Article

On Fourier Coefficients of the Symmetric Square L -Function at Piatetski-Shapiro Prime Twins

Xue Han, Xiaofei Yan and Deyu Zhang *

School of Mathematics and Statistics, Shandong Normal University, Jinan 250358, China; han_xue@stu.sdnu.edu.cn (X.H.); xfyang@sdnu.edu.cn (X.Y.)

* Correspondence: zhangdeyu@sdnu.edu.cn

Abstract: Let $\mathbb{P}_c(x) = \{p \leq x \mid p, [p^c] \text{ are primes}\}$, $c \in \mathbb{R}^+ \setminus \mathbb{N}$ and $\lambda_{\text{sym}^2 f}(n)$ be the n -th Fourier coefficient associated with the symmetric square L -function $L(s, \text{sym}^2 f)$. For any $A > 0$, we prove that the mean value of $\lambda_{\text{sym}^2 f}(n)$ over $\mathbb{P}_c(x)$ is $\ll x \log^{-A-2} x$ for almost all $c \in (\varepsilon, (\sqrt{5} + 3)/8 - \varepsilon)$ in the sense of Lebesgue measure. Furthermore, it holds for all $c \in (0, 1)$ under the Riemann Hypothesis. Furthermore, we obtain that asymptotic formula for $\lambda_f^2(n)$ over $\mathbb{P}_c(x)$ is $\sum_{p, q \text{ prime } p \leq x, q = [p^c]} \lambda_f^2(p) = \frac{x}{c \log^2 x} (1 + o(1))$, for almost all $c \in (\varepsilon, (\sqrt{5} + 3)/8 - \varepsilon)$, where $\lambda_f(n)$ is the normalized n -th Fourier coefficient associated with a holomorphic cusp form f for the full modular group.

Keywords: primes; Fourier coefficient; symmetric square L -function

MSC: 11A41; 11F11; 11F30

Citation: Han, X.; Yan, X.; Zhang, D. On Fourier Coefficients of the Symmetric Square L -Function at Piatetski-Shapiro Prime Twins. *Mathematics* **2021**, *9*, 1254. <https://doi.org/10.3390/math9111254>

Academic Editor: Luis Dieulefait

Received: 28 April 2021

Accepted: 26 May 2021

Published: 30 May 2021

1. Introduction

Let k be an even positive integer, f be a holomorphic cusp form of weight k for the full modular group and $\lambda_f(n)$ be the normalized n -th Fourier coefficient of f , i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz).$$

If we assume that f is an eigenform of all the Hecke operators, then f can be normalized such that $\lambda_f(1) = 1$ and $\lambda_f(n)$ is real. We define the Hecke L -function associated to f for $\Re s > 1$ by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1}.$$

For any prime p and all integers $\nu \geq 0$, we have

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1} \beta_f(p) + \cdots + \beta_f(p)^\nu,$$

where $\alpha_f(p), \beta_f(p)$ are the local parameters of $L(s, f)$ at prime p , satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p) \beta_f(p) = 1.$$

Then we have

$$\lambda_f^2(p) = 1 + \lambda_f(p^2).$$

Deligne [1,2] proved Ramanujan–Petersson conjecture, i.e.,

$$|\lambda_f(n)| \leq d(n) \ll n^\varepsilon$$

for all $n \geq 1$, where $d(n) = \sum_{d|n} 1$.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

In order to detect the sign changes of $\lambda_f(n)$, many authors have studied the mean value of $\lambda_f(n)$ and obtained some good results. For example, see [1,3–22]. In addition, the sums of $\lambda_f(n)$ over primes have also been studied. It is known that (see for example Section 5.6 of Iwaniec and Kowalski [23]) there exists a constant $C > 0$ such that

$$\sum_{p \leq N} \lambda_f(p) \ll_f N \exp(-C\sqrt{\log N}). \quad (1)$$

The upper bound of (1) may reach $N^{\frac{1}{2}+\varepsilon}$, assuming the Riemann Hypothesis. Furthermore, we can establish that

$$\sum_{p \leq N} \lambda_f^2(p) \sim \frac{N}{\log N}, \text{ as } N \rightarrow \infty,$$

by using the analytic properties of the Rankin–Selberg L -function $L(s, f \otimes \bar{f})$.

