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# Numerical Solution of Two Dimensional Time-Space Fractional Fokker Planck Equation With Variable Coefficients

**Elsayed I. Mahmoud [1,2] and Viktor N. Orlov [3,*]**

1 Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt; ei_abdelgalil@zu.edu.eg
2 Nikolskii Mathematical Institute, Peoples Friendship University of Russia, 117198 Moscow, Russia
3 Department of Applied Math, Moscow State University of Civil Engineering, Yaroslavskoe Shosse, 26, 129337 Moscow, Russia
* Correspondence: Orlovvn@mgsu.ru

**Abstract:** This paper presents a practical numerical method, an implicit finite-difference scheme for solving a two-dimensional time-space fractional Fokker–Planck equation with space–time depending on variable coefficients and source term, which represents a model of a Brownian particle in a periodic potential. The Caputo derivative and the Riemann–Liouville derivative are considered in the temporal and spatial directions, respectively. The Riemann–Liouville derivative is approximated by the standard Grünwald approximation and the shifted Grünwald approximation. The stability and convergence of the numerical scheme are discussed. Finally, we provide a numerical example to test the theoretical analysis.

**Keywords:** two-dimensional time–space fractional Fokker–Planck equation; standard and shifted Grünwald approximation; Riemann–Liouville fractional derivative; Caputo fractional derivative; implicit finite difference scheme; stability and convergence

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## 1. Introduction

In recent years, there has been a growing interest in the field of fractional calculus (that is, the theory of integrals and derivatives of arbitrary real or complex orders) [1]. Additionally, fractional differential equations are considered for study because they provide a better approach to describing the complex phenomena in science and engineering and they find different concepts of fractional derivatives and integration involving these models such as Grünwald–Letnikov's definition, Riemann–Liouville's definition, Caputo's definition, and Riesz's definition. Since the exact solution of fractional differential equations is difficult to find, many approximate and numerical solution methods have been developed. In [2] Aleroev T. tried to find the Analytical and Approximate Solution for Vibration String Equation with a Fractional Derivative; in [3], Elsayed A. and Orlov V. give the Numerical Scheme for Solving Time–Space Vibration String Equation of Fractional Derivative with Caputo fractional derivative in temporal direction and Riemann–Liouville fractional derivative in spatial derivative. Additionally, [4,5] give the numerical solution of a system fractional partial differential equation by extending the fractional differential transform method.

The Fokker–Planck equation (FPDE) is known for modeling various issues in electron relaxation in gases, nucleation, and quantum optics. The time evolution of the density function of the position and speed of the particle was defined by the Fokker–Planck model. Scientific appearances such as wave diffusion, constant random motion, DNA, and RNA molecules' biological code and arrangement materialization are shown by Fokker–Planck PDEs with fractional differential functions of time and space, as provided in [6,7]. Brownian motion and the approaches to reaction-kinetics of reactive fluids based on material diffusivity are currently being explored in a variety of technologies; in physicochemical systems and biological synthesis [8]. In [9], the authors present an analytical solution to some model of the Fokker–Planck equation (FPDE) with variables coefficients by the separation of variables (the Fourier method). A finite difference scheme for a solution to Fokker–Planck equation (FPDE) and their computational accuracy were discussed in several papers. In [10], the authors use Finite difference to solve Time–Space Fractional Advection–Diffusion Equations with Riesz Derivative and constant coefficients, and they convert the fractional differential equation into the equivalent integral equation. Then, fractional trapezoidal formula is used to approximate the second-order accuracy fractional integral. In [11], Shuqing Y. and Mingrong C. prove the stability and convergence of a finite difference scheme of the time–space Fokker–Planck equation (FPDE) with external force and source time, using the energy method with spatial second-order accuracy and temporal first-order accuracy.

Our reason for researching the problem of variable coefficients is its relevance in applications of light propagation in obstacle-containing tissues [12]. In this paper, we consider the following general model of the two-dimensional time–space fractional Fokker–Planck equation

$$\begin{aligned}\frac{\partial^\alpha w(x_1,x_2,t)}{\partial t^\alpha} = &-D_{x_1}^{\beta}[E(x_1,x_2,t)w(x_1,x_2,t)] - D_{x_2}^{\beta}\left[\overline{E}(x_1,x_2,t)w(x_1,x_2,t)\right] \\ &+D_{x_1}^{2\beta}[F(x_1,x_2,t)w(x_1,x_2,t)] + D_{x_2}^{2\beta}\left[\overline{F}(x_1,x_2,t)w(x_1,x_2,t)\right] \\ &+g(x_1,x_2,t),\end{aligned} \tag{1}$$

with initial and boundary condition

$$\begin{aligned} &w(x_1,x_2,0) = \varsigma_0(x_1,x_2), && (x_1,x_2) \in \omega, \\ &w(x_1,x_2,t) = 0, && (x_1,x_2) \in \partial\omega, \quad 0 \le t \le T, \end{aligned} \tag{2}$$

where $0 < \alpha < 1$ and $0.5 < \beta < 1$, are parameters describing the order of the fractional temporal and spatial derivatives, respectively, $\omega = (0,L_1)\times(0,L_2)$, $\partial\omega$ is the boundary of $\omega$, $E(x_1,x_2,t), \overline{E}(x_1,x_2,t) \ge 0$ are $x_1, x_2$-diffusion smooth function coefficients, respectively, $F(x_1,x_2,t), \overline{F}(x_1,x_2,t) \ge 0$ are $x_1, x_2$-drift smooth function coefficients, respectively, $g(x_1,x_2,t)$ represents sources and sinks, $\dfrac{\partial^\alpha w(x_1,x_2,t)}{\partial t^\alpha}$ is the Caputo fractional derivative, given by

$$\frac{\partial^\alpha w(x_1,x_2,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\nu)^{-\alpha}\frac{\partial w(x_1,x_2,\nu)}{\partial \nu}\,d\nu, & 0<\alpha<1, 0<\nu<t \\ \frac{\partial w(x_1,x_2,t)}{\partial t} & \alpha = 1, \end{cases} \tag{3}$$

while the spatial fractional derivatives $D_{x_1}^{\mu} w(x_1,x_2,t), (\mu = \beta$ or $2\beta)$, is the Riemman–Liouville derivatives, defined as

$$D_{x_1}^{\mu} w(x_1,x_2,t) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dx_1^n}\int_0^{x_1}(x_1-\nu)^{n-\mu-1}w(\nu,x_2,t)\,d\nu, \qquad n-1<\mu\le n. \tag{4}$$

Similarly, we can define the spatial fractional derivatives $D_{x_2}^{\mu} w(x_1,x_2,t)$.

