

## Article

# Modified Inertial Forward–Backward Algorithm in Banach Spaces and Its Application

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**Abstract:** In this paper, we present a new modified inertial forward–backward algorithm for finding a common solution of the quasi-variational inclusion problem and the variational inequality problem in a  $q$ -uniformly smooth Banach space. The proposed algorithm is based on descent, splitting and inertial ideas. Under suitable assumptions, we prove that the sequence generated by the iterative algorithm converges strongly to the unique solution of the abovementioned problems. Numerical examples are also given to demonstrate our results.

**Keywords:** hybrid steepest decent method; forward–backward method; quasi-variational inclusion problem; Banach space; variational inequality problem



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## 1. Introduction

Many authors are now studying algorithms for reckoning a zero point for monotone operators in a Hilbert space. This is the problem of finding a point  $x \in H$  such that

$$0 \in Tx, \quad (1)$$

where  $T$  is a monotone operator. The set of zero points of  $T$  is denoted by  $T^{-1}(0)$ . Diverse problems such as convex minimization, monotone variational inequalities over convex sets, equilibrium problems etc. can be stated in the form of (1).

The proximal point algorithm proposed by Martinet [1,2] and generalized by Rockafellar [3,4],

$$x_{n+1} = (I + \alpha_k T)^{-1} x_n, \quad \forall n \in \mathbb{N}, \quad (2)$$

is the classical method for solving (1). Here,  $(I + \alpha_k T)^{-1}$  is the resolvent operator of maximal monotone operator  $T$ ,  $I$  is the identity mapping and  $\{\alpha_n\} \subset (0, +\infty)$  is a regularization sequence. Meanwhile, Rockafellar [3] and Bruck and Reich [5] proved that the sequence  $\{x_n\}$  generated by the proximal point algorithm (2) converges weakly to a point  $x$  satisfying  $0 \in Tx$ .

Note that it is difficult to evaluate the resolvent operator, in general. One way is to decompose the given operator into the sum of two (or more) maximal monotone operators whose resolvents are easier to evaluate than the resolvents of the original ones. More precisely, define  $T = A + B$  such that the resolvent  $(I + \alpha A)^{-1}$  (implicit, backward step) and the evaluation of  $B$  (explicit, forward step) are much easier to compute than the full

resolvent  $(I + \alpha T)^{-1}$ . Then, we have to consider the following monotone inclusion problem: find  $x \in H$  such that

$$0 \in Ax + Bx. \quad (3)$$

In order to solve the problem (3), the forward–backward splitting method [6–11] is usually employed. This is defined in the following manner:  $x_1 \in H$  and

$$x_{n+1} = (I + rA)^{-1}(I - rB)x_n, \quad \forall n \in \mathbb{N}. \quad (4)$$

The forward–backward method has the advantage of being easier to compute than the backward–backward method, which ensures enhanced applicability to real-life problems. Iterations have lower computational cost and can be computed exactly. Of course, the problem decomposition is not the only consideration; the convergence rate is another (see [12]).

In 2001, Alvarez and Attouch [13] employed the heavy ball method studied in [14,15] for maximal monotone operators on the proximal point algorithm. This algorithm, called the inertial proximal point algorithm, has the following form:

$$\begin{cases} x_{n+1} = (I + r_n A)^{-1} y_n, \\ y_n = x_n + \theta_n (x_n - x_{n-1}), \quad n \geq 1. \end{cases} \quad (5)$$

They proved that if  $\{r_n\}$  is non-decreasing and  $\{r_n\} \subset [0, 1)$  with  $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ , then algorithm (5) converges weakly to a zero of  $B$ .

Recently, Lorenz and Pock [16] also proposed the following inertial forward–backward algorithm for monotone operators:

$$\begin{cases} x_{n+1} = (I + r_n A)^{-1} (y_n - r_n B y_n), \\ y_n = x_n + \theta_n (x_n - x_{n-1}), \quad n \geq 1, \end{cases} \quad (6)$$

where  $\{r_n\}$  is a positive real sequence. They proved that the algorithm involving the inertial term mentioned above has weak convergence.

However, strong convergence (norm convergence) is often much more desirable than weak convergence (see [17] and references therein). Very recently, in the spirit of the splitting forward–backward method and the hybrid steepest descent method, Liu et al. [12] proposed the following iterative scheme as a new strategy in a Hilbert space; please see Algorithm 1.

Suppose  $S: H \rightarrow H$  is strongly monotone and continuous operator,  $A: H \rightarrow 2^H$  is a maximally monotone operator and  $B: H \rightarrow H$  is a cocoercive operator. Then, they have proved—under certain appropriate assumptions on the sequences  $\{v_n\}$ ,  $\{\mu_n\}$  and  $\{\eta_n\}$ —that the sequence  $\{p_n\}$  defined by Algorithm 1 converges strongly to the unique element of some variational inequality.

The quasi-variational inclusion problem in the setting of Hilbert spaces has been extensively studied in the literature [18–21]. However, there is little work in the existing literature on this problem in the setting of Banach spaces. The main difficulties are due to the fact that the inner product structure of a Hilbert space fails to be true in a Banach space. This is the motivation of the present study.

Motivated and inspired by Alvarez and Attouch [13], Lorenz and Pock [16], Liu et al. [12] and Thong and Chalamjiak [19], we put forward the following questions:

- (1) Can the corresponding results in [12,13,16,19] in Hilbert spaces be extended to the framework of Banach spaces (e.g.,  $l_p$  for  $1 < p < \infty$ )?
- (2) Can we extend corresponding results in [12] from one strongly monotone operator to a finite family of strongly accretive operators?
- (3) Can we extend the corresponding results in [12] from one cocoercive operator to a finite family of inverse strongly accretive operators?
- (4) Can the restrictions imposed on the parameters  $\{\eta_n\}$  in [12] be relaxed?

**Algorithm 1:** The hybrid forward–backward algorithm.

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**Input:** Input the algorithm parameters  $\{\eta_i\}_{i \geq 0}$ ,  $\{v_i\}_{i \geq 0}$  and  $\delta$ ;  
**Output:** Output  $\hat{p}$ ;  
 Initialize the data  $p_0, p_1 \in H$ ;  
 Set  $n \leftarrow 1$ ;  
**while** not converged **do**  
   **if**  $\|p_n - p_{n-1}\| \neq 0$  **then**  
     Update  $\mu_n$  such that  $\mu_n = o(\frac{\eta_n}{\|p_n - p_{n-1}\|})$ ;  
   **else**  
     Choose  $\mu_n$  as any positive number;  
   **end**  
   Update  $w_n = p_n + \mu_n(p_n - p_{n-1})$ ;  
   Update  $q_n = (I + v_n A)^{-1}(I - v_n B)w_n$ ;  
   Update  $p_{n+1} = (I - \delta \eta_n S)q_n$ ;  
   Set  $n \leftarrow n + 1$ ;  
**end**  
**return**  $\hat{p} = p_n$ .

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The purpose of this work is to give affirmative answers to the questions mentioned above. We introduce a new modified inertial forward–backward algorithm for finding a common solution of the quasi-variational inclusion problem and the variational inequality problem in  $q$ -uniformly smooth Banach spaces. Then a numerical analysis is conducted.

