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Oscillation of Second-Order Differential Equations with Multiple and Mixed Delays under a Canonical Operator

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Abstract: In this work, we obtained new sufficient and necessary conditions for the oscillation of second-order differential equations with mixed and multiple delays under a canonical operator. Our methods could be applicable to find the sufficient and necessary conditions for any neutral differential equations. Furthermore, we proved the validity of the obtained results via particular examples. At the end of the paper, we provide the future scope of this study.

Keywords: oscillation; non-oscillation; nonlinear; delay argument; canonical

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1. Introduction

Currently, the study of delay differential equations is a very active area of research since it is much richer than the corresponding theory of ordinary differential equations. In particular, the delay differential equations are very useful to create mathematical models for predictions and analysis in different areas of life sciences, for example neural networks, epidemiology, population dynamics, physiology, and immunology [1–6]. Furthermore, the delay differential equations are frequently used to study the time between the infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, the stages of the life cycle, and so on [6].

Next, we highlight some current developments in oscillation theory for second-order neutral differential equations.

Santra et al. [7] considered the following highly nonlinear neutral differential equations:

$$\left(p(\vartheta)\left(h(\vartheta)+q(\vartheta)h(\vartheta-\gamma)\right)'\right)'+\sum_{k=1}^{m}r_k(\vartheta)G_k(h(\vartheta-\nu_k))=0,\quad \vartheta\geq\vartheta_0, \tag{1}$$

and studied the oscillatory behavior of (1) under a noncanonical operator with various ranges of neutral coefficient q. In another paper [8], Santra et al. considered the second-order delay differential equations with sub-linear neutral coefficients of the form:

$$\left(p(\vartheta)\left(\left(h(\vartheta)+\sum_{k=1}^{m}q_k(\vartheta)h^{\delta_k}(\gamma_k(\vartheta))\right)'\right)^{\delta_1}\right)'+r(\vartheta)h^{\delta_2}(\nu(\vartheta))=0,\quad \vartheta\geq\vartheta_0, \tag{2}$$

where $\gamma_k(\vartheta) < \vartheta$ and $\nu(\vartheta) < \vartheta$, and found new sufficient conditions for the oscillations of (2) under the canonical condition when the neutral coefficient was positive. In a recent paper [9], Santra et al. established some new oscillation theorems for the differential equations of the neutral type with mixed delays under the canonical operator with $0 \leq q < 1$. By using different methods, the following papers were concerned with the oscillation of various classes of half-linear/Emden–Fowler differential equations with different neutral coefficients (e.g., the paper [10] was concerned with neutral differential equations assuming that $0 \leq q(\vartheta) < 1$ and $q(\vartheta) > 1$ where q is the neutral coefficient; the paper [11] was concerned with neutral differential equations assuming that $0 \leq q(\vartheta) < 1$; the paper [12] was concerned with neutral differential equations assuming that $q(\vartheta)$ is non-positive; the papers [13,14] were concerned with the neutral differential equations in the case where $q(\vartheta) > 1$; the paper [15] was concerned with neutral differential equations assuming that $0 \leq q(\vartheta) \leq q_0 < \infty$ and $q(\vartheta) > 1$; the paper [16] was concerned with the neutral differential equations in the case where $0 \leq q(\vartheta) \leq q_0 < \infty$; the paper [17] was concerned with the neutral differential equations in the case when $0 \leq q(\vartheta) = q_0 \neq 1$; whereas the paper [18] was concerned with the differential equations with a nonlinear neutral term assuming that $0 \leq q(\vartheta) \leq q < 1$), which is the same research topic as that of this paper.