Another interesting question considered by many authors is the mean value of $\lambda_f(n)$ over certain sets of primes. For example, Baier and Zhao [24] studied the distribution of $\lambda_f(n)$ at Piatetski–Shapiro primes by considering estimates of exponential sum involving Hecke eigenvalue. Moreover, they conjectured that, for $1 < c < 1 + \theta$ with some suitable $\theta > 0$, there exists a constant $c_f > 0$ such that

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{R}}} \lambda_f^2([n^c]) \sim c_f \frac{N}{c \log N}, \text{ as } N \rightarrow \infty.$$

Furthermore, we can define Piatetski–Shapiro prime twins if p, q are primes and $q = [p^c]$, for any fixed $c \in \mathbb{R}^+ \setminus \mathbb{N}$. Balog [25] and Dufner [26] proved that

$$\sum_{\substack{p, q \text{ prime} \\ p \leq x, q = [p^c]}} 1 = \frac{x}{c \log^2 x} (1 + o(1)), \quad x \rightarrow \infty, \quad (2)$$

for almost all $c \in (0, 1)$ in the sense of Lebesgue measure. Furthermore, assuming the Riemann Hypothesis of automorphic L -function $L(s, f)$ is true, they found that (2) holds for all $c \in (0, 1)$. Furthermore, Zhang and Zhai [27] studied the mean value of $\lambda_f(n)$ over Piatetski–Shapiro prime twins.

Motivated by the above results, we are interested in the distribution of $\lambda_f^2(n)$ at Piatetski–Shapiro prime twins. For the form f , we know the $L(s, \text{sym}^2 f)$ is an L -function for some $GL(3, \mathbb{Z})$ automorphic representation, which is often called the symmetric-square lift of f . The n -th Fourier coefficient of $L(s, \text{sym}^2 f)$ satisfies

$$\lambda_{\text{sym}^2 f}(n) = \sum_{ml^2=n} \lambda_f(m^2).$$

The symmetric square L -function associated to f is defined by

$$L(s, \text{sym}^2 f) = \prod_p \prod_{0 \leq j \leq 2} \left(1 - \frac{\alpha_f(p)^{2-j} \beta_f(p)^j}{p^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s}$$

in the half-plane $\Re s > 1$. Then, for all $n \geq 1$, $\lambda_{\text{sym}^2 f}(n)$ is also multiplicative, real and

$$|\lambda_{\text{sym}^2 f}(n)| \leq d_3(n),$$

where $d_3(n) = \sum_{m|n} d(m)$. Furthermore, for all primes p , we have

$$\lambda_{\text{sym}^2 f}(p) = \lambda_f(p^2).$$

Many authors studied the mean value of $\lambda_{\text{sym}^2 f}(n)$. For example, see References [28–36]. In this paper, we consider the mean value of Fourier coefficients of symmetric square L -function over Piatetski–Shapiro prime twins and obtain the following results, which imply a result on the distribution of $\lambda_f^2(n)$ at Piatetski–Shapiro prime twins.

Theorem 1. For almost all $c \in (\varepsilon, (\sqrt{5} + 3)/8 - \varepsilon)$ and any $A > 0$, we have

$$\sum_{\substack{p, q \text{ prime} \\ p \leq x, q = [p^c]}} \lambda_{\text{sym}^2 f}(p) \ll \frac{x}{\log^{A+2} x}. \quad (3)$$

Theorem 2. Assuming the Riemann Hypothesis of symmetric square L -function is true, (3) holds for all $c \in (0, 1)$.