## 2. Preliminaries

To make the article self-contained, some mathematical preliminaries are necessary. In this respect, let us denote by  $E$  and  $E^*$  a real Banach space and the dual space of  $E$  respectively. We use  $\text{Fix}(T)$  to denote the set of fixed points of  $T$  and  $\mathcal{B}_r$  to denote the closed ball with center zero and radius  $r$ . Let  $C$  be a subset of  $E$  and  $q > 1$  be a real number. The (generalized) duality mapping  $J_q: E \rightarrow 2^{E^*}$  is given by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}$$

for all  $x \in E$ . Here  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing between  $E$  and  $E^*$ . If  $q = 2$ , the corresponding duality mapping is called the normalized duality mapping and is denoted by  $J$ . It is well known that if  $E$  is smooth, then  $J_q$  and  $J$  is single-valued, which is denoted by  $j_q$  and  $j$ , respectively. Clearly, the relation  $j_q(x) = \|x\|^{q-2}j(x)$ ,  $\forall x \neq 0$  holds.

Let  $C$  be a nonempty closed convex subset of a real Banach space  $E$ . Let  $A: C \rightarrow E$  be a single-valued nonlinear mapping and let  $B: C \rightarrow 2^E$  be a multivalued mapping. The so-called quasi-variational inclusion problem consists of finding a  $z \in E$  such that

$$0 \in (A + B)z. \quad (7)$$

The variational inequality problem for  $A: C \rightarrow E$  consists of finding a point  $u \in C$  such that

$$\langle Au, j_q(x - u) \rangle \geq 0, \quad \forall x \in C, \quad (8)$$

$j_q(x - y) \in J_q(x - y)$ . The set of solutions of the variational inequality problem is denoted by  $VI(C, A)$ . If  $E := H$  is a real Hilbert space, the variational inequality problem reduces to finding a point  $u \in C$  such that

$$\langle Au, x - u \rangle \geq 0, \quad \forall x \in C. \quad (9)$$

We also have that, for  $\lambda > 0$ ,  $u = P_C(I - \lambda A)u$  if and only if  $u \in VI(C, A)$ .

**Definition 1.** Let  $E$  be a Banach space. Then, a function  $\delta_E: [0, 2] \rightarrow [0, 1]$  is called the modulus of convexity of  $E$  if

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}.$$

If  $\delta_E(\epsilon) > 0$ , for all  $\epsilon \in (0, 2]$ , then  $E$  is uniformly convex.

**Definition 2.** The function  $\rho_E: [0, 1) \rightarrow [0, 1)$  is said to be the modulus of smoothness of  $E$  if

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : \|x\| = \|y\| = 1\right\}.$$

A Banach space  $E$  is called:

- (1) Uniformly smooth if  $\frac{\rho_E(t)}{t} \rightarrow 0$  as  $t \rightarrow 0$ ;
- (2)  $q$ -Uniformly smooth if there exists a fixed constant  $c > 0$  such that  $\rho_E(t) \leq ct^q$ , where  $q \in (1, 2]$ .

It is known that a uniformly convex Banach space is reflexive and strictly convex.

**Definition 3.** A mapping  $T: C \rightarrow E$  is said to be:

- (1) Nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C;$$

- (2)  $r$ -Contractive if there exists  $r \in [0, 1)$  such that

$$\|Tx - Ty\| \leq r\|x - y\| \text{ for all } x, y \in C;$$

- (3)  $\eta$ -Strongly accretive if for all  $x, y \in C$  there exists  $\eta > 0$  and  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle Tx - Ty, j_q(x - y) \rangle \geq \eta\|x - y\|^q;$$

- (4)  $\mu$ -Inverse-strongly accretive if for all  $x, y \in C$  there exists  $\mu > 0$  and  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle Tx - Ty, j_q(x - y) \rangle \geq \mu\|Tx - Ty\|^q.$$

**Definition 4.** A set-valued mapping  $T: \text{Dom}(T) \rightarrow 2^E$  is said to be:

- (1) Accretive if for any  $x, y \in \text{Dom}(T)$  there exists  $j_q(x - y) \in J_q(x - y)$ , such that for all  $u \in T(x)$  and  $v \in T(y)$ ,

$$\langle u - v, j_q(x - y) \rangle \geq 0;$$

- (2)  $m$ -Accretive if  $T$  is accretive and  $(I + \rho T)(\text{Dom}(T)) = E$  for every (equivalently, for some)  $\rho > 0$ , where  $I$  is the identity mapping.

Let  $A: \text{Dom}(A) \rightarrow 2^E$  be  $m$ -accretive. The mapping  $J_{A,\rho}: E \rightarrow \text{Dom}(A)$  defined by

$$J_{A,\rho}(u) = (I + \rho A)^{-1}(u), \quad \forall u \in E,$$

is called the resolvent operator associated with  $A$ , where  $\rho$  is any positive number and  $I$  is the identity mapping. It is well known that  $J_{A,\rho}$  is single-valued and nonexpansive.

Recall that if  $C$  and  $D$  are nonempty subsets of a Banach space  $E$  such that  $C$  is closed convex and  $D \subset C$ , then a mapping  $Q: C \rightarrow D$  is sunny provided

$$Q(x + t(x - Q(x))) = Q(x)$$

for all  $x \in C$  and  $t \geq 0$ , whenever  $Qx + t(x - Q(x)) \in C$ . A mapping  $Q: C \rightarrow D$  is called a retraction if  $Qx = x$  for all  $x \in D$ . Furthermore,  $Q$  is a sunny nonexpansive retraction from

$C$  onto  $D$  if  $Q$  is a retraction from  $C$  onto  $D$  which is also sunny and nonexpansive. A subset  $D$  of  $C$  is called a sunny nonexpansive retraction of  $C$  if there exists a sunny nonexpansive retraction from  $C$  onto  $D$ .

The following lemmas collect some properties of sunny nonexpansive retractions.

**Lemma 1.** Let  $C$  be a closed convex subset of a smooth Banach space  $E$ . Let  $D$  be a nonempty subset of  $C$ . Let  $Q: C \rightarrow D$  be a retraction and let  $j, j_q$  be the normalized duality mapping and generalized duality mapping on  $E$ , respectively ([22,23]). Then, the following are equivalent:

- (i)  $Q$  is sunny and nonexpansive;
- (ii)  $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy) \rangle, \quad \forall x, y \in C$ ;
- (iii)  $\langle x - Qx, j(y - Qx) \rangle \leq 0, \quad \forall x \in C, y \in D$ ;
- (iv)  $\langle x - Qx, j_q(y - Qx) \rangle \leq 0, \quad \forall x \in C, y \in D$ .

**Lemma 2.** Let  $E$  be a Banach space and  $J_q$  be a generalized duality mapping ([24]). Then, for any given  $x, y \in E$ , the following inequality holds:

$$\|x + y\|^q \leq \|x\|^q + q \langle y, j_q(x + y) \rangle, \quad j_q(x + y) \in J_q(x + y).$$

**Lemma 3.** Let  $C$  be a nonempty closed convex subset of a real  $q$ -uniformly smooth Banach space  $E$  ([25,26]). Let the mapping  $T: C \rightarrow E$  be a  $\iota$ -Lipschitz and  $\kappa$ -strongly accretive operator with constants  $\kappa, \iota > 0$ . Let  $0 < \delta < \left(\frac{q\kappa}{C_q \iota^q}\right)^{\frac{1}{q-1}}$  and  $\tau = \delta \left(\kappa - \frac{C_q \delta^{q-1} \iota^q}{q}\right)$ , where  $C_q$  is the  $q$ -uniform smoothness coefficient of  $E$  (see [24]). Then, for  $t \in (0, \min\{1, \frac{1}{\tau}\})$ , the mapping  $S: C \rightarrow E$  defined by  $S := (I - t\delta T)$  is a contraction with a constant  $1 - t\tau$ .

