP, and ES using the numerical values in Table 11.

7. Modeling Real Data from the Engineering and Insurance Fields

In this section, we analyzed two real datasets from the engineering and insurance fields to explore the usefulness of the Ex-LL distribution. The first data was studied by [27]. It contains 74 observations and it refers to gauge lengths of 20 mm. This data was analyzed by [28,29].

The second data set refers to losses from private passenger automobile insurance policies in United Kingdom. It consists of 32 observations and 4 variables. We particularly analyzed the variable number 4 which represents number of claims. These data is available on R©software library.

The two data sets are displayed in Tables 12 and 13.

Table 12. The observations of gauge lengths of 20 mm data.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.426	2.434	2.435	2.382	2.478	2.554	2.514	2.511	2.490	2.535
2.566	2.570	2.586	2.629	2.800	2.773	2.770	2.809	3.585	2.818
2.642	2.726	2.697	2.684	2.648	2.633	3.128	3.090	3.096	3.233
2.821	2.880	2.848	2.818	3.067	2.821	2.954	2.809	3.585	3.084
3.012	2.880	2.848	3.433						

Table 13. The losses from private passenger automobile insurance policies data.

21	40	23	5	63	171	92	44	140	343
318	129	123	448	361	169	151	479	381	166
245	970	719	304	266	859	504	162	260	578
312	96								

The Ex-LL distribution is compared with some competing distributions including the alpha power log-logistic (APLL) [30], transmuted log-logistic (TLL) [31], generalized log-logistic (GLL) [32], Marshall–Olkin log-logistic (MOLL) [13], Poisson Burr-X log-logistic (PBXLL) [33], transmuted inverse log-logistic (TILL) [34], inverse log-logistic (ILL) [34], Weibull generalized log-logistic (WGLL) [35], and LL distributions.

The competing distributions are checked using some goodness-of-fit measures including Anderson–Darling (AD), Cramér–von Mises (CM), and Kolmogorov–Smirnov (KS) with its p-value (KS-p-value).

The parameters of the competing models are estimated via the ML method. The estimates and analytical measures are obtained using the Mathematica program version 12.0. Tables 14 and 15 provide the analytical measures along with the ML estimates and their standard errors (SEs) in parenthesis. The fitted PDF, CDF, SF, and P-P plots of the Ex-LL model for the two data sets are shown in Figures 5 and 6, respectively. The results in Table 14 and 15 indicate that the Ex-LL distribution provides adequate fit than other competing models for the two datasets.

Table 14. Measures of goodness-of-fit and estimates of the Ex-LL distribution and other distributions for data set I.

Model	AD	CM	KS	KS-p-Value	Estimates (SEs)
Ex-LL	0.18708	0.02454	0.05307	0.98524	$\hat{\alpha} = 5.35622\ (6.59867)$ $\hat{\lambda} = 14044.4\ (27477.7)$ $\hat{\beta} = 6.44473\ (0.98951)$

Table 14. *Cont.*

Model	AD	CM	KS	KS-p-Value	Estimates (SEs)
LL	0.56027	0.06756	0.05931	0.95704	$\hat{\lambda} = 2407.73$ (1907.88) $\hat{\beta} = 8.63147$ (0.83741)
APLL	0.24447	0.03337	0.05432	0.98112	$\hat{\alpha} = 0.0025$ (0.0054) $\hat{a} = 7.00987$ (0.5545) $\hat{b} = 3.32516$ (0.22732)
TLL	0.28121	0.03938	0.05808	0.96415	$\hat{\alpha} = 2.79405$ (0.12052) $\hat{\beta} = 7.39915$ (0.27229) $\hat{\lambda} = 1.0000$ (0.37615)
GLL	28.2565	6.1181	0.52316	0.0000	$\hat{\alpha} = 0.41192$ (0.10455) $\hat{\beta} = 1.00000$ (0.11506) $\hat{\theta} = 0.51734$ (0.08067)
MOLL	0.56027	0.06756	0.05931	0.95704	$\hat{c} = 0.68853$ (747.74) $\hat{\lambda} = 2.57367$ (323.811) $\hat{\beta} = 8.63147$ (0.83741)
PBXLL	1.10682	0.17631	0.09159	0.56396	$\hat{c} = 24.5304$ (4.10689) $\hat{\lambda} = 5.84672 \times 10^{-8}$ (0.00018) $\hat{\beta} = 0.72432$ (0.02922)
TILL	29.9400	6.55966	0.51579	0.0000	$\hat{\lambda} = -1.00000$ (0.29515) $\hat{\alpha} = 1.96889$ (0.16327)
ILL	54.7458	11.5014	0.66589	0.0000	$\hat{\alpha} = 1.71725$ (0.15506)
WGLL	0.21443	0.02648	0.05962	0.95509	$\hat{c} = 0.28789$ (0.23415) $\hat{\lambda} = 2.36831$ (2.22062) $\hat{\beta} = 4.55449$ (1.61755)