Corollary 1. For almost all $c \in (\varepsilon, (\sqrt{5} + 3)/8 - \varepsilon)$, we have

$$\sum_{\substack{p, q \text{ prime} \\ p \leq x, q = [p^c]}} \lambda_f^2(p) = \frac{x}{c \log^2 x} (1 + o(1)), \quad x \rightarrow \infty. \quad (4)$$

Proof. The result follows easily from Theorem 1 and (2), if we notice that $\lambda_f^2(p) = 1 + \lambda_f(p^2)$. $\square$

Corollary 2. Assuming the Riemann Hypothesis of symmetric square L -function is true, (4) holds for all $c \in (0, 1)$.

Proof. The result follows from Theorem 2 and (2). $\square$

Notation. Throughout the paper, ε always denotes a sufficiently small positive constant. Let $\delta_1(\varepsilon)$ be sufficiently small and depend on ε . We write $f(x) \ll g(x)$, or $f(x) = O(g(x))$, to mean that $|f(x)| \leq Cg(x)$. Let $\rho = \sigma + i\eta$ be the nontrivial zero of the symmetric square L -function $L(s, \text{sym}^2 f)$. As usual, $\Lambda(n)$ is the von Mangoldt function.

2. Auxiliary Lemmas

Lemma 1. Let η run through a countable set of reals, with $c(\eta)$ arbitrary complex such that $S(t) = \sum_{\eta} c(\eta)e(\eta t)$ is absolutely convergent. Let $\alpha \in \mathbb{R}, \Theta \in (0, 1), \delta = \Theta/\alpha$. Then

$$\int_{-\alpha}^{\alpha} |S(t)|^2 dt \ll_{\Theta} \int_{-\infty}^{\infty} \left| \delta^{-1} \sum_{t \leq \eta \leq t+\delta} c(\eta) \right|^2 dt.$$

Proof. This lemma is Lemma 1 of Gallagher [37]. $\square$

Lemma 2. Let $T \geq T_0$, where T_0 is a sufficiently large real number. The

$$\int_T^{2T} |L(\alpha + it, \text{sym}^2 f)|^2 dt \ll \begin{cases} (10 + k + T)(\log(10 + k + T))^{17}, & \text{if } 2/3 \leq \alpha \leq 1, \\ (10 + k + T)^{3(1-\alpha)}(\log(10 + k + T))^{17}, & \text{if } 1/3 \leq \alpha \leq 2/3, \\ (10 + k + T)^{3(1-2\alpha)+1}(\log(10 + k + T))^{17}, & \text{if } 0 \leq \alpha \leq 1/3. \end{cases}$$

Proof. This lemma is Lemma 3.1 of Lü [33]. $\square$

Lemma 3. For $1/2 \leq \sigma \leq 1, T \geq 3$, define

$$N(\sigma, T) = \#\{\varrho = \beta + i\omega : L(\varrho, \text{sym}^2 f) = 0, \sigma \leq \beta \leq 1, |\omega| \leq T\}.$$

Then we have

$$N(\sigma, T) \ll T^{\frac{5(1-\sigma)}{3-2\sigma} + \varepsilon}, \text{ if } 1/2 \leq \sigma \leq 3/4,$$

and

$$N(\sigma, T) \ll T^{3(1-\sigma) + \varepsilon}, \text{ if } 3/4 < \sigma \leq 1.$$

Proof. From Lemma 2, we have

$$\int_T^{2T} |L(1/2 + it, \text{sym}^2 f)|^2 dt \ll T^{3/2 + \varepsilon}. \quad (5)$$

This combined with Theorem 1.1, 1.2 of Ye and Zhang [38] and [39] gives this lemma. $\square$

Lemma 4. For any $u \geq 2$, $T \geq 2$, we have

$$\sum_{n \leq u} \Lambda(n) \lambda_{\text{sym}^2 f}(n) = - \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T}} \frac{u^\rho}{\rho} + O\left(\log^3 u + \frac{u \log^3(uT)}{T}\right).$$

Proof. See, for example, Iwaniec and Kowalski [23]. For convenience of calculation, we reduce the summation range of n from $n \leq u$ to $n < u$. The contribution of $n = u$ is $O(\log^3 u)$, in view of $|\lambda_{\text{sym}^2 f}(p^h)| \leq d_3(p^h) \ll \log^2 u$. $\square$

