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Review

Reinstatement of the Extension Principle in Approaching Mathematical Programming with Fuzzy Numbers

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Abstract: Optimization problems in the fuzzy environment are widely studied in the literature. We restrict our attention to mathematical programming problems with coefficients and/or decision variables expressed by fuzzy numbers. Since the review of the recent literature on mathematical programming in the fuzzy environment shows that the extension principle is widely present through the fuzzy arithmetic but much less involved in the foundations of the solution concepts, we believe that efforts to rehabilitate the idea of following the extension principle when deriving relevant fuzzy descriptions to optimal solutions are highly needed. This paper identifies the current position and role of the extension principle in solving mathematical programming problems that involve fuzzy numbers in their models, highlighting the indispensability of the extension principle in approaching this class of problems. After presenting the basic ideas in fuzzy optimization, underlying the advantages and disadvantages of different solution approaches, we review the main methodologies yielding solutions that elude the extension principle, and then compare them to those that follow it. We also suggest research directions focusing on using the extension principle in all stages of the optimization process.

Keywords: fuzzy numbers; extension principle; mathematical programming

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1. Introduction

Zadeh's fuzzy set theory [1] is an accurate mathematical tool that is able to model the uncertainty widely present in real-life problems. It has a wide range of applications in various scientific fields from medicine, engineering, and computer science to artificial intelligence. Dubois [2] emphasized that one of the roles of the fuzzy set theory is to facilitate a joint functionality of the numerical and qualitative approaches in decision-making.

Dzitac et al. [3] recently presented several important aspects of Zadeh's fuzzy logic theory that were proved to have useful applications. A discussion on the need of fuzzy logic and a nonstandard perspective on it was given in [4]. Wu and Xu [5] presented a wide range of applications of the fuzzy logic in decision making that proved fuzzy logic's ability in handling uncertain linguistic information. Shi [6] introduced several results from fuzzy group's theory that could represent a good foundation when the multivalued computer systems will be redeveloped in the future. Nădăban [7] presented a concise and unitary general view on the algebraic connections between classic, fuzzy, and quantum logics.

In this study, we restrict our attention to fuzzy mathematical programming. Zimmerman [8,9] emphasized the role of the fuzzy set theory in mathematical programming, introducing a solution approach to multiple objective optimization problems based on

aggregation of fuzzy goals and fuzzy constraints. Verdegay [10] emphasized that the fuzzy linear programming is one of the most studied topics in the theory of the fuzzy sets and systems. We focus especially on optimization problems that involve fuzzy numbers as coefficients and/or variables aiming to rehabilitate the position of Zadeh's extension principle [1] in approaching such problems.

When a solution approach to fuzzy optimization problems strictly follows the extension principle, the ranking of the involved fuzzy quantities is avoided. From our perspective, this fact is a real advantage since there are many ranking functions defined in the literature (Abbasbandy [11] mentioned more than thirty); each of them might generate a solution approach to certain classes of optimization problems, and any comparison of their effectiveness is almost impossible.

After a brief presentation of the basic notation and terminology related to fuzzy sets and mathematical programming given in Section 2, we include in Section 3 a discussion on the indispensability of the extension principle in solving mathematical programming problems with fuzzy numbers. In Section 4, we survey the main methodologies that address full fuzzy optimization problems and analyze the effects of neglecting the extension principle in some of their optimization steps. In Section 5, we suggest research directions focusing on using the extension principle in all stages of the optimization process. Our concluding remarks are presented in Section 6.

2. Notation and Terminology

2.1. Fuzzy Sets

Zadeh [1] introduced the concept of fuzzy set $\widetilde{A}$ over the universe X as a collection of pairs $\left(x, \mu_{\widetilde{A}}(x)\right)$ such that the first component of each pair $x \in X$ is an element of the universe, while the second element $\mu_{\widetilde{A}}(x) \in [0,1]$ is its corresponding membership degree. Function $\mu_{\widetilde{A}} : X \to [0,1]$ is called the membership function of the fuzzy set $\widetilde{A}$.

Atanassov [12] introduced the intuitionistic fuzzy sets as a generalization to the fuzzy sets. An intuitionistic fuzzy set $\widetilde{A}^I$ of a universe X is a set of triples

$$\left(x, \mu_{\widetilde{A}^I}(x), \nu_{\widetilde{A}^I}(x)\right) \tag{1}$$

such that $x \in X$, $\mu_{\widetilde{A}^I}(x), \nu_{\widetilde{A}^I}(x) \in [0,1]$, and $0 \leq \mu_{\widetilde{A}^I}(x) + \nu_{\widetilde{A}^I}(x) \leq 1$. The membership function of $\widetilde{A}^I$ is $\mu_{\widetilde{A}^I} : X \to [0,1]$, and the nonmembership function of $\widetilde{A}^I$ is $\nu_{\widetilde{A}^I} : X \to [0,1]$ of $\widetilde{A}^I$ in X. For each $x \in X$ the value $\mu_{\widetilde{A}^I}(x)$, called the membership degree, the value $\nu_{\widetilde{A}^I}(x)$ is called the nonmembership degree, and the value

$$h(x) = 1 - \mu_{\widetilde{A}^I}(x) - \nu_{\widetilde{A}^I}(x) \tag{2}$$

is called the degree of hesitancy of x in $\widetilde{A}^I$. Deng [13] proposed a new way to measure the information volume of fuzzy and intuitionistic fuzzy membership functions.

2.1.1. Fuzzy Numbers

Fuzzy numbers (FNs) are special cases of fuzzy sets. A fuzzy set $\widetilde{A}$ of the universe of real numbers R is called a fuzzy number if and only if: (i) it is fuzzy normal and fuzzy convex; (ii) the membership function $\mu_{\widetilde{A}}$ is upper semicontinuous; and (iii) its support, i.e., $\left\{x \in R | \mu_{\widetilde{A}}(x) > 0\right\}$ is bounded. Similarly, an intuitionistic fuzzy set of R is an intuitionistic fuzzy number (IFN) if and only if its membership function fulfills all conditions in the definition of a fuzzy number; the nonmembership function is fuzzy concave and lower semicontinuous; and its support $\left\{x \in R | \nu_{\widetilde{A}}(x) < 1\right\}$ is bounded.

