mathematics

MDPI

Article

On Riemann–Liouville and Caputo Fractional Forward Difference Monotonicity Analysis

Pshtiwan Othman Mohammed [1,*], Thabet Abdeljawad [2,3,4,*] and Faraidun Kadir Hamasalh [1]

1. Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Kurdistan Region, Iraq; faraidun.hamasalh@univsul.edu.iq
2. Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
3. Department of Medical Research, China Medical University, Taichung 40402, Taiwan
4. Department of Computer Science and Information Engineering, Asia University, Taichung 41354, Taiwan

* Correspondence: pshtiwan.muhammad@univsul.edu.iq (P.O.M.); tabdeljawad@psu.edu.sa (T.A.)

Abstract: Monotonicity analysis of delta fractional sums and differences of order $\upsilon \in (0,1]$ on the time scale $h\mathbb{Z}$ are presented in this study. For this analysis, two models of discrete fractional calculus, Riemann–Liouville and Caputo, are considered. There is a relationship between the delta Riemann–Liouville fractional h-difference and delta Caputo fractional h-differences, which we find in this study. Therefore, after we solve one, we can apply the same method to the other one due to their correlation. We show that $y(z)$ is υ-increasing on $\mathrm{M}_{a+\upsilon h,h}$, where the delta Riemann–Liouville fractional h-difference of order υ of a function $y(z)$ starting at $a+\upsilon h$ is greater or equal to zero, and then, we can show that $y(z)$ is υ-increasing on $\mathrm{M}_{a+\upsilon h,h}$, where the delta Caputo fractional h-difference of order υ of a function $y(z)$ starting at $a+\upsilon h$ is greater or equal to $-\frac{1}{\Gamma(1-\upsilon)}(z-(a+\upsilon h))_h^{(-\upsilon)}y(a+\upsilon h)$ for each $z \in \mathrm{M}_{a+h,h}$. Conversely, if $y(a+\upsilon h)$ is greater or equal to zero and $y(z)$ is increasing on $\mathrm{M}_{a+\upsilon h,h}$, we show that the delta Riemann–Liouville fractional h-difference of order υ of a function $y(z)$ starting at $a+\upsilon h$ is greater or equal to zero, and consequently, we can show that the delta Caputo fractional h-difference of order υ of a function $y(z)$ starting at $a+\upsilon h$ is greater or equal to $-\frac{1}{\Gamma(1-\upsilon)}(z-(a+\upsilon h))_h^{(-\upsilon)}y(a+\upsilon h)$ on $\mathrm{M}_{a,h}$. Furthermore, we consider some related results for strictly increasing, decreasing, and strictly decreasing cases. Finally, the fractional forward difference initial value problems and their solutions are investigated to test the mean value theorem on the time scale $h\mathbb{Z}$ utilizing the monotonicity results.

Keywords: discrete fractional calculus; υ-monotonicity analysis; discrete delta fractional operators; mean value theorem

check for updates

Citation: Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. On Riemann–Liouville and Caputo Fractional Forward Difference Monotonicity Analysis. *Mathematics* **2021**, *9*, 1303. https://doi.org/10.3390/math9111303

Academic Editor: António M. Lopes

Received: 13 May 2021
Accepted: 3 June 2021
Published: 7 June 2021

1. Introduction

Fractional differentiation and integration have opened many new doors for researchers in recent decades due to their wide and novel applicability in many fields of science including mathematical analysis, technology, and engineering (see [1–7]). Many techniques are used to deal with these new differential and integral operators; for instance, some researchers used analytical techniques including Laplace transform, spline interpolation, Green function, Crank–Nicolson approximation method, method of separation of variable, and many others to derive exact solutions to linear differential or integral equations (see [8–14]). Using the fixed-point technique, some researchers provided the conditions under which differential and integral equations have unique solutions. Some others provided numerical schemes that could be used to solve numerically differential and integral equations with fractional order. Very recently, fractional differentiation and integration found application in image processing, where the fractional kernel is used to remove noise in a given image.

On the other hand, fractional operators were employed in fuzzy theory. In fact, so far, researchers have developed a new class of differential and integral equations called fuzzy fractional differential and integral equations. This topic is highly regarded as its applications are found in many fields too. We point out that fractional differential and integral operators can be represented differently in continuous form, discrete form and discretized form. Discrete fractional calculus has been the focus of many researchers in recent years. For recent research on this topic, we advise the readers to refer to [15–27].

The aim of this study is to investigate the υ-monotonicity analysis on h-discrete delta fractional models in the sense of Riemann–Liouville (RL) and Caputo fractional operators on the time scale $h\mathbb{Z}$. The remainder of our article is structured as follows. In Section 2, we provide some notations and make some preparations. In Section 3, the monotonicity results and some corollaries are presented. In Section 4, some results related to the RL fractional forward difference equation (RL $-$ FFDE) (Section 4.1) and the Caputo fractional forward difference equation (Caputo$-$FFDE) (Section 4.2) are prepared. Additionally, we discuss the mean value theorem (MVT) later as an application of our monotonicity results. The paper is concluded in Section 5.

2. Preliminaries

Related concepts regarding the discrete fractional operators used in the current article are shown in this section.

Definition 1 (see [20–22])**.** *Let f be defined on the time scale $h\mathbb{Z}$; then, the forward h-difference operator is given by*

$$\Delta_h f(\mathbf{z}) = \frac{f(\sigma_h(\mathbf{z})) - f(\mathbf{z})}{h} \quad \left(\forall\, \mathbf{z} \in \mathrm{M}_{a,h} := \{a, a+h, a+2h, \dots\}\right),$$

and the backward h-difference operator is given by

$$\nabla_h f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(\rho_h(\mathbf{z}))}{h} \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a,h}),$$

where $\sigma_h(\mathbf{z}) := \mathbf{z} + h$ and $\rho_h(\mathbf{z}) := \mathbf{z} - h$.

Definition 2 (see [20–22])**.** *Let $\upsilon > 0$ and $f : \mathrm{M}_{a,h} \to \mathbb{R}$ with a starting point a. Then, the delta left* RL *fractional h-sum of order υ is given by*

$$\left({}_{a}\Delta_h^{-\upsilon} f\right)(\mathbf{z}) = \frac{1}{\Gamma(\upsilon)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}-\upsilon} \left(\mathbf{z} - \sigma_h(rh)\right)_h^{(\upsilon-1)} f(rh)h \quad \left(\forall\, \mathbf{z} \in \mathrm{M}_{a+\upsilon h,h}\right), \tag{1}$$

and for a function $f : {}_{b,h}\mathrm{M} := \{\dots, b-2h, b-h, b\} \to \mathbb{R}$ with an end point b, the delta right RL *fractional h-sum of order υ is given by*

$$\left({}_{h}\Delta_b^{-\upsilon} f\right)(\mathbf{z}) = \frac{1}{\Gamma(\upsilon)} \sum_{r=\frac{\mathbf{z}}{h}+\upsilon}^{\frac{b}{h}} \left(rh - \sigma_h(\mathbf{z})\right)_h^{(\upsilon-1)} f(rh)h \quad \left(\forall\, \mathbf{z} \in {}_{b-\upsilon h,h}\mathrm{M}\right), \tag{2}$$

where the h-falling factorial function $\mathbf{z}_h^{(\upsilon)}$ is defined by

$$\mathbf{z}_h^{(\upsilon)} = h^{\upsilon} \frac{\Gamma\left(\frac{\mathbf{z}}{h} + 1\right)}{\Gamma\left(\frac{\mathbf{z}}{h} + 1 - \upsilon\right)} \quad (\forall\, \mathbf{z}, \upsilon \in \mathbb{R}), \tag{3}$$

and we use the convention that division at a pole yields 0.