Lemma 5. Let $L(s, f)$ be an L -function of degree k such that Rankin–Selberg convolutions $L(s, f \otimes f)$ and $L(s, f \otimes \bar{f})$ exist, and the latter has a simple pole at $s = \alpha + it = 1$ while the former is entire if $f \neq \bar{f}$. Suppose that the ramified primes $|\alpha_f(p)| \leq p/2$. There exists an absolute constant $c > 0$ such that $L(s, f)$ has no zeros in the region

$$\alpha \geq 1 - \frac{c}{k^4 \log(q(f)|t| + 3)},$$

where $q(f, s)$ is the analytic conductor and $q(f) = q(f, 0)$.

Proof. This lemma is Theorem 5.10 of Iwaniec and Kowalski [23]. $\square$

Lemma 6. Let $x, T \in \mathbb{R}^+$, $x \rightarrow \infty$ and

$$S := \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T}} x^\sigma,$$

we have

$$S \ll \begin{cases} x^{1/2} T^{5/4 + \varepsilon}, & \text{if } T \geq x^{4/5}; \\ x^{3/2} T^{5/2 + \varepsilon} e^{-\sqrt{5 \log M \log T}}, & \text{if } x^{9/20} \leq T < x^{4/5}; \\ x^{3/4} T^{5/6 + \varepsilon} + x^{1-\beta} T^{3\beta + \varepsilon}, & \text{if } T < x^{9/20}. \end{cases}$$

Proof. Note that

$$S = \int_{1/2}^{1-\beta} x^\sigma d_\sigma N(\sigma, T),$$

where $\beta = \beta(T)$ is the width of the zero-free region of the symmetric square L -function in Lemma 5. Using integration by parts and Lemma 3, we obtain

$$\begin{aligned} S &\ll_l x^{1/2}T + \int_{1/2}^{3/4} \exp(f_1(\sigma))d\sigma + \int_{3/4}^{1-\beta} \exp(f_2(\sigma))d\sigma \\ &:= x^{1/2}T + S_1 + S_2, \end{aligned} \quad (6)$$

where $\ll_l$ is fulfilled apart from a fixed number of log-factors,

$$f_1(\sigma) = \frac{5(1-\sigma)}{3-2\sigma} \log T + \sigma \log x + \varepsilon \log T$$

and

$$f_2(\sigma) = 3(1-\sigma) \log T + \sigma \log x + \varepsilon \log T.$$

We estimate S_1 first. The first and second derivatives of $f_1(\sigma)$ are

$$f_1'(\sigma) = -\frac{5 \log T}{(3-2\sigma)^2} + \log x$$

and

$$f_1''(\sigma) = -\frac{20 \log T}{(3-2\sigma)^3} < 0, \text{ if } \sigma < 3/2.$$

Then $f_1(\sigma)$ is a concave function as $\sigma < 3/2$. Let $f_1'(\sigma_1) = 0$, we have $\sigma_1 = \frac{3}{2} - \frac{1}{2} \sqrt{\frac{5 \log T}{\log x}}$. So we have the following three cases.

Case 1. When $T \geq x^{4/5}$. In this case, we have $\sigma_1 \leq 1/2$ and $f_1(\sigma)$ is monotonically decreasing in $[1/2, 3/4]$. Hence

$$\max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(1/2) = (5 \log T)/4 + (\log x)/2 + \varepsilon \log T.$$

Case 2. When $x^{9/20} \leq T < x^{4/5}$. In this case, the function $f_1(\sigma)$ takes the extreme value at σ_1 , which gives

$$\max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(\sigma_1) = (5 \log T)/2 + (3 \log x)/2 - \sqrt{5 \log T \log x} + \varepsilon \log T.$$

Case 3. When $T < x^{9/20}$. In this case, we have $\sigma_1 > 3/4$ and $f_1(\sigma)$ is monotonically increasing in $[1/2, 3/4]$. Hence,

$$\max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(3/4) = (5 \log T)/6 + (3 \log x)/4 + \varepsilon \log T.$$