**Lemma 4.** Assume that  $E$  is a real uniformly convex and  $q$ -uniformly smooth Banach space ([26]). Suppose that  $B: C \rightarrow E$  is an  $a$ -inverse-strongly accretive operator for some  $a > 0$  and  $A: C \subseteq \text{Dom}(A) \rightarrow 2^E$  is an  $m$ -accretive operator. Moreover, denote  $J_r$  by

$$J_r := J_{A,r} = (I + rA)^{-1}$$

and  $T_r$  by

$$T_r := J_r(I - rB) = (I + rA)^{-1}(I - rB).$$

Then, it holds for all  $r > 0$  that  $\text{Fix}(T_r) = (A + B)^{-1}(0)$ .

**Lemma 5.** Assume that  $C$  is a nonempty closed subset of a real uniformly convex and  $q$ -uniformly smooth Banach space  $E$  ([9]). Suppose that  $B: C \rightarrow E$  is  $\alpha$ -inverse-strongly accretive and  $A$  is  $m$ -accretive in  $E$ , with  $\text{Dom}(M) \subseteq C$ . Then, it holds that:

- (1) Given  $0 < s \leq r$  and  $x \in E$ ,

$$\|T_s x - T_r x\| \leq \left|1 - \frac{s}{r}\right| \|x - T_r x\| \quad \text{and} \quad \|x - T_s x\| \leq 2\|x - T_r x\|;$$

- (2) Given  $k > 0$ , there exists a continuous, strictly increasing and convex function  $\phi_q: [0, \infty) \rightarrow [0, \infty)$  with  $\phi_q(0) = 0$  such that for all  $x, y \in \mathcal{B}_k$ ,

$$\begin{aligned} \|T_r x - T_r y\|^q &\leq \|x - y\|^q - r(\alpha q - r^{q-1} C_q) \|Bx - By\|^q \\ &\quad - \phi_q(\|(I - J_r)(I - rB)x - (I - J_r)(I - rB)y\|). \end{aligned}$$

To develop the results in this article, some technical results are necessary, as stated in the three lemmas below.

**Lemma 6.** Let  $\{\alpha_n\}$  be a sequence of nonnegative numbers satisfying the property ([27]):

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N},$$

where  $\{\gamma_n\}, \{b_n\}, \{c_n\}$  satisfy the restrictions:

- (i)  $\sum_{n=1}^{\infty} \gamma_n = \infty, \lim_{n \rightarrow \infty} \gamma_n = 0;$
- (ii)  $b_n \geq 0, \sum_{n=1}^{\infty} b_n < \infty;$
- (iii)  $\limsup_{n \rightarrow \infty} c_n \leq 0.$

Then,  $\lim_{n \rightarrow \infty} \alpha_n = 0.$

**Lemma 7.** Let  $q > 1$  ([28]). Then, the following inequality holds for arbitrary positive real numbers  $a, b$ :

$$a^q - b^q \leq qa^{q-1}(a - b).$$

**Lemma 8.** Let  $\{\alpha_n\}$  be a sequence of real numbers such that there exists a subsequence  $\{n_i\}$  of  $\{n\}$  such that  $\alpha_{n_i} < \alpha_{n_i+1}$  for all  $i \in \mathbb{N}$  ([29]). Then there exists a nondecreasing sequence  $\{m_k\} \subseteq \mathbb{N}$  such that  $m_k \rightarrow \infty$  and the following properties are satisfied for all (sufficiently large) numbers  $k \in \mathbb{N}$ :

$$\alpha_{m_k} \leq \alpha_{m_k+1} \text{ and } \alpha_k \leq \alpha_{m_k+1}.$$

In fact,  $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}.$

We make complete these preliminaries with two results on classes of accretive operators.

**Lemma 9.** Let  $C$  be a nonempty closed convex subset of a real smooth Banach space  $E$  ([30]). Let  $N \geq 1$  be some positive integer,  $A$  be  $m$ -accretive in  $E$  with  $\text{Dom}(A) \subseteq C$ ,  $B_i : C \rightarrow E$  be  $\beta_i$ -inverse-strongly accretive with  $\beta = \min\{\beta_1, \beta_2, \dots, \beta_N\}$  and  $\bigcap_{i=1}^N (A + B_i)^{-1}(0) \neq \emptyset$ . Let  $\{\gamma_i\}$  be a real number sequence in  $(0, 1)$  with  $\sum_{i=1}^N \gamma_i = 1$ . Then  $\sum_{i=1}^N \gamma_i B_i$  is  $\beta$ -inverse-strongly accretive, and

$$(A + \sum_{i=1}^N \gamma_i B_i)^{-1}(0) = \bigcap_{i=1}^N (A + B_i)^{-1}(0).$$

**Lemma 10.** Let  $C$  be a nonempty closed convex subset of a real smooth Banach space  $E$ . Let  $N \geq 1$  be some positive integer,  $S_i : C \rightarrow E$  be  $\kappa_i$ -strongly accretive with  $\kappa = \min\{\kappa_1, \kappa_2, \dots, \kappa_N\}$ . Let  $\{\lambda_i\}$  be a real number sequence in  $(0, 1)$  with  $\sum_{i=1}^N \lambda_i = 1$ . Then, the mapping  $\sum_{i=1}^N \lambda_i S_i$  is  $\kappa$ -strongly accretive.

**Proof.** Let  $x, y \in C$ . It follows that

$$\begin{aligned} \langle \sum_{i=1}^N \lambda_i S_i x - \sum_{i=1}^N \lambda_i S_i y, j_q(x - y) \rangle &= \sum_{i=1}^N \lambda_i \langle S_i x - S_i y, j_q(x - y) \rangle \\ &\geq \sum_{i=1}^N \lambda_i \kappa_i \|x - y\|^q \\ &\geq \kappa \|x - y\|^q. \end{aligned}$$

Consequently, the mapping  $\sum_{i=1}^N \lambda_i S_i$  is  $\kappa$ -strongly accretive.  $\square$

Now, we are ready to state and prove our results.

### 3. Main Results

In this section we present an algorithm for solving quasi-variational inclusion problems and variational inequality problems in Banach spaces. We assume that:

- (A1)  $E$  is a uniformly convex and  $q$ -uniformly smooth Banach space, and  $A$  is  $m$ -accretive;
- (A2)  $B_i$  is  $\beta_i$ -inverse-strongly accretive with constants  $\beta_i > 0$  for each  $i = 1, 2, \dots, N$ , and  $\beta = \min\{\beta_i : 1 \leq i \leq N\};$
- (A3)  $S_i : E \rightarrow E$  is  $\iota_i$ -Lipschitzian and  $\kappa_i$ -strongly accretive with constants  $\iota_i, \kappa_i > 0$  for each  $i = 1, 2, \dots, N, \iota = \min\{\iota_i : 1 \leq i \leq N\}$  and  $\kappa = \min\{\kappa_i : 1 \leq i \leq N\};$

(A4)  $0 < \delta < \left(\frac{q\kappa}{C_q t^q}\right)^{\frac{1}{q-1}}$  and  $\Omega = \bigcap_{i=1}^N (A + B_i)^{-1}(0) \neq \emptyset$ .