An LR flat fuzzy number is defined using two reference functions for the left and right sides of the fuzzy number, respectively. The reference functions L and R are both defined on the interval $[0, \infty)$, take values from the interval $[0,1]$, and have two essential characteristics: (i) $L(0) = R(0) = 1$; and (ii) both L and R are nonincreasing on $[0, \infty)$. We refer the reader to the book of Dubois and Prade [14] for more details.

In what follows, we are interested in triangular and trapezoidal fuzzy and intuitionistic fuzzy numbers. The graph of the nonzero piece of the membership function of a triangular fuzzy number (TFN) forms a triangle with the abscissa, and is generally expressed by its components, as a triple (a_1, a_2, a_3), $a_1 \leq a_2 \leq a_3$. The interval (a_1, a_3) is the support of the fuzzy set and the component a_2 is the value with the maximal amplitude. Similarly, the graph of the nonzero piece of the membership function of a trapezoidal fuzzy number (TrFN) forms a trapezoid with the abscissa, and is generally expressed by its components, as a quadruple (a_1, a_2, a_3, a_4), $a_1 \leq a_2 \leq a_3 \leq a_4$. The interval (a_1, a_4) is the support of the fuzzy set and all values in the interval $[a_1, a_4]$ have the maximal amplitude.

A triangular intuitionistic fuzzy number (TIFN) is generally denoted by

$$\widetilde{A}^I = (a_1, a_2, a_3; a_1', a_2', a_3'). \tag{3}$$

Its first three components are related to the membership function (that is identical to a membership function of a triangular fuzzy number), and last three related to the nonmembership function. Similarly, a trapezoidal intuitionistic fuzzy number (TrIFN) is generally described by

$$\widetilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4'), \tag{4}$$

first four components being related to the membership function that is in fact a membership function of a trapezoidal fuzzy number.

2.1.2. The Extension Principle

Bellman and Zadeh [15] introduced the concepts of fuzzy decisions and fuzzy constraints, and proposed a principle to aggregate them. The fuzzy arithmetic was developed with the help of the extension principle mentioned by Zadeh from the beginning in [1].

According to this principle, the fuzzy set $\widetilde{B}$ of the universe Y that is the result of evaluating the function f at the fuzzy sets $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r$ over their universes $X_1, X_2, \ldots, X_r$ is defined through its membership function as

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r)\in f^{-1}(y)} \left(\min\left\{\mu_{\widetilde{A}_1}(x_1), \ldots, \mu_{\widetilde{A}_r}(x_r)\right\}\right), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

See also Zimmerman [16] for more details.

Fuzzy addition and subtraction of triangular and trapezoidal fuzzy numbers both yield triangular and trapezoidal numbers, respectively. Strictly following the extension principle, neither fuzzy multiplication nor division of triangular/trapezoidal fuzzy numbers yield triangular/trapezoidal numbers. This issue is generally overcome using a relative innocent approximation that replaces the exact results by those triangular/trapezoidal numbers that keep the extreme values (i.e., the endpoints of their support, and the values with maximal amplitude) the same.

Diniz et al. [17] discussed the optimization of a fuzzy-valued function using Zadeh's extension principle. The objective function was a Zadeh's extension of a function with respect to a parameter and an independent variable. Kupka [18] introduced some results on the approximation of Zadeh's extension of a given function, and studied the quality of the approximation with respect to the choice of the metric on the space of the fuzzy sets.

2.2. Mathematical Programming Models

2.2.1. The General Crisp Model

Model (6) generally defines a crisp mathematical programming problem that consists in maximizing the objective function f that depends on the coefficients c, and the decision variables x over the feasible set $X(b)$ that depends on the coefficients b.

$$\begin{array}{ll} \max & f(x, c), \\ \text{s.t.} & x \in X(b), \end{array} \tag{6}$$

Under additional assumptions (imposed on the objective function and/or the constraints), Model (6) becomes convex (if the objective function is convex over the feasible set that is also convex); linear (if both the objective function and constraint system are defined by linear expressions, i.e., $f(x, c) = c^T x$, and $X(A, b) = \{x | Ax \leq b, x \geq 0\}$, with A, b, and c being matrices of certain dimensions); linear fractional (when the objective function is a ratio of linear functions); mixed integer (if some decision variables take integer and/or binary values); multiple objective (if the values of the objective function are vectors), etc. Specific models have specific solution methods. The basic references here are [19] for linear programming, [20] for multiple objective programming, [21] for linear fractional programming, and [22] for integer programming.

2.2.2. Fuzzified Models

Model (7) generally defines the class of fuzzy mathematical programming problems that use the fuzzy coefficients $\widetilde{b}$ and $\widetilde{c}$ to describe the uncertainty of the real system on which the feasible set and objective function depend, respectively.

$$\begin{array}{ll} \max & f(x, \widetilde{c}), \\ \text{s.t.} & x \in X(\widetilde{b}), \end{array} \tag{7}$$

Problems belonging to this class are known as mathematical programming problems with fuzzy coefficients. Depending on the type of the constraints that define the feasible set $X(\widetilde{b})$ a solution approach to (7) provides either crisp optimal values (real numbers) or "optimal" fuzzy sets values for the decision variables. In the second case, the fuzziness of the decision variables cannot be concluded from the model; sometimes, it is ignored, especially when the focus is exclusively put on the objective function's fuzzy set value, and sometimes, it is derived with respect to the endpoints of the support of the fuzzy set of optimal objective values.