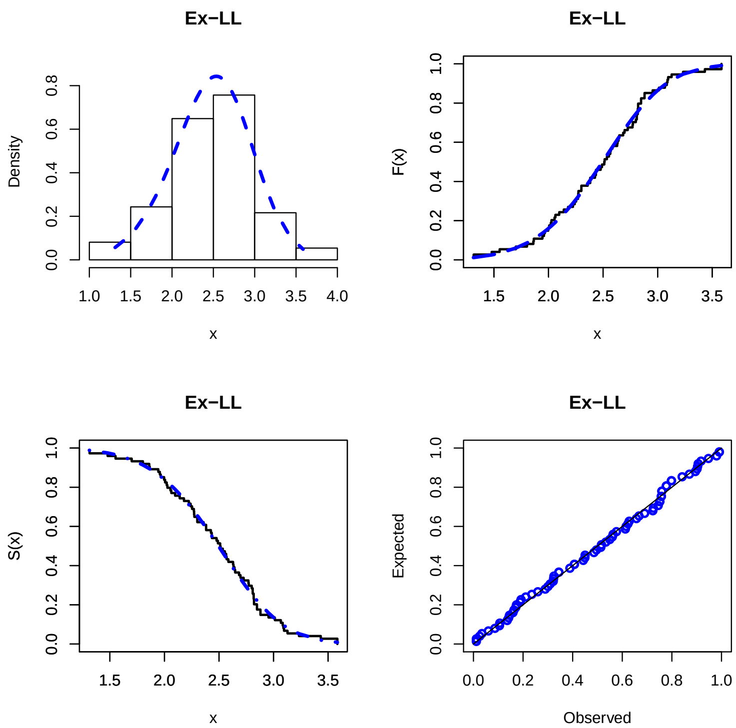

Figure 5. Histogram of data set I along with the fitted functions of the Ex-LL model.