Combining all the above cases, we have

$$S_1 = \int_{1/2}^{3/4} \exp(f_1(\sigma))d\sigma \ll \begin{cases} x^{1/2}T^{5/4+\varepsilon}, & \text{if } T \geq x^{4/5}; \\ x^{3/2}T^{5/2+\varepsilon}e^{-\sqrt{5 \log x \log T}}, & \text{if } x^{9/20} \leq T < x^{4/5}; \\ x^{3/4}T^{5/6+\varepsilon}, & \text{if } T < x^{9/20}. \end{cases} \quad (7)$$

Next, we need to estimate S_2 . It is easy to see that $f_2(\sigma)$ is linear function in $[3/4, 1-\beta]$, hence

$$\begin{aligned} f_2(\sigma) &\ll \max(f_2(3/4), f_2(1-\beta)) \\ &= \max((3 \log T)/4 + (3 \log x)/4, 3\beta \log T + (1-\beta) \log x) + \varepsilon \log T. \end{aligned} \quad (8)$$

From (6), (7) and (8), we obtain

$$S \ll_l \begin{cases} x^{1/2}T + x^{1/2}T^{5/4+\varepsilon} + x^{3/4}T^{3/4+\varepsilon}, & \text{if } T \geq x^{4/5}; \\ x^{1/2}T + x^{3/2}T^{5/2+\varepsilon}e^{-\sqrt{5\log x \log T}} + x^{3/4}T^{3/4+\varepsilon}, & \text{if } x^{9/20} \leq T < x^{4/5}; \\ x^{1/2}T + x^{3/4}T^{5/6+\varepsilon} + x^{1-\beta}T^{3\beta+\varepsilon}, & \text{if } T < x^{9/20}, \end{cases}$$

which proves this lemma. $\square$

Lemma 7. Let $x_0 := 2$, $x_{v+1} := x_v + x_v \log^{-2} x_v$. Let γ_0 be a constant and

$$D(\gamma_0, x_v, T) = \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} \left| \sum_{x_v^{1/\gamma} < m \leq x_v^{1/\gamma}} \Lambda(m) \sum_{\substack{\rho = \sigma + i\eta \\ T/2 < |\eta| \leq T}} \frac{(m+1)^{\gamma\rho} - m^{\gamma\rho}}{\rho} \right|^2 d\gamma.$$

If $T > x_v^{1/\gamma_0 - \delta}$, we have

$$D(\gamma_0, x_v, T) \ll x_v^{2/\gamma_0} T^{-3} \left(\sum_{\substack{\rho = \sigma + i\eta \\ T/2 < |\eta| \leq T}} x_v^\sigma \right)^2. \quad (9)$$

If $T \leq x_v^{1/\gamma_0 - \delta}$, we have

$$D(\gamma_0, x_v, T) \ll x_v^{2/\gamma_0} T^{-3} \left(\sum_{\substack{\rho = \sigma + i\eta \\ T/2 < |\eta| \leq T}} x_v^\sigma \right)^2 \left(T x_v^{-1/\gamma_0} \right)^2. \quad (10)$$

Proof. This lemma follows from (13) and (16) of Dufner [26]. $\square$

3. Proof of Theorem 2

In this section, we write $\gamma = 1/c$. Then we have

$$\Pi_{2,c}(x) := \sum_{\substack{p,q \text{ prime} \\ p \leq x, q = [p^c]}} \lambda_{\text{sym}^2 f}(p) = \sum_{\substack{p,q \text{ prime} \\ p \leq x, q \leq p^c < (q+1)}} \lambda_{\text{sym}^2 f}(p) = \sum_{q \leq x^c} \sum_{\substack{p \leq x \\ q^\gamma \leq p < (q+1)^\gamma}} \lambda_{\text{sym}^2 f}(p).$$

