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Orbit Growth of Shift Spaces Induced by Bouquet Graphs and Dyck Shifts

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Abstract: For a discrete dynamical system, the prime orbit and Mertens' orbit counting functions describe the growth of its closed orbits in a certain way. The asymptotic behaviours of these counting functions can be determined via Artin–Mazur zeta function of the system. Specifically, the existence of a non-vanishing meromorphic extension of the zeta function leads to certain asymptotic results. In this paper, we prove the asymptotic behaviours of the counting functions for a certain type of shift spaces induced by directed bouquet graphs and Dyck shifts. We call these shift spaces as the bouquet-Dyck shifts. Since their respective zeta function involves square roots of polynomials, the meromorphic extension is difficult to be obtained. To overcome this obstacle, we employ some theories on zeros of polynomials, including the well-known Eneström–Kakeya Theorem in complex analysis. Finally, the meromorphic extension will imply the desired asymptotic results.

Keywords: bouquet-Dyck shift; Artin–Mazur zeta function; prime orbit counting function; Mertens' orbit counting functions; Eneström–Kakeya Theorem

MSC: 37C35; 37C30; 37C15; 37B10

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1. Introduction

Let (X, T) be a discrete dynamical system, where X is a topological space and $T : X \to X$ is a continuous map. A point $x \in X$ is said to be periodic with period $n \in \mathbb{N}$ if $T^n(x) = x$. Moreover, if $T^k(x) \neq x$ for any $k \in \{1, 2, \ldots, n-1\}$, then x has least period n.

The orbit of a point x is the set $\{T^k(x) \mid k \in \mathbb{N}_0\}$. The orbit is finite with size n if and only if x is periodic with least period n. Such orbit is called a (prime) closed or periodic orbit. We denote τ as a closed orbit of size $|\tau|$.

Closed orbits have been a subject of research in the field of dynamical systems, especially in ergodic theory. One aspect to be studied is the growth of closed orbits in a system. The growth can be described through the following counting functions: for $N \in \mathbb{N}$,

(i) prime orbit counting function
$$\pi(N) = \sum_{\substack{\tau \\ |\tau| \leq N}} 1 \tag{1}$$
which specifies the cumulative number of closed orbits based on the period;

(ii) Mertens' orbit counting functions
$$\mathcal{M}(N) = \prod_{\substack{\tau \\ |\tau| \leq N}} \left(1 - \frac{1}{e^{h|\tau|}}\right) \quad \text{and} \quad \mathscr{M}(N) = \sum_{\substack{\tau \\ |\tau| \leq N}} \frac{1}{e^{h|\tau|}} \tag{2}$$
where h is the topological entropy of the system (with assumption that $h > 0$).

These counting functions were introduced as the dynamical analogues of the counting functions for primes in number theory [1]. Specifically, Prime Number Theorem and

Mertens' Theorem provide the asymptotic results for the following counting functions: for $N \in \mathbb{N}$,

$$\sum_{\substack{p \\ p \leq N}} 1 \sim \frac{N}{\ln N}, \quad \prod_{\substack{p \\ p \leq N}} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\ln N} \quad \text{and} \quad \sum_{\substack{p \\ p \leq N}} \frac{1}{p} = \ln\ln N + M + o(1) \tag{3}$$

where γ and M are Euler–Mascheroni constant and Meissel–Mertens constant, respectively, and p runs through primes. Inspired by these results, the aim here is to obtain analogous asymptotic behaviours for the orbit counting functions for a given system. We refer these behaviours as the (asymptotic) orbit growth of the system.

The earlist works on this research problem were actually done for dynamical flows (see [2] for a brief history on this topic). These include the works by Parry and Pollicott [3,4] and Sharp [5] on suspension flows and Axiom A flows, respectively. From both works, similar results can be deduced for a mixing shift of finite type with topological entropy $h > 0$, which are

$$\pi(N) \sim \frac{e^{h(N+1)}}{(e^h - 1)N}, \quad \mathcal{M}(N) \sim \frac{e^{-\gamma}}{\alpha N} \quad \text{and} \quad \mathscr{M}(N) = \ln N + \gamma + \ln\alpha - C + o(1) \tag{4}$$

for some positive constants α and C (which can be specified in Theorem 1). The results above are obtained through its Artin–Mazur zeta function.