Definition 3 (see [22]). *For $0 \leq v < 1$, the delta left* RL *fractional h-difference of order v is defined by*

$$\begin{aligned}\left({}^{RL}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) &= \left(\Delta_h\, {}_{a}\Delta_h^{-(1-v)}f\right)(\mathbf{z}) \\ &= \frac{1}{\Gamma(1-v)}\Delta_h\left(\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v)}f(rh)h\right) \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a+(1-v)h,h}),\end{aligned}$$

and the delta right RL *fractional h-difference of order v is defined by*

$$\begin{aligned}\left({}^{RL}_{h}\Delta^{v}_{b}f\right)(\mathbf{z}) &= \left(-\Delta_h\, {}_{h}\Delta_b^{-(1-v)}f\right)(\mathbf{z}) \\ &= \frac{1}{\Gamma(1-v)}\Delta_h\left(\sum_{r=\frac{\mathbf{z}}{h}+1-v}^{\frac{b}{h}}(\rho_h(\mathbf{z})-rh)_h^{(-v)}f(rh)h\right) \quad (\forall\, \mathbf{z} \in {}_{b-(1-v)h,h}\mathrm{M}).\end{aligned}$$

Definition 4 (see [22]). *For $0 \leq v < 1$, the delta left Caputo fractional h-difference of order v is defined by*

$$\begin{aligned}\left({}^{C}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) &= \left({}_{a}\Delta_h^{-(1-v)}\,\Delta_h f\right)(\mathbf{z}) \\ &= \frac{1}{\Gamma(1-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v)}(\Delta_h f)(rh)h \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a+(1-v)h,h}),\end{aligned}$$

and the delta right Caputo fractional h-difference of order v is defined by

$$\begin{aligned}\left({}^{C}_{h}\Delta^{v}_{b}f\right)(\mathbf{z}) &= \left({}_{h}\Delta_b^{-(1-v)}(-\Delta_h f)\right)(\mathbf{z}) \\ &= \frac{1}{\Gamma(1-v)}\sum_{r=\frac{\mathbf{z}}{h}+1-v}^{\frac{b}{h}}(\rho_h(\mathbf{z})-rh)_h^{(-v)}(\Delta_h f)(rh)h \quad (\forall\, \mathbf{z} \in {}_{b-(1-v)h,h}\mathrm{M}).\end{aligned}$$

In the following lemma, we show that $\mathbf{z}_h^{(v)}$ increases on $\mathrm{M}_{0,h}$.

Lemma 1. *Let $v > 0$ and $h > 0$, then $\Delta_h\left(\mathbf{z}_h^{(v)}\right) = v\,\mathbf{z}_h^{(v-1)}$. Moreover, $\mathbf{z}_h^{(v)}$ increases on $\mathrm{M}_{0,h}$.*

Proof. From Definition 1 and Equation (3), we have

$$\begin{aligned}\Delta_h\left(\mathbf{z}_h^{(v)}\right) = \frac{(\mathbf{z}+h)_h^{(v)}-\mathbf{z}_h^{(v)}}{h} &= \frac{1}{h}\left(h^v\frac{\Gamma\left(\frac{\mathbf{z}}{h}+2\right)}{\Gamma\left(\frac{\mathbf{z}}{h}+2-v\right)} - h^v\frac{\Gamma\left(\frac{\mathbf{z}}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}}{h}+1-v\right)}\right) \\ &= h^{v-1}\frac{\Gamma\left(\frac{\mathbf{z}}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}}{h}+1-v\right)}\left(\frac{\frac{\mathbf{z}}{h}+1}{\frac{\mathbf{z}}{h}+1-v}-1\right) \\ &= v\,h^{v-1}\frac{\Gamma\left(\frac{\mathbf{z}}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}}{h}+1-(v-1)\right)} = v\,\mathbf{z}_h^{(v-1)}.\end{aligned}$$

Since $v, h > 0$, it follows that

$$\Delta_h\left(\mathbf{z}_h^{(v)}\right) = \frac{(\mathbf{z}+h)_h^{(v)}-\mathbf{z}_h^{(v)}}{h} = v\,\mathbf{z}_h^{(v-1)} \geq 0,$$

which implies that $(\mathbf{z}+h)_h^{(v)} \geq \mathbf{z}_h^{(v)}$, and this completes the proof. □

The following theorem can be seen as an equivalence definition to the delta RL fractional h-differences.

Theorem 1. *Let $0 < v < 1$. Then, the delta left and delta right* RL *fractional h-differences of order v defined on $\mathrm{M}_{a+(1-v)h,h}$ and ${}_{b-(1-v)h,h}\mathrm{M}$, respectively, are defined by*

$$\left({}^{RL}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) = \frac{1}{\Gamma(-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}f(rh)h, \tag{4}$$

and

$$\left({}^{RL}_{h}\Delta^{v}_{b}f\right)(\mathbf{z}) = \frac{1}{\Gamma(-v)}\sum_{r=\frac{\mathbf{z}}{h}-v}^{\frac{b}{h}}(\rho_h(\mathbf{z})-rh)_h^{(-v-1)}f(rh)h. \tag{5}$$

Proof. From Definitions 1 and 3, we have

$$\begin{aligned}
&\left({}^{RL}_{a}\Delta^{v}_{h}f\right)(\mathbf{z})\\
&= \frac{1}{\Gamma(1-v)}\Delta_h\left(\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v)}f(rh)h\right)\\
&= \frac{1}{h\Gamma(1-v)}\left(\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v}(\mathbf{z}+h-\sigma_h(rh))_h^{(-v)}f(rh)h - \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v)}f(rh)h\right)\\
&= \frac{1}{h\Gamma(1-v)}\left(\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}+h-\sigma_h(rh))_h^{(-v)}f(rh)h - \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v)}f(rh)h\right)\\
&+ \frac{1}{h\Gamma(1-v)}(\mathbf{z}+h-\sigma_h(\mathbf{z}+v\,h))_h^{(-v)}f(\mathbf{z}+v\,h)h\\
&= \frac{1}{\Gamma(1-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\Delta_h\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v)}f(rh)h + \frac{-(v\,h)_h^{(-v)}}{h\Gamma(1-v)}f(\mathbf{z}+v\,h)h\\
&= \frac{1}{(-v)\Gamma(-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\Delta_h\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v)}f(rh)h + h^{1-v}f(\mathbf{z}+v\,h)h.
\end{aligned}$$

Then by using Lemma 1, we get

$$\begin{aligned}
\left({}^{RL}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) &= \frac{1}{\Gamma(-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1}\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v-1)}f(rh)h + h^{-v-1}f(\mathbf{z}+v\,h)h\\
&= \frac{1}{\Gamma(-v)}\sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v}\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v-1)}f(rh)h,
\end{aligned}$$

which is the required Equation (4). In the same manner as Equation (4), we can prove Equation (5), and thus, the proof is completed. □

A relationship between the delta RL fractional and delta Caputo fractional h-differences are presented in the following proposition.

Proposition 1. *Let $0 \le v < 1$, then*

$$\left({}^{C}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) = \left({}^{RL}_{a}\Delta^{v}_{h}f\right)(\mathbf{z}) - \frac{1}{\Gamma(1-v)}(\mathbf{z}-a)_h^{(-v)}f(a) \tag{6}$$

for $\mathbf{z} \in \mathrm{M}_{a+(1-v)h,h}$, *and*

$$\left({}^{C}_{h}\Delta^{v}_{b} f\right)(\mathbf{z}) = \left({}^{RL}_{h}\Delta^{v}_{b} f\right)(\mathbf{z}) - \frac{1}{\Gamma(1-v)}(b-\mathbf{z})^{(-v)}_{h} f(b) \tag{7}$$

for $\mathbf{z} \in {}_{b-(1-v)h,h}\mathrm{M}$.