We split the summation range of q into two parts: $q \leq x^c - 1$ and $x^c - 1 < q \leq x^c$ to get

$$\begin{aligned} \Pi_{2,c}(x) &= \sum_{q \leq x^c - 1} \sum_{q^\gamma \leq p < (q+1)^\gamma} \lambda_{\text{sym}^2 f}(p) + \sum_{x^c - 1 < q \leq x^c} \sum_{q^\gamma \leq p \leq x} \lambda_{\text{sym}^2 f}(p) \\ &= \sum_{q \leq x^c} \sum_{q^\gamma \leq p < (q+1)^\gamma} \lambda_{\text{sym}^2 f}(p) - \sum_{x^c - 1 < q \leq x^c} \sum_{x < p < (q+1)^\gamma} \lambda_{\text{sym}^2 f}(p) \\ &:= E_1 - E_2. \end{aligned} \quad (11)$$

For E_2 , there is at most one prime q_x satisfying $x^c - 1 < q_x \leq x^c$, hence

$$E_2 \ll \sum_{q_x^\gamma < p \leq (q_x+1)^\gamma} |\lambda_{\text{sym}^2 f}(p)| \ll (q_x + 1)^\gamma - q_x^\gamma \ll x^{1-c}, \quad (12)$$

if we notice that $|\lambda_{\text{sym}^2 f}(p)| \leq d_3(p) \ll 1$.

For E_1 , by the definition of the von Mangoldt function, we have

$$E_1 = \sum_{q \leq x^c} \sum_{q^\gamma \leq n < (q+1)^\gamma} \frac{\Lambda(n) \lambda_{\text{sym}^2 f}(n)}{\log n} + O \left(\sum_{q \leq x^c} \sum_{\substack{q^\gamma \leq n = p^h < (q+1)^\gamma \\ h \geq 2}} \frac{\lambda_{\text{sym}^2 f}(p^h) \log p}{\log p^h} \right). \quad (13)$$

The error term of the above formula contributes

$$\begin{aligned}
 & \ll \sum_{q \leq x^c} \sum_{\substack{q^\gamma \leq p^h < (q+1)^\gamma \\ h \geq 2}} |\lambda_{\text{sym}^2 f}(p^h)| \\
 &= \sum_{q \leq x^c} \sum_{q^\gamma \leq p^2 < (q+1)^\gamma} |\lambda_{\text{sym}^2 f}(p^2)| + \sum_{q \leq x^c} \sum_{\substack{q^\gamma \leq p^h < (q+1)^\gamma \\ h \geq 3}} |\lambda_{\text{sym}^2 f}(p^h)| \\
 &\ll \sum_{q \leq x^c} \sum_{q^\gamma \leq n^2 < (q+1)^\gamma} 1 + \sum_{q \leq x^c} \sum_{\substack{q^\gamma \leq n^h < (q+1)^\gamma \\ h \geq 3}} \log^2 x \\
 &\ll \sum_{q \leq x^c} \left((q+1)^{\gamma/2} - q^{\gamma/2} \right) + \sum_{q \leq x^c} \left((q+1)^{\gamma/3} - q^{\gamma/3} \right) \log^3 x \\
 &\ll \sum_{q \leq x^c} q^{\gamma/2-1} + \sum_{q \leq x^c} q^{\gamma/3-1} \log^3 x \ll x^{1/2}.
 \end{aligned}$$

We use the same method to deal with the main term of (13) and get

$$E_1 = \sum_{2 \leq m \leq x^c} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq n < (m+1)^\gamma} \frac{\Lambda(n) \lambda_{\text{sym}^2 f}(n)}{\log n} + O(x^{1-c/2}) + O(x^{1/2}), \quad (14)$$

where the first O -term comes from

$$\begin{aligned}
 & \sum_{\substack{m=q^h \leq x^c \\ h \geq 2}} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq n < (m+1)^\gamma} \frac{\Lambda(n) \lambda_{\text{sym}^2 f}(n)}{\log n} \\
 &= \sum_{\substack{m=q^h \leq x^c \\ h \geq 2}} \frac{\log q}{\log q^h} \sum_{m^\gamma \leq p^r < (m+1)^\gamma} \frac{\lambda_{\text{sym}^2 f}(p^r) \log p}{\log p^r} \\
 &\ll \sum_{\substack{m=q^h \leq x^c \\ h \geq 2}} \sum_{m^\gamma \leq p^r < (m+1)^\gamma} |\lambda_{\text{sym}^2 f}(p^r)| \ll x^{1-c/2}.
 \end{aligned}$$

