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Lie Symmetry Analysis, Self-Adjointness and Conservation Law for a Type of Nonlinear Equation

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Abstract: In the present paper, we mainly focus on the symmetry of the solutions of a given PDE via Lie group method. Meanwhile we transfer the given PDE to ODEs by making use of similarity reductions. Furthermore, it is shown that the given PDE is self-adjoining, and we also study the conservation law via multiplier approach.

Keywords: lie symmetry analysis; nonlinear self-adjointness; conservation law

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1. Introduction

When investigating the classification of the solutions of differential equations, Sophus Lie introduced the notion of the continuous transformation group, which was named the Lie group by the following people in memory of him. Afterwards, the application of Lie groups has been developed by lots of mathematicians, such as Ovsiannikov [1], Ibragimov [2], Olver [3] and Bluman et al. [4,5]. Symmetry is one of the intrinsic features of a partial differential equation (PDE). Lie group analysis is one of the most powerful methods to study the symmetry of their solutions. Based on the symmetries of a PDE, many other properties of the equation, such as exact solutions to conservation laws can be successively considered. In any case, another method to solve the nonlinear evolution equation (such as Burgers equation) is to use Hopf cole transformation in a high order spectral column setting [6,7].

The conservation laws (Cls) have drawn great attention from the mathematical physicists. In the past few decades, many methods for dealing with the Cls were derived, the most famous being Noether's approach [8], multiplier approach and Ibragimov's method [9]. Moreover, Cls for nonlinearly self-adjoint systems can be constructed systematically by pairs of symmetries and adjoint symmetries. In particular, ref. [10] supplies the most general connection between Cls and pairs of symmetries and adjoint symmetries for non-self-adjoint systems. On the contribution of the Lie symmetry method, significant studies have been performed on the integrability of the nonlinear PDEs, group classification, optimal system, reduced solutions and conservation laws, such as [11–23], and references therein.

In [24,25], a new completely integrable equation

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x. \tag{1}$$

was proposed. It was shown that this equation has no smooth solitons and it has bi-Hamiltonian structure and Lax pair. At the end of [25], Qiao introduced a more general PDE

$$m_t = \frac{1}{2}\left(\frac{1}{m^k}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^k}\right)_x. \tag{2}$$

with a constant $k \in \mathbb{R}$.

If $k = 0$, it is a trivial case; if $k = -1$, it is a linear case; if $k = \frac{1}{2}$, it is a Harry–Dym case; if $k = 2$, it is reduced to (1). Unless explicitly stated, throughout this paper we always assume $k \neq 0, -1$. Sakovich in [26] supplied a transformation which was used to derive smooth soliton solutions of the new equation from the known rational and soliton solutions of the old one. To the authors' knowledge, the Lie symmetry analysis, self-adjointness and conservation laws of (2) have not been studied. In this paper, we mainly focus on the above aspects on (2) for $k \neq 0, -1$.

If we perform the transformation $u(x,t) = \frac{1}{m(x,t)}$, Equation (2) is reduced to

$$u_t + \frac{k(k-1)(k-2)}{2}u^{k-1}u_x^3 + \frac{3k(k-1)}{2}u^k u_x u_{xx} + \frac{k}{2}u^{k+1}u_{xxx} - \frac{k}{2}u^{k+1}u_x = 0. \tag{3}$$

From now on, we will focus on the study of Equation (2), which is equivalent to that of Equation (2).

2. Lie Symmetries of Equation (2)

In this section, we shall investigate the Lie symmetry analysis of Equation (2).

First of all, let us consider the Lie group of point transformations

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{4}$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above transformation group of transformations can expressed as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{5}$$

where $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ are coefficient functions of the vector field to be determined.