For a discrete dynamical system, its Artin–Mazur zeta function [6] is the generating function

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{F(n)}{n} z^n\right) \tag{5}$$

where $F(n)$ is the number of periodic points of period n. If the zeta function has a non-vanishing meromorphic extension beyond its radius of convergence, then the asymptotic behaviours of the counting functions can be obtained as follows.

Theorem 1 ([7]). *Let (X, T) be a discrete dynamical system with topological entropy $h > 0$ and Artin–Mazur zeta function $\zeta(z)$. Suppose that there exists a function $\alpha(z)$ such that it is analytic and non-zero for $|z| < Re^{-h}$ for some $R > 1$, and*

$$\zeta(z) = \frac{\alpha(z)}{(1 - e^{hp}z^p)^m} \tag{6}$$

for $|z| < e^{-h}$ for some $m, p \in \mathbb{N}$. Then,

(a) (Prime Orbit Theorem)

$$\pi(N) \sim mp \cdot \frac{e^{hp\left(\left\lfloor \frac{N}{p} \right\rfloor + 1\right)}}{(e^{hp} - 1)N}, \tag{7}$$

(b) (Mertens' Orbit Theorem)

$$\mathcal{M}(N) \sim \frac{p^m e^{-m\gamma}}{\alpha(e^{-h}) \cdot N^m} \quad \textit{and} \quad \mathscr{M}(N) = m \ln\left\lfloor \frac{N}{p} \right\rfloor + m\gamma + \ln\alpha(e^{-h}) - C + o(1) \tag{8}$$

where γ is Euler–Mascheroni constant and C is a positive constant that can be specified as

$$C = \sum_{\tau} \left(\ln\left(\frac{1}{1 - e^{-h|\tau|}}\right) - \frac{1}{e^{h|\tau|}}\right). \tag{9}$$

Equation (6) is equivalent to saying that

(i) $\zeta(z)$ is a meromorphic function for $|z| < Re^{-h}$, and

(ii) there are exactly p poles inside this region. Each pole is located on the radius e^{-h}, has order m and of the form $\omega^k e^{-h}$ for $k \in \{0, 1, \ldots, p-1\}$, where $\omega \in \mathbb{C}$ is the principal pth root of unity.

In recent years, the above approach had been demonstrated to obtain the orbit growth of periodic-finite-type shifts [7], and also Dyck and Motzkin shifts [8]. However, the tools used to analyse their respective zeta function are different for each case. These involve some theories on graphs and matrices for the periodic-finite-type shifts, and basic complex analysis for the Dyck and Motzkin shifts. For the Motzkin shift $\mathfrak{M}_{M,P}$ (see Section 2 for notation), it was found that

$$\pi(N) \sim \frac{2(M+P+1)^{(N+1)}}{(M+P)N}, \quad \mathcal{M}(N) \sim \frac{e^{-2\gamma}}{N^2} \cdot \frac{M(M+P+1)}{(M-1)^2} \quad \text{and} \tag{10}$$

$$\mathscr{M}(N) = 2\ln N + 2\gamma + \ln\frac{(M-1)^2}{M(M+P+1)} - C + o(1). \tag{11}$$

The results for the Dyck shift $\mathfrak{D}_M$ can be deduced by setting $P = 0$. These results are more precise than the previous results obtained via the estimation of number of periodic points in [9,10]. Akhatkulov et al. [11] later found sharper results for Dyck shifts, but it did not include similar results for Motzkin shifts.

There are other systems which use the approach via zeta function to obtain their orbit growth, such as ergodic toral automorphisms [12,13]. In fact, certain types of shift spaces in the literature have been shown to have a non-vanishing meromorphic extension of their respective zeta function, and this implies the orbit growth as in Theorem 1. However, those results on orbit growth are not stated therein. Examples are shifts of quasi-finite type [14], beta shifts [15] and negative beta shifts [16].