Let

$$W(x) := \sum_{2 \leq m \leq x^c} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq n < (m+1)^\gamma} \frac{\Lambda(n) \lambda_{\text{sym}^2 f}(n)}{\log n}.$$

By the range of n and Taylor's formula, we have

$$\log n = \gamma \log m + O(m^{-1})$$

and

$$\frac{1}{\log n} = \frac{1}{\gamma \log m} + O\left(\frac{1}{m \log^2 m}\right).$$

Then

$$W(x) = \gamma^{-1} \sum_{2 \leq m \leq x^c} \frac{\Lambda(m)}{\log^2 m} \sum_{m^\gamma \leq n < (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}^2 f}(n) + O(x^{1-c} \log^3 x), \quad (15)$$

where the O -term comes from

$$\begin{aligned} & \sum_{2 \leq m \leq x^c} \frac{\Lambda(m)}{m \log^3 m} \sum_{m^\gamma \leq n < (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}^2 f}(n) \\ & \ll \log^3 x \sum_{2 \leq m \leq x^c} m^{\gamma-2} \log^{-3} m \\ & \ll x^{1-c} \log^3 x \sum_{2 \leq m \leq x^c} \frac{1}{m \log^3 m} \\ & \ll x^{1-c} \log^3 x. \end{aligned}$$

From (14), (15) and partial summation, we obtain

$$E_1 \ll \log^{-2} x \cdot \max_{x_0 \leq x} |\Psi_{2,c}(x_0)| + x^{1-c} \log^3 x + x^{1-c/2} + x^{1/2}, \quad (16)$$

where

$$\Psi_{2,c}(x) = \sum_{m \leq x^c} \Lambda(m) \sum_{m^\gamma \leq n < (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}^2 f}(n).$$

To get (3), it suffices to prove that

$$|\Psi_{2,c}(x)| \ll x \log^{-A} x, \text{ for any } A > 0. \quad (17)$$

The inequality (17) will be proved from a variant of (17) for short intervals. Let $x_0 := 2$, $x_{v+1} := x_v + x_v \log^{-2} x_v$, we can see that the inequality

$$|\Delta \Psi_{2,c}(x_v)| = \left| \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \sum_{m^\gamma \leq n < (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}^2 f}(n) \right| \ll x_v \log^{-A-2} x_v \quad (18)$$

implies (17). We use Lemma 4 and get

$$\sum_{m^\gamma \leq n < (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}^2 f}(n) = - \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T_v}} \frac{(m+1)^{\gamma\rho} - m^{\gamma\rho}}{\rho} + O\left(\log^3 m + \frac{(m+1)^\gamma \log^3(mT)}{T}\right). \quad (19)$$

Taking $T_v = x_v \log^{-2A-3} x_v$, then we obtain

$$|\Delta \Psi_{2,c}(x_v)| \ll |\Phi(\gamma, x_v, T_v)| + x_v^c \log^{2A+4} x_v, \quad (20)$$

where

$$\Phi(\gamma, x_v, T_v) = \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T_v}} \frac{(m+1)^{\gamma\rho} - m^{\gamma\rho}}{\rho}. \quad (21)$$

Under the Riemann Hypothesis, we have

$$\begin{aligned} \Phi(\gamma, x_v, T_v) &= \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \int_{m^\gamma}^{(m+1)^\gamma} \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T_v}} t^{-1/2+i\eta} dt \\ &= \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \int_{m^\gamma}^{(m+1)^\gamma} \sum_{\substack{\rho = \sigma + i\eta \\ |\eta| \leq T_v}} e^{(-1/2+i\eta) \log t} dt. \end{aligned}$$