For more details on the oscillation theory for second-order neutral differential equations, we refer the reader to the papers [19–25]. We may note that most of the works considered sufficient conditions only, and only a few considered both sufficient and necessary condition for the oscillation of the considered differential equations. Hence, in this study, we established both sufficient and necessary conditions for the oscillation of second-order differential equations of the form:

$$\left(p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}\right)' + \sum_{k=1}^{m} r_k(\vartheta) h^{\delta_2}(\nu_k(\vartheta)) = 0, \quad \vartheta \geq \vartheta_0, \tag{3}$$

where:

$$g(\vartheta) = h(\vartheta) + q(\vartheta)h(\gamma(\vartheta))$$

such that:

(C1) $\nu_k \in C([0,\infty),\mathbb{R})$, $\gamma \in C^2([0,\infty),\mathbb{R})$, if we consider the simple delay then $\nu_k(\vartheta) < \vartheta$ for $k = 1,2,\cdots,m$, $\gamma(\vartheta) < \vartheta$, $\lim_{\vartheta\to\infty} \nu_k(\vartheta) = \infty$, and $\lim_{\vartheta\to\infty} \gamma(\vartheta) = \infty$;
(C2) $\nu_k \in C([0,\infty),\mathbb{R})$, $\gamma \in C^2([0,\infty),\mathbb{R})$, if we consider the advanced delay, then (C1) can be modified by $\nu_k(\vartheta) > \vartheta$ for $k = 1,2,\cdots,,m$, $\gamma(\vartheta) < \vartheta$, and $\lim_{\vartheta\to\infty} \gamma(\vartheta) = \infty$;
(C3) $p \in C^1([0,\infty),\mathbb{R})$, $0 < p(\vartheta)$; $r_k \in C([0,\infty),\mathbb{R})$, $r_k(\vartheta) \geq 0$ for $\vartheta \geq 0$, and $k = 1,2,\ldots,m$; $\lim_{\vartheta\to\infty} P(\vartheta) = \infty$ where $P(\vartheta) = \int_0^{\vartheta} p^{-1/\delta_1}(s)\,\mathrm{d}s$;
(C4) δ_1 and δ_2 are the quotient of two odd positive integers;
(C5) $q \in C^2([0,\infty),\mathbb{R}_+)$ and $0 \leq q(\vartheta) \leq q < 1$;
(C6) $\int_{\vartheta_0}^{\infty} \sum_{k=1}^{m} r_k(\eta)d\eta = \infty$;

2. Preliminaries

In this section, we provide some preliminary lemmas, which we need for our further work.

Lemma 1. *Considering* (C1)–(C5) *for $\vartheta \geq \vartheta_0$ and h as an eventually positive solution of* (3), *we have:*

$$g(\vartheta) > 0, \quad g'(\vartheta) > 0, \quad \textit{and} \quad \left(p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}\right)' \leq 0 \quad \textit{for some} \quad \vartheta \geq \vartheta_1 \geq \vartheta_0. \tag{4}$$

Proof. We considered $h(\vartheta)$ to be an eventually positive solution of (3). Therefore, $g(\vartheta) > 0$, and for $\vartheta_0 \geq 0$, we have $h(\vartheta) > 0$, $h(\nu_k(\vartheta)) > 0$, $h(\gamma(\vartheta)) > 0$ for $\vartheta \geq \vartheta_0$ and $k = 1, 2, \ldots, m$. From (3), we obtain that:

$$\left(p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}\right)' = -\sum_{k=1}^{m} r_k(\vartheta) h^{\delta_2}(\nu_k(\vartheta)) \leq 0 \tag{5}$$

which proves that $p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}$ is non-increasing for $\vartheta \geq \vartheta_0$. Next, to prove $g > 0$ and $p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}$ is positive for $\vartheta \geq \vartheta_1 > \vartheta_0$, we assumed that $p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1} \leq 0$ for $\vartheta \geq \vartheta_1$, and we can find $c_1 > 0$ such that:

$$p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1} \leq -c_1\,,$$

that is,

$$g'(\vartheta) \leq (-c_1)^{1/\delta_1} p^{-1/\delta_1}(\vartheta)\,.$$

Upon integration from ϑ_1 to ϑ, we obtain:

$$g(\vartheta) - g(\vartheta_1) \leq (-c_1)^{1/\delta_1}\left(P(\vartheta) - P(\vartheta_1)\right).$$

If we take the limit on both sides as $\vartheta \to \infty$, we obtain $\lim_{\vartheta\to\infty} g(\vartheta) \leq -\infty$, which contradicts $g(\vartheta) > 0$. Hence, $p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1} > 0$ for $\vartheta \geq \vartheta_1$, i.e., $g'(\vartheta) > 0$ for $\vartheta \geq \vartheta_1$.