On related notes, there are other approaches to determine the orbit growth of a system, such as using orbit monoids in [17] and orbit Dirichlet series in [18]. The choice of such approach depends on the nature of the system in question. However, our machinery here is the approach via zeta function. For a detailed exposure on the topic of orbit growth, we encourage the interested readers to refer to our survey paper in [19]. We shall also mention that there is a similar research problem of counting finite orbits for group actions, and some asymptotic results have been obtained for finitely-generated torsion-free nilpotent group shifts [20], algebraic flip systems [21] and flip systems of shifts of finite type [22].

Now, we focus our attention to shift dynamical systems. In fact, many results mentioned above are regarding certain types of shift spaces. Over the years, new types of shift spaces have been introduced together with their zeta functions, but their orbit growth is left undetermined. Krieger and Matsumoto [23] constructed a type of shift spaces called Markov-Dyck shifts by using inverse graph semigroups. Later, Inoue and Krieger [24] introduced another type of shift spaces constructed as a combination of sofic shifts and Dyck shifts. Both types of shift spaces are examples of sofic-Dyck shifts [25]. The zeta functions for these shift spaces had been found, though expressed implicitly and very sophisticated to be studied.

Inoue and Krieger [24] also provided an example of shift spaces constructed from directed bouquet graphs and Dyck shifts. We call these as bouquet-Dyck shifts. These shift spaces include Dyck and Motzkin shifts, and a class of shift subspaces from Dyck shifts in [26]. Their respective zeta function is available in explicit form, thus is simpler and easier to be studied for its meromorphic extension. However, the zeta function involves square roots of certain polynomials, and the meromorphic extension is still difficult to be determined.

Hence, the aim of our paper is to obtain the orbit growth of the bouquet-Dyck shifts via their respective zeta function. We will determine the meromorphic extension in the form of (6) and then apply Theorem 1. Regarding the meromorphic extension, we will utilise some theories in complex analysis, especially on zeros of complex polynomials, to obtain bounds for the zeros and singularities of the zeta function. This idea includes the well-known Eneström–Kakeya Theorem in the field of complex analysis.

In Section 2, we define the bouquet-Dyck shifts and state their zeta function and topological entropy. Later in Section 3, we find the meromorphic extension of the zeta function, and then deduce the orbit growth in Theorem 4. We also demonstrate these results on the shift spaces found in [26] as an example.

2. Bouquet-Dyck Shifts

In this section, we introduce the bouquet-Dyck shifts as a type of shift spaces. More details on general shift spaces can be found in [27].

For an integer $M \geq 2$, define the set

$$\mathcal{D} = \{l_k, r_k \mid 1 \leq k \leq M\}. \tag{12}$$

We generate a monoid $\mathcal{S}$ with identity $\mathbf{1}$ and zero $\mathbf{0}$ from $\mathcal{D}$ through the following operation:

(i) $l_k \circ r_m = \begin{cases} \mathbf{1} & \text{if } k = m; \\ \mathbf{0} & \text{otherwise;} \end{cases}$

(ii) $a \circ \mathbf{1} = \mathbf{1} \circ a = a$ for any $a \in \mathcal{D} \cup \{\mathbf{0}, \mathbf{1}\}$;

(iii) $a \circ \mathbf{0} = \mathbf{0} \circ a = \mathbf{0}$ for any $a \in \mathcal{D} \cup \{\mathbf{0}, \mathbf{1}\}$.

This is called a Dyck monoid. The condition $M \geq 2$ is necessary to generate $\mathbf{0}$ in the monoid. In fact, we can see later that if $M = 1$, then the constructed shift space is a shift of finite type (which is trivial).

Let P and Q be non-negative integers such that both are either zero or non-zero simultaneously. Define a directed bouquet graph G as follows:

(i) the set of vertices is

$$\mathcal{V} = \{v_0\} \cup \{v_{ij} \mid 1 \leq i \leq P,\ 1 \leq j < Q\}; \tag{13}$$

(ii) the set of edges is

$$\mathcal{E} = \left\{e_{ij} = \left(v_{i(j-1)}, v_{ij}\right) \mid 1 \leq i \leq P,\ 1 \leq j \leq Q\right\} \tag{14}$$

where $v_{i0} = v_{iQ} = v_0$ for any i.