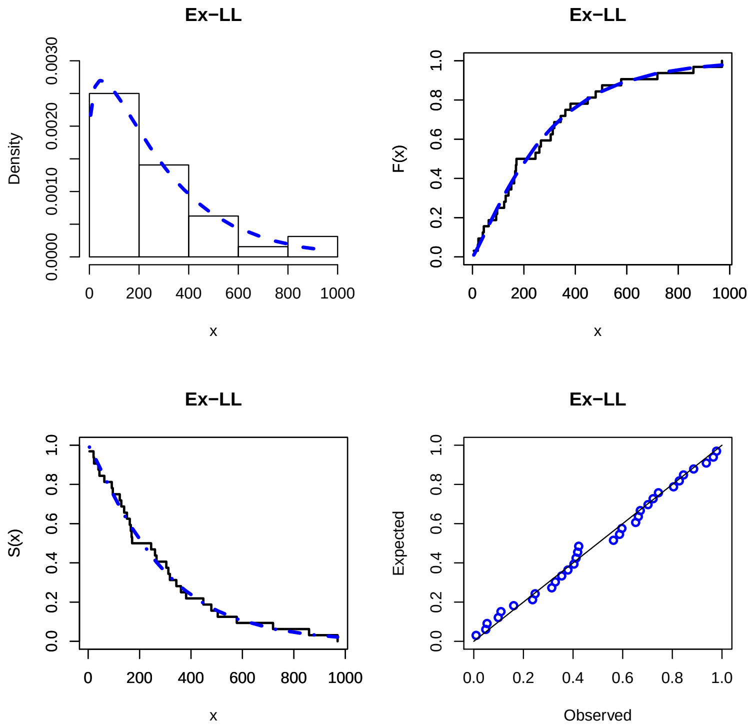

Figure 6. Histogram of data set II along with the fitted functions of the Ex-LL model.

Table 15. Measures of goodness-of-fit and estimates of the Ex-LL distribution and other distributions for data set II.

Model	AD	CM	KS	KS-p-Value	Estimates (SEs)
Ex-LL	0.14686	0.02332	0.07721	0.99108	$\hat{\alpha} = 28.8527\ (63.2613)$ $\hat{\lambda} = 558573\ (2.27773 \times 10^6)$ $\hat{\beta} = 1.15130\ (0.17685)$
LL	0.43828	0.04824	0.08713	0.96832	$\hat{\lambda} = 4708.8\ (6204.98)$ $\hat{\beta} = 1.60132\ (0.23852)$
APLL	0.43812	0.04816	0.08704	0.96861	$\hat{\alpha} = 1.3504\ (7.91583)$ $\hat{a} = 1.5993\ (0.25100)$ $\hat{b} = 179.106\ (326.737)$
TLL	0.43816	0.04818	0.08706	0.96853	$\hat{\alpha} = 187.685\ (198.581)$ $\hat{\beta} = 1.59980\ (0.24797)$ $\hat{\lambda} = -0.07504\ (1.68415)$
GLL	6.05879	1.23007	0.34740	0.00088	$\hat{\alpha} = 12.7009\ (9.22939)$ $\hat{\beta} = 1.00000\ (0.16110)$ $\hat{\theta} = 0.36290\ (0.09022)$
MOLL	0.43828	0.04824	0.08712	0.96832	$\hat{c} = 113.207\ (7.31161 \times 10^6)$ $\hat{\lambda} = 10.258\ (1.76675 \times 10^6)$ $\hat{\beta} = 1.60132\ (0.57739)$
PBXLL	0.84684	0.14436	0.14782	0.48651	$\hat{c} = 26.6401\ (6.83793)$ $\hat{\lambda} = 6.65389 \times 10^{-8}\ (0.00095)$ $\hat{\beta} = 0.12657\ (0.00757)$

Table 15. *Cont.*

Model	AD	CM	KS	KS-p-Value	Estimates (SEs)
TILL	13.4133	2.93189	0.51187	0.00000	$\hat{\lambda} = -1.00000$ (0.44793) $\hat{\alpha} = 0.33840$ (0.04258)
ILL	24.0976	5.04201	0.67941	0.00000	$\hat{\alpha} = 0.29516$ (0.04048)
WGLL	0.30483	0.04463	0.08940	0.96016	$\hat{c} = 0.00178$ (8.97093) $\hat{\lambda} = 69.1613$ (347363) $\hat{\beta} = 4.26012$ (0.62882)

8. Conclusions

In this article, we introduce a flexible extension of the log-logistic distribution called the extended log-logistic (Ex-LL) distribution. The EX-LL distribution can be adopted to model heavy-tailed actuarial data. The EX-LL distribution exhibits increasing, reversed-J, decreasing, upside-down bathtub, and J-shapes hazard rate functions. We derive some of its basic mathematical properties. Some risk measures are obtained for the Ex-LL distribution along with their detailed simulation results which illustrate that the tail of the Ex-LL distribution is heavier than the tails of the log-logistic and exponentiated log-logistic distributions. The parameters of the Ex-LL model are estimated using five classical estimation approaches and simulation results are conducted to explore their behavior for different sample sizes. The simulation results show that the maximum likelihood is the best estimation method for the Ex-LL parameters. The practical importance of the Ex-LL distribution is illustrated by two real data sets from the engineering and insurance sciences, showing its superiority fit as compared by nine competing distributions. We hope that the Ex-LL model will attract wider applications in other applied fields such as medicine, economics, reliability, life testing, and survival analyses.