Theorem 1. *For the arbitrary real parameter k, if $k \neq 0, -1$, the complete group classification of Equation (2) is as follows:* (1) *If $k \neq \frac{1}{2}$, the vector field of Equation (2) is*

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial t},\ V_3 = -(k+1)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{6}$$

(2) *If $k = \frac{1}{2}$, the vector field of Equation (2) is*

$$\begin{aligned} &V_1 = \tfrac{\partial}{\partial x},\ V_2 = \tfrac{\partial}{\partial t},\ V_3 = -\tfrac{3}{2}t\tfrac{\partial}{\partial t} + u\tfrac{\partial}{\partial u}, \\ &V_4 = \tfrac{1}{2}e^x\tfrac{\partial}{\partial x} + e^x u\tfrac{\partial}{\partial u},\ V_5 = -\tfrac{1}{2}e^{-x}\tfrac{\partial}{\partial x} + e^{-x}u\tfrac{\partial}{\partial u}. \end{aligned} \tag{7}$$

Proof. The third prolongation of V is

$$\begin{aligned} \mathrm{pr}^{(3)}V = V &+ \phi^x\tfrac{\partial}{\partial u_x} + \phi^t\tfrac{\partial}{\partial u_t} + \phi^{xx}\tfrac{\partial}{\partial u_{xx}} + \phi^{xt}\tfrac{\partial}{\partial u_{xt}} + \phi^{tt}\tfrac{\partial}{\partial u_{tt}} \\ &+ \phi^{xxx}\tfrac{\partial}{\partial u_{xxx}} + \phi^{xtt}\tfrac{\partial}{\partial u_{xtt}} + \phi^{xxt}\tfrac{\partial}{\partial u_{xxt}} + \phi^{ttt}\tfrac{\partial}{\partial u_{ttt}}, \end{aligned} \tag{8}$$

where the explicit expressions of unknown functions $\phi^t, \phi^x, \phi^{xx}, \phi^{xxx}$ are as follows

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \quad (9)$$
$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, \quad (10)$$
$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \quad (11)$$
$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}. \quad (12)$$

In (9)–(12), D_x and D_t are both total derivatives with respect to x, t, respectively. We denote the left hand side of Equation (2) by Δ. In view of $\mathrm{pr}^{(3)}V(\Delta)|_{\Delta=0} = 0$, we obtain

$$\begin{aligned} 0 \;=\; & \phi^t + \tfrac{k(k-1)^2(k-2)}{2}u^{k-2}\phi u_x^3 + \tfrac{3k(k-1)(k-2)}{2}u^{k-1}\phi^x u_x^2 \\ & + \tfrac{3k^2(k-1)}{2}u^{k-1}\phi u_x u_{xxx} + \tfrac{3k(k-1)}{2}u^k\phi^x u_{xx} + \tfrac{3k(k-1)}{2}u^k\phi^{xx}u_x \\ & + \tfrac{k(k+1)}{2}u^k\phi u_{xxx} + \tfrac{k}{2}u^{k+1}\phi^{xxx} - \tfrac{k(k+1)}{2}u^k\phi u_x - \tfrac{k}{2}u^{k+1}\phi^x. \end{aligned} \quad (13)$$

We substitute (9)–(12) into (8) and replace u_t by

$$-\left(\frac{k(k-1)(k-2)}{2}u^{k-1}u_x^3 + \frac{3k(k-1)}{2}u^k u_x u_{xx} + \frac{k}{2}u^{k+1}u_{xxx} - \frac{k}{2}u^{k+1}u_x\right)$$

whenever it appears in (13) and divide it into several cases to discuss it.

Firstly we assume $k \neq 1, 2$. By solving (13), one can get following forms for the infinitesimal elements ξ, τ, ϕ:

$$\begin{cases} \tau_x = \tau_u = 0,\ \tau = \tau(t), \\ \xi_u = 0,\ \phi_{uu} = 0, \\ u\tau_t + (k+1)\phi - 3u\xi_x = 0, \\ u\phi_u + u\tau_t + k\phi - 3u\xi_x = 0, \\ 2u\phi_u + u\tau_t + (k-1)\phi - 3u\xi_x = 0. \end{cases} \quad (14)$$

If $k \neq \frac{1}{2}$, Equation (14) implies $\xi = c,\ \tau = -(k+1)at + b,\ \phi = au$, which yields the vector fields in (6).

If $k = \frac{1}{2}$, Equation (14) implies

$$\begin{aligned} \xi &= \tfrac{1}{2}c_1e^x - \tfrac{1}{2}c_2e^{-x} + c_4, \\ \tau &= -\tfrac{3}{2}c_3t + c_5, \\ \phi &= (c_1e^x + c_2e^{-x} + c_3)u, \end{aligned}$$

which yields the vector fields in (7).