Proof. From Definition 4 and the fact that $\left(\mathbf{z} - \sigma_h(rh)\right)^{(-v)}_{h} = 0$ at $r = \frac{\mathbf{z}}{h} + v$, we have

$$\begin{aligned}
\left({}^{C}_{a}\Delta^{v}_{h} f\right)(\mathbf{z}) &= \left({}_{a}\Delta^{-(1-v)}_{h}\, \Delta_h f\right)(\mathbf{z}) = {}_{a}\Delta^{-(1-v)}_{h}\left(\frac{f(\mathbf{z}+h) - f(\mathbf{z})}{h}\right) \\
&= \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a+h}{h}}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} + h - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h \\
&\quad - \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v-1} \left(\mathbf{z} - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h \\
&= \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a}{h}+1}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} + h - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h \\
&\quad - \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h \\
&= \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} + h - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h - \frac{1}{\Gamma(1-v)}(\mathbf{z}-a)^{(-v)}_{h} f(a) \\
&\quad - \frac{1}{h\Gamma(1-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h \\
&= \frac{1}{\Gamma(1-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v} \Delta_h \left(\mathbf{z} - \sigma_h(rh)\right)^{(-v)}_{h} f(rh)h - \frac{1}{\Gamma(1-v)}(\mathbf{z}-a)^{(-v)}_{h} f(a).
\end{aligned}$$

By using Lemma 1 and Theorem 1, we get

$$\begin{aligned}
\left({}^{C}_{a}\Delta^{v}_{h} f\right)(\mathbf{z}) &= \frac{1}{\Gamma(-v)} \sum_{r=\frac{a}{h}}^{\frac{\mathbf{z}}{h}+v} \left(\mathbf{z} - \sigma_h(rh)\right)^{(-v-1)}_{h} f(rh)h - \frac{1}{\Gamma(1-v)}(\mathbf{z}-a)^{(-v)}_{h} f(a) \\
&= \left({}^{RL}_{a}\Delta^{v}_{h} f\right)(\mathbf{z}) - \frac{1}{\Gamma(1-v)}(\mathbf{z}-a)^{(-v)}_{h} f(a),
\end{aligned}$$

which is the required Equation (6). Using the same technique used for Equation (6), we can prove Equation (7), and thus, the proof is completed. □

In the following lemma, we prove and modify a power rule that appeared in ([22], Lemma 4). We state and prove the modified result in a simpler way as follows:

Lemma 2. *Let* $v > 0$, $\mu > -1$ *and* $h > 0$, *then*

$$ {}_{a+\mu h}\Delta^{-v}_{h}(\mathbf{z}-a)^{(\mu)}_{h} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}(\mathbf{z}-a)^{(v+\mu)}_{h} \tag{8}$$

for $\mathbf{z} \in \mathrm{M}_{a+(\mu+v)h,h}$.

Proof. Following ([4], Lemma 1), we have

$$_{\frac{a}{h}+\mu}\Delta^{-v}\left(\frac{\mathbf{z}}{h}-\frac{a}{h}\right)^{(\mu)}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}\left(\frac{\mathbf{z}}{h}-\frac{a}{h}\right)^{(v+\mu)} \tag{9}$$

for $\mathbf{z}\in \mathrm{M}_{a+(\mu+v)h,h}$. Calculating both sides of Equation (9), we get

$$\begin{aligned}
\mathtt{LHS} := {}_{\frac{a}{h}+\mu}\Delta^{-v}\left(\frac{\mathbf{z}}{h}-\frac{a}{h}\right)^{(\mu)} &= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\left(\frac{\mathbf{z}}{h}-\sigma_h(rh)\right)^{(v-1)}\left(r-\frac{a}{h}\right)^{(\mu)} \\
&= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\left(\frac{\mathbf{z}-rh-h}{h}\right)^{(v-1)}\left(\frac{rh-a}{h}\right)^{(\mu)} \\
&= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\frac{\Gamma\left(\frac{\mathbf{z}-rh-h}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}-rh-h}{h}+1-(v-1)\right)}\cdot\frac{\Gamma\left(\frac{rh-a}{h}+1\right)}{\Gamma\left(\frac{rh-a}{h}+1-\mu\right)} \\
&= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\frac{(\mathbf{z}-rh-h)_h^{(v-1)}}{h^{v-1}}\cdot\frac{(rh-a)_h^{(\mu)}}{h^{\mu}} \\
&= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\frac{1}{h^{v+\mu}}(\mathbf{z}-\sigma_h(rh))_h^{(v-1)}(rh-a)_h^{(\mu)}h,
\end{aligned}$$

and

$$\begin{aligned}
\mathtt{RHS} := \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}\left(\frac{\mathbf{z}}{h}-\frac{a}{h}\right)^{(v+\mu)} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1-(v+\mu)\right)} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}\frac{(\mathbf{z}-a)_h^{(v+\mu)}}{h^{v+\mu}}.
\end{aligned}$$

Substituting the `LHS` and `RHS` results into Equation (9), we get

$$\frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+\mu}^{\frac{\mathbf{z}}{h}-v}\frac{1}{h^{v+\mu}}(\mathbf{z}-\sigma_h(rh))_h^{(v-1)}(rh-a)_h^{(\mu)}h=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}\frac{(\mathbf{z}-a)_h^{(v+\mu)}}{h^{v+\mu}}.$$

Multiplying by a positive constant $h^{v+\mu}$ on both sides of the equality, we get the desired Equation (8). □

3. The Monotonicity Results

This section illustrates the monotonicity of a discrete function. Monotonicity analysis of discrete functions defined on M_1^0 was originally introduced in [27], and there is extensive literature on monotonicity analysis techniques and its extensions on $\mathrm{M}_{a,h}$; for example, see [22,23,26].

Definition 5 (see [22,23,26])**.** *Let $0<h,v\le 1$, and $y:\mathrm{M}_{a,h}\to \mathrm{R}$ be a function satisfying $y(a)\ge 0$. Then, y is called an v-increasing function on $\mathrm{M}_{a,h}$ if*

$$y(\mathbf{z}+h)\ge v\,y(\mathbf{z})\quad (\forall\,\mathbf{z}\in\mathrm{M}_{a,h}).$$

Observe that, if $y(\mathbf{z})$ is increasing on $\mathrm{M}_{a,h}$, then $y(\mathbf{z}+h)\ge y(\mathbf{z})$ for all $\mathbf{z}\in\mathrm{M}_{a,h}$, and thus, $y(\mathbf{z})$ is v-increasing on $\mathrm{M}_{a,h}$.

Definition 6 (see [22,23,26])**.** *Let $0 < h, v \leq 1$, and $y : \mathrm{M}_{a,h} \to \mathrm{R}$ be a function satisfying $y(a) \geq 0$. Then, y is called an v-decreasing function on $\mathrm{M}_{a,h}$ if*

$$y(\mathbf{z}+h) \leq v\, y(\mathbf{z}) \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a,h}).$$

Observe that, if $y(\mathbf{z})$ is decreasing on $\mathrm{M}_{a,h}$, then $y(\mathbf{z}+h) \leq y(\mathbf{z})$ for all $\mathbf{z} \in \mathrm{M}_{a,h}$, and thus, $y(\mathbf{z})$ is v-decreasing on $\mathrm{M}_{a,h}$.

Remark 1. *Note that, if $v = 1$ in Definition 5, then the increasing and v-increasing concepts coincide and that, if $v = 1$ in Definition 6, then the decreasing and v-decreasing concepts coincide.*

To provide motivation for the above monotonicity definitions, we prove a few fundamental results of the discrete RL and Caputo fractional operators.

Theorem 2. *Let $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\,h) \geq 0$. Suppose that $\left({}^{RL}_{a+vh}\Delta^{v}_{h} y\right)(\mathbf{z}) \geq 0$ for $0 < h \leq 1$, $0 < v < 1$ and $\mathbf{z} \in \mathrm{M}_{a+h,h}$. Then, $y(\mathbf{z})$ is v-increasing on $\mathrm{M}_{a+vh,h}$.*

Proof. From the assumption and proof of Theorem 1, we have

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h} y\right)(\mathbf{z}) = h^{-v} y(\mathbf{z} + v\,h) + \frac{1}{\Gamma(-v)} \sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1} (\mathbf{z} - \sigma_h(rh))_h^{(-v-1)} y(rh)h \geq 0. \tag{10}$$

For $\mathbf{z} = a$, we see that

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h} y\right)(a) = h^{-v} y(a + v\,h) \geq 0,$$

and, thus, $y(a + vh) \geq 0$.