Thus, the lemma is proven. □

Lemma 2. *Assuming* (C1)–(C5) *for* $\vartheta \geq \vartheta_0$ *and* h *as an eventually positive solution of* (3), *we have:*

$$h(\vartheta) \geq (1-q)g(\vartheta) \quad for \quad \vartheta \geq \vartheta_1. \tag{6}$$

Proof. We considered $h(\vartheta)$ to be an eventually positive solution of (3). Therefore, $g(\vartheta) > 0$, and for $\vartheta \geq \vartheta_1 > \vartheta_0$ we have:

$$\begin{aligned}
(1-q)g(\vartheta) &\leq (1-q(\vartheta))g(\vartheta) \\
&\leq g(\vartheta) - q(\vartheta)g(\gamma(\vartheta)) \\
&= h(\vartheta) + q(\vartheta)h(\gamma(\vartheta)) - q(\vartheta)\Big(h(\gamma(\vartheta)) + q(\gamma(\vartheta))h(\gamma(\gamma(\vartheta)))\Big) \\
&\leq h(\vartheta) - q(\vartheta)q(\gamma(\vartheta))h(\gamma(\gamma(\vartheta))) \\
&\leq h(\vartheta).
\end{aligned}$$

Hence, g satisfies (6) for $\vartheta \geq \vartheta_1$. □

Remark 1. *Lemmas 1 and 2 hold for* $\delta_1 > \delta_2$ *or* $\delta_1 < \delta_2$.

Lemma 3. *Assuming* (C1)–(C5) *for* $\vartheta \geq \vartheta_0$ *and* h *as an eventually positive solution of* (3), *we have:*

$$g(\vartheta) \geq \frac{1}{2} P(\vartheta) \Delta^{1/\delta_1}(\vartheta) \quad for \quad \vartheta \geq \vartheta_3 \tag{7}$$

where:

$$\Delta(\vartheta) = \int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(\eta)\Big((1-q)g(\nu_k(\eta))\Big)^{\delta_2} \mathrm{d}\eta\,.$$

Proof. We assumed that $h(\vartheta)$ is an eventually positive solution of (3). Therefore, $g(\vartheta) > 0$, and for $\vartheta_0 \geq 0$, we have that $h(\vartheta) > 0$, $h(\nu_k(\vartheta)) > 0$, and $h(\gamma(\vartheta)) > 0$ for $\vartheta \geq \vartheta_0$ and

$k = 1, 2, \ldots, m$. Thus, Lemmas 1 and 2 hold for $\vartheta \geq \vartheta_1$. By Lemma 1 and for $\vartheta_2 > \vartheta_1$, we have $g'(\vartheta) > 0$ for $\vartheta \geq \vartheta_2$. Therefore, for $\vartheta_3 > \vartheta_2$ and $c_2 > 0$, we have $g(\vartheta) \geq c_2$ where $\vartheta \geq \vartheta_3$. By Lemma 2, we find $h(\vartheta) \geq (1-q)g(\vartheta)$ for $\vartheta \geq \vartheta_3$, and (3) gives:

$$\Big(p(\vartheta)\big(g'(\vartheta)\big)^{\delta_1}\Big)' + \sum_{k=1}^{m} r_k(\vartheta)\Big((1-q)g\big(\nu_k(\vartheta)\big)\Big)^{\delta_2} \leq 0\,. \tag{8}$$

Integrating (8) from ϑ to $+\infty$, we obtain:

$$[p(s)\big(g'(s)\big)^{\delta_1}]_{\vartheta}^{\infty} + \int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(s)\Big((1-q)g\big(\nu_k(s)\big)\Big)^{\delta_2} \mathrm{d}s \leq 0\,.$$