Author Contributions: Investigation, N.M.A. A.M.G. and H.M.A.; Methodology, N.M.A., A.M.G. and H.M.A.; Project administration, A.Z.A.; Resources, A.Z.A.; Software, A.M.G. and A.Z.A.; Writing—original draft, A.M.G. and A.Z.A.; Writing—review & editing, H.M.A. and A.Z.A. All authors have read and agreed to the published version of the manuscript.

Funding: This study was funded by Taif University Researchers Supporting Project number (TURSP-2020/279), Taif University, Taif, Saudi Arabia.

Acknowledgments: The authors would like to thank the Editorial Board, and two reviewers for their constructive suggestions which greatly improved the final version of this manuscript. The authors extend their thanks to Taif University Researchers Supporting Project number (TURSP-2020/279), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cooray, K.; Ananda, M.M.A. Modeling actuarial data with a composite lognormal-Pareto model. *Scand. Actuar. J.* **2005**, *2005*, 321–334. [CrossRef]
2. Lane, M.N. Pricing risk transfer transactions 1. *ASTIN Bull. J. IAA* **2000**, *30*, 259–293. [CrossRef]
3. Klugman, S.A.; Panjer, H.H.; Willmot, G.E. *Loss Models: From Data to Decisions*; John Wiley & Sons: Hoboken, NJ, USA, 2012; Volume 715.
4. Ibragimov, R.; Prokhorov, A. *Heavy Tails and Copulas: Topics in Dependence Modeling in Economics and Finance*; World Scientific: Singapore, 2017.
5. Punzo, A.; Mazza, A.; Maruotti, A. Fitting insurance and economic data with outliers: A flexible approach based on finite mixtures of contaminated gamma distributions. *J. Appl. Stat.* **2018**, *45*, 2563–2584. [CrossRef]
6. Fisk, P.R. The graduation of income distributions. *Econom. J. Econom. Soc.* **1961**, *29*, 171–185. [CrossRef]
7. Dagum, C. A model of income distribution and the conditions of existence of moments of finite order. *Bull. Int. Stat. Inst.* **1975**, *46*, 199–205.
8. Shoukri, M.M.; Mian, I.U.H.; Tracy, D.S. Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. *Can. J. Stat.* **1988**, *16*, 223–236. [CrossRef]