The proposed algorithm is of the form:

**Theorem 1.** Assume that  $\{\eta_n\}$ ,  $\{v_n\}$ ,  $\{\gamma_n\}$ ,  $\{\lambda_n\}$  and  $\{\mu_n\}$  are real number sequences in  $[0, 1]$  satisfying:

- (i)  $0 < \liminf_{n \rightarrow \infty} v_n \leq \limsup_{n \rightarrow \infty} v_n < \left(\frac{q\beta}{C_q t^q}\right)^{\frac{1}{q-1}}$ ;
- (ii)  $\lim_{n \rightarrow \infty} \eta_n = 0$ ,  $\sum_{i=1}^{\infty} \eta_n = \infty$ ;
- (iii)  $\lim_{n \rightarrow \infty} \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| = 0$ ;
- (iv)  $\sum_{i=1}^N \gamma_n = 1$ ,  $\sum_{i=1}^N \lambda_n = 1$ .

Then  $\{p_n\}$  generated by Algorithm 2 converges strongly to a point  $z^* \in \Omega$ , which is also the unique solution of the variational inequality problem

$$\langle Sz^*, j_q(x - z^*) \rangle \geq 0, \forall x \in \Omega. \quad (10)$$

**Remark 1.** We note that condition (iii) can be easily implemented since the value of  $\|p_n - p_{n-1}\|$  is known before choosing  $\mu_n$ . Indeed, the parameter  $\mu_n$  can be chosen such that

$$\mu_n = \begin{cases} w, & p_n = p_{n-1}, \\ \frac{\xi_n}{\|p_n - p_{n-1}\|}, & p_n \neq p_{n-1}, \end{cases}$$

where  $w \geq 0$  and  $\{\xi_n\}$  is a positive sequence such that  $\xi_n = o(\eta_n)$ .

We now prove Theorem 1.

**Proof.** We first prove that the sequence  $\{p_n\}$  is bounded. Writing  $B = \sum_{i=1}^N \gamma_i B_i$ , one can then deduce from Lemma 9 that  $B$  is  $\beta$ -inverse-strongly accretive. Additionally, letting  $S = \sum_{i=1}^N \lambda_i S_i$ , we infer from Lemma 10 and Assumption (A3) that  $S : C \rightarrow E$  is  $t$ -Lipschitzian and  $\kappa$ -strongly accretive. Taking any  $z \in \Omega$ , we infer from Algorithm 2, Lemma 5 (2) and (i) that

$$\begin{aligned} & \|q_n - z\|^q \\ &= \left\| (I + v_n A)^{-1} (I - v_n B) w_n - (I + v_n A)^{-1} (I - v_n B) z \right\|^q \\ &\leq \|w_n - z\|^q - v_n (\beta q - v_n^{q-1} C_q) \|B w_n - B z\|^q \\ &\quad - \phi_q(\|(I - J_{v_n})(I - v_n B) w_n - (I - J_{v_n})(I - v_n B) z\|), \end{aligned} \quad (11)$$

which implies

$$v_n (\beta q - v_n^{q-1} C_q) \|B w_n - B z\|^q \leq \|w_n - z\|^q - \|q_n - z\|^2 \quad (12)$$

and

$$\|q_n - z\| \leq \|w_n - z\|. \quad (13)$$

From the definition of  $w_n$ , we deduce

$$\|w_n - z\| \leq \|p_n - z\| + \mu_n \|p_n - p_{n-1}\|. \quad (14)$$

Invoking (iii), there exists a positive constant  $M_1 < \infty$  such that

$$\frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| \leq M_1.$$

**Algorithm 2:** The modified hybrid forward–backward algorithm.

---

**Input:** Input the algorithm parameters  $\{\eta_i\}_{i \geq 1}$ ,  $\{v_i\}_{i \geq 1}$ ,  $\{\lambda_i\}_{i \geq 1}$ ,  $\{\gamma_i\}_{i \geq 1}$  and  $\delta$ ;  
**Output:** Output  $\hat{p}$ ;  
 Initialize the data  $p_0, p_1 \in E$ ;  
 Set  $n \leftarrow 1$ ;  
**while** not converged **do**  
   **if**  $\|p_n - p_{n-1}\| \neq 0$  **then**  
     Update  $\mu_n$  such that  $\mu_n = o(\frac{\eta_n}{\|p_n - p_{n-1}\|})$ ;  
   **else**  
     Choose  $\mu_n$  as any positive number;  
   **end**  
   Update  $w_n = p_n + \mu_n(p_n - p_{n-1})$ ;  
   Update  $q_n = (I + v_n A)^{-1}(I - v_n \sum_{i=1}^N \gamma_i B_i)w_n$ ;  
   Update  $p_{n+1} = (I - \eta_n \delta \sum_{i=1}^N \lambda_i S_i)q_n$ ;  
   Set  $n \leftarrow n + 1$ ;  
**end**  
**return**  $\hat{p} = p_n$ .

---

From (13) and (14), Lemma 3, Assumption (A4) and (ii), one obtains

$$\begin{aligned}
 & \|p_{n+1} - z\| \\
 &= \|(I - \delta \eta_n S)q_n - (I - \delta \eta_n S)z - \delta \eta_n Sz\| \\
 &\leq (1 - \eta_n \tau) \|q_n - z\| + \delta \eta_n \|Sz\| \\
 &\leq (1 - \eta_n \tau) \|w_n - z\| + \delta \eta_n \|Sz\| \\
 &\leq (1 - \eta_n \tau) (\|p_n - z\| + \eta_n M_1) + \delta \eta_n \|Sz\| \\
 &\leq (1 - \eta_n \tau) \|p_n - z\| + \eta_n M_1 + \delta \eta_n \|Sz\| \\
 &\leq \max\{\|p_n - z\|, \frac{1}{\tau} (M_1 + \delta \|Sz\|)\} \\
 &\leq \dots \leq \max\{\|p_1 - z\|, \frac{1}{\tau} (M_1 + \delta \|Sz\|)\},
 \end{aligned} \tag{15}$$

where  $\tau = \delta \left( \kappa - \frac{C_q \delta^{q-1} \iota^q}{q} \right) > 0$ . This implies that sequence  $\{p_n\}$  is bounded. At the same time, by putting together (13) and (14), one concludes that  $\{q_n\}$  and  $\{w_n\}$  are bounded. According to Lemma 2, we obtain

$$\begin{aligned}
 & \|w_n - z\|^q \\
 &= \|p_n - z + \mu_n(p_n - p_{n-1})\|^q \\
 &\leq \|p_n - z\|^q + q \mu_n \langle p_n - p_{n-1}, j_q(w_n - z) \rangle \\
 &\leq \|p_n - z\|^q + q \mu_n \|p_n - p_{n-1}\| \|w_n - z\|^{q-1} \\
 &\leq \|p_n - z\|^q + \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_2,
 \end{aligned} \tag{16}$$

where  $M_2 = \sup_{n \geq 1} \{q \eta_n \|w_n - z\|^{q-1}\} < \infty$ . By combining (11) with (16), one immediately concludes that

$$\begin{aligned}
 & \|q_n - z\|^q \\
 &\leq \|p_n - z\|^q + \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_2 - v_n (\beta q - v_n^{q-1} C_q) \|Bw_n - Bz\|^q \\
 &\quad - \phi_q(\|(I - J_{v_n})(I - v_n B)w_n - (I - J_{v_n})(I - v_n B)z\|).
 \end{aligned} \tag{17}$$