Since $p(\vartheta)\big(g'(\vartheta)\big)^{\delta_1}$ is non-decreasing and positive, so $\lim_{\vartheta\to\infty} p(\vartheta)\big(g'(\vartheta)\big)^{\delta_1}$ finitely exists and is positive.

$$p(\vartheta)\big(g'(\vartheta)\big)^{\delta_1} \geq \int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(s)\Big((1-q)g\big(\nu_k(s)\big)\Big)^{\delta_2} \mathrm{d}s\,,$$

that is,

$$\begin{aligned} g'(\vartheta) &\geq p^{-1/\delta_1}(\vartheta)\Big[\int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(s)\Big((1-q)g\big(\nu_k(s)\big)\Big)^{\delta_2} \mathrm{d}s\Big]^{1/\delta_1} \\ &= (1-q)^{\delta_2/\delta_1} p^{-1/\delta_1}(\vartheta)\Big[\int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(s) g^{\delta_2}\big(\nu_k(s)\big)\,\mathrm{d}s\Big]^{1/\delta_1}. \end{aligned} \tag{9}$$

Using (C3), there exists $\vartheta_3 > \vartheta_2$ for which $P(\vartheta) - P(\vartheta_3) \geq \frac{1}{2}P(\vartheta)$ for $\vartheta \geq \vartheta_3$. Integrating (9) from ϑ_3 to ϑ, we have:

$$\begin{aligned} g(\vartheta) - g(\vartheta_3) &\geq \int_{\vartheta_3}^{\vartheta} p^{-1/\delta_1}(\kappa)\Big[\int_{\kappa}^{\infty} \sum_{k=1}^{m} r_k(\theta)\Big((1-q)g\big(\nu_k(\theta)\big)\Big)^{\delta_2} \mathrm{d}\theta\Big]^{1/\delta_1} \mathrm{d}\kappa \\ &\geq \int_{\vartheta_3}^{\vartheta} p^{-1/\delta_1}(\kappa)\Big[\int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(\theta)\Big((1-q)g\big(\nu_k(\theta)\big)\Big)^{\delta_2} \mathrm{d}\theta\Big]^{1/\delta_1} \mathrm{d}\kappa, \end{aligned}$$

that is,

$$\begin{aligned} g(\vartheta) &\geq (P(\vartheta) - P(\vartheta_3))\Big[\int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(\theta)\Big((1-q)g\big(\nu_k(\theta)\big)\Big)^{\delta_2} \mathrm{d}\theta\Big]^{1/\delta_1} \\ &\geq \frac{1}{2}P(\vartheta)\Big[\int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(\theta)\Big((1-q)g\big(\nu_k(\theta)\big)\Big)^{\delta_2} \mathrm{d}\theta\Big]^{1/\delta_1}. \end{aligned} \tag{10}$$

Hence,

$$g(\vartheta) \geq \frac{1}{2}P(\vartheta)\Delta^{1/\delta_1}(\vartheta) \quad \text{for} \quad \vartheta \geq \vartheta_3$$

where:

$$\Delta(\vartheta) = \int_{\vartheta}^{\infty} \sum_{k=1}^{m} r_k(\eta)\Big((1-q)g\big(\nu_k(\eta)\big)\Big)^{\delta_2} \mathrm{d}\eta\,.$$

Hence, g satisfies (7) for $\vartheta \geq \vartheta_3$. □

3. Oscillation Theorems

In this section, we present our main results from which we found the necessary and sufficient conditions for the oscillation of (3).