For $\mathbf{z} = a + h$, we see that

$$\begin{aligned}\left({}^{RL}_{a+vh}\Delta^{v}_{h} y\right)(a+h) &= h^{-v} y(a + v\,h + h) + \frac{1}{\Gamma(-v)} (-vh)_h^{(-v-1)} y(a + v\,h)h \\ &= h^{-v} y(a + v\,h + h) - vh^{-v} y(a + v\,h) \geq 0,\end{aligned}$$

and, thus, $y(a + vh + h) \geq vy(a + vh)$ since $h^{-v} > 0$ and $y(a + vh) \geq 0$.

Now, inductively, we show that

$$y(\mathbf{z} + h) \geq vy(\mathbf{z}), \quad \forall \mathbf{z} \in \mathrm{M}_{a+vh,h}.$$

Suppose that $y(k + h) \geq vy(k), \forall\, k < \mathbf{z}$ provided that $k, \mathbf{z} \in \mathrm{M}_{a+vh,h}$. Then, we have to show that $y(\mathbf{z} + h) \geq vy(\mathbf{z})$.

Replace $\mathbf{z}$ by $\mathbf{z} + h$ in Equation (10) to get

$$\begin{aligned}&h^{-v} y(\mathbf{z} + v\,h + h) + \frac{1}{\Gamma(-v)} \sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v} (\mathbf{z} + h - \sigma_h(rh))_h^{(-v-1)} y(rh)h \\ &= h^{-v} y(\mathbf{z} + v\,h + h) - \frac{v}{\Gamma(1-v)} \sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v} (\mathbf{z} + h - \sigma_h(rh))_h^{(-v-1)} y(rh)h \geq 0.\end{aligned}$$

It follows that

$$h^{-v}y(\mathbf{z}+v\,h+h)-\frac{v}{\Gamma(1-v)}\Bigg[(\mathbf{z}+h-\sigma_h(a+vh))_h^{(-v-1)}y(a+vh)h$$
$$+(\mathbf{z}+h-\sigma_h(a+vh+h))_h^{(-v-1)}y(a+vh+h)h+\cdots$$
$$+(\mathbf{z}+h-\sigma_h(\mathbf{z}+vh))_h^{(-v-1)}y(\mathbf{z}+vh)h\Bigg]\geq 0,$$

or equivalently,

$$h^{-v}y(\mathbf{z}+v\,h+h)\geq\frac{v}{\Gamma(1-v)}\Bigg[(\mathbf{z}+h-\sigma_h(a+vh))_h^{(-v-1)}y(a+vh)h$$
$$+(\mathbf{z}+h-\sigma_h(a+vh+h))_h^{(-v-1)}y(a+vh+h)h+\cdots$$
$$+(\mathbf{z}+h-\sigma_h(\mathbf{z}+vh))_h^{(-v-1)}y(\mathbf{z}+vh)h\Bigg]$$
$$\geq\frac{v}{\Gamma(1-v)}(\mathbf{z}+h-\sigma_h(\mathbf{z}+vh))_h^{(-v-1)}y(\mathbf{z}+vh)h$$
$$=\frac{v}{\Gamma(1-v)}\,h^{-v-1}\Gamma(1-v)y(\mathbf{z}+vh)h$$
$$=v\,h^{-v}\Gamma(1-v)y(\mathbf{z}+vh).$$

This implies that $y(\mathbf{z}+v\,h+h)\geq y(\mathbf{z}+v\,h)$, and this completes the proof. □

Corollary 1. *Let $y:\mathrm{M}_{a+vh,h}\to\mathrm{R}$ be a function satisfying $y(a+v\,h)\geq 0$. Suppose that*

$$\left({}^{C}_{a+vh}\Delta_h^v y\right)(\mathbf{z})\geq-\frac{1}{\Gamma(1-v)}(\mathbf{z}-(a+vh))_h^{(-v)}y(a+vh),$$

for $0<h\leq 1$, $0<v<1$, and $\mathbf{z}\in\mathrm{M}_{a+h,h}$. Then, $y(\mathbf{z})$ is v-increasing on $\mathrm{M}_{a+vh,h}$.

Proof. The proof follows from Proposition 1 and Theorem 2. □

Theorem 3. *Let $0<h\leq 1$, $0<v<1$, and $y:\mathrm{M}_{a+vh,h}\to\mathrm{R}$ be a function satisfying $y(a+v\,h)\geq 0$. If $y(\mathbf{z})$ is increasing on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{RL}_{a+vh}\Delta_h^v y\right)(\mathbf{z})\geq 0\quad(\forall\mathbf{z}\in\mathrm{M}_{a,h}).$$

Proof. For each $\mathbf{z}\in\mathrm{M}_{a+h,h}$, we have to show that

$$\left({}^{RL}_{a+vh}\Delta_h^v y\right)(\mathbf{z})=\frac{1}{\Gamma(-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(rh)h\geq 0.$$

From Equation (10) with $\mathbf{z}=a$, we have

$$\left({}^{RL}_{a+vh}\Delta_h^v y\right)(a)=h^{-v}y(a+v\,h)\geq 0,$$

since $h^{-v}>0$ and $y(a+v\,h)\geq 0$. Assume that $\left({}^{RL}_{a+vh}\Delta_h^v y\right)(i)\geq 0,\forall\, i<\mathbf{z}$; then, we have to show that $\left({}^{RL}_{a+vh}\Delta_h^v y\right)(\mathbf{z})\geq 0$.

From Equation (10), we see that

$$\begin{aligned}\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) &= h^{-v}y(\mathbf{z}+v\,h)+\frac{1}{\Gamma(-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(rh)h\\ &= h^{-v}y(\mathbf{z}+v\,h)-\frac{v}{\Gamma(1-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(rh)h\\ &= h^{-v}y(\mathbf{z}+v\,h)-vh^{-v}y(\mathbf{z}+v\,h-h)\\ &\quad-\frac{v}{\Gamma(1-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-2}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(rh)h.\end{aligned}\tag{11}$$

Write $r=\frac{a}{h}+v+\ell$, $\ell=0,1,\ldots,\frac{\mathbf{z}}{h}-\frac{a}{h}+v-2$. We see that

$$\begin{aligned}\frac{v\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v-1)}}{\Gamma(1-v)}\cdot h &= \frac{v\,h^{-v}}{\Gamma(1-v)}\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v-\ell\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}-\ell+1\right)}\\ &= \frac{v\,h^{-v}}{\Gamma(1-v)}\frac{\Gamma(m-v)}{\Gamma(m+1)},\quad \left(\text{denoting } m=\frac{\mathbf{z}-a}{h}-\ell\right),\\ &= -\frac{h^{-v}}{\Gamma(-v)}\frac{\Gamma(m-v)}{\Gamma(m+1)}.\end{aligned}$$

Since $\ell\leq\frac{\mathbf{z}-a}{h}+v-2$, we get $m>2-v>v$ for $v\in(0,1)$ and, hence, $m>m-v>0$. Additionally, $h^{-v}>0$, and from the graph of the gamma function, we see that $\Gamma(-v)<0$ for $v\in(0,1)$. Therefore,

$$\frac{v\big(\mathbf{z}-\sigma_h(rh)\big)_h^{(-v-1)}}{\Gamma(1-v)}\cdot h>0.\tag{12}$$

From the assumption that $y(\mathbf{z})$ increases, then $y(\mathbf{z}+vh)\geq y(\mathbf{z}+vh-h)\geq y(a+vh)$ for each $\mathbf{z}\in\mathrm{M}_{a,h}$. It follows that

$$y(\mathbf{z}+vh-h)\geq y(rh),\quad \forall r=\frac{a}{h}+v,\frac{a}{h}+v,\ldots,\frac{\mathbf{z}}{h}+v-2.\tag{13}$$