Whenever the values of the decision variables are fuzzy sets the model should be formalized as

$$\begin{array}{ll} \max & f(\widetilde{x}, \widetilde{c}), \\ \text{s.t.} & \widetilde{x} \in X(\widetilde{b}), \end{array} \tag{8}$$

in order to explicitly show the fuzziness of x-es. In fact, Model (8) describes the so-called full fuzzy (FF) mathematical programming problems. Again, particular forms of the objective function and/or constraints in Model (8) contribute to classifying it as full fuzzy linear programming model, full fuzzy multiple objective model, full fuzzy fractional model, etc. Similar concepts and models exist in an intuitionistic fuzzy (IF) environment.

We will use the following abbreviations related to mathematical programming: LP for linear programming; LFP for linear fractional programming; MO for multiple objective; TP for transportation problem. More elaborate abbreviations can be obtained by concatenating some of the basic abbreviations. For instance FIF-LP stands for full intuitionistic fuzzy linear programming.

3. The Necessity of the Extension Principle

The extension principle was created to generalize the crisp mathematical concepts to concepts compatible with the fuzzy set theory. Neglecting the extension principle when dealing with optimization problems whose models use fuzzy numbers either move the derived fuzzy solutions out of their proper bounds or damage their shapes.

The solution approaches to Model (8) a priori impose certain shapes for the decision variables (see FIS in Figure 1), mainly the same shape as the shapes used for the coefficients, generally fuzzy numbers. Further on, they formally use the fuzzy arithmetic theory to evaluate the expressions of the objective function and constraints with respect to the fuzzy coefficients and variables, and then propose different methods relying on fuzzy numbers optimization and ranking.

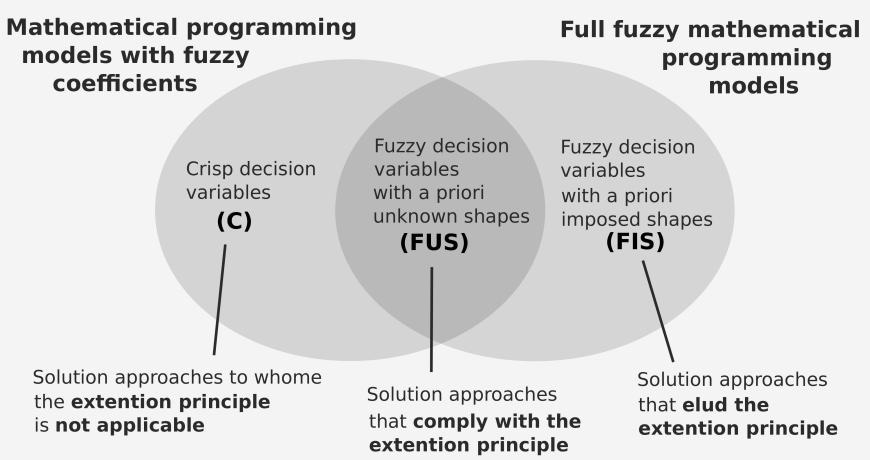

Figure 1. Decision variables shapes with respect to the two types of fuzzy mathematical programming and how they comply with the extension principle.

Figure 2 visually presents the current position on which the extension principle is mainly involved in the solution approaches to mathematical programming with fuzzy numbers; and the desired position on which the extension principle should be generally reinstated.

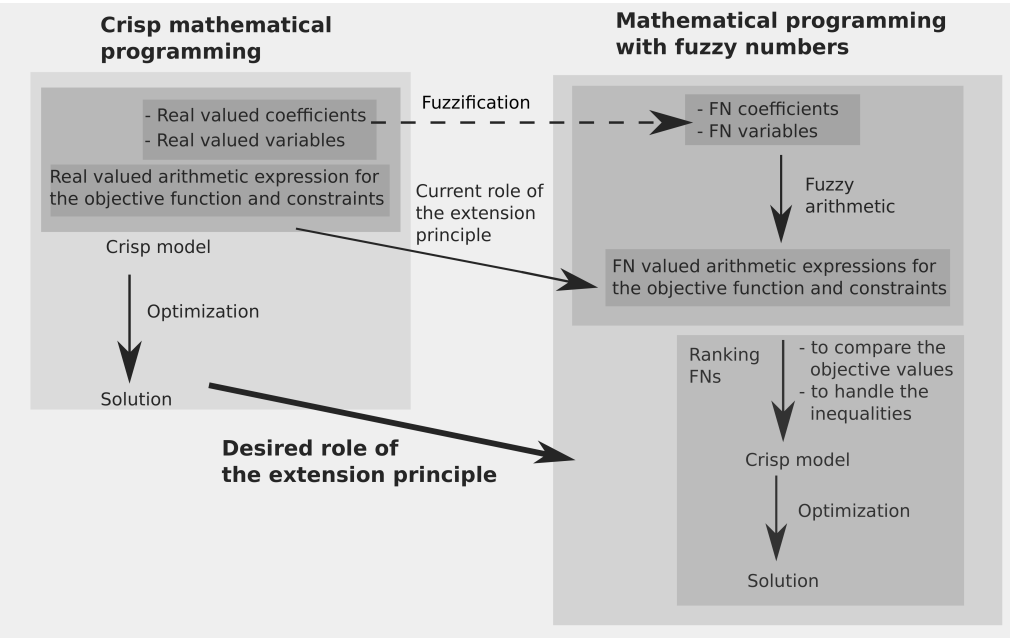

Figure 2. Current versus desired roles of the extension principle in mathematical programming with fuzzy numbers.