In the case of $\alpha = \beta = 1$, Equation (1) reduces to the, two-dimensional classical advection–dispersion equation, which is considered a useful model for explaining transport dynamics in complex systems governed by irregular diffusion and non-exponential relaxation patterns [13,14].

The structure of the paper is organized as follows. In Section 2, the implicit finite difference method is considered to solve two-dimensional time–space fractional Fokker–Planck equation Equations (1) and (2). In Section 3, the unconditional stability of the implicit finite difference method is proved. In Section 4, we study the convergence of proposed numerical scheme, to the exact solution of equation Equations (1) and (2). In

Section 5, some numerical examples are taken to confirm the theoretical results. Finally, we give our conclusions in Section 6.

## 2. Numerically Implicit Finite Difference Scheme

In this section we introduce numerical method to solve Equations (1) and (2).

For spatial discretization of intervals $[0, L_1]$ and $[0, L_2]$, let $h_1 = \frac{L_1}{M_1}$ and $h_2 = \frac{L_2}{M_2}$ for two positive integers $M_1$ and $M_2$ with $x_{1i} = ih_1, i = 0, 1, .., M_1$ and $x_{2j} = jh_2, j = 0, 1, .., M_2$. For temporal discretization of interval $[0, T]$, let $\tau = \dfrac{T}{N}$ for two positive integer $N$ with $t_k = k\tau, 0 \le k \le N$.

Let $w_{ij}^k$ be the numerical approximation to $w(x_{1i}, x_{2j}, t_k)$. We can find Caputo time fractional derivative $\dfrac{\partial^\alpha w(x_1, x_2, t)}{\partial t^\alpha}$ for $t = t_{k+1} = (k+1)\tau$, as follows

$$\begin{aligned}
\frac{\partial^\alpha w(x_{1i}, x_{2j}, t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{k} \int_{t_{k-l}}^{t_{k-l+1}} (t_{k+1} - \nu)^{-\alpha} \frac{\partial w(x_{1i}, x_{2j}, \nu)}{\partial \nu} \, d\nu \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{k} \frac{w(x_{1i}, x_{2j}, t_{k-l+1}) - w(x_{1i}, x_{2j}, t_{k-l})}{\tau} \int_{t_{k-l}}^{t_{k-l+1}} (t_{k+1} - \nu)^{-\alpha} \, d\nu \\
&+ O(\tau^{2-\alpha}) \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{l=0}^{k} \xi_{\alpha,l} \big( w(x_{1i}, x_{2j}, t_{k-l+1}) - w(x_{1i}, x_{2j}, t_{k-l}) \big) + O(\tau^{2-\alpha}),
\end{aligned} \tag{5}$$

where $\xi_{\alpha,l} = (l+1)^{1-\alpha} - (l)^{1-\alpha}$, follows from Equation (5), for $0 < \alpha < 1$, it is easy to deduce that

$$\frac{\partial^\alpha w(x_{1i}, x_{2j}, t_{k+1})}{\partial t^\alpha} = \begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \Big( w(x_{1i}, x_{2j}, t_{k+1}) - \xi_{\alpha,k} w(x_{1i}, x_{2j}, 0) + \\ \qquad \sum_{l=0}^{k-1} (\xi_{\alpha,l+1} - \xi_{\alpha,l}) w(x_{1i}, x_{2j}, t_{k-l}) \Big) + O(\tau^{2-\alpha}), & k \ge 1, \\ \\ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \big( w(x_{1i}, x_{2j}, t_1) - w(x_{1i}, x_{2j}, 0) \big) + O(\tau^{2-\alpha}), & k = 0. \end{cases} \tag{6}$$

For the $x_1-$spatial fractional derivatives $D_{x_1}^{\beta}(E(x_1, x_2, t) w(x_1, x_2, t))$ and $D_{x_1}^{2\beta}(F(x_1, x_2, t) w(x_1, x_2, t))$, we used the standard Grünwald approximation and shifted Grünwald approximation, respectively, as

$$\begin{aligned}
&D_{x_1}^{\beta}(E(x_{1i}, x_{2j}, t_{k+1}) w(x_{1i}, x_{2j}, t_{k+1})) = \\
&\qquad h_1^{-\beta} \sum_{l=0}^{i} \eta_{\beta,l} E(x_{1(i-l)}, x_{2j}, t_{k+1}) w(x_{1(i-l)}, x_{2j}, t_{k+1}) + O(h_1^2),
\end{aligned} \tag{7}$$

$$\begin{aligned}
&D_{x_1}^{2\beta}(F(x_{1i}, x_{2j}, t_{k+1}) w(x_{1i}, x_{2j}, t_{k+1})) = \\
&\qquad h_1^{-2\beta} \sum_{l=0}^{i+1} \eta_{2\beta,l} F(x_{1(i-l+1)}, x_{2j}, t_{k+1}) w(x_{1(i-l+1)}, x_{2j}, t_{k+1}) + O(h_1^2),
\end{aligned} \tag{8}$$

where

$$\begin{array}{lll} \eta_{\beta,0} = 1, & \eta_{\beta,l} = \binom{\beta}{l}, & l = 1, 2, \dots \\ \eta_{2\beta,0} = 1, & \eta_{2\beta,l} = \binom{2\beta}{l}, & l = 1, 2, \dots \end{array}. \tag{9}$$

Similarly, we can define the $x_2$-spatial fractional derivatives $D_{x_2}^{\beta}(\overline{E}(x_1, x_2, t) w(x_1, x_2, t))$ and $D_{x_2}^{2\beta}(\overline{F}(x_1, x_2, t) w(x_1, x_2, t))$.