9. Arnold, B.C. *Pareto Distributions Fairland*; International Cooperative Publishing House: Silver Spring, MD, USA, 1983.
10. Kleiber, C.; Kotz, S. *Statistical Size Distributions in Economics and Actuarial Sciences*; John Wiley & Sons: Hoboken, NJ, USA, 2003; Volume 470.
11. De Santana, T.V.F.; Ortega, E.M.M.; Cordeiro, G.M.; Silva, G.O. The Kumaraswamy-log-logistic distribution. *J. Stat. Theory Appl.* **2012**, *11*, 265–291.
12. Lemonte, A.J. The beta log-logistic distribution. *Braz. J. Probab. Stat.* **2014**, *28*, 313–332. [CrossRef]
13. Gui, W. Marshall–Olkin extended log-logistic distribution and its application in minification processes. *Appl. Math. Sci.* **2013**, *7*, 3947–3961. [CrossRef]
14. Tahir, M.H.; Mansoor, M.; Zubair, M.; Hamedani, G. McDonald log-logistic distribution with an application to breast cancer data. *J. Stat. Theory Appl.* **2014**, *13*, 65–82. [CrossRef]
15. Hamedani, G. The Zografos-Balakrishnan log-logistic distribution: Properties and applications. *J. Stat. Theory Appl.* **2013**, *12*, 225–244.
16. Cordeiro, G.M.; Afify, A.Z.; Ortega, E.M.; Suzuki, A.K.; Mead, M.E. The odd Lomax generator of distributions: Properties, estimation and applications. *J. Comput. Appl. Math.* **2019**, *347*, 222–237. [CrossRef]
17. Zhao, W.; Khosa, S.K.; Ahmad, Z.; Aslam, M.; Afify, A.Z. Type-I heavy tailed family with applications in medicine, engineering and insurance. *PLoS ONE* **2020**, *15*, e0237462.
18. Lorenz, M.O. Methods of measuring the concentration of wealth. *Publ. Am. Stat. Assoc.* **1905**, *9*, 209–219. [CrossRef]
19. Bonferroni, C.E. *Elementi di Statistica Generale*; Seeber: Firenze, Italy, 1930.
20. Arcagni, A.; Porro, F. The graphical representation of inequality. *Rev. Colomb. De Estad.* **2014**, *37*, 419–437. [CrossRef]
21. Morales, D.; Pardo, L.; Vajda, I. Some new statistics for testing hypotheses in parametric models. *J. Multivar. Anal.* **1997**, *62*, 137–168. [CrossRef]
22. Kurths, J.; Voss, A.; Saparin, P.; Witt, A.; Kleiner, H.J.; Wessel, N. Quantitative analysis of heart rate variability. *Chaos Interdiscip. J. Nonlinear Sci.* **1995**, *5*, 88–94. [CrossRef]
23. Song, K.S. Rényi information, loglikelihood and an intrinsic distribution measure. *J. Stat. Plan. Inference* **2001**, *93*, 51–69. [CrossRef]
24. Galambos, J. *The Asymptotic Theory of Extreme Order Statistics*; R.E. Krieger Pub. Co.: Malabar, FL, USA, 1987.
25. Artzner, P. Application of coherent risk measures to capital requirements in insurance. *N. Am. Actuar. J.* **1999**, *3*, 11–25. [CrossRef]
26. Landsman, Z. On the tail mean—variance optimal portfolio selection. *Insur. Math. Econ.* **2010**, *46*, 547–553. [CrossRef]
27. Kundu, D.; Raqab, M.Z. Estimation of $R = P(Y < X)$ for three parameter Weibull distribution. *Stat. Probab. Lett.* **2009**, *79*, 1839–1846.
28. Afify, A.; Cordeiro, G.; Butt, N.; Ortega, E.; Suzuki, A. A new lifetime model with variable shapes for the hazard rate. *Braz. J. Probab. Stat.* **2017**, *31*, 516–541. [CrossRef]
29. Afify, A.; Mohamed, O. A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications. *Mathematics* **2020**, *8*, 135. [CrossRef]
30. Aldahlan, M.A. Alpha power transformed log-logistic distribution with application to breaking stress data. *Adv. Math. Phys.* **2020**, *2020*, 2193787. [CrossRef]
31. Granzotto, D.C.T.; Louzada, F. The transmuted log-logistic distribution: Modeling, inference, and an application to a polled tabapua race time up to first calving data. *Commun. Stat. Theory Methods* **2015**, *44*, 3387–3402. [CrossRef]
32. Adeyinka, F.S.; Olapade, A.K. On transmuted four parameters generalized log-logistic distribution. *Int. J. Stat. Distrib. Appl.* **2019**, *5*, 32–37.
33. Almamy, J.A. Extended Poisson log-logistic distribution. *Int. J. Stat. Probab.* **2019**, *8*, 56–69. [CrossRef]
34. Para, B.A.; Jan, T.R. Transmuted inverse log logistic model: Properties and application in medical sciences and engineering. *Math. Theory Model.* **2017**, *7*, 157–181.
35. Abouelmagd, T.H.M.; Hamed, M.S.; Almamy, J.A.; Ali, M.M.; Yousof, H.M.; Korkmaz, M.C. Extended Weibull log-logistic distribution. *J. Nonlinear Sci. Appl.* **2019**, *12*, 523–534. [CrossRef]