Comparing Models (7) and (8) and the solution approaches found in the literature, it is noticeable that problems with fuzzy coefficients and their corresponding full fuzzy problems share some hidden characteristics visible through certain solution approaches. To derive solution values to Model (7), one must use crisp decision variables (see C in Figure 1) through the computation, and can easily ignore the ranges of their values whenever the main focus is put on the objective function, i.e., on finding the fuzzy set that in the best way describes the possible values of the objective function. In some cases, paying more attention to the values of the decision variables, one can see that they describe fuzzy sets that easily fit in Model (8). Consequently, the border between the class of problems with fuzzy coefficients and the class of full fuzzy problems cannot be clearly stated, and solution approaches to problems with fuzzy coefficients might be extremely useful in solving full fuzzy problems, especially when they strictly comply with the extension principle. Figure 1

visually presents how the shapes of the decision variables in Models (7) and (8) comply with the extension principle.

By the literature review given in the next section, we aim to present the existing approaches to both classes of fuzzy mathematical programming problems, grouping them by the models they solve and their methodological particularities, and also conclude about the negative effects of eluding the extension principle in the crucial stages of the optimization process.

4. Literature Review from the Extension-Principle-Based Perspective

LP models both crisp and fuzzy are widely studied due to their simplicity. Ghanbari et al. [23] surveyed the literature on fuzzy LP problems discussing both the mathematical models and the solutions. Fractional programming models are the next most used models since they offer a relatively simple generalization of linear programming toward hard nonlinearity. Stanojević et al. [24] reviewed the literature on LFP problems with fuzzy numbers. Transportation problems in both forms linear and fractional are also widely studied in the literature due to their particularities that bring consistent simplifications to the general solution approaches.

The main facts that make a difference among solution approaches to full fuzzy optimization problems are related to the way of ranking the fuzzy numbers when optimizing the objective function and/or evaluating the fuzzy constraints.

We organize our literature review in three subsections that cover the three classes of problems mentioned above and restrict our attention to those papers that are relevant to our discussion, which is focused on how the presence of the extension principle in formulating the solution concept contributes to the relevance of the derived fuzzy set solutions.

4.1. Fuzzy Transportation Problems

A full fuzzy linear TP consists of minimizing the fuzzy cost of transferring fuzzy amounts of goods from sources to destinations with fuzzy demands. Model (9)

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{c}_{ij}\widetilde{x}_{ij} \\ \text{s.t.} \quad & \\ & \sum_{j=1}^{n}\widetilde{x}_{ij} = \widetilde{a}_i, \quad i=\overline{1,m}, \\ & \sum_{i=1}^{m}\widetilde{x}_{ij} = \widetilde{b}_j, \quad j=\overline{1,n}, \\ & \widetilde{x}_{ij} \geq \widetilde{0} \qquad i=\overline{1,m}, j=\overline{1,n}, \end{aligned} \tag{9}$$

summarizes the essential details, assuming that there are m sources with the amount of available goods expressed by the fuzzy quantities $\widetilde{a}_i$, $i=\overline{1,m}$ and n destinations with the fuzzy demands $\widetilde{b}_j$, $j=\overline{1,n}$. The decision variables $\widetilde{x}_{ij}$, $i=\overline{1,m}$, $j=\overline{1,n}$, represent the amount of goods transported from the source i to the destination j, with a transport cost per unit equal to $\widetilde{c}_{ij}$.

Liu and Kao [25] and Liu [26] proposed solution approaches to linear and linear fractional TPs with fuzzy coefficients, respectively, that fit to Model (7). Their solution concept is based on defining the fuzzy set of the optimal solution values of the objective function through a membership function that complies to the extension principle. Equation (10) shows their membership function adapted to the notation used in Model (7).

$$\mu_{\widetilde{f}}(z) = \begin{cases} \max\limits_{(b,c)|z=\max\limits_{x\in X(b)} f(x,c)} \left(\mu_{(\widetilde{b},\widetilde{c})}(b,c)\right), & \exists b,c|z=\max\limits_{x\in X(b)} f(x,c), \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $\mu_{(\widetilde{b},\widetilde{c})}(b,c) = \min\{\mu_{\widetilde{b}}(b), \mu_{\widetilde{c}}(c)\}$, and $\mu_{\widetilde{b}}(b)$ and $\mu_{\widetilde{c}}(c)$ represent the membership functions of the coefficients $\widetilde{b}$ and $\widetilde{c}$, respectively.

Neither Liu and Kao [25] nor Liu [26] reported the shapes of the fuzzy sets of the optimal values of the decision variables. Later on, based on Liu and Kao's solution concept [25], the optimal values of the decision variables were empirically disclosed by a Monte Carlo simulation in [27] to be trapezoidal-like FNs whenever the problem's coefficients were TrFNs. In that way, it was proved that the problems solved by Liu and Kao [25] belong to the class of problems that can be modeled using fuzzy decision variables with a priori unknown shapes (see FUS in Figure 1). Stanojević and Stanojević [28] introduced an extension-principle-based formulation for the fuzzy set value of each decision variable and proposed mathematical models able to derive numerical approximations to the membership functions of the fuzzy sets representing the best values for the decision variables. Formulation (11) represents their definition for each decision variable $\widetilde{x}_i$, $i = \overline{1,n}$ adapted to the notation used in Model (7).

$$\mu_{\widetilde{x}_i}(t) = \begin{cases} \max\limits_{(b,c)|t = \arg_i \max\limits_{y \in X(b)} f(y,b)} \left(\mu_{(\widetilde{b},\widetilde{c})}(b,c)\right), & \exists b,c|t = \arg_i \max\limits_{y \in X(b)} f(y,b), \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where $\arg_i \max\limits_{y \in X(b)} f(y,b)$ represents the optimal value of the crisp scalar decision variable y_i obtained when maximizing $f(y,b)$ over the feasible set $X(b)$, and $\mu_{(\widetilde{b},\widetilde{c})}(b,c)$ has the same meaning as in the definition of the membership function of optimal solution values (10). The membership function of the fuzzy set $\widetilde{x}$ that collects the crisp values of the vector x of the decision variables, i.e.,

$$\mu_{\widetilde{x}}(t) = \begin{cases} \max\limits_{(b,c)|t = \arg \max\limits_{y \in X(b)} f(y,b)} \left(\mu_{(\widetilde{b},\widetilde{c})}(b,c)\right), & \exists b,c|t = \arg \max\limits_{y \in X(b)} f(y,b), \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

was first given in [29] for the general linear case. The graph of the membership function given in (11) represents in fact the projection of the graph of the membership function (12) on the plane $(Ox_i, O\alpha)$, $i = \overline{1,n}$, where the axes Ox_i and $O\alpha$ contain the values of the variable x_i and the membership degrees α, respectively.