So, from Equations (1), (2), (6)–(8), we have

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(w(x_{1i},x_{2j},t_{k+1})-\xi_{\alpha,k}w(x_{1i},x_{2j},0)+\sum_{l=0}^{k-1}(\xi_{\alpha,l+1}-\xi_{\alpha,l})(x_{1i},x_{2j},t_{k-l})\right) \tag{10}$$
$$\begin{aligned}
&=-h_1^{-\beta}\sum_{l=0}^{i}\eta_{\beta,l}E(x_{1(i-l)},x_{2j},t_{k+1})w(x_{1(i-l)},x_{2j},t_{k+1})\\
&\quad-h_2^{-\beta}\sum_{l=0}^{j}\eta_{\beta,l}\overline{E}(x_{1i},x_{2(j-l)},t_{k+1})w(x_{1i},x_{2(j-l)},t_{k+1})\\
&\quad+h_1^{-2\beta}\sum_{l=0}^{i+1}\eta_{2\beta,l}F_(x_{1(i-l+1)},x_{2j},t_{k+1}w(x_{1(i-l+1)},x_{2j},t_{k+1}\\
&\quad+h_2^{-2\beta}\sum_{l=0}^{j+1}\eta_{2\beta,l}\overline{F}(x_{1i},x_{2(j-l+1)},t_{k+1})w(x_{1i},x_{2(j-l+1)},t_{k+1})\\
&\quad+g(x_{1i},x_{2j},t_{k+1})+R_{ij}^{k+1},
\end{aligned}$$

where $|R_{ij}^k|\le C\tau^{\alpha}(\tau^{2-\alpha}+h_1^2+h_2^2), C>0$.

Let $E_{ij}^k=E(x_{1i},x_{2j},t_k),\overline{E}_{ij}^k=\overline{E}(x_{1i},x_{2j},t_k),F_{ij}^k=F(x_{1i},x_{2j},t_k),\overline{F}_{ij}^k=\overline{F}(x_{1i},x_{2j},t_k),c_1=\Gamma(2-\alpha)\tau^{\alpha}/h_1^{\beta}$, $\bar{c}_1=\Gamma(2-\alpha)\tau^{\alpha}/h_2^{\beta}$, $c_2=\Gamma(2-\alpha)\tau^{\alpha}/h_1^{2\beta}$, $\bar{c}_2=\Gamma(2-\alpha)\tau^{\alpha}/h_2^{2\beta}$, $G_{ij}^{k+1}=\Gamma(2-\alpha)\tau^{\alpha}g_(x_{1i},x_{2j},t_{k+1})$, then we can obtain the following implicit finite difference scheme

$$w_{ij}^{k+1}+\sum_{l=0}^{k-1}(\xi_{\alpha,l+1}-\xi_{\alpha,l})w_{ij}^{k-l}+c_1\sum_{l=0}^{i}\eta_{\beta,l}E_{(i-l)j}^{k+1}w_{(i-l)j}^{k+1}+\bar{c}_1\sum_{l=0}^{j}\eta_{\beta,l}\overline{E}_{i(j-l)}^{k+1}w_{i(j-l)}^{k+1} \tag{11}$$
$$\begin{aligned}
&-c_2\sum_{l=0}^{i+1}\eta_{2\beta,l}F_{(i-l+1)j}^{k+1}w_{(i-l+1)j}^{k+1}-\bar{c}_2\sum_{l=0}^{j+1}\eta_{2\beta,l}\overline{F}_{i(j-l+1)}^{k+1}w_{i(j-l+1)}^{k+1}\\
&=\xi_{\alpha,k}w_{ij}^0+G_i^{k+1},
\end{aligned}$$

where $i=1,2,\dots,M_1-1, j=1,2,\dots,M_2-1, k=0,1,\dots,N-1$, with the initial and boundary condition $w_{ij}^0=\varsigma_0(x_{1i},x_{2j}), w_{0j}^{k+1}=w_{M_1j}^{k+1}=w_{i0}^{k+1}=w_{iM_2}^{k+1}=0, 0\le i\le M_1, 0\le j\le M_2, 0\le k\le N-1$, respectively.

To study the stability and convergence of Equation (11), we use the following lemmas, which have been proven previously [15–17].

**Lemma 1.** *The coefficients $\xi_{\alpha,l},\eta_{\beta,l}$ and $\eta_{2\beta,l}$ satisfy*

1. $\xi_{\alpha,0}=1,\quad \xi_{\alpha,l}>0,\quad \xi_{\alpha,l+1}>\xi_{\alpha,l},\quad l=0,1,2,\dots,N.$
2. $\sum_{l=1}^{k-1}(\xi_{\alpha,l}-\xi_{\alpha,l+1})=\xi_{\alpha,1}-\xi_{\alpha,k}$
3. $\eta_{\beta,0}=1,\quad \eta_{\beta,l}<0\quad (\forall l\ge 1),$
   $\sum_{l=0}^{\infty}\eta_{\beta,l}=0,\quad \sum_{l=0}^{i}\eta_{\beta,l}>0,\quad \sum_{l=0}^{i+1}\eta_{\beta,l}<\sum_{l=0}^{i}\eta_{\beta,l},\quad l=0,1,2,\dots,M.$
4. $\eta_{2\beta,0}=1,\quad \eta_{2\beta,1}=-2\beta\quad \eta_{2\beta,l}>0\quad (\forall l\ge 2),$
   $\sum_{l=0}^{\infty}\eta_{2\beta,l}=0,\quad \sum_{l=0}^{i}\eta_{2\beta,l}<0,\quad \sum_{l=0}^{i+1}\eta_{2\beta,l}>\sum_{l=0}^{i}\eta_{2\beta,l},\quad l=0,1,2,\dots,M.$