It follows from Lemma 2, Lemma 3, (17) and (ii) that

$$\begin{aligned}
 & \|p_{n+1} - z\|^q \\
 = & \|(I - \delta\eta_n S)q_n - (I - \delta\eta_n S)z - \delta\eta_n Sz\|^q \\
 \leq & \|(I - \delta\eta_n S)q_n - (I - \delta\eta_n S)z\|^q - q\delta\eta_n \langle Sz, j_q(p_{n+1} - z) \rangle \\
 \leq & (1 - \eta_n \tau)^q \|q_n - z\|^q + q\delta\eta_n \langle Sz, j_q(z - p_{n+1}) \rangle \\
 \leq & \|q_n - z\|^q + q\delta\eta_n \langle Sz, j_q(z - p_{n+1}) \rangle \\
 \leq & \|p_n - z\|^q + \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_2 - v_n(\beta q - v_n^{q-1} C_q) \|Bw_n - Bz\|^2 \\
 & - \phi_q(\|(I - J_{v_n})(I - v_n B)w_n - (I - J_{v_n})(I - v_n B)z\|) + \eta_n M_3, \quad (18)
 \end{aligned}$$

where  $M_3 = \sup_{n \geq 1} q\delta \langle Sz, j_q(z - p_{n+1}) \rangle < \infty$ . Let us rewrite (18) as

$$v_n(\beta q - v_n^{q-1} C_q) \|Bw_n - Bz\|^q \leq \|p_n - z\|^q - \|p_{n+1} - z\|^q + \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_2 + \eta_n M_3, \quad (19)$$

and

$$\begin{aligned}
 & \phi_q(\|(I - J_{v_n})(I - v_n B)w_n - (I - J_{v_n})(I - v_n B)z\|) \\
 \leq & \|p_n - z\|^q - \|p_{n+1} - z\|^q + \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_2 + \eta_n M_3. \quad (20)
 \end{aligned}$$

Next, we show that  $\{p_n\}$  converges by considering two possible cases.

Case 1. Suppose that there exists  $N \in \mathbb{N}$  such that the sequence  $\{\|p_n - z^*\|\}_{n \geq N}$  is monotonically decreasing; thus,  $\lim_{n \rightarrow \infty} \|p_n - z^*\|$  exists. Using (i), (ii), (iii) and letting  $n$  tend to infinity in (19), one finds that

$$\lim_{n \rightarrow \infty} \|Bw_n - Bz^*\| = 0. \quad (21)$$

It follows from (ii), (iii), (20) and the properties of  $\phi_q$  (see Lemma 5 (2)) that

$$\lim_{n \rightarrow \infty} \|(I - J_{v_n})(I - v_n B)w_n - (I - J_{v_n})(I - v_n B)z^*\| = 0. \quad (22)$$

We deduce from Lemma 4 and (22) that

$$\lim_{n \rightarrow \infty} \|(I - v_n B)w_n - q_n - (I - v_n B)z^* + z^*\| = 0. \quad (23)$$

Notice that

$$\|w_n - q_n\| \leq \|(I - v_n B)w_n - (I - v_n B)z^* - q_n + z^*\| + v_n \|Bw_n - Bz^*\|.$$

Together with (21) and (23), this implies that

$$\lim_{n \rightarrow \infty} \|w_n - q_n\| = 0. \quad (24)$$

It follows from (ii), (iii) and the definitions of  $w_n$  and  $p_{n+1}$  that

$$\lim_{n \rightarrow \infty} \|w_n - p_n\| = \lim_{n \rightarrow \infty} \mu_n \|p_n - p_{n-1}\| = 0 \quad (25)$$

and

$$\lim_{n \rightarrow \infty} \|p_{n+1} - q_n\| = \lim_{n \rightarrow \infty} \delta\eta_n \|Sq_n\| = 0. \quad (26)$$

Resorting to (24)–(26), one deduces that

$$\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| \leq \lim_{n \rightarrow \infty} (\|p_{n+1} - q_n\| + \|q_n - w_n\| + \|w_n - p_n\|) = 0. \quad (27)$$

Denote  $T_{v_n} = (I + v_n A)^{-1}(I - v_n B)$ . Then, it immediately holds that

$$\lim_{n \rightarrow \infty} \|T_{v_n} w_n - w_n\| = \lim_{n \rightarrow \infty} \|q_n - w_n\| = 0. \quad (28)$$

In view of Lemma 5 (1) and (28), there exists  $r > 0$  such that  $r \leq v_n$  for all  $n \geq 1$  and

$$\lim_{n \rightarrow \infty} \|T_r w_n - w_n\| \leq 2 \lim_{n \rightarrow \infty} \|T_{v_n} w_n - w_n\| = 0. \quad (29)$$

Setting  $\tau = \delta \left( \kappa - \frac{C_q \delta^{q-1} l^q}{q} \right)$  and taking a real number  $\eta \in (0, \min\{1, \frac{1}{\tau}\})$ , we find by Lemma 3 that  $(I - \eta \delta S)$  is contractive with a constant  $1 - \eta \tau$ . Let  $z_t$  satisfy  $z_t = t(I - \eta \delta S)z_t + (1 - t)T_r z_t$ . By Lemma 1, Lemma 9 and Xu's theorem (4.1) [31], we obtain that  $z_t$  converges strongly to a point  $z^* \in \text{Fix}(T_r)$ , which is also the unique solution of variational inequality problem

$$\langle Sz^*, j_q(x - z^*) \rangle \geq 0, \forall x \in \text{Fix}(T_r).$$

Apply Lemma 4 to obtain that  $z^*$  is the unique solution of variational inequality problem (10).

Next we show that

$$\limsup_{n \rightarrow \infty} \langle Sz^*, j_p(z^* - p_n) \rangle \leq 0, \quad (30)$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \langle Sz^*, j(z^* - p_n) \rangle \leq 0.$$

Using Lemma 2, we deduce that

$$\begin{aligned} & \|z_t - w_n\|^2 \\ = & \|t((I - \eta \delta S)z_t - w_n) + (1 - t)(T_r z_t - w_n)\|^2 \\ \leq & (1 - t)^2 \|T_r z_t - w_n\|^2 + 2t \langle (I - \eta \delta S)z_t - w_n, j(z_t - w_n) \rangle \\ \leq & (1 - t)^2 (\|T_r z_t - T_r w_n\| + \|T_r w_n - w_n\|)^2 + 2t \langle (I - \eta \delta S)z_t - z_t, j(z_t - w_n) \rangle + 2t \|z_t - w_n\|^2 \\ \leq & (1 - t)^2 (\|z_t - w_n\| + \|T_r w_n - w_n\|)^2 - 2t \eta \delta \langle Sz_t, j(z_t - w_n) \rangle + 2t \|z_t - w_n\|^2 \\ \leq & (1 - t)^2 \|z_t - w_n\|^2 + M_{n,t} - 2t \eta \delta \langle Sz_t, j(z_t - w_n) \rangle + 2t \|z_t - w_n\|^2, \end{aligned}$$

where  $M_{n,t} = \|T_r w_n - w_n\| (2\|z_t - w_n\| + \|T_r w_n - w_n\|) (\rightarrow 0 \text{ as } n \rightarrow \infty)$ . This implies

$$\langle Sz_t, j(z_t - w_n) \rangle \leq \frac{t}{2\eta \delta} \|z_t - w_n\|^2 + \frac{1}{2t\eta \delta} M_{n,t}.$$

It follows that

$$\limsup_{n \rightarrow \infty} \langle Sz_t, j(z_t - w_n) \rangle \leq \frac{t}{2\eta \delta} M_4, \quad (31)$$

where  $M_4 > 0$  is a constant such that  $M_4 \geq \|z_t - w_n\|^2$  for all  $n \geq 1$  and  $t \in (0, 1)$ . Taking the  $\limsup$  as  $t \rightarrow 0$  in (31) and noting the fact that the two limits are interchangeable because the duality map  $J$  is norm-to-norm uniformly continuous on bounded sets, we obtain (30).