The first step in solving a general crisp TP is reserved to a simple balancing procedure. In the case of solving fuzzy TPs, the situation is quite different: some approaches can be applied only to balanced problems; several approaches focus only on balancing a general fuzzy TP; and also some approaches can be applied to unbalanced problems. Unfortunately, the solution provided by an approach applied to an unbalanced problem is far from being similar to the solution derived after a balancing procedure is applied. Mishra and Kumar [30] proposed a balancing procedure for FIF-TP that is an adaptation of Kumar and Kaur's method [31] to balance FF-TPs. Any solution approach to fuzzy TP based on a solution concept that is in accordance to the extension principle does not need a fuzzy balancing procedure, since crisp TPs are solved and their optimal solutions build the final fuzzy set solution to the original fuzzy problem. The methodology proposed in Stanojević and Stanojević [28] to solve FIF-TPs discussed this issue in detail.

Kumar and Kaur [31], after balancing the fuzzy transportation problem, proposed two approaches to solved it. In both approaches, they first applied algebraic operators on the fuzzy numbers to derive the fuzzy number values of the objective function and left-hand sides of the constraints; and then focused on the optimization of the objective function. In the first method, Kumar and Kaur applied a ranking function on the TrFN representing the objective function, obtained a crisp expression of the components of the involved TrFNs, and optimized it subject to a conjunctive constraint system. In the second

method, they solved four crisp TPs, one for each component of the TrFN representing the objective function. None of these methods respects the extension principle.

Singh and Yadav [32] proposed several algorithms to determine a basic feasible solution to a balanced FIF-TP. All their algorithms are analogous to the well known methods developed for crisp TPs. This simple analogy also discards any possibility to follow the extension principle.

Stanojević and Stanojević [28] improved Liu and Kao's approach [25] by proposing a mathematical model with disjunctive constraint system and then extended it to solving TPs with TrIFN as parameters. They compare their results with the results reported by Kumar and Hussain [33], Ebrahimnejad and Verdegay [34] (who formulated their solution approach based on an accuracy function used for ordering the TrIFNs), and Mahmoodirad et al. [35]. All of these papers addressed FIF-TPs. The experiments showed that, due to its compliance with the extension principle, the method proposed in [28] found a fuzzy set solution with a wider support and smaller optimal values for the objective function that had to be minimized than the methods introduced in [33–35].

Further investigations are necessary to conclude about Mahajan and Gupta's approach [36] for solving a FIF-MO-TP problem. Using an accuracy function on each objective, they first reduced the problem to a crisp MO-TP, and developed an algorithm to solve it. To handle the intuitionistic fuzzy constraints, they used linear, exponential, and hyperbolic membership functions. Unifying the arithmetic and optimization, and formulating an extension-principle-based approach to solve the same problem, possibly better results can be derived.

Table 1 reports results found in the literature for three numerical examples of full fuzzy TPs. The first example uses trapezoidal fuzzy numbers, whereas the next two of them consider intuitionistic fuzzy numbers for all coefficients and decision variables. The results derived with a full respect to the extension principle correspond to the references written in bold. All full fuzzy TPs were minimization problems, and all optimal fuzzy numbers that comply to the extension principle are clearly smaller than those that elude it. For the first example, the values with maximal amplitude are the same in both references, but the second reference reports smaller minimal values with nonzero membership function. For the second and third examples, one can notice that smaller minimal values with nonzero membership function were obtained by the solution approach that strictly followed the extension principle. Figure 3 shows graphically the results reported in Table 1 for the second example.

Table 1. Comparative numerical results for full fuzzy and full intuitionistic fuzzy TPs.

Ex.	Fuzzy Type	Ref.	$\widetilde{z}_{\min}$
1	trapezoidal	[25]	$(2100, 2900, 3500, 5800)$
		[28]	(1500, 2900, 3500, 5800)
2	trapezoidal intuitionistic	[34]	(3300, 5800, 9100, 13,200; 2350, 4450, 11,050, 15,550)
		[28]	(1400, 3760, 8750, 13,500; 700, 2280, 9550, 16,000)
3	triangular intuitionistic	[33]	$(137, 292, 502; 12, 292, 961)$
		[34]	$(63, 313, 773; 2, 313, 1726)$
		[35]	$(63, 310, 757; 2, 310, 1806)$
		[28]	$(32, 305.4, 765; 0, 305.4, 1697)$

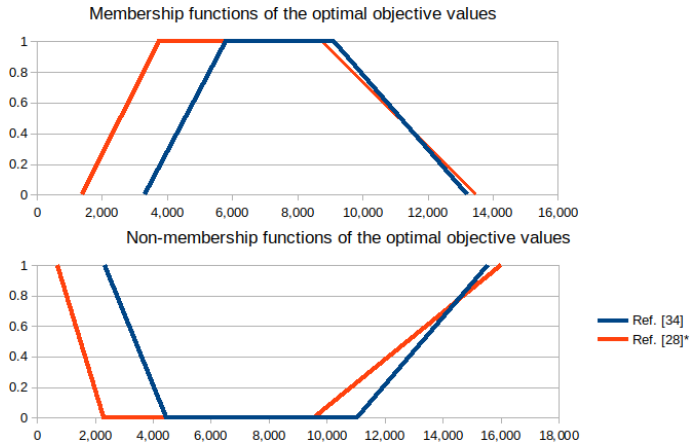

Figure 3. Graphic representation of the results reported in Table 1, example 2. Symbol * marks the reference that introduced the approach that comply to the extension principle.