We can see that if $P = Q = 0$, then G is simply the vertex v_0. Nevertheless, we then form a new graph $\mathcal{G}$ from G by attaching $2M$ loops on vertex v_0, and each loop is assigned uniquely with a label from $\mathcal{D}$. We also denote $\mathcal{P}$ as the set of finite paths in $\mathcal{G}$.

Now, let $\mathcal{A} = \mathcal{D} \cup \mathcal{E}$ be equipped with the discrete topology. Its product $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology. Let $\mathcal{L} = \bigcup_{s=1}^{\infty} \mathcal{A}^s$. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $w \in \mathcal{L}$ of length $|w|$, we denote $w \prec x$ if there exists $t \in \mathbb{Z}$ such that $x_t x_{t+1} \ldots x_{t+|w|-1} = w$.

Define a map $\phi : \mathcal{A} \to \mathcal{D} \cup \{\mathbf{1}\}$ as

$$\phi(a) = \begin{cases} a & \text{if } a \in \mathcal{D}, \\ \mathbf{1} & \text{otherwise} \end{cases} \tag{15}$$

and another map $red : \mathcal{L} \to \mathcal{S}$ as

$$red(a_1 a_2 \ldots a_{|w|}) = \phi(a_1) \circ \phi(a_2) \circ \ldots \circ \phi(a_{|w|}) \tag{16}$$

for any $w = a_1 a_2 \ldots a_{|w|} \in \mathcal{L}$. The *bouquet-Dyck shift* is the set

$$\mathcal{X}_{M,P,Q} = \{x \in \mathcal{A}^{\mathbb{Z}} \mid \forall\, w \prec x,\ w \in \mathcal{P} \text{ and } red(w) \neq \mathbf{0}\} \tag{17}$$

equipped with the shift map $\sigma : \mathcal{X}_{M,P,Q} \to \mathcal{X}_{M,P,Q}$ where

$$\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}. \tag{18}$$

In graph terminology, $\mathcal{X}_{M,P,Q}$ consists of all bi-infinite paths such that every subpath is not reduced to **0** under the Dyck monoid operation. Notice that the $2M$ loops on vertex v_0 produce a Dyck shift, while the bouquet graph G produces a shift of finite type (which is a sofic shift). So, $\mathcal{X}_{M,P,Q}$ is constructed through the combination of both types of shift spaces. Examples of $\mathcal{X}_{M,P,Q}$ are the Motzkin shift $\mathfrak{M}_{M,P}$ with $Q = 1$, and the Dyck shift $\mathfrak{D}_M$ with $P = Q = 0$.

In [24], the Artin–Mazur zeta function of $\mathcal{X}_{M,P,Q}$ is given as

$$\zeta(z) = \frac{2\left(1 - Pz^Q + \sqrt{(1-Pz^Q)^2 - 4Mz^2}\right)}{\left(1 - Pz^Q - 2Mz + \sqrt{(1-Pz^Q)^2 - 4Mz^2}\right)^2} \tag{19}$$

and its topological entropy is

$$h = -\ln\lambda, \tag{20}$$

where λ is the unique positive solution of the polynomial equation

$$Pz^Q + (M+1)z - 1 = 0. \tag{21}$$

The equation above arises from solving the denominator of $\zeta(z)$ for its roots. Note also that the numerator of $\zeta(z)$ is non-zero on the whole complex plane.

3. Orbit Growth of Bouquet-Dyck Shifts

Throughout this section, we consider a bouquet-Dyck shift $\mathcal{X}_{M,P,Q}$ with the value of λ in (20). Our plan is to construct the function $\alpha(z)$ as in Theorem 1. Based on (6), we can expect that our zeta function $\zeta(z)$ in (19) is a meromorphic function for $|z| < R\lambda$, where R is to be found. Furthermore, it shall have only one pole, which is λ, in this region. Therefore, we need to prove the existence of such pole, and find the suitable value of R.