Considering Equations (11)–(13), it follows that

$$\begin{aligned}\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) &= h^{-v}y(\mathbf{z}+v\,h)-vh^{-v}y(\mathbf{z}+v\,h-h)\\ &\quad-\frac{v}{\Gamma(1-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-2}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(\mathbf{z}+v\,h-h)h\\ &= h^{-v}y(\mathbf{z}+v\,h)-\frac{v}{\Gamma(1-v)}\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}y(\mathbf{z}+v\,h-h)h\\ &= h^{-v}y(\mathbf{z}+v\,h)-\frac{(-vh)_h^{(-v)}}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)+\frac{(-vh)_h^{(-v)}}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\\ &\quad-\frac{vh}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}\\ &= h^{-v}\big(y(\mathbf{z}+v\,h)-y(\mathbf{z}+v\,h-h)\big)\\ &\quad+\frac{1}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\left[(-vh)_h^{(-v)}-vh\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}(\mathbf{z}-\sigma_h(rh))_h^{(-v-1)}\right].\end{aligned}$$

Since $y(\mathbf{z})$ increases, it follows that

$$
\begin{aligned}
\left({}_{a+vh}^{RL}\Delta_h^{v}y\right)(\mathbf{z}) &\geq \frac{1}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\left[(-vh)_h^{(-v)}-vh\sum_{r=\frac{a}{h}+v}^{\frac{\mathbf{z}}{h}+v-1}\left(\mathbf{z}-\sigma_h(rh)\right)_h^{(-v-1)}\right]\\
&=\frac{1}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\Big[(-vh)_h^{(-v)}-vh(\mathbf{z}-a-vh-h)_h^{(-v-1)}\\
&\quad -vh(\mathbf{z}-a-vh-2h)_h^{(-v-1)}-\cdots-vh(-vh)_h^{(-v-1)}\Big]\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\Bigg[\frac{\Gamma(1-v)}{\Gamma(1)}-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v-1\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}\right)}\\
&\quad -\cdots-v\frac{\Gamma(2-v)}{\Gamma(3)}-v\frac{\Gamma(1-v)}{\Gamma(2)}\Bigg]\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\Bigg[\frac{\Gamma(1-v)}{\Gamma(1)}-v\frac{\Gamma(1-v)}{\Gamma(2)}-v\frac{\Gamma(2-v)}{\Gamma(3)}-v\frac{\Gamma(3-v)}{\Gamma(4)}\\
&\quad -\cdots-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\Bigg[\frac{\Gamma(1-v)}{\Gamma(2)}(1-v)-v\frac{\Gamma(2-v)}{\Gamma(3)}-v\frac{\Gamma(3-v)}{\Gamma(4)}\\
&\quad -\cdots-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\Bigg]\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\left[\frac{\Gamma(2-v)}{\Gamma(3)}(2-v)-v\frac{\Gamma(3-v)}{\Gamma(4)}-\cdots-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\right]\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\left[\frac{\Gamma(3-v)}{\Gamma(4)}(3-v)-\cdots-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\right].
\end{aligned}
$$

Continue this process to get

$$
\begin{aligned}
\left({}_{a+vh}^{RL}\Delta_h^{v}y\right)(\mathbf{z}) &\geq \frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\left(\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}\right)}-v\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\right)\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}\right)}\left(1-v\frac{1}{\frac{\mathbf{z}-a}{h}}\right)\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\frac{\left(\frac{\mathbf{z}-a}{h}-v\right)\Gamma\left(\frac{\mathbf{z}-a}{h}-v\right)}{\left(\frac{\mathbf{z}-a}{h}\right)\Gamma\left(\frac{\mathbf{z}-a}{h}\right)}\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\frac{\Gamma\left(\frac{\mathbf{z}-a}{h}-v+1\right)}{\Gamma\left(\frac{\mathbf{z}-a}{h}+1\right)}\\
&=\frac{h^{-v}y(\mathbf{z}+v\,h-h)}{\Gamma(1-v)}\frac{\Gamma\left(\frac{\mathbf{z}-a-vh}{h}+1\right)}{\Gamma\left(\frac{\mathbf{z}-a-vh}{h}+1-(-v)\right)}\\
&=\frac{(\mathbf{z}-(a+vh))_h^{(-v)}}{\Gamma(1-v)}y(\mathbf{z}+v\,h-h)\geq 0,
\end{aligned}
$$

which completes the proof. □

Corollary 2. *Let $0 < h \leq 1$, $0 < v < 1$, and $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) \geq 0$. If $y(\mathbf{z})$ increases on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{C}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \geq -\frac{1}{\Gamma(1-v)}(\mathbf{z} - (a + vh))^{(-v)}_{h} y(a + vh) \quad (\forall \mathbf{z} \in \mathrm{M}_{a,h}).$$

Proof. The proof follows from Proposition 1 and Theorem 3. □

Theorem 4. *Let $0 < h \leq 1$, $0 < v < 1$, and $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) > 0$. If $y(\mathbf{z})$ strictly increases on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) > 0 \quad (\forall\ \mathbf{z} \in \mathrm{M}_{a,h}).$$

Proof. The proof is similar to Theorem 3, so it is omitted. □

Corollary 3. *Let $0 < h \leq 1$, $0 < v < 1$, and $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) > 0$. If $y(\mathbf{z})$ strictly increases on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{C}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) > -\frac{1}{\Gamma(1-v)}(\mathbf{z} - (a + vh))^{(-v)}_{h} y(a + vh) \quad (\forall \mathbf{z} \in \mathrm{M}_{a,h}).$$

Theorem 5. *Let $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) \leq 0$. Suppose that $\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \leq 0$ for $0 < h \leq 1$, $0 < v < 1$ and $\mathbf{z} \in \mathrm{M}_{a+h,h}$. Then, $y(\mathbf{z})$ is v-decreasing on $\mathrm{M}_{a+vh,h}$.*

Proof. Let $g : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function such that $g(\mathbf{z}) = -y(\mathbf{z})$. Therefore,

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h}g\right)(\mathbf{z}) = \left({}^{RL}_{a+vh}\Delta^{v}_{h}(-y)\right)(\mathbf{z}) = -\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \geq 0.$$

Apply Theorem 2 to $g(\mathbf{z})$, and thus, the proof is completed. □

Corollary 4. *Let $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) \geq 0$. Suppose that $\left({}^{C}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \leq -\frac{1}{\Gamma(1-v)}(\mathbf{z} - (a + vh))^{(-v)}_{h} y(a + vh)$ for $0 < h \leq 1$, $0 < v < 1$ and $\mathbf{z} \in \mathrm{M}_{a+h,h}$. Then, $y(\mathbf{z})$ is v-decreasing on $\mathrm{M}_{a+vh,h}$.*

Theorem 6. *Let $0 < h \leq 1$, $0 < v < 1$, and $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) \leq 0$. If $y(\mathbf{z})$ decreases on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \leq 0 \quad (\forall\ \mathbf{z} \in \mathrm{M}_{a,h}).$$

Proof. The proof is obtained by applying Theorem 4 to $g(\mathbf{z}) = -y(\mathbf{z})$. □

Corollary 5. *Let $0 < h \leq 1$, $0 < v < 1$, and $y : \mathrm{M}_{a+vh,h} \to \mathrm{R}$ be a function satisfying $y(a + v\, h) \leq 0$. If $y(\mathbf{z})$ decreases on $\mathrm{M}_{a+vh,h}$, then*

$$\left({}^{C}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) \leq -\frac{1}{\Gamma(1-v)}(\mathbf{z} - (a + vh))^{(-v)}_{h} y(a + vh) \quad (\forall \mathbf{z} \in \mathrm{M}_{a,h}).$$

4. Fractional Forward Difference Initial Value Problem and Mean Value Theorem

In this section, we move on from monotonicity analysis to the `MVT`. Having established the monotonicity analysis for the discrete `RL` and Caputo fractional operators, we now obtain the `MVT` using those discrete monotonicity results.