4.2. Fuzzy Linear Programming Problems

A full fuzzy LP problem is generally described by Model (13)

$$\begin{array}{ll} \max & \widetilde{c}^T \widetilde{x} \\ \text{s.t.} & \widetilde{A}\widetilde{x} \leq \widetilde{b}, \\ & \widetilde{x} \geq 0. \end{array} \tag{13}$$

Several variants of Model (13) are sometimes addressed in the literature—for instance, those that minimize the objective function and/or use equality constraints to define the feasible set, but they are not different in nature, and their approaches can be easily adapted to solve Model (13) as well.

Perez-Canedo et al. [37] surveyed the literature on fuzzy linear programming, paying attention to those papers that used a lexicographic method to rank the fuzzy numbers. Their up-to-date review shows that fuzzy LP relying on lexicographic methods is an active research area with a wide range of applications in practice.

We briefly survey several papers on fuzzy linear programming, including notes from the extension-principle-based perspective.

Hosseinzadeh Lotfi et al. [38] used lexicographic method and fuzzy approximate solutions to FF-LP problems with TFNs. They transformed all TFNs coefficients in symmetric TFNs and assumed that all decision variables were symmetric TFNs. They also used a special ranking function on fuzzy numbers, and obtained a crisp MO-LP to solve. They did not involve the extension principle in the optimization step, and derived only one efficient solution to the crisp MO-LP.

Khan et al. [39] introduced a simplex-like technique for solving FF-LP problems. They involved a ranking function in the Gaussian elimination process and developed a flexible easy and reasonable solution algorithm. The use of any ranking function supposes defuzzification before optimization that is not in the desired accordance to the extension principle.

To transform FF-LP problems into crisp LPs, Kumar et al. [40] used a ranking function in order to compare the FN values of the objective function and a component-wise comparison of the left and right hand sides of the constraints. They finally solved one single objective linear programming problem deriving optimal values for all components of all decision variables. The use of two distinct methods to compare fuzzy quantities embedded in the same model is against the essence of the extension principle.

Ezzati et al. [41] addressed FF-LP problems by applying first fuzzy arithmetic to the TFN values of the decision variables and coefficients in order to derive the TFN value of the objective function; then, they constructed a three-objective crisp problem and solved it by a lexicographic method. Stanojević and Stanojević [29] proposed empirical solutions to FF-LP problems based on a Monte Carlo simulation that fully comply with the extension

principle. They analyzed one of Ezzati et al.'s examples and conclude that Ezzati et al.'s methodology fails to follow the extension principle.

Noticing a shortcoming in the solution approach introduced in [41], Bhardwaj and Kumar [42] corrected and improved that approach by proposing a proper transformation able to replace the fuzzy inequality constraints by equivalent equality constraints. The effort put to rewrite the inequalities in a convenient form is unnecessary whenever the extension principle is applied.

The subsequent papers have come to our attention and require further examination. Das et al. [43] introduced a new method to solve FF-LP problems with TrFNs. Their method was based on solving a mathematical model derived from the MO-LP problem and lexicographic ordering method. They solved real-life problems as production planning and diet problem to illustrate the applicability of their approach.

Pérez-Cañedo and Concepción-Morales [44] derived a unique optimal fuzzy value to a FF-LP problem with inequality constraints containing unrestricted LR flat fuzzy coefficients and decision variables. In [45], they ranked LR-type IFNs using a lexicographic criterion, and introduced a method to derive solutions to FIF-LP problems with unique optimal values.

Khalifa [46] addressed FF-LP problems with coefficients and variables expressed by LR fuzzy numbers. He transformed the original fuzzy problem into a three-objective crisp LP problem, and solved it using a classic weighted sum method. The derived crisp solution was next used to construct the fuzzy solution to the original problem.

Khalili Goudarzi et al. [47] proposed a solution approach to FF-MILP problems. First, they formulate a crisp three-objective problem, and find the positive and negative ideal solutions to each objective. Then, they determined a linear membership function for each objective, and constructed a new achievement function defined as a convex combination of the lower bound for satisfaction degree of all objectives, and the weighted sum of the satisfaction degrees of all objectives. In this way, their approach aimed to ensure a balanced compromise solution.

Hamadameen and Hassan [48] introduced a compromise solution approach to FF-MO-LP problems based on a revised simplex method and a Gaussian elimination method.

Table 2 presents the numerical results derived in the literature to a maximization full fuzzy LP problem with triangular fuzzy coefficients and decision variables. Analyzing the reported results one can notice the improvement brought by the approach that complies to the extension principle: the optimal objective function value with maximal amplitude is significantly greater for the third reference than for the other two references. A significant difference can be seen observing the optimal values of the decision variables: the supports of the fuzzy numbers derived by the approach described in the third reference are much wider than those derived by the approaches introduced in the first two references. All values belonging to the supports of the reported fuzzy numbers are feasible values with certain nonzero membership degrees. The narrow supports seen for the first two references show that many relevant feasible values of the decision variables were ignored by the approaches that eluded the extension principle in the optimization step. Figure 4 gives a graphic representation of the results reported in Table 2 for the optimal objective values.

Table 2. Comparative numerical results for a full fuzzy LP problem.