Theorem 1. *Under Assumptions* (C2)–(C5) *for* $\vartheta \geq \vartheta_0$ *and* $\delta_2 > \delta_1$*, all solutions of* (3) *oscillate if and only if:*

$$\int_0^\infty p^{-1/\delta_1}(s)\Big[\int_s^\infty \sum_{k=1}^m r_k(\eta)\,\mathrm{d}\eta\Big]^{1/\delta_1}\mathrm{d}s = \infty\,. \tag{11}$$

Proof. To prove the sufficient part by the contradiction, we assumed $h(\vartheta)$ is an eventually positive solution of (3). Therefore, $g(\vartheta) > 0$, and for $\vartheta_0 \geq 0$, we have that $h(\vartheta) > 0$, $h(\nu_k(\vartheta)) > 0$, and $h(\gamma(\vartheta)) > 0$ for $\vartheta \geq \vartheta_0$ and $k = 1, 2, \ldots, m$. Thus, Lemmas 1 and 2 hold for $\vartheta \geq \vartheta_1$. By Lemma 1 and $\vartheta_2 > \vartheta_1$, we have $g'(\vartheta) > 0$ where $\vartheta \geq \vartheta_2$. Again, by Lemma 2, it follows that $h(\vartheta) \geq (1-q)g(\vartheta)$ for $\vartheta \geq \vartheta_3$. Then, preceding as in the proof of Lemma 3, we have (9). Using (C2) and that $g(\vartheta)$ is non-decreasing on (9), we obtain:

$$g'(\vartheta) \geq (1-q)^{\delta_2/\delta_1} p^{-1/\delta_1}(\vartheta)\Big[\int_\vartheta^\infty \sum_{k=1}^m r_k(s)\,\mathrm{d}s\Big]^{1/\delta_1} g^{\delta_2/\delta_1}(\vartheta)\,,$$

that is,

$$\frac{g'(\vartheta)}{g^{\delta_2/\delta_1}(\vartheta)} \geq (1-q)^{\delta_2/\delta_1} p^{-1/\delta_1}(\vartheta)\Big[\int_\vartheta^\infty \sum_{k=1}^m r_k(s)\,\mathrm{d}s\Big]^{1/\delta_1}.$$

Since $\delta_2 > \delta_1$ and after integration on both sides from ϑ_3 to $+\infty$, we have:

$$(1-q)^{\delta_2/\delta_1}\int_{\vartheta_3}^\infty p^{-1/\delta_1}(s)\Big[\int_s^\infty \sum_{k=1}^m r_k(\psi)\,\mathrm{d}\psi\Big]^{1/\delta_1}\mathrm{d}s \leq \int_{\vartheta_3}^\infty \frac{g'(\eta)}{g^{\delta_2/\delta_1}(\eta)}\,\mathrm{d}\eta < \infty\,,$$

which contradicts (11); hence, the proof of the sufficient part is complete.

Next, we prove the necessary part by a contrapositive argument. If (11) does not hold, then for $\varepsilon > 0$ and $\vartheta' \geq \vartheta_0$, we can obtain:

$$\int_{\vartheta'}^\infty p^{-1/\delta_1}(s)\Big[\int_s^\infty \sum_{k=1}^m r_k(\eta)\,\mathrm{d}\eta\Big]^{1/\delta_1}\mathrm{d}s < \varepsilon \quad \text{for} \quad \vartheta \geq \vartheta',$$

where $2\varepsilon = \left[\frac{1}{1-q}\right]^{-\delta_2/\delta_1} > 0$. Consider a set:

$$S = \left\{h \in C([0,\infty)) : \frac{1}{2} \leq h(\vartheta) \leq \frac{1}{1-q} \text{ for } \vartheta \geq \vartheta'\right\}$$

and $\Omega : S \to S$ as:

$$(\Omega h)(\vartheta) = \begin{cases} 0 & \text{if } \vartheta \leq \vartheta', \\ \frac{1+a}{2(1-q)} - q(\vartheta)h(\gamma(\vartheta)) & \\ + \int_T^\vartheta p^{-1/\delta_1}(s)\Big[\int_s^\infty \sum_{k=1}^m r_k(\psi)h^{\delta_2}(\nu_k(\psi))\,\mathrm{d}\psi\Big]^{1/\delta_1}\mathrm{d}s & \text{if } \vartheta > \vartheta'\,. \end{cases}$$