Secondly, we assume $k = 1$ or $k = 2$, similar process yields the vector fields in (6). □

It is necessary to check that the vector fields form a Lie algebra, respectively. Taking (7) as an example,

$$\begin{aligned} &[V_1, V_2] = 0,\ [V_1, V_3] = 0,\ [V_1, V_4] = V_4,\ [V_1, V_5] = -V_5, \\ &[V_2, V_3] = -\frac{3}{2}V_2,\ [V_2, V_4] = 0,\ [V_2, V_5] = 0,\ [V_3, V_4] = 0, \\ &[V_3, V_5] = 0,\ [V_4, V_5] = \frac{1}{2}V_1. \end{aligned}$$

We denote this Lie algebra by $\mathfrak{L}$. If we drop the relation in (7) of $V_1, \cdots, V_5$, it will be an abstract Lie algebra. The vector fields corresponding to $V_1, \cdots, V_5$ supply a Lie algebra representation of $\mathfrak{L}$ (see [27]). In addition, $\mathfrak{L}$ can be decomposed as $\mathfrak{L} = \mathfrak{s} \oplus \mathfrak{n}$, where $\mathfrak{s} = \mathrm{span}\{V_1, V_4, V_5\}$ denotes the simple ideal of $\mathfrak{L}$, $\mathfrak{n} = \mathrm{span}\{V_2, V_3\}$ stands for the nilpotent subalgebra of $\mathfrak{L}$.

As we know, infinitesimal generators V_i generate the one-parameter transform groups, by solving the following ordinary differential equations with the initial conditions:

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= \xi(x^*, t^*, u^*), x^*|_{\varepsilon=0} = x, \\ \frac{dt^*}{d\varepsilon} &= \tau(x^*, t^*, u^*), t^*|_{\varepsilon=0} = t, \\ \frac{du^*}{d\varepsilon} &= \phi(x^*, t^*, u^*), u^*|_{\varepsilon=0} = u, \end{aligned}$$

it follows the case $k \neq \frac{1}{2}$

$$G_1 : (x, t, u) \longmapsto (x + \varepsilon, t, u), \tag{15}$$
$$G_2 : (x, t, u) \longmapsto (x, t + \varepsilon, u), \tag{16}$$
$$G_3 : (x, t, u) \longmapsto (e^{\varepsilon} x, e^{-(k+1)\varepsilon} t, e^{\varepsilon} u), \tag{17}$$

and case $k = \frac{1}{2}$

$$G_1 : (x, t, u) \longmapsto (x + \varepsilon, t, u), \tag{18}$$
$$G_2 : (x, t, u) \longmapsto (x, t + \varepsilon, u), \tag{19}$$
$$G_3 : (x, t, u) \longmapsto (e^{\varepsilon} x, e^{-\frac{3}{2}\varepsilon} t, e^{\varepsilon} u), \tag{20}$$
$$G_4 : (x, t, u) \longmapsto \left(x - \ln(2 - \varepsilon e^{x}) + \ln 2, t, \frac{4u}{(2 - \varepsilon e^{x})^2} \right), \tag{21}$$
$$G_5 : (x, t, u) \longmapsto \left(x + \ln(2 - \varepsilon e^{-x}) - \ln 2, t, \frac{4u}{(2 - \varepsilon e^{-x})^2} \right). \tag{22}$$

Remark 1. *Any solution of Equation (2) when $k \neq \frac{1}{2}$ is invariant under the operation of G_1, G_2, G_3 and their products in (15)–(17).*

Any solution of Equation (2) when $k = \frac{1}{2}$ is invariant under the operation of G_1–G_5 and their products in (18)–(22).