**Lemma 2.** *Let the following conditions hold*

*a.* $\frac{1}{2}\le\beta\le 1,$

*b.* *The functions $E(x_1,x_2,t)$ and $\overline{E}(x_1,x_2,t)$ are positive definite monotone increasing functions in $\omega$,*

*c.* *The functions $F(x_1,x_2,t)$ and $\overline{F}(x_1,x_2,t)$ are positive definite convex monotone decreasing functions in $\omega$,*

*then we have,*

1. $D_{x_1}^{\beta} E(x_{1i}, x_{2j}, t_k) \geq 0$ *and* $D_{x_2}^{\beta} \overline{E}(x_{1i}, x_{2j}, t_k) \geq 0$ . *i.e.,*

$$h_1^{-\beta} \sum_{l=0}^{i} \eta_{\beta,l} E(x_{1(i-l)}, x_{2j}, t_k) \geq 0,$$

$$h_2^{-\beta} \sum_{l=0}^{j} \eta_{\beta,l} \overline{E}(x_{1i}, x_{2(j-l)}, t_k) \geq 0.$$

2. $D_{x_1}^{2\beta} F(x_{1i}, x_{2j}, t_k) \leq 0$ *and* $D_{x_2}^{2\beta} \overline{F}(x_{1i}, x_{2j}, t_k) \leq 0$. *i.e.,*

$$h_1^{-2\beta} \sum_{l=0}^{i+1} \eta_{2\beta,l} F(x_{1(i-l+1)}, x_{2j}, t_k) \leq 0,$$

$$h_2^{-2\beta} \sum_{l=0}^{j+1} \eta_{2\beta,l} \overline{F}(x_{1i}, x_{2(j-l+1)}, t_k) \leq 0.$$

**Lemma 3.** *For* $0 < \alpha < 1$, *there exists a constant* $C > 0$, *where*

$$\xi_{\alpha,k}^{-1} \leq C k^{\alpha}$$

## 3. Stability of Implicit Finite Difference Scheme

In this section, we will discuss the stability of numerical scheme Equation (11), let $\tilde{w}_{ij}^k, (0 \leq i \leq M_1, 0 \leq j \leq M_2, 0 \leq k \leq N)$ be the approximate solution of scheme Equation (11), the error $\varepsilon_{ij}^k = \tilde{w}_{ij}^k - w_{ij}^k$ satisfies,

$$\begin{aligned} &\varepsilon_{ij}^{k+1} + c_1 \sum_{l=0}^{i} \eta_{\beta,l} E_{(i-l)j}^{k+1} \varepsilon_{(i-l)j}^{k+1} + \overline{c}_1 \sum_{l=0}^{j} \eta_{\beta,l} \overline{E}_{i(j-l)}^{k+1} \varepsilon_{i(j-l)}^{k+1} \\ &- c_2 \sum_{l=0}^{i+1} \eta_{2\beta,l} F_{(i-l+1)j}^{k+1} \varepsilon_{(i-l+1)j}^{k+1} - \overline{c}_2 \sum_{l=0}^{j+1} \eta_{2\beta,l} \overline{F}_{i(j-l+1)}^{k+1} \varepsilon_{i(j-l+1)}^{k+1} \\ &= \xi_{\alpha,k} \varepsilon_{ij}^0 + \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) \varepsilon_{ij}^{k-l}, \qquad k \geq 1, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\varepsilon_{ij}^{1} + c_1 \sum_{l=0}^{i} \eta_{\beta,l} E_{(i-l)j}^{1} \varepsilon_{(i-l)j}^{1} + \overline{c}_1 \sum_{l=0}^{j} \eta_{\beta,l} \overline{E}_{i(j-l)}^{1} \varepsilon_{i(j-l)}^{1} \\ &- c_2 \sum_{l=0}^{i+1} \eta_{2\beta,l} F_{(i-l+1)j}^{1} \varepsilon_{(i-l+1)j}^{1} - \overline{c}_2 \sum_{l=0}^{j+1} \eta_{2\beta,l} \overline{F}_{i(j-l+1)}^{1} \varepsilon_{i(j-l+1)}^{1} \\ &= \varepsilon_{ij}^0, \qquad k = 0. \end{aligned} \tag{13}$$

Let $\Xi^k = (\Xi_1^k, \Xi_2^k, \ldots, \Xi_{M_1-1}^k)^T$, where $\Xi_i^k = (\varepsilon_{i1}^k, \varepsilon_{i2}^k, \ldots, \varepsilon_{i(M_2-1)}^k)^T$ are the $k-$error vectors, $\|\Xi^k\|_\infty = \max\limits_{1 \leq i \leq M_1-1, 1 \leq j \leq M_2-1} |\varepsilon_{ij}^k|$.

**Theorem 1.** *For every space-positive definite monotone increasing functions* $E(x_1, x_2, t), \overline{E}(x_1, x_2, t)$ *and space-positive definite monotone decreasing convex functions* $F(x_1, x_2, t), \overline{F}(x_1, x_2, t)$ *the fractional implicit finite difference schemes Equation (11), for* $0 < \alpha < 1$ *satisfies*

$$\|\Xi^{k+1}\|_\infty \leq \|\Xi^0\|_\infty. \tag{14}$$

**Proof.** Using the mathematical induction method, we can prove Equation (14).

Firstly, when $k = 0$. Assume that $|\varepsilon^1_{i'j'}| = \max\limits_{1\le i\le M_1-1, 1\le j\le M_2-1} |\varepsilon^1_{ij}|$. Then, by using Lemmas 1 and 2 and Equation (14), we have