Note that if the denominator of $\zeta(z)$ is analytic at λ, then λ is a pole of $\zeta(z)$ if and only if it is a zero of the denominator. So, it is sufficient to consider λ as a possible zero of the denominator. However, the denominator contains the expression $\sqrt{(1-Pz^Q)^2 - 4Mz^2}$ which may not be analytic in certain region, or even at λ. Hence, we need to solve the following tasks:

(i) To find the region of analyticity of $\sqrt{(1-Pz^Q)^2 - 4Mz^2}$, and thus the denominator;
(ii) To show that λ is a zero, and it is the closest to the origin among the zeros of the denominator. This will help us to determine the suitable value of R.

We will see that both tasks involve finding the roots of polynomials. Except quadratic polynomials, it may be impossible to determine the roots in exact for the polynomials of higher degree. Due to that, we will estimate the roots by using certain bounds. The following two theorems will be helpful for this purpose. Here, the notation "gcd" refers to the greatest common divisor.

Theorem 2 ([28])**.** *Let $p(z)$ be a complex trinomial in the form*

$$p(z) = z^n - cz^k - 1 \tag{22}$$

where $k, n \in \mathbb{N}$ with $n \geq 3$, $1 \leq k < n$ and $\gcd(k,n) = 1$, and $c \in \mathbb{C}$ with $c \neq 0$. Define the following complex trinomials:

$$\phi(z) = z^n + |c|z^k - 1, \qquad \psi(z) = z^n - |c|z^k - 1. \tag{23}$$

Then,

(a) $\phi(z)$ *and* $\psi(z)$ *have unique positive roots* λ_ϕ *and* λ_ψ*, respectively, where* $0 < \lambda_\phi < 1 < \lambda_\psi$*, and*

(b) *if* μ *is a root of* $p(z)$*, then* $\lambda_\phi \leq |\mu| \leq \lambda_\psi$*.*

Theorem 3 (Eneström–Kakeya Theorem [29])**.** *Let* $p(z)$ *be a complex polynomial with degree* $n \in \mathbb{N}$ *and positive coefficients* $c_0, c_1, \dots, c_n$ *in the form*

$$p(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0. \tag{24}$$

Define

$$\underline{K} = \min_{0 \leq i < n} \left\{ \frac{c_i}{c_{i+1}} \right\} \quad \text{and} \quad \overline{K} = \max_{0 \leq i < n} \left\{ \frac{c_i}{c_{i+1}} \right\}. \tag{25}$$

By setting $i \in \{1, 2, \dots, n+1\}$ *and* $c_{-1} = c_{n+1} = 0$*, define also*

$$\underline{L} = \gcd\{ i \mid c_{i-1} - \underline{K} c_i > 0 \} \quad \text{and} \quad \overline{L} = \gcd\{ i \mid \overline{K} c_{n+1-i} - c_{n-i} > 0 \}. \tag{26}$$

Then, for any root μ *of* $p(z)$*,*

(a) $\underline{K} \leq |\mu| \leq \overline{K}$*, and*

(b) *if* $\underline{L} = 1$*, then* $|\mu| > \underline{K}$*, and if* $\overline{L} = 1$*, then* $|\mu| < \overline{K}$*.*

We begin with finding the region of analyticity for the denominator of $\zeta(z)$.

Proposition 1. *If* μ *is a root of the polynomial* $(1 - Pz^Q)^2 - 4Mz^2$*, then* $|\mu| > \lambda$*.*

Proof. The cases for $P = Q = 0$ and $Q = 1$ are straightforward because we can obtain the exact values of λ and the roots μ. For $Q \geq 2$, observe that

$$(1 - Pz^Q)^2 - 4Mz^2 \equiv (Pz^Q + 2\sqrt{M}z - 1)(Pz^Q - 2\sqrt{M}z - 1). \tag{27}$$

Denote $\Phi(z)$ and $\Psi(z)$ as the first and second trinomials, respectively, in the factorization above. It is easy to check that $\Phi(z)$ and $\Psi(z)$ have unique positive roots λ_Φ and λ_Ψ, respectively.