4.1. Establishing the Riemann–Liouville case

The following is the main results to start off the `MVT` here.

Lemma 3 ([2,3]). *Let $y : \mathrm{M}_{a,h} \to \mathrm{R}$ and $0 < h \leq 1$, $0 < v < 1$, then*

$$ {}_a\Delta_h^{-v}(\Delta_h y(\mathbf{z})) = \Delta_h\big({}_a\Delta_h^{-v} y\big)(\mathbf{z}) - \frac{(\mathbf{z}-a)_h^{(v-1)}}{\Gamma(v)} y(a). $$

Lemma 4 ([2,3]). *Let $y : \mathrm{M}_{a,h} \to \mathrm{R}$ and $0 < h \leq 1$, $0 < v < 1$, then*

$$ {}_{a+\beta h}\Delta_h^{-v}\Big({}_a\Delta_h^{-\beta} y\Big)(\mathbf{z}) = \Big({}_a\Delta_h^{-(v+\beta)} y\Big)(\mathbf{z}) = {}_{a+vh}\Delta_h^{-\beta}\big({}_a\Delta_h^{-v} y\big)(\mathbf{z}), \quad \mathbf{z} \in \mathrm{M}_{a+(v+\beta)h,h}. $$

Theorem 7. *Let $y : \mathrm{M}_{a,h} \to \mathrm{R}$ and $0 < h \leq 1$, $0 < v < 1$, then*

$$ {}_{a+h}\Delta_h^{-v}\Big({}^{RL}_{a+vh}\Delta_h^{v} y\Big)(\mathbf{z}) = y(\mathbf{z}) - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} y(a+vh). $$

Proof. Let $g(\mathbf{z}) := {}_{a+h}\Delta_h^{-v}\Big({}^{RL}_{a+vh}\Delta_h^{v} y\Big)(\mathbf{z})$, then we have

$$ \begin{aligned} g(\mathbf{z}) &= {}_{a+h}\Delta_h^{-v}\Big({}^{RL}_{a+vh}\Delta_h^{v} y\Big)(\mathbf{z}) = {}_{a+h}\Delta_h^{-v}\Big(\Delta_h\Big({}_{a+vh}\Delta_h^{-(1-v)} y\Big)(\mathbf{z})\Big) \\ &= {}_{a+h}\Delta_h^{-v}(\Delta_h f(\mathbf{z})), \quad \text{where } f(\mathbf{z}) = \Big({}^{RL}_{a+vh}\Delta_h^{-(1-v)} y\Big)(\mathbf{z}). \end{aligned} $$

Apply Lemma 3 but replace a by $a+h$ to get

$$ g(\mathbf{z}) = \Delta_h\big({}_{a+h}\Delta_h^{-v} f\big)(\mathbf{z}) - \frac{(\mathbf{z}-(a+h))_h^{(v-1)}}{\Gamma(v)} f(a+h). \tag{14} $$

Calculating $f(a+h)$ using Definition 2, we get

$$ \begin{aligned} f(a+h) &= \Big({}_{a+vh}\Delta_h^{-(1-v)} y\Big)(a+h) = \frac{1}{\Gamma(1-v)} \sum_{r=\frac{a}{h}+v}^{\frac{a+h}{h}-(1-v)} (a+h-\sigma_h(rh))_h^{(v-1)} y(rh)h \\ &= \frac{1}{\Gamma(1-v)}(a+h-\sigma_h(a+vh))_h^{(-v)} y(a+vh)h \\ &= \frac{1}{\Gamma(1-v)}(-vh)_h^{(-v)} y(a+vh)h \\ &= \frac{h}{\Gamma(1-v)} h^{-v}\frac{\Gamma(1-v)}{\Gamma(1)} y(a+vh) = h^{1-v} y(a+vh). \end{aligned} $$

Substituting this and the value of $f(\mathbf{z})$ in Equation (14), we get

$$ g(\mathbf{z}) = \Delta_h\Big({}_{a+h}\Delta_h^{-v}\Big({}_{a+vh}\Delta_h^{-(1-v)} y\Big)(\mathbf{z})\Big) - \frac{(\mathbf{z}-(a+h))_h^{(v-1)}}{\Gamma(v)} h^{1-v} y(a+vh). $$

Applying Lemma 4 using $\beta = 1-v$, we get

$$ \begin{aligned} g(\mathbf{z}) &= \Delta_h\Big({}_{a+h}\Delta_h^{-(v+1-v)} y\Big)(\mathbf{z}) - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} y(a+vh) \\ &= y(\mathbf{z}) - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} y(a+vh). \end{aligned} $$

This completes the proof. □

Consider $\mathtt{RL}-\mathtt{FFDE}$:

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) = f(\mathbf{z}, y(\mathbf{z})) \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a+h,h}) \tag{15}$$

with the initial condition

$$\left({}^{RL}_{a+vh}\Delta^{-(1-v)}_{h}y\right)(\mathbf{z})\Big|_{\mathbf{z}=a+h} = h^{1-v}y(a+vh) = c_1, \tag{16}$$

where $a \in \mathrm{R}$, $h, v \in (0,1)$, and c_1 is a constant.

Theorem 8. *y is a solution of $\mathtt{RL}-\mathtt{FFDE}$ in Equation (15) with the initial condition in Equation (16) if and only if it has the representation*

$$y(\mathbf{z}) = \frac{(\mathbf{z}-(a+h))_h^{(v-1)}}{\Gamma(v)}c_1 + {}_{a+h}\Delta_h^{-v} f(\mathbf{z}, y(\mathbf{z})).$$

Proof. This proof follows from Theorem 7. □

According to Theorem 7, we can write

$${}_{a+h}\Delta_h^{-v}\left({}^{RL}_{a+vh}\Delta^{v}_{h}y\right)(\mathbf{z}) = y(\mathbf{z}) - \mathfrak{N}_h(v, \mathbf{z}, a)y(a+vh), \tag{17}$$

where $\mathfrak{N}_h(v, \mathbf{z}, a) = \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)}$.

It is worthwhile to mention that analyzing the monotonicity property of such fractional difference operators is useful for better understanding the qualitative properties of solutions of different discrete fractional dynamic equations. The monotonicity properties are part of the basics of the discrete fractional calculus, where for example, we used them in this article to prove a discrete fractional version of $\mathtt{MVT}$. Then, it is of interest to obtain the following $\mathtt{MVT}$ for $\mathtt{RL}-\mathtt{FFDE}$ in Equation (17).

Theorem 9. *Let f and g be functions defined on $\mathrm{I} := \mathrm{M}_{a+h,h} \cap {}_{b-h,h}\mathrm{M} = \{a+h, a+2h, a+3h, \ldots, b-2h, b-h, b\}$, where $b = a + kh$ for some $k \in \mathrm{M}$. Assume that g strictly increases, $g(a+vh) > 0$, and $0 < v < 1$, $0 < h \leq 1$. Then, there exist $s_1, s_2 \in \mathrm{I}$ such that*

$$\frac{\left({}^{RL}_{a+vh}\Delta^{v}_{h}f\right)(s_1)}{\left({}^{RL}_{a+vh}\Delta^{v}_{h}g\right)(s_1)} \leq \frac{f(b) - \mathfrak{N}_h(v,b,a)f(a+vh)}{g(b) - \mathfrak{N}_h(v,b,a)g(a+vh)} \leq \frac{\left({}^{RL}_{a+vh}\Delta^{v}_{h}f\right)(s_2)}{\left({}^{RL}_{a+vh}\Delta^{v}_{h}g\right)(s_2)}. \tag{18}$$

Proof. First, we prove that the denominators in the inequality in Equation (18) are not zero. Since g strictly increases with $g(a+vh) > 0$, then by Theorem 4, we have

$$\left({}^{RL}_{a+vh}\Delta^{v}_{h}g\right)(\mathbf{z}) > 0 \quad (\forall\, \mathbf{z} \in \mathrm{I}). \tag{19}$$

Apply the fractional sum operator ${}_{a+h}\Delta_h^{-v}$ on both sides in Equation (19) to get

$${}_{a+h}\Delta_h^{-v}\left({}^{RL}_{a+vh}\Delta^{v}_{h}g\right)(\mathbf{z}) > {}_{a+h}\Delta_h^{-v}(0).$$

This together with Equation (17) imply that

$$g(\mathbf{z}) - \mathfrak{N}_h(v, \mathbf{z}, a)g(a+vh) > 0 \quad (\forall\, \mathbf{z} \in \mathrm{I}),$$

and particularly $g(b) - \mathfrak{N}_h(v, b, a)g(a+vh) > 0$.