$$
\begin{aligned}
|\varepsilon^1_{i'j'}| \le& \left[1 + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E^1_{(i'-l)j'} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}^1_{i'(j'-l)} - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F^1_{(i'-l+1)j'} \right.\\
&\left. - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}^1_{i'(j'-l+1)} \right] |\varepsilon^1_{i'j'}| \\
\le& \left[ |\varepsilon^1_{i'j'}| + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E^1_{(i'-l)j'} |\varepsilon^1_{i'j'}| + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}^1_{i'(j'-l)} |\varepsilon^1_{i'j'}| - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F^1_{(i'-l+1)j'} |\varepsilon^1_{i'j'}| \right.\\
&\left. - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}^1_{i'(j'-l+1)} |\varepsilon^1_{i'j'}| \right] \\
\le& \left[ (1 + c_1 E^1_{i'j'} + \bar{c}_1 \overline{E}^1_{i'j'} + 2\beta c_2 F^1_{i'j'} + 2\beta \bar{c}_2 \overline{F}^1_{i'j'}) |\varepsilon^1_{i'j'}| + c_1 \sum_{l=1}^{i'} \eta_{\beta,l} E^1_{(i'-l)j'} |\varepsilon^1_{(i'-l)j'}| \right.\\
& + \bar{c}_1 \sum_{l=1}^{j'} \eta_{\beta,l} \overline{E}^1_{i'(j'-l)} |\varepsilon^1_{i'(j'-l)}| - c_2 \sum_{l=0, l\ne 1}^{i'+1} \eta_{2\beta,l} F^1_{(i'-l+1)j'} |\varepsilon^1_{(i'-l+1)j'}| \\
&\left. - \bar{c}_2 \sum_{l=0, l\ne 1}^{j'+1} \eta_{2\beta,l} \overline{F}^1_{i'(j'-l+1)} |\varepsilon^1_{i'(j'-l+1)}| \right] \\
\le& \left| \varepsilon^1_{i'j'} + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E^1_{(i'-l)j'} \varepsilon^1_{(i'-l)j'} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}^1_{i'(j'-l)} \varepsilon^1_{i'(j'-l)} \right.\\
&\left. - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F^1_{(i'-l+1)j'} \varepsilon^1_{(i'-l+1)j'} - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}^1_{i'(j'-l+1)} \varepsilon^1_{i'(j'-l+1)} \right| \\
\le& |\varepsilon^0_{i'j'}| \le \|\Xi^0\|_\infty.
\end{aligned}
$$

Thus, $\|\Xi^1\|_\infty \le \|\Xi^0\|_\infty$.

Secondly, let's that $\|\Xi^n\|_\infty \le \|\Xi^0\|_\infty, n = 1, 2 \ldots, k$, and letting $|\varepsilon^{k+1}_{i'j'}| = \max\limits_{1\le i\le M_1-1, 1\le j\le M_2-1} |\varepsilon^{k+1}_{ij}|$.

Therefore, by using Lemmas 1 and 2 and Equation (12), we can deduce

$$
\begin{aligned}
|\varepsilon_{i'j'}^{k+1}| \leq & \left[1 + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} \right. \\
& \left. -\bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} \right] |\varepsilon_{i'j'}^{k+1}| \\
\leq & \left[ |\varepsilon_{i'j'}^{k+1}| + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} |\varepsilon_{i'j'}^{k+1}| + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} |\varepsilon_{i'j'}^{k+1}| - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} |\varepsilon_{i'j'}^{k+1}| \right. \\
& \left. -\bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} |\varepsilon_{i'j'}^{k+1}| \right] \\
\leq & \left[ (1 + c_1 E_{i'j'}^{k+1} + \bar{c}_1 \overline{E}_{i'j'}^{k+1} + 2\beta c_2 F_{i'j'}^{k+1} + 2\beta \bar{c}_2 \overline{F}_{i'j'}^{k+1}) |\varepsilon_{i'j'}^{k+1}| + c_1 \sum_{l=1}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} |\varepsilon_{(i'-l)j'}^{k+1}| \right. \\
& + \bar{c}_1 \sum_{l=1}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} |\varepsilon_{i'(j'-l)}^{k+1}| - c_2 \sum_{l=0,l\neq 1}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} |\varepsilon_{(i'-l+1)j'}^{k+1}| \\
& \left. -\bar{c}_2 \sum_{l=0,l\neq 1}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} |\varepsilon_{i'(j'-l+1)}^{k+1}| \right] \\
\leq & \left| \varepsilon_{i'j'}^{k+1} + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} \varepsilon_{(i'-l)j'}^{k+1} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} \varepsilon_{i'(j'-l)}^{k+1} \right. \\
& \left. -c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} \varepsilon_{(i'-l+1)j'}^{k+1} - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} \varepsilon_{i'(j'-l+1)}^{k+1} \right| \\
= & \left| \xi_{\alpha,k} \varepsilon_{i'j'}^{0} + \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) \varepsilon_{i'j'}^{k-l} \right| \\
\leq & \left[ \xi_{\alpha,k} |\varepsilon_{i'j'}^{0}| + \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) |\varepsilon_{i'j'}^{k-l}| \right] \\
\leq & [\xi_{\alpha,k} + \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1})] \|\Xi^0\|_\infty \leq \|\Xi^0\|_\infty.
\end{aligned}
$$

Thus, $\|\Xi^{k+1}\|_\infty \leq \|\Xi^0\|_\infty$. □

## 4. Convergence of Implicit Finite Difference Scheme

In this section, we will study the convergence of numerical scheme Equation (11), to the exact solution of equation Equations (1) and (2).