Remark 2. *If $k \neq \frac{1}{2}$, Equation (2) is also invariant under the operation of the prolongations of V_1–V_3. We take V_3 as an example, since the prolongations of V_1 and V_2 are trivial.*

$$\mathrm{pr}^{(3)} V_3(\Delta) = (k + 2)\Delta.$$

Remark 3. *If $k = \frac{1}{2}$, Equation (2) is invariant under the operation of the prolongations of V_1–V_5. For instance*

$$\mathrm{pr}^{(3)} V_4(\Delta) = e^{x}\Delta, \ \mathrm{pr}^{(3)} V_5(\Delta) = e^{-x}\Delta.$$

3. Similarity Reductions for Equation (2)

In this section, we shall construct the similarity variables so as to deal with the symmetry reduction, which transfer the PDE into ODE. Considering the fact that the vector fields of $k \neq \frac{1}{2}$ and $k = \frac{1}{2}$ are partly the same, we shall discuss them together.

(1) For the generator V_1, we assume $\zeta = t$, $u = f(\zeta)$ and obtain the trivial solution $f = c$, where c is an arbitrary nonzero constant.

(2) For the linear combination V_2, we have

$$u = f(\zeta), \tag{23}$$

where $\zeta = x$. Substituting (23) into Equation (2), we can get

$$(k - 1)(k - 2) f'^3 - 3(k - 1) f f' f'' + f^2 f''' - f^2 f' = 0, \tag{24}$$

where $f^{'} = \frac{df}{d\zeta}$.

(3) For the generator V_3, we have

$$u = t^{-\frac{1}{k+1}} f(\zeta), \tag{25}$$

where $\zeta = x$. Substituting (25) into Equation (2), we arrive at

$$-\frac{1}{k+1}f + \frac{k(k-1)(k-2)}{2} f^{k-1} f'^3 - \frac{3k(k-1)}{2} f^k f' f'' + \frac{k}{2} f^{k+1} f''' - \frac{k}{2} f^{k+1} f' = 0, \tag{26}$$

where $f' = \frac{df}{d\zeta}$.

(4) For the generator V_4, we have

$$u = f(\zeta)e^{2x}, \tag{27}$$

where $\zeta = t$. By substituting (27) into Equation (2), we get the trivial solution $f = c$, where c is an arbitrary nonzero constant.

(5) For the generator V_5, we have

$$u = f(\zeta)e^{-2x}, \tag{28}$$

where $\zeta = t$, we get the trivial solution $f = c$, where c is an arbitrary nonzero constant.

The ODEs (24) and (26) hold for arbitrary k, including $\frac{1}{2}$, but the similarity reductions of V_4 and V_5 only belong to the equation when $k = \frac{1}{2}$.

In the above, we sketch the graphs of $f(\zeta)$ in Equations (24) and (26) and 3D-plot of $u(x,t)$ in Equations (23)–(26) under initial conditions $f(0) = \frac{3}{2}$, $f(1) = 1$, $f'(0) = 0$. Please refer to Figures 1–6.

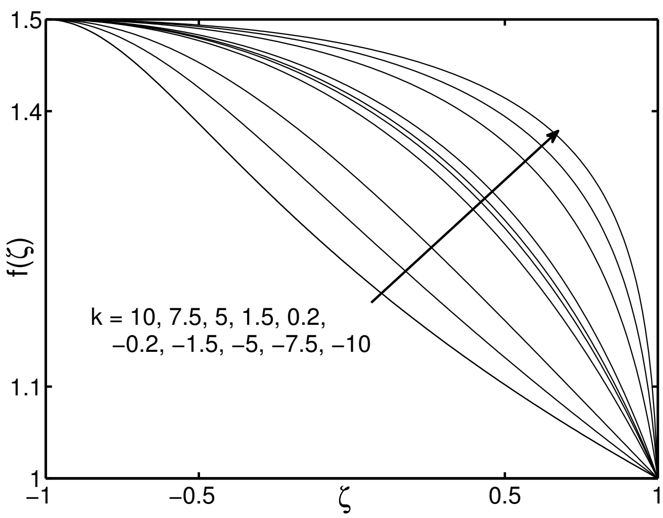

Figure 1. The graph of $f(\xi)$ given by Equation (24) as k near 0.