It follows from Lemma 2, Lemma 3 and (13) that

$$\begin{aligned} & \|p_{n+1} - z^*\|^q \\ &= \|(I - \eta_n \delta S)q_n - (I - \eta_n \delta S)z^* - \eta_n \delta Sz^*\|^q \\ &\leq \|(I - \eta_n \delta S)q_n - (I - \eta_n \delta S)z^*\|^q - q\eta_n \delta \langle Sz^*, j_q(p_{n+1} - z^*) \rangle \\ &\leq (1 - \eta_n \tau) \|q_n - z^*\|^q + \delta \eta_n \langle Sz^*, j_q(z^* - p_{n+1}) \rangle \\ &\leq (1 - \eta_n \tau) \|w_n - z^*\|^q + \delta \eta_n \langle Sz^*, j_q(z^* - p_{n+1}) \rangle \\ &\leq (1 - \eta_n \tau) (\|p_n - z^*\| + \mu_n \|p_n - p_{n-1}\|)^q + \delta \eta_n \langle Sz^*, j_q(z^* - p_{n+1}) \rangle. \end{aligned} \quad (32)$$

Noticing Lemma 7, we infer

$$\begin{aligned} & (\|p_n - z^*\| + \mu_n \|p_n - p_{n-1}\|)^q \\ &\leq \|p_n - z^*\|^q + q\mu_n \|p_n - p_{n-1}\| (\|p_n - z^*\| + \mu_n \|p_n - p_{n-1}\|)^{q-1}. \end{aligned} \quad (33)$$

Substituting (33) into (32) and simplifying, we have

$$\begin{aligned} & \|p_{n+1} - z^*\|^q \\ &\leq (1 - \eta_n \tau) (\|p_n - z^*\|^q + \mu_n \|p_n - p_{n-1}\| M_4) + \delta \eta_n \langle Sz^*, j_q(z^* - p_{n+1}) \rangle \\ &\leq (1 - \eta_n \tau) \|p_n - z^*\|^q + \mu_n \|p_n - p_{n-1}\| M_4 + \delta \eta_n \langle Sz^*, j_q(z^* - p_{n+1}) \rangle \\ &= (1 - \eta_n \tau) \|p_n - z^*\|^q + \eta_n \left( \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| M_5 + \delta \langle Sz^*, j_q(z^* - p_{n+1}) \rangle \right), \end{aligned} \quad (34)$$

where

$$M_5 = \sup_{n \geq 1} q (\|p_n - z^*\| + \mu_n \|p_n - p_{n-1}\|)^{q-1} < \infty.$$

So doing, Lemma 6 asserts that  $\lim_{n \rightarrow \infty} \|p_{n+1} - z^*\| = 0$ . Coming back to (25)–(27), one concludes that  $\{p_n\}$ ,  $\{q_n\}$  and  $\{w_n\}$  converge strongly to  $z^*$ .

Case 2. Assume that  $\{\|p_n - z^*\|\}$  is not monotonically decreasing. Then, there exists a subsequence  $\{\|p_{n_i} - z^*\|\}$  of  $\{\|p_n - z^*\|\}$  such that

$$\|p_{n_i} - z^*\| < \|p_{n_i+1} - z^*\|, \quad \forall i \in \mathbb{N}. \quad (35)$$

From Lemma 8, there exists a nondecreasing sequence  $\{m_k\} \subset \mathbb{N}$  such that

$$\max\{\|p_{m_k} - z^*\|, \|p_k - z^*\|\} \leq \|p_{m_k+1} - z^*\|. \quad (36)$$

Thanks to (19), (20) and (36), one infers that

$$\lim_{k \rightarrow \infty} \|Bw_{m_k} - Bz^*\| = 0 \quad (37)$$

and

$$\lim_{k \rightarrow \infty} \phi_q \left( \|(I - J_{v_{m_k}})(I - v_{m_k} B)w_{m_k} - (I - J_{v_{m_k}})(I - v_{m_k} B)z^*\| \right) = 0 \quad (38)$$

Repeating the argument for (24), we deduce

$$\lim_{k \rightarrow \infty} \|w_{m_k} - q_{m_k}\| = 0$$

A calculation similar to the proof in Case 1 guarantees that

$$\lim_{k \rightarrow \infty} \|p_{m_k+1} - p_{m_k}\| = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle \leq 0.$$

Because the duality map  $J_q$  is norm-to-norm uniformly continuous on bounded sets, we obtain that from (11), Lemmas 2 and 3,

$$\limsup_{k \rightarrow \infty} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle \leq 0. \quad (39)$$

From (11), Lemmas 2 and 3, one obtains

$$\begin{aligned} & \|p_{m_k+1} - z^*\|^q \\ &= \|(I - \eta_{m_k} \delta S)q_{m_k} - (I - \eta_{m_k} \delta S)z^* - \eta_{m_k} \delta Sz^*\|^q \\ &\leq \|(I - \eta_{m_k} \delta S)q_{m_k} - (I - \eta_{m_k} \delta S)z^*\|^q - q\eta_{m_k} \delta \langle Sz^*, j_q(p_{m_k+1} - z^*) \rangle \\ &\leq (1 - \eta_{m_k} \tau) \|q_{m_k} - z^*\|^q + q\delta\eta_{m_k} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle \\ &\leq (1 - \eta_{m_k} \tau) \|w_{m_k} - z^*\|^q + q\delta\eta_{m_k} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle \\ &\leq (1 - \eta_{m_k} \tau) (\|p_{m_k} - z^*\| + \mu_{m_k} \|p_n - p_{m_k-1}\|)^q + q\delta\eta_{m_k} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle. \end{aligned} \quad (40)$$

From (36) and Lemma 7, we derive

$$\begin{aligned} & (\|p_{m_k} - z^*\| + \mu_n \|p_{m_k} - p_{m_k-1}\|)^q \\ &\leq \|p_{m_k} - z^*\|^q + q\mu_{m_k} \|p_{m_k} - p_{m_k-1}\| (\|p_{m_k} - z^*\| + \mu_n \|p_{m_k} - p_{m_k-1}\|)^{q-1} \\ &\leq \|p_{m_k+1} - z^*\|^q + q\mu_{m_k} \|p_{m_k} - p_{m_k-1}\| (\|p_{m_k} - z^*\| + \mu_n \|p_{m_k} - p_{m_k-1}\|)^{q-1}. \end{aligned} \quad (41)$$