Let, $w(x_{1i}, x_{2j}, t_k)$ be the exact solution of Equations (1) and (2) at a point $(x_{1i}, x_{2j}, t_k)$, $e_{ij}^k = w(x_{1i}, x_{2j}, t_k) - w_{ij}^k$ be the error between the exact and the numerical solution, which satisfies

$$
\begin{aligned}
& e_{ij}^{k+1} + c_1 \sum_{l=0}^{i} \eta_{\beta,l} E_{(i-l)j}^{k+1} e_{(i-l)j}^{k+1} + \bar{c}_1 \sum_{l=0}^{j} \eta_{\beta,l} \overline{E}_{i(j-l)}^{k+1} e_{i(j-l)}^{k+1} \\
& \quad - c_2 \sum_{l=0}^{i+1} \eta_{2\beta,l} F_{(i-l+1)j}^{k+1} e_{(i-l+1)j}^{k+1} - \bar{c}_2 \sum_{l=0}^{j+1} \eta_{2\beta,l} \overline{F}_{i(j-l+1)}^{k+1} e_{i(j-l+1)}^{k+1} \\
& \quad = \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) e_{ij}^{k-l} + R_{ij}^{k+1}, \qquad k \geq 1
\end{aligned}
\tag{15}
$$

and

$$\begin{aligned} & e_{ij}^1+c_1\sum_{l=0}^{i}\eta_{\beta,l}E_{(i-l)j}^1e_{(i-l)j}^1+\bar{c}_1\sum_{l=0}^{j}\eta_{\beta,l}\overline{E}_{i(j-l)}^1e_{i(j-l)}^1 \\ & -c_2\sum_{l=0}^{i+1}\eta_{2\beta,l}F_{(i-l+1)j}^1e_{(i-l+1)j}^1-\bar{c}_2\sum_{l=0}^{j+1}\eta_{2\beta,l}\overline{F}_{i(j-l+1)}^1e_{i(j-l+1)}^1 \\ & =R_{ij}^1, \qquad\qquad k=0, \end{aligned} \tag{16}$$

hence, $e_i^0=w(s_i,0)-w_i^0=0$.

Letting, $V^k=(V_1^k,V_2^k,\dots,V_{M_1-1}^k)^T$, where, $V_i^k=(e_{i1}^k,e_{i2}^k,\dots,V_{i(M_2-1)}^k)^T$ and $\|V^k\|_\infty=\max\limits_{1\le i\le M_1-1,1\le j\le M_2-1}|e_{ij}^k|$, follow that, we obtain the convergence of numerical scheme by applying the following theorem.

**Theorem 2.** *For every space positive definite monotone increasing functions $E(x_1,x_2,t)$, $\overline{E}(x_1,x_2,t)$ and space positive definite monotone decreasing convex functions $F(x_1,x_2,t)$, $\overline{F}(x_1,x_2,t)$, then the fractional implicit finite difference schemes Equation (11), for $0<\alpha<1$, satisfy*

$$\|V^{k+1}\|_\infty\le C\xi_{\alpha,k}^{-1}\tau^\alpha(\tau^{2-\alpha}+h_1^2+h_2^2), \tag{17}$$

*and the errors between the exact solutions and numerical solutions are valid*

$$|w(x_{1i},x_{2j},t_k)-w_{ij}^k|\le C^*(\tau^{2-\alpha}+h_1^2+h_2^2), \tag{18}$$

*where, $i=1,2,\dots,M_1-1$, $j=1,2,\dots,M_2-1$, $k=0,2,\dots,N$, and $C^*\ge 0$ is constant.*

**Proof.** Similar to the proof of Theorem 1, by using the mathematical induction to prove Equation (17).

When $k=0$, it follows from Lemmas 1 and 2 and Equation (16) under consideration of $\|V^1\|_\infty=|e_{i'j'}^1|=\max\limits_{1\le i\le M_1-1,1\le j\le M_2-1}|e_{ij}^1|$, it is easy to prove

$$|e_{ij}^1|\le C\xi_{\alpha,0}^{-1}\tau^\alpha(\tau^{2-\alpha}+h_1^2+h_2^2), \tag{19}$$

then, suppose that, $\|V^s\|_\infty\le C\xi_{\alpha,s-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h_1^2+h_2^2)\le C\xi_{\alpha,k}^{-1}\tau^\alpha(\tau^{2-\alpha}+h_1^2+h_2^2)$ for $s=1,2,\dots,k$, and letting $|e_{i'j'}^{k+1}|=\max\limits_{1\le i\le M_1-1,1\le j\le M_2-1}|e_{ij}^{k+1}|$.

Therefore, from Lemmas 1 and 2 and Equation (15), we have

$$
\begin{aligned}
|e_{i'j'}^{k+1}| \le & \left[ 1 + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} \right. \\
& \left. - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} \right] |e_{i'j'}^{k+1}| \\
\le & \left[ |e_{i'j'}^{k+1}| + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} |e_{i'j'}^{k+1}| + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} |e_{i'j'}^{k+1}| \right. \\
& \left. - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} |e_{i'j'}^{k+1}| - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} |e_{i'j'}^{k+1}| \right] \\
\le & \left[ (1 + c_1 E_{i'j'}^{k+1} + \bar{c}_1 \overline{E}_{i'j'}^{k+1} + 2\beta c_2 F_{i'j'}^{k+1} + 2\beta \bar{c}_2 \overline{F}_{i'j'}^{k+1}) |e_{i'j'}^{k+1}| \right. \\
& + c_1 \sum_{l=1}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} |e_{(i'-l)j'}^{k+1}| + \bar{c}_1 \sum_{l=1}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} |e_{i'(j'-l)}^{k+1}| \\
& \left. - c_2 \sum_{l=0,l\neq 1}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} |e_{(i'-l+1)j'}^{k+1}| - \bar{c}_2 \sum_{l=0,l\neq 1}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} |e_{i'(j'-l+1)}^{k+1}| \right] \\
\le & \left| e_{i'j'}^{k+1} + c_1 \sum_{l=0}^{i'} \eta_{\beta,l} E_{(i'-l)j'}^{k+1} e_{(i'-l)j'}^{k+1} + \bar{c}_1 \sum_{l=0}^{j'} \eta_{\beta,l} \overline{E}_{i'(j'-l)}^{k+1} e_{i'(j'-l)}^{k+1} \right. \\
& \left. - c_2 \sum_{l=0}^{i'+1} \eta_{2\beta,l} F_{(i'-l+1)j'}^{k+1} e_{(i'-l+1)j'}^{k+1} - \bar{c}_2 \sum_{l=0}^{j'+1} \eta_{2\beta,l} \overline{F}_{i'(j'-l+1)}^{k+1} e_{i'(j'-l+1)}^{k+1} \right| \\
= & \left| \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) e_{i'j'}^{k-l} + R_{i'j'}^{k+1} \right| \\
\le & \left[ \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) |e_{i'j'}^{k-l}| + C\tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) \right] \\
\le & \left[ \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) C\xi_{\alpha,k}^{-1} \tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) + C\tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) \right] \\
= & \left[ \sum_{l=0}^{k-1} (\xi_{\alpha,l} - \xi_{\alpha,l+1}) + \xi_{\alpha,k} \right] C\xi_{\alpha,k}^{-1} \tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) \\
\le & C\xi_{\alpha,k}^{-1} \tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2).
\end{aligned}
$$