For $Q = 2$, we solve the relevant quadratic equations to obtain that

$$\lambda = \frac{-(M+1) + \sqrt{(M+1)^2 + 4P}}{2P} \quad \text{and} \quad |\mu| \geq \frac{-2\sqrt{M} + \sqrt{4M + 4P}}{2P}. \tag{28}$$

Consider the function

$$f(y) = \frac{-y + \sqrt{y^2 + 4P}}{2P} \tag{29}$$

for $y \in \mathbb{R}_{\geq 0}$. We can check that $f'(y) < 0$ for $y \geq 0$, so $f(y)$ is strictly decreasing on $\mathbb{R}_{\geq 0}$. Since $M + 1 > 2\sqrt{M}$ for $M \geq 2$, we obtain that $f(M+1) < f\left(2\sqrt{M}\right)$, and consequently, $\lambda < |\mu|$.

Now, suppose that $Q \geq 3$. By using substitution $z = P^{-\frac{1}{Q}} w$, the trinomials $\Phi(z)$ and $\Psi(z)$ are transformed into trinomials

$$\phi(w) = w^Q + 2P^{-\frac{1}{Q}}\sqrt{M}w - 1, \qquad \psi(w) = w^Q - 2P^{-\frac{1}{Q}}\sqrt{M}w - 1. \tag{30}$$

Based on Theorem 2, $\phi(w)$ and $\psi(w)$ have unique positive roots λ_ϕ and λ_ψ, respectively. Furthermore, if $\tilde{u}$ is a root of $\phi(w)$ or $\psi(w)$, then $\lambda_\phi \leq |\tilde{u}| \leq \lambda_\psi$. By re-substitution, we obtain that $\lambda_\Phi \leq |\mu| \leq \lambda_\Psi$ for any root μ of the polynomial $(1 - Pz^Q)^2 - 4Mz^2$.

It remains to compare λ and λ_Φ. Since $Pz^2 + (M+1)z - 1 > \Phi(z)$ for $z \in \mathbb{R}_{>0}$, it is easy to deduce that $\lambda < \lambda_\Phi$. $\square$

Proposition 2. *The function* $\sqrt{(1-Pz^Q)^2-4Mz^2}$ *is analytic for* $|z| < R_1\lambda$, *with*

$$R_1 = \min_{\mu}\left\{\frac{|\mu|}{\lambda}\right\} \tag{31}$$

where μ *runs through the zeros of the polynomial* $(1-Pz^Q)^2-4Mz^2$.

Proof. We consider the cases for $Q \geq 2$ first. The polynomial $(1-Pz^Q)^2-4Mz^2$ can be factored as

$$(1-Pz^Q)^2-4Mz^2 \equiv P^2 \cdot \prod_{\mu}(z-\mu) \cdot \prod_{\mu',\,\overline{\mu'}}\left(z^2-(\mu'+\overline{\mu'})z+\mu'\overline{\mu'}\right) \tag{32}$$

where μ runs through the real roots, and μ' and $\overline{\mu'}$ run through the conjugate pairs of non-real roots of the polynomial. Therefore, it is sufficient to check for the analyticity of the square root of each factor for $|z| < R_1\lambda$. Since the function $\sqrt{z}$ is analytic anywhere except for non-positive real values of z, it remains to check that each factor does not result in those values for $|z| < R_1\lambda$.

Suppose that there exists a negative root μ. Its factor $z-\mu$ produces a non-positive real value only if z is real with $z \leq \mu$, and consequently, $|z| \geq |\mu|$. By contrapositive, if $|z| < |\mu|$, then $z-\mu$ cannot be a non-positive real value.

For other real roots, recall above that there are exactly two positive roots λ_Φ and λ_Ψ. Consider the product $(z-\lambda_\Phi)(z-\lambda_\Psi)$. Suppose that for some $z \in \mathbb{C}$, the product produces a non-positive real value, i.e.,

$$z^2-(\lambda_\Phi+\lambda_\Psi)z+\lambda_\Phi\lambda_\Psi = -\epsilon \tag{33}$$

for some $\epsilon \geq 0$. By expressing $z = u+iv$ where $u, v \in \mathbb{R}$ and comparing real and imaginary parts, we obtain

$$u^2-v^2-(\lambda_\Phi+\lambda_\Psi)u+\lambda_\Phi\lambda_\Psi = -\epsilon \quad \text{and} \quad 2uv-(\lambda_\Phi+\lambda_\Psi)v = 0. \tag{34}$$