	[41]	[40]	[29]
$\widetilde{z}_{\max}$	$(304.58, 509.79, 704.37)$	$(301.83, 503.23, 724.15)$	$(279.37, 579.32, 985.13)$
$\widetilde{x}_1^{\max}$	$(17.27, 17.27, 17.27)$	$(15.28, 15.28, 15.28)$	$(0, 38.28, 61.24)$
$\widetilde{x}_2^{\max}$	$(2.16, 2.16, 2.16)$	$(2.40, 2.40, 9.10)$	$(0; 1.32, 53.08)$
$\widetilde{x}_3^{\max}$	$(4.64, 9.97, 16.36)$	$(6.00, 11.25, 11.25)$	$(0, 1.28, 51.54)$
$\widetilde{x}_4^{\max}$	$(6.36, 6.36, 6.36)$	$(6.49, 6.49, 9.49)$	$(0, 1.14, 45.60)$

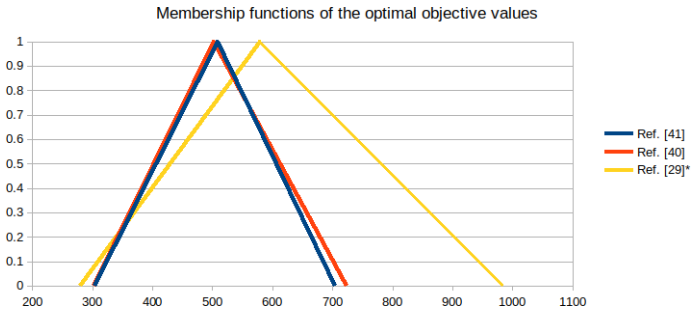

Figure 4. Graphic representation of optimal objective values reported in Table 2. Symbol * marks the reference that introduced the approach that comply to the extension principle.

4.3. Fuzzy Linear Fractional Programming Problems

A full fuzzy LFP problem differs from a full fuzzy LP problem only through its objective function. However, the nonlinearity of the objective function brings certain complications to the solution approaches. Model (14)

$$\begin{array}{ll} \max & \dfrac{\widetilde{c}^T\widetilde{x}+\widetilde{c}_0}{\widetilde{d}^T\widetilde{x}+\widetilde{d}_0} \\ \text{s.t.} & \widetilde{A}\widetilde{x}\leq\widetilde{b}, \\ & \widetilde{x}\geq 0, \end{array} \tag{14}$$

has a linear fractional objective function that has to be maximized subject to linear constraints. Due to the presence of the denominator, an additional condition assuming its strictly positiveness over the feasible set has to be imposed. This condition does not reduce the generality of the problem, since the case of an objective function with strictly negative denominator can be equivalently transformed to a case complying to the initially imposed condition.

Except the methodology for solving a linear fractional transportation problem proposed by Liu [26], no other attempts were made in the literature to approach the fractional programming problems with an exclusive focus on the extension principle. All of the results reported in the papers that are briefly described below might be amended by an approach that fully comply to the extension principle. The analogical treatment of the FF-LFP problems inspired from the approaches to FF-LP problems is the main fact that underpins this conclusion.

Pop and Stancu-Minasian [49] transformed a FF-LFP problem into a crisp MO-LFP problem, and applied Buckley and Feuring's approach [50] to derive the final solution. Stanojević and Stancu-Minasian [51] evaluated the fuzzy inequalities of a FF-LFP problem, and using a generalized form of Charnes-Cooper transformation linearized the original problem. Zadeh's principle was used exclusively for the fuzzy arithmetic between coefficients and variables. Das et al. [52] developed an algorithm for solving the FF-LFP problem. They used a generalized form of Charnes–Cooper transformation and obtained a crisp MO-LP model solved by a lexicographic method. For their numerical experiments, they used real-life problems, and for comparison, they used the approaches introduced in [49,51]. Any linearization of a fractional expression in fuzzy environment is not conformed with the extension principle if the same fuzzy quantity appears in both numerator and denominator of the ratio.

Stanojević et al. [53] evaluated the approximation error that arises when the exact membership function of a ratio of TFNs is replaced by the membership function of a TFN, and its applications to decision-making. They mentioned for the first time the importance of using crisp decision variables in the optimization models even when fuzzy solutions were desired, which was a step toward unification of full fuzzy models with no a priori shapes for the decision variables and models with fuzzy coefficients and constraints that dictate the fuzziness of the decision variables.

Chinnadurai and Muthukumar [54] proposed a numerical approach to solving LFP problems in a fuzzy environment. They obtained (α, r)-acceptable optimal values by solving crisp bi-objective LFP problems. Their procedure was improved by Ebrahimnejad et al. [55], who modified the optimization model, and ensured the non-negativity of the fuzzy valued decision variables. A new extension principle-based solution concept able to embed the (α, r)-acceptable optimality is desired in order to assure further improvements.

Kaur and Kumar [56] discussed the shortcoming of papers [49,51] that resides in an improper interpretation of the fuzzy constraints of FF-LFP models. To avoid the constraint interpretation, they used Yager's ranking approach [57]. An effective way to remove the constraints interpretation from a solution approach is to transfer them in crisp environment, as proved in [28] for unbalanced FF-TP problems.

Arya et al. [58] are the first authors that addressed FF-MO-LFP problems. Their approach analogically follows Chakraborty and Gupta's method [59] designed to solve MO-LFP problems via fuzzy goals, and it is essentially based on a generalized Charnes–Cooper transformation whose arithmetic is in discordance with the extension principle.

Loganathan and Ganesan [60] transformed the original FF-LFP problem into a FF-LP problem and then replaced all coefficients and variables by their parametric forms derived from the α-cuts of triangular fuzzy numbers. They solved the parametric problem using a simplex-like algorithm, and derived parametric solutions. The fuzzy solutions were further derived numerically for different values of the parameter $\alpha \in [0, 1]$. Because of the linearization step, the paper by Loganathan and Ganesan [60] might be affected by not complying to the extension principle.

4.4. Summary

A summary of the surveyed papers can be found in Table 3. In this table, the references are grouped with respect to the class of problems they address. The references marked with a superscript ^{r} are review papers, those written in bold introduce solution approaches that strictly comply to the extension principle, and those marked with a superscript * need further investigations in order to conclude whether the presented methodologies strictly followed the extension principle or not. The rest of them elude the extension principle within the optimization step.