Thus, Equation (17) is proved.

Hence, from Lemma 3, where, $\xi_{\alpha,k}^{-1} \le C_1 k^\alpha, C_1 \ge 0$ is constant and $k\tau \le T$ is finite, then from Equation (17),

$$
\begin{aligned}
|w(x_{1i}, x_{2j}, t_k) - w_{ij}^k| \le & |e_{i'j'}^k| \\
\le & C\xi_{\alpha,k}^{-1} \tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) \\
\le & CC_1 k^\alpha \tau^\alpha (\tau^{2-\alpha} + h_1^2 + h_2^2) \\
\le & C^* (\tau^{2-\alpha} + h_1^2 + h_2^2).
\end{aligned} \tag{20}
$$

This proves the convergent of numerical scheme Equation (11). □

## 5. Numerical Experiments

In this section, some numerical experiments are presented to observe the accuracy and efficiency of the proposed implicit finite difference scheme Equation (11) by considering the following example of two-dimensional Time–Space Fractional Fokker–Planck Equation with variable coefficients and $(0 < \alpha < 1, \frac{1}{2} < \beta < 1)$.

**Example 1.** *Let's consider*

$$\begin{aligned}\frac{\partial^\alpha w(x_1,x_2,t)}{\partial t^\alpha} = & -D^\beta_{x_1}\left[t x_1^\beta x_2^\beta w(x_1,x_2,t)\right] - D^\beta_{x_2}\left[t x_1^\beta x_2^\beta w(x_1,x_2,t)\right] \\ & + D^{2\beta}_{x_1}\left[(1-x_1^{2\beta})(1-x_2^{2\beta}) w(x_1,x_2,t)\right] \\ & + D^{2\beta}_{x_2}\left[(1-x_1^{2\beta})(1-x_2^{2\beta}) w(x_1,x_2,t)\right] \\ & + g(x_1,x_2,t),\end{aligned} \tag{21}$$

*with initial and boundary condition*

$$\begin{aligned} w(x_1,x_2,0) &= 0, && (x_1,x_2) \in (0,1)\times(0,1), \\ w(x_1,x_2,t) &= 0, && (x_1,x_2) \in \partial((0,1)\times(0,1)), \quad 0 \le t \le T, \end{aligned} \tag{22}$$

*where*

$$\begin{aligned} g(s,t) = & \Gamma(\alpha+2) t x_1^2 x_2^2 (1-x_1)^2(1-x_2)^2 \\ & + t^{\alpha+2} x_2^{\beta+2}(1-x_2)^2\left(\frac{\Gamma(3+\beta)x_1^2}{\Gamma(3)} - \frac{2\Gamma(4+\beta)x_1^3}{\Gamma(4)} + \frac{\Gamma(5+\beta)x_1^4}{\Gamma(5)}\right) \\ & + t^{\alpha+2} x_1^{\beta+2}(1-x_1)^2\left(\frac{\Gamma(3+\beta)x_2^2}{\Gamma(3)} - \frac{2\Gamma(4+\beta)x_2^3}{\Gamma(4)} + \frac{\Gamma(5+\beta)x_2^4}{\Gamma(5)}\right) \\ & - t^{\alpha+1} x_2^2(1-x_2)^2(1-x_2^{2\beta})\left(\frac{\Gamma(3)x_1^{2-2\beta}}{\Gamma(3-2\beta)} - \frac{2\Gamma(4)x_1^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{\Gamma(5)x_1^{4-2\beta}}{\Gamma(5-2\beta)} \right. \\ & \qquad \left. - \frac{\Gamma(3+2\beta)x_1^2}{\Gamma(3)} + \frac{2\Gamma(4+2\beta)x_1^3}{\Gamma(4)} - \frac{\Gamma(5+2\beta)x_1^4}{\Gamma(5)}\right) \\ & - t^{\alpha+1} x_1^2(1-x_1)^2(1-x_1^{2\beta})\left(\frac{\Gamma(3)x_2^{2-2\beta}}{\Gamma(3-2\beta)} - \frac{2\Gamma(4)x_2^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{\Gamma(5)x_2^{4-2\beta}}{\Gamma(5-2\beta)} \right. \\ & \qquad \left. - \frac{\Gamma(3+2\beta)x_2^2}{\Gamma(3)} + \frac{2\Gamma(4+2\beta)x_2^3}{\Gamma(4)} - \frac{\Gamma(5+2\beta)x_2^4}{\Gamma(5)}\right), \end{aligned}$$

*the exact solution is* $w(x_1,x_2,t) = t^{\alpha+1}x_1^2x_2^2(1-x_1)^2(1-x_2)^2$.