Next, we prove $(\Omega h)(\vartheta) \in S$. For $h(\vartheta) \in S$,

$$\begin{aligned}(\Omega h)(\vartheta) &\leq \frac{1+a}{2(1-q)} + \int_T^\vartheta p^{-1/\delta_1}(s)\Big[\int_s^\infty \sum_{k=1}^m r_k(\psi)\Big(\frac{1}{1-q}\Big)^{\delta_2}\mathrm{d}\psi\Big]^{1/\delta_1}\mathrm{d}s \\ &\leq \frac{1+a}{2(1-q)} + \Big(\frac{1}{1-q}\Big)^{\delta_2/\delta_1} \times \varepsilon \\ &= \frac{1+a}{2(1-q)} + \frac{1}{2} = \frac{1}{1-q}.\end{aligned}$$

Again, for $h(\vartheta) \in S$:

$$(\Omega h)(\vartheta) \geq \frac{1+a}{2(1-q)} - q(\vartheta) \times \frac{1}{1-q} + 0 \geq \frac{1+a}{2(1-q)} - \frac{a}{1-q} = \frac{1}{2}.$$

Hence, Ω maps from S to S.

Next, we plan to search a fixed point of Ω in S that is a non-oscillatory solution (specifically eventually positive) of (3) for which we define a sequence in S by:

$$\begin{aligned} h_0(\vartheta) &= 0 \quad \text{for } \vartheta \geq \vartheta_0, \\ h_1(\vartheta) &= (\Omega u_0)(\vartheta) = \begin{cases} 0 & \text{if } \vartheta < \vartheta' \\ \frac{1}{2} & \text{if } \vartheta \geq \vartheta' \end{cases}, \\ h_{n+1}(\vartheta) &= (\Omega u_n)(\vartheta) \quad \text{for } n \geq 1, \vartheta \geq \vartheta'. \end{aligned}$$

Here, we see $h_1(\vartheta) \geq h_0(\vartheta)$ for each fixed ϑ and $\frac{1}{2} \leq h_{n-1}(\vartheta) \leq h_n(\vartheta) \leq \frac{1}{1-q}, \quad \vartheta \geq \vartheta'$ for $n \geq 1$. Therefore, h_n converges pointwise to a function h, i.e., $\Omega h = h \in S$, and hence, h is an eventually positive solution.

Thus, the theorem is proven. □

Theorem 2. *Under Assumptions* (C1) *and* (C3)–(C6) *for $\vartheta \geq \vartheta_0$, every solution of* (3) *oscillates.*

Proof. Let $h(\vartheta)$ be an eventually positive solution of (3). Therefore, $g(\vartheta) > 0$, and for $\vartheta_0 \geq 0$, we have that $h(\vartheta) > 0$, $h(\nu_k(\vartheta)) > 0$, and $h(\gamma(\vartheta)) > 0$ for $\vartheta \geq \vartheta_0$ and $k = 1, 2, \ldots, m$. Thus, Lemmas 1 and 2 hold for $\vartheta \geq \vartheta_1$. By Lemma 1 and $\vartheta_2 > \vartheta_1$, we find $g'(\vartheta) > 0$ where $\vartheta \geq \vartheta_2$. Then, for $\vartheta_3 > \vartheta_2$ and $c_2 > 0$, we obtain $g(\vartheta) \geq c_2$ for $\vartheta \geq \vartheta_3$. By Lemma 2, it follows that $h(\vartheta) \geq (1-q)g(\vartheta) \geq (1-q)c_2$ for $\vartheta \geq \vartheta_3$, and (3) gives:

$$\left(p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}\right)' + \sum_{k=1}^{m} r_k(\vartheta)\left((1-q)c_2\right)^{\delta_2} \leq 0. \tag{12}$$

Integrating (13) from ϑ_3 to ϑ, we obtain:

$$\left((1-q)c_2\right)^{\delta_2} \int_{\vartheta_3}^{\vartheta} \sum_{k=1}^{m} r_k(\eta) d\eta \leq -\left[p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}\right]_{\vartheta_3}^{\vartheta}. \tag{13}$$

Since $\lim_{\vartheta \to \infty} p(\vartheta)\left(g'(\vartheta)\right)^{\delta_1}$ exists finitely, letting $\vartheta \to \infty$, we have:

$$\left((1-q)c_2\right)^{\delta_2} \int_{\vartheta_3}^{\vartheta} \sum_{k=1}^{m} r_k(\eta) d\eta < \infty \tag{14}$$

which contradicts (C6).