Table 3. A summary of the surveyed papers (the symbol ^{r} marks the review papers, the symbol * marks the papers that need further investigations, and the references written in bold strictly comply to the extension principle).

Problem	References
Fuzzy TP	[30], [31], [32], [33], [34], [35], **[26]**, **[25]**, **[27]**, **[28]**, [36]*
Fuzzy LP	[50], [37], [38], [39], [40], [41], **[29]**, [42], [43]*, [44]*, [45]*, [46]*, [47]*, [48]*, [23]^r
Fuzzy LFP	[49], [51], [53]^r, [54], [55], [56], [58], [60], [24]^r

We restricted our attention to about thirty papers from the literature that proposed solution approaches to full fuzzy mathematical programming problems, i.e., problems with both coefficients and variables assumed to be fuzzy quantities. This study is not meant to be an extensive survey of the above mentioned topic, but we hope that it succeeds to provide a hint on the extent to which the extension principle is currently ignored in the literature. We focused on basic methodologies for optimizing fuzzy number valued objective functions rather than on case studies dealing with fuzzy optimization problems involving fuzzy goals and/or fuzzy decisions.

5. Research Directions

Our main reason for writing this review paper is our belief that the extension principle, which is one of the main principles in fuzzy sets theory, is neglected within mathematical programming with fuzzy numbers.

As can be seen from the literature review presented in previous section, there are various solving methods that comply with the extension principle when using fuzzy arithmetic but globally neglect it, since they rely on the ranking of fuzzy numbers when optimizing the objective function and interpreting the fuzzy constraints.

In the recent literature, a Monte Carlo simulation algorithm [29] was proposed to derive extension-principle-based fuzzy set solutions to linear programming problems with fuzzy numbers. Deriving empirically such fuzzy set solutions to a fuzzy problem has the same complexity as solving the crisp variant of that problem. Therefore, whenever a crisp problem is generalized to a problem with fuzzy numbers, the Monte Carlo simulation can be employed to disclose the shapes of the fuzzy set solutions. This method works for a wide range of problems; its application is not limited to the optimization problems. After disclosing the shapes of the fuzzy sets solutions, the next step is to develop algorithms able to derive numerically the membership functions of those fuzzy sets solutions.

Due to its simplicity, the class of linear programming problems, including the subclass of transportation problems, were already solved numerically using nonlinear models (see [28,29]). Two important research directions arise and deserve attention: (i) generalize the methodology to solve other classes of problems (for instance, problems from the field of data envelopment analysis), and (ii) find replacements for the nonlinear models in order to simplify the numerical approaches. Moreover, everything already concluded for mathematical optimization problems with fuzzy and/or intuitionistic fuzzy numbers is worth generalizing to another level of uncertainty, e.g., to Pythagorean [61] and Fermatean fuzzy numbers [62].

An argument that supports the relevance of the above mentioned research directions is illustrated in Tables 1 and 2. They disclose the nature of the differences among solutions that follow the extension principle and those that do not comply to it when solving transportation problems and general linear programming problems, respectively. Further on, due to the similarities of the existing solving approaches to full fuzzy LFP problems to those for LP problems, similar improvements are expected when solving full fuzzy LFP problems applying an extension principle-based approach. Such approaches have been lacking in the literature thus far.

Finally, inducing the same logic to a wider class of problems either relying or not on fuzzy LP but using fuzzy quantities within parameters, a similar increase in the accuracy and reliability of the derived solutions is expected when a methodology that strictly follows the extension principle is applied.

6. Conclusions

In this paper, we surveyed the literature on mathematical programming with fuzzy numbers from a perspective that emphasizes the necessity of using the extension principle within the whole optimization process, not only within the fuzzy arithmetic evaluation.

Since our review showed that the extension principle was widely present through the fuzzy arithmetic, but much less involved in the foundations of the solution concepts, we advanced the idea that the rehabilitation of the extension principle when deriving relevant fuzzy descriptions to optimal solutions is highly needed. Our opinion was supported by several recent studies showing that better and more effective solutions can be obtained when the algebraic aggregation of the fuzzy quantities and optimization are unified and together conduce to a solution concept that complies to the extension principle.

We discussed the similarities between optimization problems with fuzzy coefficients and full fuzzy optimization problems, and concluded about the unification of their subclass of optimization problems with fuzzy coefficients that assume fuzzy decision variables but with a priori unknown shapes. The solution approaches to this subclass of optimization problems with fuzzy numbers are expected to follow the extension principle, and a posteriori disclose the shapes of the decision variables.

We believe that this study advances relevant ideas to initiate the revitalization of the extension principle in mathematical optimization with fuzzy numbers. Due to its critical

view on the basis of the methodology used to solve certain classes of fuzzy optimization problems, and since its implementation in various fields is able to provide more accurate and reliable optimal solutions, we expect that our conclusions will have an impact on solving practical industrial problems. As it has been said, the impact should be effective on the methodology level, and could be seen as a subtle attempt to filling the gap between the academic researches with their theoretical results and the relevant solutions needed in practice.

Our future research will be focused on deriving analytic solutions to full fuzzy optimization problems that are in accordance to the extension principle, and also simple enough to be able to replace the existing solution approaches. We will follow the research directions proposed in the previous section in order to enlarge the class of problems approachable via the extension principle.

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Abbreviations

The following abbreviations are used in this manuscript:

LP	Linear Programming
LFP	Linear Fractional Programming
TP	Transportation Problem
MO	Multiple Objective
FN	Fuzzy Number
LR-FN	Left-Right flat Fuzzy Number
IFN	Intuitionistic Fuzzy Number
TFN	Triangular Fuzzy Number
TIFN	Triangular Intuitionistic Fuzzy Number
TrFN	Trapezoidal Fuzzy Number
TrIFN	Trapezoidal Intuitionistic Fuzzy Number
FF	Full Fuzzy
FIF	Full Intuitionistic Fuzzy

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