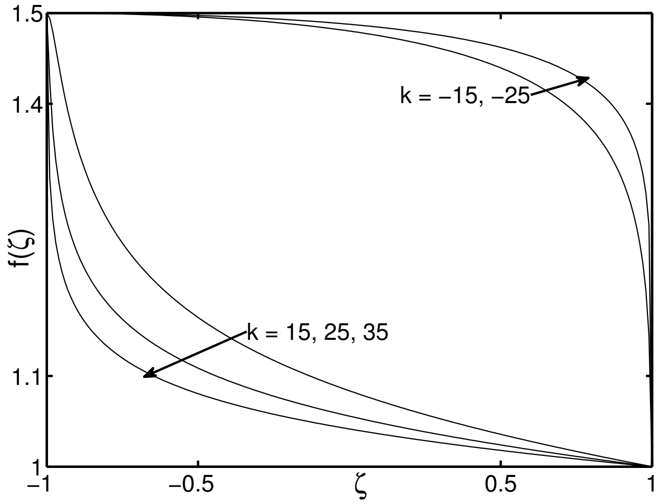

Figure 2. The graph of $f(\xi)$ given by Equation (24) as k approaching $-\infty$ and $+\infty$.

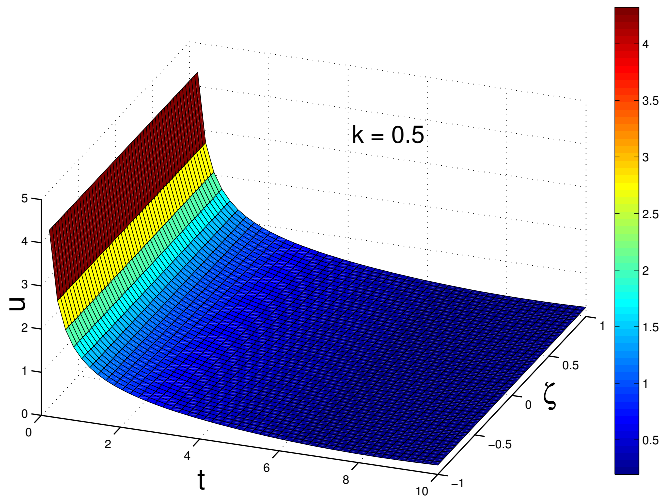

Figure 3. The graph of $u(x,t)$ given by Equations (23) and (24) for $k = 0.5$.

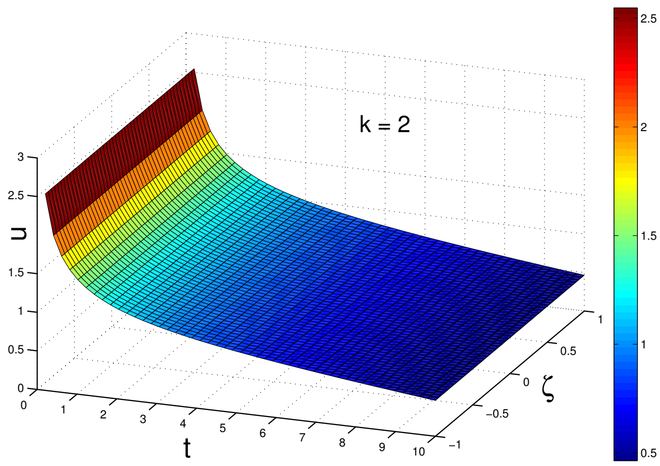

Figure 4. The graph of $u(x,t)$ given by Equations (23) and (24) for $k = 2$.

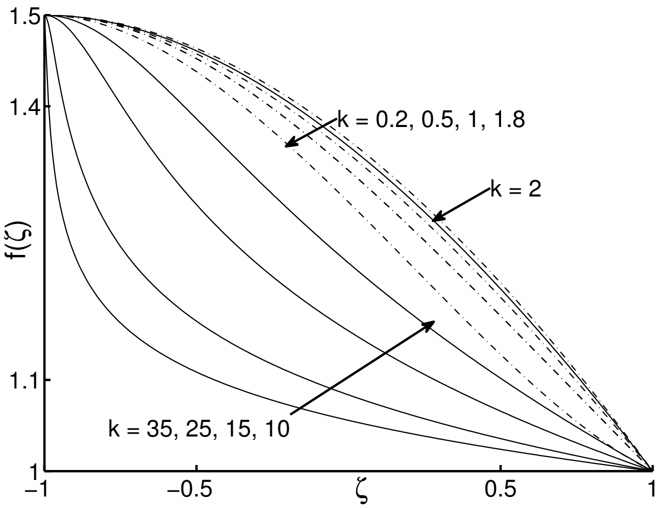

Figure 5. The graph of $f(\xi)$ given by Equation (26) as k near 0.