*As an analyzing the error in numerical solution we use the same step of step size in each spatial direction i.e.,* $h = h_1 = h_2$, *consider the* $k-$*max of* ${}_kL_2$*-norms*

$$\varepsilon(\tau,h_)= \max_{1\le k\le N} h\left(\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}|w(x_{1i},x_{2j},t_k) - w_{ij}^k|^2\right)^{\frac{1}{2}}, \tag{23}$$

*where we can roughly calculate the convergence rate as*

$$Rate \simeq \log_2[\varepsilon(2\tau,2h)/\varepsilon(\tau,h)].$$

*We can see that the numerical solution of our numerical scheme at* $h = 0.1, \tau = 0.05$ *and* $T = 1, L_1 = L_2 = 1$ *can match the exact solution, as shown in Figures 1 and 2, when* $\alpha = 0.7$ *and* $\beta = 0.6$.

*In Table 1, we use the format described in Section 2 with a spatial and temporal mesh with $\tau = h^{\frac{2}{2-\alpha}}$ to analyze how the error $\varepsilon(\tau, h)$ and convergence rate changes at different spatial and temporal steps, for four different choices of $\alpha$ and $\beta$, with an observation for the $(2-\alpha)$-th temporal order convergence rate, the presented results reflecting the relationship of the mesh size with the rate of convergence depending on the parameter $\alpha$ and $\beta$*

**Remark 1.** *The numerical implicit finite difference scheme Equation (11) still work well for the two-dimensional time–space fractional Foker–Planck PDE with the order of its spatial and temporal derivatives $\beta$ and $\alpha$ satisfies $0 < \alpha \leq 1, 0.5 < \beta \leq 1$; as a result, all the theoretical analyses are still valid.*

**Table 1.** The error and convergence rate for Numerical scheme solution of (21) when $T = 1, \tau = h^{\frac{2}{2-\alpha}}$.

| | $\alpha = 0.2, \beta = 0.9$ | | $\alpha = 0.4, \beta = 0.8$ | | $\alpha = 0.7, \beta = 0.7$ | | $\alpha = 0.9, \beta = 0.6$ | |
|---|---|---|---|---|---|---|---|---|
| $\tau, h$ | $\varepsilon(\tau, h)$ | *Rate* | $\varepsilon(\tau, h)$ | *Rate* | $\varepsilon(\tau, h)$ | *Rate* | $\varepsilon(\tau, h)$ | *Rate* |
| 1/4 | $1.8251 \times 10^{-4}$ | | $1.2760 \times 10^{-4}$ | | $5.5186 \times 10^{-5}$ | | $3.0542 \times 10^{-5}$ | |
| 1/8 | $5.1971 \times 10^{-5}$ | 1.8121 | $4.1884 \times 10^{-5}$ | 1.6071 | $2.2395 \times 10^{-5}$ | 1.3011 | $1.4228 \times 10^{-5}$ | 1.1020 |
| 1/16 | $1.4893 \times 10^{-5}$ | 1.8030 | $1.3645 \times 10^{-5}$ | 1.6180 | $9.0354 \times 10^{-6}$ | 1.3095 | $6.6308 \times 10^{-6}$ | 1.1014 |
| 1/32 | $4.2617 \times 10^{-6}$ | 1.80516 | $4.2503 \times 10^{-6}$ | 1.6827 | $3.6635 \times 10^{-6}$ | 1.3023 | $3.0860 \times 10^{-6}$ | 1.1034 |

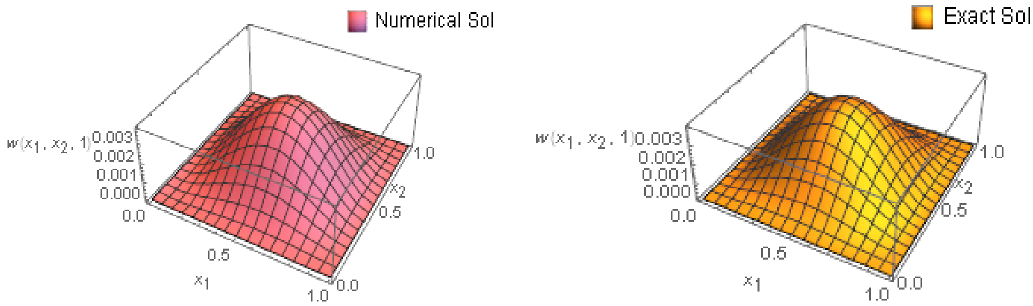


**Figure 1.** Exact and Numerical scheme solution of (21) at $\alpha = 0.7, \beta = 0.8, \tau = 0.05, h = 0.1$ and $t = T = 1$.

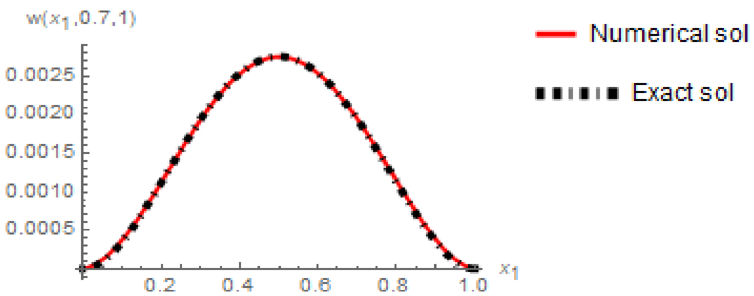


**Figure 2.** Comparison the exact and numerical solution of (21) at $t = T = 1, x_2 = 0.7, \tau = 0.05$, $h = 0.1, \beta = 0.6, \alpha = 0.7$ for different $\alpha$.

## 6. Conclusions

In this article, we present the implicit finite difference scheme for two dimensional time–space fractional Fokker–Planck equation with time–space-dependent function coefficients and source term. We use two fractional derivative operators (the Caputo derivative with order $0 < \alpha \leq 1$ and the Riemann–Liouville derivative for two orders $\beta, 2\beta$, $(0.5 < \beta \leq 1)$. The unconditional stable and convergence for proposed numerical scheme is proven, and some numerical examples for the implicit finite difference method are given, which agree with our theoretical analysis.

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