Thus, the theorem is proven. □

Theorem 3. *Under Assumptions* (C1) and (C3)–(C5) *for $\vartheta \geq \vartheta_0$ and $\delta_2 < \delta_1$, each solution of* (3) *oscillates if:*

$$\int_0^{\infty} \sum_{k=1}^{m} r_k(\zeta)[(1-q)P(\nu_k(\zeta))]^{\delta_2} \, \mathrm{d}\zeta = \infty. \tag{15}$$

Proof. On the contrary, we assumed $h(\vartheta)$ to be an eventually positive solution of (3). By the same argument used in the proof of Lemma 3, we have (7) for $\vartheta \geq \vartheta_2 \geq \vartheta_3$. Using (C3), there exists $\vartheta_3 > \vartheta_2$ for which $P(\vartheta) - P(\vartheta_3) \geq \frac{1}{2}P(\vartheta)$ for $\vartheta \geq \vartheta_3$. Now,

$$\begin{aligned}\Delta'(\vartheta) &= -\sum_{k=1}^{m} r_k(\vartheta)\Big((1-q)g\big(\nu_k(\vartheta)\big)\Big)^{\delta_2} \\ &\leq -\frac{1}{2^{\delta_2}}\sum_{k=1}^{m} r_k(\vartheta)[(1-q)P\big(\nu_k(\vartheta)\big)]^{\delta_2}\Delta^{\delta_2/\delta_1}\big(\nu_k(\vartheta)\big) \leq 0 \end{aligned} \tag{16}$$

which shows that $h(\vartheta)$ is decreasing on $[\vartheta_4, \infty)$ and $\lim_{\vartheta\to\infty}\Delta(\vartheta)$ exists. Using (16) and (C1), we obtain:

$$\begin{aligned}\Big[\Delta^{1-\delta_2/\delta_1}(\vartheta)\Big]' &= (1-\delta_2/\delta_1)\Delta^{-\delta_2/\delta_1}(\vartheta)\Delta'(\vartheta) \\ &\leq -\frac{1-\delta_2/\delta_1}{2^{\delta_2}}\sum_{k=1}^{m} r_k(\vartheta)[(1-q)P\big(\nu_k(\vartheta)\big)]^{\delta_2}\Delta^{\delta_2/\delta_1}\big(\nu_k(\vartheta)\big)\Delta^{-\delta_2/\delta_1}(\vartheta) \\ &\leq -\frac{1-\delta_2/\delta_1}{2^{\delta_2}}\sum_{k=1}^{m} r_k(\vartheta)[(1-q)P\big(\nu_k(\vartheta)\big)]^{\delta_2}. \end{aligned} \tag{17}$$

Integrating (17) from ϑ_4 to ϑ, we have:

$$\Big[h^{1-\delta_2/\delta_1}(s)\Big]_{\vartheta_4}^{\vartheta} \leq -\frac{1-\delta_2/\delta_1}{2^{\delta_2}}\int_{\vartheta_4}^{\vartheta}\sum_{k=1}^{m} r_k(s)[(1-q)P\big(\nu_k(s)\big)]^{\delta_2}\mathrm{d}s,$$

that is,

$$\begin{aligned}\frac{1-\delta_2/\delta_1}{2^{\delta_2}}\Big[\int_0^{\infty}\sum_{k=1}^{m} r_k(s)[(1-q)P\big(\nu_k(s)\big)]^{\delta_2}\,\mathrm{d}s\Big] &\leq -\Big[h^{1-\delta_2/\delta_1}(s)\Big]_{\vartheta_4}^{\vartheta} \\ &< h^{1-\delta_2/\delta_1}(\vartheta_4) < \infty\end{aligned}$$

which contradicts (15).

Thus, the theorem is proven. □

4. Examples

In this section, we present two examples to illustrate the results.