To end the proof, we use the contradiction technique: we assume that Equation (18) is not true. Then, either

$$\frac{f(b)-\mathfrak{N}_h(v,b,a)f(a+vh)}{g(b)-\mathfrak{N}_h(v,b,a)g(a+vh)} < \frac{\left({}^{RL}_{a+vh}\Delta^v_h f\right)(\mathbf{z})}{\left({}^{RL}_{a+vh}\Delta^v_h g\right)(\mathbf{z})} \quad (\forall\, \mathbf{z} \in \mathrm{I}), \tag{20}$$

or

$$\frac{f(b)-\mathfrak{N}_h(v,b,a)f(a+vh)}{g(b)-\mathfrak{N}_h(v,b,a)g(a+vh)} > \frac{\left({}^{RL}_{a+vh}\Delta^v_h f\right)(\mathbf{z})}{\left({}^{RL}_{a+vh}\Delta^v_h g\right)(\mathbf{z})} \quad (\forall\, \mathbf{z} \in \mathrm{I}). \tag{21}$$

By means of Equation (19), we can multiply both sides of Equation (20) by a positive constant, and thus, we get

$$\frac{f(b)-\mathfrak{N}_h(v,b,a)f(a+vh)}{g(b)-\mathfrak{N}_h(v,b,a)g(a+vh)}\left({}^{RL}_{a+vh}\Delta^v_h g\right)(\mathbf{z}) < \left({}^{RL}_{a+vh}\Delta^v_h f\right)(\mathbf{z}) \quad (\forall\, \mathbf{z} \in \mathrm{I}).$$

Applying the fractional sum operator ${}_{a+h}\Delta_h^{-v}$ on both sides of this inequality, we get

$$\frac{f(b)-\mathfrak{N}_h(v,b,a)f(a+vh)}{g(b)-\mathfrak{N}_h(v,b,a)g(a+vh)}\left({}_{a+h}\Delta_h^{-v}\left({}^{RL}_{a+vh}\Delta^v_h g\right)(\mathbf{z})\right) < {}_{a+h}\Delta_h^{-v}\left({}^{RL}_{a+vh}\Delta^v_h f\right)(\mathbf{z}).$$

This at $\mathbf{z} = b$ and Equation (17) imply

$$f(b)-\mathfrak{N}_h(v,b,a)f(a+vh) < f(b)-\mathfrak{N}_h(v,b,a)f(a+vh),$$

and this implies that $f(b) < f(b)$, which is a contradiction. In the similar manner, the inequality in Equation (21) leads to a contradiction. The proof follows. □

4.2. Establishing the Caputo Case

Let us discuss the following results.

Lemma 5. *For any $a \in \mathrm{R}$ and $h, v > 0$, we have*

$${}_{a+h}\Delta_h^{-v}(\mathbf{z}-(a+vh))_h^{(-v)} = \Gamma(1-v)\left[1-\frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)}\right] \quad (\forall\, \mathbf{z} \in \mathrm{M}_{a+v\,h,h}).$$

Proof. Let f be a function defined on $\mathrm{M}_{a,h}$; then, from the Definition 2, we have

$$\begin{aligned}\left({}_{a+h}\Delta_h^{-v} f\right)(\mathbf{z}) &= \frac{1}{\Gamma(v)}\sum_{r=\frac{a}{h}+1}^{\frac{\mathbf{z}}{h}-v}(\mathbf{z}-\sigma_h(rh))_h^{(v-1)} f(rh)h \\ &= \left({}_a\Delta_h^{-v} f\right)(\mathbf{z}) - \frac{h}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} f(a).\end{aligned} \tag{22}$$

Considering $f(\mathbf{z}) = (\mathbf{z}-(a+vh))_h^{(-v)}$ in Equation (22), we get

$${}_{a+h}\Delta_h^{-v}(\mathbf{z}-(a+vh))_h^{(-v)} = {}_a\Delta_h^{-v}(\mathbf{z}-(a+vh))_h^{(-v)} - \frac{h}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)}(-vh)_h^{(-v)}$$

Then, by applying Lemma 2 and the identity (3), it follows that

$$\begin{aligned}
{}_{a+h}\Delta_h^{-v}(\mathbf{z}-(a+vh))_h^{(-v)} &= \frac{\Gamma(1-v)}{\Gamma(1-v+v)}(\mathbf{z}-(a+vh))_h^{(0)} \\
&\quad - \frac{h}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} h^{-v} \frac{\Gamma\left(1-\frac{vh}{h}\right)}{\Gamma\left(1-\frac{vh}{h}+v\right)} \\
&= \Gamma(1-v) - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)}\Gamma(1-v).
\end{aligned}$$

Hence, the proof is complete. □

Theorem 10. *Let* $y : \mathrm{M}_{a,h} \to \mathrm{R}$ *and* $0 < h \leq 1$, $0 < v < 1$; *then,*

$$ {}_{a+h}\Delta_h^{-v}\left({}_{a+vh}^{\;\;C}\Delta_h^{v} y\right)(\mathbf{z}) = y(\mathbf{z}) - y(a+vh). \tag{23}$$

Proof. We use Proposition 1 in order to proceed

$$\left({}_{a+vh}^{\;\;C}\Delta_h^{v} y\right)(\mathbf{z}) = \left({}_{a+vh}^{\;\;RL}\Delta_h^{v} y\right)(\mathbf{z}) - \frac{1}{\Gamma(1-v)}(\mathbf{z}-(a+vh))_h^{(-v)} y(a+vh). \tag{24}$$

Taking ${}_{a+h}\Delta_h^{-v}$ on both sides of Equation (24) and then using Theorem 7 and Lemma 5, we get

$$\begin{aligned}
{}_{a+h}\Delta_h^{-v}\left({}_{a+vh}^{\;\;C}\Delta_h^{v} y\right)(\mathbf{z}) &= {}_{a+h}\Delta_h^{-v}\left({}_{a+vh}^{\;\;RL}\Delta_h^{v} y\right)(\mathbf{z}) - \frac{y(a+vh)}{\Gamma(1-v)}\,{}_{a+h}\Delta_h^{-v}(\mathbf{z}-(a+vh))_h^{(-v)} \\
&= y(\mathbf{z}) - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)} y(a+vh) \\
&\quad - \frac{y(a+vh)}{\Gamma(1-v)} \cdot \Gamma(1-v)\left[1 - \frac{h^{1-v}}{\Gamma(v)}(\mathbf{z}-(a+h))_h^{(v-1)}\right] \\
&= y(\mathbf{z}) - y(a+vh),
\end{aligned}$$

and after some simplifications, the result is obtained. □

Consider the following Caputo−FFDE:

$$\left({}_{a+vh}^{\;\;C}\Delta_h^{v} y\right)(\mathbf{z}) = f(\mathbf{z}, y(\mathbf{z})) \quad (\forall\ \mathbf{z} \in \mathrm{M}_{a+h,h}) \tag{25}$$

with the initial condition

$$\left({}_{a+vh}^{\;\;RL}\Delta_h^{-(1-v)} y\right)(\mathbf{z})\Big|_{\mathbf{z}=a+h} = h^{1-v} y(a+vh) = h^{1-v} c_2, \tag{26}$$

where $a \in \mathrm{R}$, $h, v \in (0,1)$, and c_2 is a constant.