Substituting (41) into (40) yields

$$\|p_{m_k+1} - z^*\|^q \leq \frac{\mu_{m_k}}{\eta_{m_k}} \|p_{m_k} - p_{m_k-1}\| M_6 + \frac{q\delta}{\tau} \langle Sz^*, j_q(z^* - p_{m_k+1}) \rangle,$$

where  $M_6 = \sup_{n \geq 1} q(\|p_{m_k} - z^*\| + \mu_n \|p_{m_k} - p_{m_k-1}\|)^{q-1} < \infty$ . By applying (iii) and (39), we obtain

$$\lim_{k \rightarrow \infty} \|p_{m_k+1} - z^*\|^q = 0.$$

Hence,

$$\lim_{k \rightarrow \infty} \|p_{m_k+1} - z^*\| = 0.$$

It thus follows from (36) that

$$\lim_{k \rightarrow \infty} \|p_k - z^*\| = 0.$$

From the above, one can conclude that the sequences generated by Algorithm 2 converge strongly to a point  $z^* \in \Omega$ , which is also the unique solution of the variational inequality problem (1). This completes the proof.  $\square$

**Remark 2.** Theorem 1 extends, improves and develops corresponding results in [12,13,16,19] in the following respects:

- The results in this paper improve and extend corresponding results in [12,13,16,19] in Hilbert spaces to more general  $q$ -uniformly smooth Banach spaces.
- The results in this paper extend corresponding results in [12] from one strongly monotone operator to a finite family of strongly accretive operators.
- The results in this paper extend corresponding results in [12] from one cocoercive operator to a finite family of inverse-strongly accretive operators.
- We omit the condition  $\eta_{n+1} = O(\eta_n)$ , which is very necessary in Theorem 1 of Liu et al. [12].
- The proof of our Theorem 1 is very different from the proof of Theorem 1 of Liu et al. [12].

The following result can be obtained from Theorem 1 immediately.

**Corollary 1.** Let  $N = 1$  in Theorem 1. Then,  $\{p_n\}$  generated by Algorithm 2 converges strongly to a point  $z^* \in \Omega (= (A + B_1)^{-1}(0))$ , which is also the unique solution of the variational inequality problem

$$\langle S_1 z^*, j_q(x - z^*) \rangle \geq 0, \forall x \in \Omega.$$

#### 4. Application to Constrained Convex Minimization Problems

In this section, we shall apply our main results to an approximate constrained convex minimization problem. Let  $E$  be the real Euclidean  $n$ -space  $\mathbb{R}^n$  and  $\Omega$  be a nonempty closed convex subset of  $H$ . We consider the following constrained convex minimization problem:

$$\min_{x \in \Omega} \varphi(x), \quad (42)$$

where  $\varphi : \Omega \rightarrow \mathbb{R}$  is a real-valued convex function and assumes that the problem (42) is consistent (i.e., its solution set is nonempty). Let  $D$  denote its solution set. For the minimization problem (42), if  $\varphi$  is differentiable, then we have the following lemma.

**Lemma 11.** (Optimality condition) ([32]) A necessary condition of optimality for a point  $z \in \Omega$  to be a solution of the minimization problem (42) is that  $z$  solves the variational inequality

$$\langle \nabla \varphi(z), x - z \rangle \geq 0, \quad \forall x \in \Omega. \quad (43)$$

Equivalently,  $x \in \Omega$  solves the fixed point equation

$$z = Q_\Omega(z - \alpha \nabla \varphi(z)) \quad (44)$$

for every constant  $\alpha > 0$ . If, in addition,  $\varphi$  is convex, then the optimality condition (43) is also sufficient.

**Theorem 2.** Let  $E$  be the real Euclidean  $n$ -space  $\mathbb{R}^n$ . Assume that  $\varphi$  is differentiable and the gradient  $\nabla \varphi$  is  $\iota$ -Lipschitzian and  $\kappa$ -strongly monotone with constants  $\iota, \kappa > 0$ . Let  $A$  be  $m$ -accretive and  $B_i$  be  $\beta_i$ -inverse-strongly accretive with constants  $\beta_i > 0$  for each  $i = 1, 2, \dots, N$ , and  $\beta = \min\{\beta_i : 1 \leq i \leq N\}$ . Assume  $0 < \delta < \frac{2\kappa}{\iota^2}$  and  $\Omega = \bigcap_{i=1}^N (A + B_i)^{-1}(0) \neq \emptyset$ . Let  $\{p_n\}$  be a sequence generated by

Assume that  $\{\eta_n\}$ ,  $\{v_n\}$ ,  $\{\gamma_n\}$  and  $\{\mu_n\}$  are real number sequences in  $[0, 1]$  satisfying:

- $0 < \liminf_{n \rightarrow \infty} v_n \leq \limsup_{n \rightarrow \infty} v_n < \frac{2\beta}{\iota^2}$ ;
- $\lim_{n \rightarrow \infty} \eta_n = 0, \sum_{i=1}^{\infty} \eta_n = \infty$ ;
- $\lim_{n \rightarrow \infty} \frac{\mu_n}{\eta_n} \|p_n - p_{n-1}\| = 0$ ;
- $\sum_{i=1}^N \gamma_n = 1$ .

Then,  $\{p_n\}$  generated by Algorithm 3 converges strongly to a point  $z \in \Omega$ , which is also the unique solution of the minimization problem (42).

**Algorithm 3:** The modified hybrid forward–backward algorithm.

---

**Input:** Input the algorithm parameters  $\{\eta_i\}_{i \geq 1}$ ,  $\{v_i\}_{i \geq 1}$ ,  $\{\gamma_i\}_{i \geq 1}$  and  $\delta$ ;  
**Output:** Output  $\hat{p}$ ;  
Initialize the data  $p_0, p_1 \in E$ ;  
Set  $n \leftarrow 1$ ;  
**while** not converged **do**  
    **if**  $\|p_n - p_{n-1}\| \neq 0$  **then**  
        Update  $\mu_n$  such that  $\mu_n = o(\frac{\eta_n}{\|p_n - p_{n-1}\|})$ ;  
    **else**  
        Choose  $\mu_n$  as any positive number;  
    **end**  
    Update  $w_n = p_n + \mu_n(p_n - p_{n-1})$ ;  
    Update  $q_n = (I + v_n A)^{-1}(I - v_n \sum_{i=1}^N \gamma_i B_i)w_n$ ;  
    Update  $p_{n+1} = (I - \eta_n \delta \nabla \varphi)q_n$ ;  
    Set  $n \leftarrow n + 1$ ;  
**end**  
**return**  $\hat{p} = p_n$ .

---

**5. Numerical Examples**

The purpose of this section is to give a numerical example supporting Theorem 1.