Example 1. *Assume the differential equations:*

$$\Big(\big((h(\vartheta) + e^{-\vartheta}h(\gamma(\vartheta)))'\big)^{101/3}\Big)' + \vartheta(h(\vartheta-2))^{111/3} + (\vartheta+1)(h(\vartheta-3))^{111/3} = 0. \tag{18}$$

Here, $\delta_2 = 111/3 > \delta_1 = 101/3$, $p(\vartheta) = 1$, $0 < q(\vartheta) = e^{-\vartheta} < 1$, $\nu_k(\vartheta) = \vartheta - (k+1)$ *with index* $k = 1, 2$. *To check* (11), *we have:*

$$\begin{aligned}\int_{\vartheta_0}^{\infty}\Big[\frac{1}{p(s)}\Big[\int_s^{\infty}\sum_{k=1}^{m} r_k(\psi)\,\mathrm{d}\psi\Big]\Big]^{1/\delta_1}\mathrm{d}s &\geq \int_{\vartheta_0}^{\infty}\Big[\frac{1}{p(s)}\Big[\int_s^{\infty} r_1(\psi)\,d\psi\Big]\Big]^{1/\delta_1}\mathrm{d}s \\ &\geq \int_2^{\infty}\Big[\int_s^{\infty}\psi\,d\psi\Big]^{3/101}\mathrm{d}s = \infty.\end{aligned}$$

Therefore, by Theorem 1, all solutions of (18) *are oscillatory.*

Example 2. *Consider the differential equations:*

$$\left(e^{-\vartheta}\left(\left(h(\vartheta)+e^{-\vartheta}h(\gamma(\vartheta))\right)'\right)^{1101/3}\right)' + \frac{1}{\vartheta+1}(h(\vartheta-2))^{113/3} + \frac{1}{\vartheta+2}(h(\vartheta-3))^{113/3} = 0, \tag{19}$$

where $\delta_2 = 113/3 < \delta_1 = 1101/3$, $p(\vartheta) = e^{-\vartheta}$, $0 < q(\vartheta) = e^{-\vartheta} < 1$, *and* $\nu_k(\vartheta) = \vartheta - (k+1)$ *with index* $k = 1, 2$ *and* $P(\vartheta) = \int_0^\vartheta e^{3s/1101} ds = \frac{1101}{3}(e^{3\vartheta/1101} - 1)$. *To check* (15), *we have:*

$$\begin{aligned}
&\frac{1}{(2)^{\delta_2}}\Big[\int_0^\infty \sum_{k=1}^m r_k(\zeta)[(1-q)P(\nu_k(\zeta))]^{\delta_2}\, d\zeta\Big] \\
&\geq \frac{1}{(2)^{113/3}}\int_0^\infty r_1(\zeta)[(1-q)P(\nu_1(\zeta))]^{\delta_2}\, d\zeta \\
&= \frac{1}{(2)^{113/3}}\int_0^\infty \frac{1}{\zeta+1}\Big[(1-q)\frac{1101}{3}\left(e^{3(\zeta-2)/1101}-1\right)\Big]^{113/3} d\zeta = \infty.
\end{aligned}$$

Therefore, by Theorem 3, every solution of (19) *oscillates.*

5. Conclusions and Open Problem

By this work, we obtained sufficient and necessary conditions for the oscillation of a highly nonlinear neutral differential Equation (3) when $q \in [0,1)$. In [26], we obtained the the sufficient and necessary conditions for the oscillatory or asymptotic behavior of a nonlinear impulsive differential system of the neutral type when the neutral coefficient lies in $(-1, 0]$. Therefore, we can claim that the method adopted in the current paper could be applicable for different kinds of second-order nonlinear neutral delay differential equations when the neutral coefficient lies in either $(-1, 0]$ or $[0, 1)$. Based on this paper and [19,20,25,27–31] an open problem can arise: *"Is it possible to study the oscillation of all solutions of* (3) *to obtain necessary and sufficient conditions when* $q \in (-\infty, -1]$ *and* $q \in (1, \infty)$*?"*

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