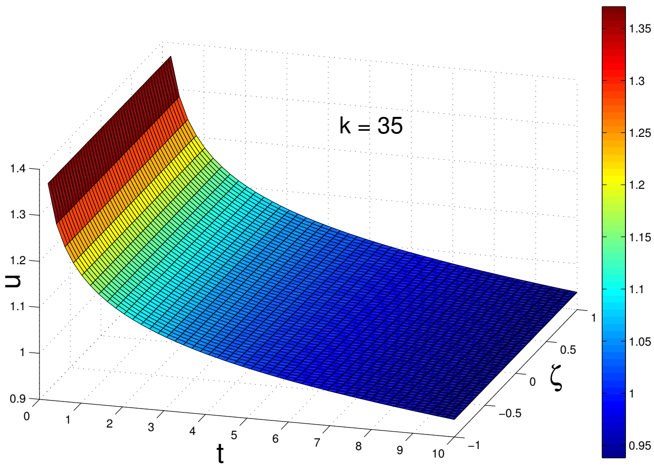

Figure 6. The graph of $u(x,t)$ given by Equations (25) and (26) for $k = 35$.

4. Nonlinear Self-Adjointness and Conservation Law

For given PDEs

$$R^{\beta}(x, u, u_{(1)}, \cdots, u_{(k)}) = 0, \tag{29}$$

define the Euler–Lagrange operator

$$\frac{\delta}{\delta u^{\alpha}} \equiv \frac{\partial}{\partial u^{\alpha}} + \sum_{j=1}^{\infty} (-1)^{j} D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial u^{\alpha}_{i_1 \cdots i_j}}, \alpha = 1, 2, \cdots, m, \tag{30}$$

and the formal Lagrangian

$$\mathcal{L} = \sum_{\beta=1}^{m} v^{\beta} R^{\beta}(x, u, u_{(1)}, \cdots, u_{(k)}). \tag{31}$$

The adjoint equations is given by

$$(R^{\alpha})^{*}(x, u, u_{(1)}, \cdots, u_{(k)}) = \frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0, \ \alpha = 1, 2, \cdots, m, \ v = v(x). \tag{32}$$

Theorem 2. *The differential Equation (2) is self-adjoining if and only if $k \neq 1$.*

Proof. It is obvious that $\alpha = 1$ and $R^\alpha(x, u, u_{(1)}, \cdots, u_{(k)}) = \Delta(x, t, u, u_t, u_x, u_{xx}, u_{xxx})$. The formal Lagrangian is

$$\mathcal{L} = v\left(u_t + \frac{k(k-1)(k-2)}{2}u^{k-1}u_x^3 + \frac{3k(k-1)}{2}u^k u_x u_{xx} + \frac{k}{2}u^{k+1}u_{xxx} - \frac{k}{2}u^{k+1}u_x\right). \quad (33)$$

Substituting it into $\frac{\delta}{\delta u} = 0$, we have the adjoint equation of Equation (2)

$$\begin{aligned} &k(k-1)(k-2)u^{k-2}vu_x^3 - \tfrac{k}{2}u^{k+1}v_{xxx} - 3ku^{k-1}v_x u_x^2 + 3k(k-2)u^{k-1}vu_x u_{xx} \\ &+\tfrac{k}{2}u^{k+1}v_x - 3ku^k v_{xx}u_x - 3ku^k v_x u_{xx} - v_t = 0. \end{aligned} \quad (34)$$