The second equation implies the following solutions:

(i) $u = \dfrac{\lambda_\Phi+\lambda_\Psi}{2}$. This implies further that $v^2 = -\dfrac{(\lambda_\Phi-\lambda_\Psi)^2}{4}+\epsilon$ from the first equation. If $v^2 < 0$, then there is no solution for z. Otherwise,

$$|z|^2 = u^2+v^2 = \lambda_\Phi\lambda_\Psi+\epsilon \geq \lambda_\Phi^2 \tag{35}$$

and thus, $|z| \geq \lambda_\Phi$;

(ii) $v = 0$. This implies further that $\lambda_\Phi \leq u \leq \lambda_\Psi$ from the first equation. Thus, $|z| \geq \lambda_\Phi$.

Overall, we deduce that if $(z-\lambda_\Phi)(z-\lambda_\Psi)$ produces a non-positive real value, then $|z| \geq \lambda_\Phi$. By contrapositive, if $|z| < \lambda_\Phi$, then $(z-\lambda_\Phi)(z-\lambda_\Psi)$ cannot be a non-positive real value.

On the other hand, suppose that there exists a conjugate pair μ' and $\overline{\mu'}$ of non-real compls. We can use similar argument above to show that if $|z| < |\mu'|$, then its corresponding quadratic factor cannot be a non-positive real value.

Since $R_1\lambda \leq |\mu|$ for any root μ, we reach our conclusion. The cases for $P = Q = 0$ and $Q = 1$ are done similarly by applying the argument above on the quadratic polynomial $(1-Pz^Q)^2-4Mz^2$ directly. $\square$

The previous proposition implies that both numerator and denominator of $\zeta(z)$ are analytic for $|z| < R_1\lambda$. Since λ is a zero of the denominator, it is indeed a pole of $\zeta(z)$. It remains to find its order.

Proposition 3. *For the polynomial $Pz^Q + (M+1)z - 1$, the root λ is simple, and if μ is another root of the polynomial, then $|\mu| > \lambda$.*

Proof. The cases for $P = Q = 0$ and $Q = 1$ are straightforward since the polynomial is linear. Now, suppose that $Q \geq 2$. Since λ is a root, the polynomial $Pz^Q + (M+1)z - 1$ can be factored as

$$Pz^Q + (M+1)z - 1 \equiv P(z-\lambda)\left(\sum_{k=0}^{Q-1} \lambda^k z^{Q-k-1} + \frac{M+1}{P}\right). \tag{36}$$

Denote the last polynomial in the factorization above as $q(z)$. We will apply Theorem 3 on $q(z)$. Based on the coefficients, it is easy to check that $\underline{K} = \lambda$ and $\underline{L} = 1$. Hence, for any root μ of $q(z)$, we obtain that $|\mu| > \lambda$. This implies that λ is not a root of $q(z)$. So, λ is a simple root of $Pz^Q + (M+1)z - 1$. Since the remaining roots of $Pz^Q + (M+1)z - 1$ are the roots of $q(z)$, we obtain the desired inequality. $\square$

The previous proposition implies that λ is a pole of $\zeta(z)$ of order 2, since the denominator has power 2. Furthermore, it is the closest to the origin among the poles of $\zeta(z)$.

Now, set

$$R_2 = \min_{\mu}\left\{\frac{|\mu|}{\lambda}\right\} \tag{37}$$

where μ runs through the roots of $Pz^Q + (M+1)z - 1$ except λ itself. Set further $R = \min\{R_1, R_2\}$ and consider the region where $|z| < R\lambda$. Observe that λ is the only pole of $\zeta(z)$ inside this region. Furthermore, since the numerator and denominator are analytic for $|z| < R\lambda$, now we know that $\zeta(z)$ is a meromorphic function for $|z| < R\lambda$. In other word, there exists an analytic function $\eta(z)$ for $|z| < R\lambda$ such that

$$\zeta(z) = \frac{\eta(z)}{(z-\lambda)^2} \tag{38}$$

and $\eta(\lambda) \neq 0$. Finally, we set $\alpha(z) = \lambda^{-2} \cdot \eta(z)$ and apply Theorem 1 to deduce the orbit growth of the bouquet-Dyck shifts.