Theorem 11. *y is a solution of* Caputo−FFDE *in Equation* (25) *with the initial condition in Equation* (26) *if and only if it has the representation*

$$y(\mathbf{z}) = c_2 + {}_{a+h}\Delta_h^{-v} f(\mathbf{z}, y(\mathbf{z})).$$

Proof. The proof follows from Theorem 10 directly. □

Remark 2. *In the case of Caputo,* MVT *for Caputo*−FFDE *in Equation* (23)*,*

$$\frac{\left({}^{C}_{a+vh}\Delta^{v}_{h}f\right)(s_1)}{\left({}^{C}_{a+vh}\Delta^{v}_{h}g\right)(s_1)} \le \frac{f(b)-f(a+vh)}{g(b)-g(a+vh)} \le \frac{\left({}^{C}_{a+vh}\Delta^{v}_{h}f\right)(s_2)}{\left({}^{C}_{a+vh}\Delta^{v}_{h}g\right)(s_2)}, \tag{27}$$

does not hold, where the assumptions of Theorem 9 are supposed to be given. The reason for this is that we do not know whether $\left({}^{C}_{a+vh}\Delta^{v}_{h}g\right)(\mathbf{z}) > 0$ *when, by assumption and Corollary 3, we have*

$$\begin{aligned}\left({}^{C}_{a+vh}\Delta^{v}_{h}g\right)(\mathbf{z}) &> -\frac{1}{\Gamma(1-v)}(\mathbf{z}-(a+vh))_h^{(-v)}g(a+vh) \quad (\forall\, \mathbf{z}\in \mathrm{I}) \\ &< 0,\end{aligned}$$

since the three quantities $\frac{1}{\Gamma(1-v)}$, $(\mathbf{z}-(a+vh))_h^{(-v)}$*, and* $g(a+vh)$ *are all positive for* $v\in(0,1)$ *and* $\mathbf{z}\in\mathrm{I}$*.*

5. Conclusions

In this paper, a discrete v-monotonicity analysis for discrete functions defined on $\mathrm{M}_{a,h}$ in the framework of the discrete RL fractional sums, and RL and Caputo fractional differences on the time scale $h\,\mathbb{Z}$ were successfully studied. The relation between delta RL and delta Caputo fractional h-differences was established. The RL − FFDE and Caputo−FFDE were considered, and their solutions were discussed. Utilizing the monotonicity results and RL − FFDE solution, the MVT was presented. However, MVT for Caputo−FFDE was not valid using its corresponding monotonicity results and Caputo−FFDE.

A further extension of this study includes improving our findings to study other classes of discrete fractional sums and differences, including those of exponential kernel $\left({}^{CFR}_{\ a}\Delta^{v}_{h}y\right)(\mathbf{z})$ or Mittag–Liffler kernel $\left({}^{ABR}_{\ a}\Delta^{v}_{h}y\right)(\mathbf{z})$, defined in [22]. Fortunately, some research is in progress in this area.

Author Contributions: Conceptualization, P.O.M. and T.A.; data curation, F.K.H.; formal analysis, P.O.M. and T.A.; funding acquisition, T.A.; investigation, P.O.M.; methodology, F.K.H.; project administration, P.O.M. and T.A.; software, P.O.M. and T.A.; supervision, T.A. and F.K.H.; validation, F.K.H.; visualization, F.K.H.; writing—original draft, P.O.M.; writing—review and editing, T.A. and F.K.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The second author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: New York, NY, USA, 1993.
2. Daftardar-Gejji, V. *Fractional Calculus and Fractional Differential Equations*; Springer: Singapore, 2019.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier B.V.: Amsterdam, The Netherlands, 2006.
4. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Berlin, Germany, 2010.
5. Wu, J.; Yuan, J.; Gao, W. Analysis of fractional factor system for data transmission in SDN. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 191–196. [CrossRef]

6. Kurt, A.; Şenol, M.; Tasbozan, O.; Chand, M. Two reliable methods for the solution of fractional coupled Burgers' equation arising as a model of Polydispersive sedimentation. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 523–534. [CrossRef]
7. Touchent, K.A.; Hammouch, Z.; Mekkaoui, T. A modified invariant subspace method for solving partial differential equations with non-singular kernel fractional derivatives. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 35–48. [CrossRef]
8. Hamasalh, F.K.; Mohammed, P.O. Generalized quartic fractional spline approximation function with applications. *IJOPCM* **2015**, *8*, 67–80. [CrossRef]
9. Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.G.; Hamed, Y.S. Some higher-degree Lacunary fractional splines in the approximation of fractional differential equations. *Symmetry* **2021**, *13*, 422. [CrossRef]
10. Onal, M.; Esen, A. A Crank-Nicolson approximation for the time fractional Burgers equation. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 177–184. [CrossRef]
11. Martinez, M.; Mohammed, P.O.; Valdes, J.E.N. Non-conformable fractional Laplace transform. *Kragujev. J. Math.* **2022**, *46*, 341–354.
12. Wang, Y. The Green's function of a class of two-term fractional differential equation boundary value problem and its applications. *Adv. Differ. Equ.* **2020**, *2020*, 80. [CrossRef]
13. Zhang, S.; Hong, S. Variable separation method for a nonlinear time fractional partial differential equation with forcing term. *J. Comput. Appl. Math.* **2018**, *339*, 297–305. [CrossRef]
14. Tarasov, V. *Handbook of Fractional Calculus with Applications*; Applications in Physics, Part A; De Gruyter: Boston, MA, USA; Berlin, Germany, 2019; Volume 4.
15. Goodrich, C.; Peterson, A. *Discrete Fractional Calculus*; Springer: Berlin, Germany, 2015.
16. Goodrich, C. Existence of a positive solution to a system of discrete fractional boundary value problems. *Appl. Math. Comput.* **2011**, *217*, 4740–4753. [CrossRef]
17. Atici, F.; Eloe, P. A transform method in discrete fractional calculus. *Internat. J. Differ. Equ.* **2007**, *2*, 165–176.
18. Atici, F.; Eloe, P. Initial value problems in discrete fractional calculus. *Proc. Am. Math. Soc.* **2009**, *137*, 981–989. [CrossRef]
19. Mohammed, P.O. A generalized uncertain fractional forward difference equations of Riemann-Liouville type. *J. Math. Res.* **2019**, *11*, 43–50. [CrossRef]
20. Abdeljawad, T. On delta and nabla caputo fractional differences and dual identities. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 12. [CrossRef]
21. Abdeljawad, T.; Jarad, F.; Atangana, A.; Mohammed, P.O. On a new type of fractional difference operators on h-step isolated time scales. *J. Fract. Calc. Nonlinear Sys.* **2021**, *1*, 46–74.
22. Abdeljawad, T. Different type kernel h–fractional differences and their fractional h–sums. *Chaos Solit. Fract.* **2018**, *116*, 146–156. [CrossRef]
23. Mohammed, P.O.; Hamasalh, F.K.; Abdeljawad, T. Difference monotonicity analysis on discrete fractional operators with discrete generalized Mittag-Leffler kernels. *Adv. Differ. Equ.* **2021**, *2021*, 213. [CrossRef]
24. Mohammed, P.O.; Abdeljawad, T. Discrete generalized fractional operators defined using h-discrete Mittag-Leffler kernels and applications to AB fractional difference systems. *Math. Meth. Appl. Sci.* **2020**, 1–26. [CrossRef]
25. Abdeljawad, T.; Atici, F. On the Definitions of Nabla Fractional Operators. *Abstr. Appl. Anal.* **2012**, *2012*, 406757. [CrossRef]
26. Suwan, I.; Owies, S.; Abdeljawad, T. Monotonicity results for h-discrete fractional operators and application. *Adv. Differ. Equ.* **2018**, *2018*, 207. [CrossRef]
27. Atici, F.; Uyanik, M. Analysis of discrete fractional operators. *Appl. Anal. Discr. Math.* **2015**, *9*, 139–149. [CrossRef]