By means of

$$\begin{aligned} &\big[k(k-1)(k-2)u^{k-2}vu_x^3 - \tfrac{k}{2}u^{k+1}v_{xxx} - 3ku^{k-1}v_x u_x^2 + 3k(k-2)u^{k-1}vu_x u_{xx} \\ &+\tfrac{k}{2}u^{k+1}v_x - 3ku^k v_{xx}u_x - 3ku^k v_x u_{xx} - v_t\big]|_{v=\varphi(x,t,u)} = \lambda\Delta, \end{aligned}$$

we have

$$\varphi = \begin{cases} (c_1e^x + c_2e^{-x} + c_3)u^{-2}, \text{ if } k \neq 1, \\ 0, \text{ if } k = 1, \end{cases} \quad (35)$$

which completes the proof. □

We will now investigate the conservation law for Equation (2) by multiplier. For the differential Equation (29) with $\beta = 1$, we suppose $\Lambda = \Lambda(x, u)$ is its multiplier and T^i's are its conserve vectors, then Λ satisfies the property

$$D_i T^i = \Lambda E. \quad (36)$$

By (35) and (36), we have

$$\begin{aligned} &\big[(c_1e^x + c_2e^{-x} + c_3)u^{-2}\big]\big[u_t + \tfrac{k(k-1)(k-2)}{2}u^{k-1}u_x^3 + \tfrac{3k(k-1)}{2}u^k u_x u_{xx} \\ &+\tfrac{k}{2}u^{k+1}u_{xxx} - \tfrac{k}{2}u^{k+1}u_x\big] \\ &= D_x\Big[c_1\Big(\tfrac{k(k-1)}{2}u^{k-2}e^x u_x^2 + \tfrac{k}{2}u^{k-1}e^x u_{xx} - \tfrac{k}{2}u^{k-1}e^x u_x\Big) \\ &+c_2\Big(\tfrac{k(k-1)}{2}u^{k-2}e^{-x}u_x^2 + \tfrac{k}{2}u^{k-1}e^{-x}u_{xx} + \tfrac{k}{2}u^{k-1}e^{-x}u_x\Big) \\ &+c_3\Big(\tfrac{k(k-1)}{2}u^{k-2}u_x^2 + \tfrac{k}{2}u^{k-1}u_{xx} - \tfrac{1}{2}u^k\Big)\Big] \\ &+D_t\big[-(c_1e^x + c_2e^{-x} + c_3)u^{-1}\big]. \end{aligned}$$

Therefore the conserved vectors for Equation (2) are

$$\begin{aligned} T_1^x &= \tfrac{k(k-1)}{2}u^{k-2}e^x u_x^2 + \tfrac{k}{2}u^{k-1}e^x u_{xx} - \tfrac{k}{2}u^{k-1}e^x u_x,\ T_1^t = -e^x u^{-1}, \\ T_2^x &= \tfrac{k(k-1)}{2}u^{k-2}e^{-x}u_x^2 + \tfrac{k}{2}u^{k-1}e^{-x}u_{xx} + \tfrac{k}{2}u^{k-1}e^{-x}u_x,\ T_2^t = -e^{-x}u^{-1}, \\ T_3^x &= \tfrac{k(k-1)}{2}u^{k-2}u_x^2 + \tfrac{k}{2}u^{k-1}u_{xx} - \tfrac{1}{2}u^k,\ T_3^t = -u^{-1}. \end{aligned}$$

5. Discussion

We suggest a more general PDE:

$$m_t = c\left(\frac{1}{m^k}\right)_{xxx} - a\left(\frac{1}{m}\right)_x - b\left(\frac{1}{m^k}\right)_x, \quad (37)$$

with $k \in \mathbb{R}$ and $bc \neq 0$. If $a = 0$, $b = c = \dfrac{1}{2}$, it becomes Equation (2). The Lie symmetry and similarity reductions and the soliton solutions can be researched in the near future.

6. Conclusions

1. In this paper, the vector fields which make the equation under consideration symmetry are obtained. The Lie algebras and Lie transformation groups are performed. Moreover, it is pointed out that the vector fields supply a representation of the Lie algebra.

2. By the similarity reductions the equation under consideration is transferred to ODEs.

3. It is shown that the equation under consideration is nonlinear adjoint if and only if $k \neq 1$. The conserved vectors are obtained by multiplier method.

4. The vector fields generate the equation under consideration supply a representation of a Lie algebra. However, for a given finitely dimensional Lie algebra, such as nine types of simply Lie algebras, how to get its representation via vector fields? If we have already obtained the vector fields, can we get the differential equation which generates the vector field? If the differential equation is obtained, is it unique? All of them are the aims that we will study in the near future.

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