Theorem 4. *For bouquet-Dyck shift $\mathcal{X}_{M,P,Q}$ with Artin–Mazur zeta function $\zeta(z)$ and topological entropy h,*

$$\pi(N) \sim \frac{2e^{h(N+1)}}{(e^h - 1)N}, \quad \mathcal{M}(N) \sim \frac{e^{-2\gamma}}{\varphi N^2} \quad \text{and} \quad \mathscr{M}(N) = 2\ln N + 2\gamma + \ln\varphi - C + o(1) \tag{39}$$

where γ is the Euler–Mascheroni constant, φ is a positive constant defined by

$$\varphi = \lim_{z \to e^{-h}} (1 - e^h z)^2 \cdot \zeta(z) \tag{40}$$

and C is another positive constant as in (9).

As a corollary, the orbit growths of Dyck and Motzkin shifts can be deduced by setting $P = Q = 0$ and $Q = 1$, respectively, and calculating the exact value of φ based on Theorem 4. These results agree with [8].

Example 1. *We demonstrate the result of Theorem 4 on the shift subspaces of Dyck shifts in [26]. These shift spaces were originally constructed from Dyck languages, but here, we provide an alternative definition.*

Let $\mathcal{D}_M$ be the Dyck shift over $\mathcal{A} = \{l_k, r_k \mid 1 \leq k \leq M\}$. For $J \in \{0, 1, \ldots, M\}$, define the shift subspace $\mathcal{D}_M(J) \subseteq \mathcal{D}_M$ as follows: $x \in \mathcal{D}_M(J)$ if and only if for $ab \in \mathcal{A}^2$ with $ab \prec x$, we *have $a = l_k$ if and only if $b = r_k$ for $1 \leq k \leq J$. In other word, $l_k r_k$ can only appear together in x for $1 \leq k \leq J$.*

Note that $\mathcal{D}_M(0)$ is the Dyck shift $\mathcal{D}_M$. Moreover, it is easy to check that $\mathcal{D}_M(M-1)$ and $\mathcal{D}_M(M)$ are shifts of finite type. For $1 \leq J \leq M-2$, $\mathcal{D}_M(J)$ is the bouquet-Dyck shifts $\mathcal{X}_{M-J,J,2}$ with the graph $\mathcal{G}$ as follows:

(i) each of the $2(M-J)$ loops at the vertex v_0 is assigned uniquely with a label from $\{l_k, r_k \mid J+1 \leq k \leq M\}$;

(ii) for $1 \leq k \leq J$, the edges e_{k1} and e_{k2} are assigned with the labels l_k and r_k, respectively.

In this case, its topological entropy can be calculated in exact, which is

$$h = \ln \frac{(M-J+1) + \sqrt{(M-J+1)^2 + 4J}}{2}. \tag{41}$$

Hence, the orbit growth of $\mathcal{D}_M(J)$ is given as in Theorem 4.

4. Conclusions

In this paper, we have determined the orbit growth of a bouquet-Dyck shift via its Artin–Mazur zeta function in Theorem 4. The results include the cases for Dyck shifts, Motzkin shifts and a certain class of shift subspaces from Dyck shifts. Although the approach via zeta function is straightforward due to Theorem 1, the difficulty arises due to the form of its zeta function in (19). Since the zeta function contains square roots of polynomials, some tools in complex analysis are needed to determine the meromorphic extension in (6).

In general setting, this approach is applicable to any discrete dynamical system, as long as its zeta function satisfies the conditions in Theorem 1. However, the meromorphic extension may be difficult to be obtained and some advanced theories in other mathematical fields may be required for this purpose.

Bouquet-Dyck shifts are a small class of sofic-Dyck shifts [25]. The next aim is to obtain the orbit growth of the sofic-Dyck shifts, or if not, their shift subspaces such as Markov-Dyck shifts [23] and shift spaces introduced in [24]. However, their zeta functions are a lot more sophisticated than for bouquet-Dyck shifts. We hope that our work here provides a new interest, insight and idea to the readers to tackle the research problem on the orbit growth of those shift spaces.

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