**Example 3.** Let  $E = \mathbb{R}$  with the inner product defined by  $\langle x, y \rangle = xy$  for all  $x, y \in \mathbb{R}$  and the standard norm  $|\cdot|$ . Let  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $Ax = 3x$  for all  $x \in \mathbb{R}$ . Let  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $B_i x = \frac{i}{6}x$  for each  $i = 1, 2, \dots, 30$ . Let  $S_i : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $S_i x = x$  for each  $i = 1, 2, \dots, 30$ . It is easy to check that  $\Omega = \{0\}$  and  $A$  is maximal monotone. Additionally, it is easy to check that  $B_i$  is  $\frac{6}{i}$ -inverse-strongly accretive for each  $i = 1, 2, \dots, 30$  and  $S_i$  is 1-Lipschitzian and 1-strongly monotone for each  $i = 1, 2, \dots, 30$ . Let us choose  $\mu_n = 0$ ,  $\eta_n = \frac{1}{n}$ ,  $v_n = \frac{1}{3}$ ,  $\delta = \frac{1}{4}$  and  $\gamma_i = \frac{2}{3^i} + \frac{1}{30 \times 3^{30}}$  for each  $i \in \{1, 2, \dots, 30\}$ . Then  $\{\mu_n\}$ ,  $\{\eta_n\}$ ,  $\{v_n\}$ ,  $\{\gamma_n\}$  and  $\delta$  satisfy all the conditions of Theorem 1. Therefore Algorithm 2 becomes

$$p_{n+1} = \frac{1}{2} \left( 1 - \frac{1}{4n} \right) \left( 1 - \frac{1}{3} \sum_{i=1}^{30} \left( \frac{2}{3^i} + \frac{1}{30 \times 3^{30}} \right) \frac{i}{6} \right) p_n, \quad n \geq 1. \quad (45)$$

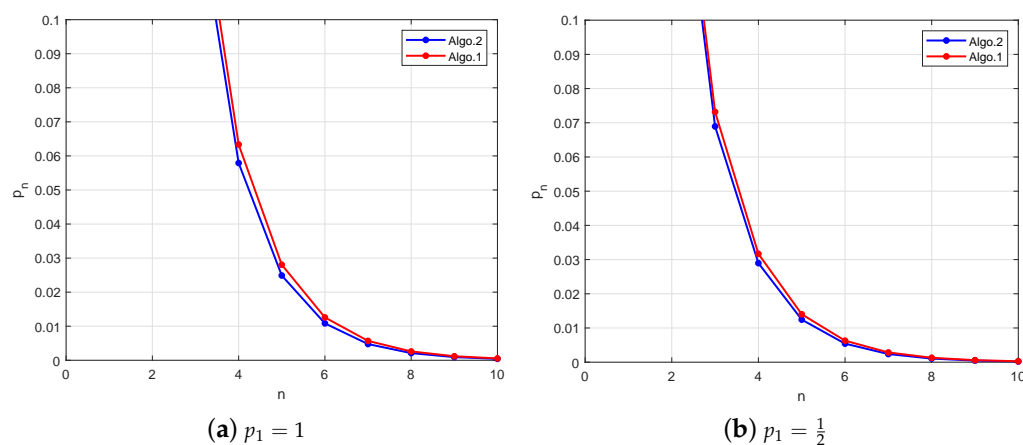
In this example, we compare the efficiency of Algorithm 2 with Algorithm 1 and the efficiency of Algorithm 2 with (6). It is easily seen that Algorithm 2 is more general while Algorithm 1 and (6) are its special cases. As a matter of fact, if  $B_i = B_j$  and  $S_i = S_j$  for all  $i, j \in \{1, 2, \dots, 30\}$ , then Algorithm 2 is reduced to Algorithm 1. Furthermore, if  $\delta = 0$ , then Algorithm 2 collapses to (6).

In Algorithm 1, let  $\{\mu_n\}$ ,  $\{\eta_n\}$ ,  $\{v_n\}$ ,  $\{\gamma_n\}$ ,  $\delta$  and  $A$  be same as those in Example 3. Let  $B, S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Bx = \frac{1}{6}x$  and  $S(x) = x$ , respectively. As shown in Table 1 and Figure 1, it is trivial to see that Algorithm 2 clearly converges much more quickly than Algorithm 1.

In addition, if we set  $\delta = 0$ , then Algorithm 1 is reduced to (6). As shown in Table 2 and Figure 2, Algorithm 2 still converges more quickly than (6).

**Table 1.** Comparison results 1.

Iter	$p_1 = 1$		$p_1 = \frac{1}{2}$	
	Algo. 2	Algo. 1	Algo. 2	Algo. 1
n	$p_n$	$p_n$	$p_n$	$p_n$
1	1.000000000000	1.000000000000	0.500000000000	0.500000000000
2	0.343750000000	0.354166666667	0.171875000000	0.177083333333
3	0.137858072917	0.146339699074	0.068929036458	0.073169849537
4	0.057919537580	0.063346119738	0.028959768790	0.031673059869
5	0.024887301304	0.028043855092	0.012443650652	0.014021927546
6	0.010836345776	0.012580784993	0.005418172888	0.006290392496
7	0.004759714377	0.005693387653	0.002379857189	0.002846693827
8	0.002103623765	0.002592524735	0.001051811882	0.001296262367
9	0.000934030864	0.001185990048	0.000467015432	0.000592995024
10	0.000416205883	0.000544493888	0.000208102941	0.000272246944

**Figure 1.** Comparison results 2.**Table 2.** Comparison results 2.

Iter	$p_1 = \frac{1}{3}$		$p_1 = \frac{1}{6}$	
	Algo. 2	Algo. (6)	Algo. 2	Algo. (6)
n	$p_n$	$p_n$	$p_n$	$p_n$
1	0.333333333333	0.333333333333	0.166666666667	0.166666666667
2	0.114583333333	0.157407407407	0.057291666667	0.078703703704
3	0.045952690972	0.074331275720	0.022976345486	0.037165637860
4	0.019306512527	0.035100880201	0.009653256263	0.017550440101
5	0.008295767101	0.016575415651	0.004147883551	0.008287707825
6	0.003612115259	0.007827279613	0.001806057629	0.003913639806
7	0.001586571459	0.003696215373	0.000793285730	0.001848107686
8	0.000701207922	0.001745435037	0.000350603961	0.000872717519
9	0.000311343621	0.000824233212	0.000155671811	0.000412116606
10	0.000138735294	0.000389221239	0.000069367647	0.000194610619

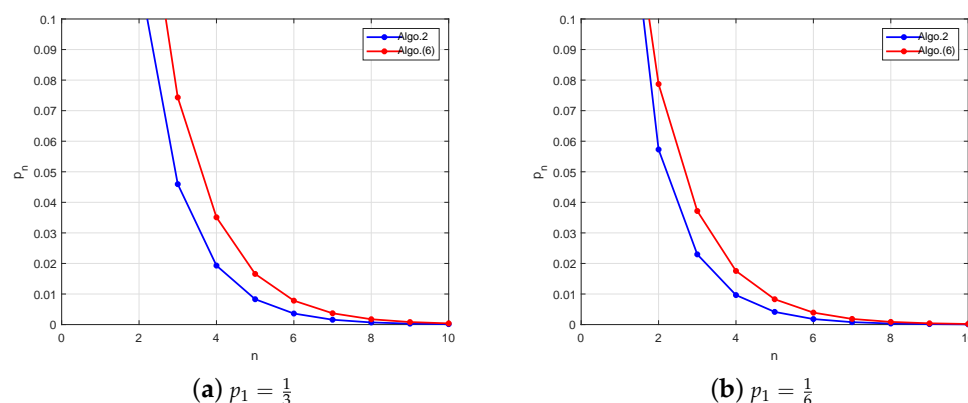


Figure 2. Comparison results 2.

## 6. Conclusions

In this paper, we study a new modified inertial forward–backward algorithm for finding a common solution of the quasi-variational inclusion problem and the variational inequality problem in a  $q$ -uniformly smooth Banach space. Under suitable assumptions, we established a strong convergence theorem for approximating the unique solution of the abovementioned problems. Some applications and numerical experiments of the established results are given to further illustrate the applicability of our results. In our future work, we plan to extend our results to more general Banach spaces.

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