Making the change of variables $u = \log t$, we deduce that

$$\Phi(\gamma, x_v, T_v) = \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \int_{\gamma \log m}^{\gamma \log(m+1)} e^{u/2} \sum_{\substack{\rho=\sigma+i\eta \\ |\eta| \leq T_v}} e^{i\eta u} du.$$

Using the Cauchy–Schwarz inequality twice, we get

$$\begin{aligned} |\Phi(\gamma, x_v, T_v)|^2 &\ll \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda^2(m) \sum_{x_v^c < m \leq x_{v+1}^c} \left| \int_{\gamma \log m}^{\gamma \log(m+1)} e^{u/2} \sum_{\substack{\rho=\sigma+i\eta \\ |\eta| \leq T_v}} e^{i\eta u} du \right|^2 \\ &\ll \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda^2(m) \sum_{x_v^c < m \leq x_{v+1}^c} m^{\gamma-1} \int_{\gamma \log m}^{\gamma \log(m+1)} \left| \sum_{\substack{\rho=\sigma+i\eta \\ |\eta| \leq T_v}} e^{i\eta u} \right|^2 du \quad (22) \\ &\ll x_v \log^{-1} x_v \cdot \int_{\log x_v}^{\gamma + \log x_{v+1}} \left| \sum_{\substack{\rho=\sigma+i\eta \\ |\eta| \leq T_v}} e^{i\eta u} \right|^2 du. \end{aligned}$$

Making the change of variables $t = u(2\pi)^{-1} - (\gamma + \log x_{v+1} + \log x_v)(4\pi)^{-1}$, we deduce that the last integral in (22) can be written as

$$2\pi \int_{-\alpha}^{\alpha} \left| \sum_{|\eta| \leq T_v} c(\eta) e(\eta t) \right|^2 dt, \quad (23)$$

where $\alpha = (\gamma + \log x_{v+1} - \log x_v)(4\pi)^{-1}$ and

$$c(\eta) = \begin{cases} e^{i\eta(\gamma + \log x_{v+1} + \log x_v)/2}, & \text{if } |\eta| \leq T_v; \\ 0, & \text{otherwise.} \end{cases}$$

Applying Lemma 1 with $\Theta = 1/2$ and $\delta = \Theta/\alpha \sim 2\pi/\gamma$ to estimate the integral in (23), we have

$$\begin{aligned} &\ll \int_{-\infty}^{\infty} \left| \delta^{-1} \sum_{\substack{t \leq \eta \leq t+\delta \\ |\eta| \leq T_v}} c(\eta) \right|^2 dt \\ &= \int_{-\infty}^{\infty} \delta^{-2} \sum_{\substack{t \leq \eta_1, \eta_2 \leq t+\delta \\ |\eta_1|, |\eta_2| \leq T_v}} e^{i(\eta_1 - \eta_2)(\gamma + \log x_{v+1} + \log x_v)/2} dt \\ &\ll \delta^{-2} \cdot \delta^2 \cdot T_v \ll x_v \log^{-2A-3} x_v. \end{aligned} \quad (24)$$

From (20), (22) and (24), we get

$$|\Delta \Psi_{2,c}(x_v)| \ll x_v \log^{-A-2} x_v + x_v^c \log^{2A+4} x_v.$$

This combined with (16) and (17) gives

$$E_1 \ll x \log^{-A-2} x. \quad (25)$$

Theorem 2 follows from (11), (12) and (25).

4. Proof of Theorem 1

To prove Theorem 1, we need Lemma 6 and Lemma 7. Furthermore, we only need to estimate $\Phi(\gamma, x_v, T_v)$ unconditionally. Note that

$$\Phi(\gamma, x_v, T_v) \ll \log x_v \cdot \max_{1 \leq T \leq T_v} |\Phi_1(\gamma, x_v, T)|,$$

where

$$\Phi_1(\gamma, x_v, T) = \sum_{x_v^c < m \leq x_{v+1}^c} \Lambda(m) \sum_{\substack{\rho = \sigma + i\eta \\ T/2 < |\eta| \leq T}} \frac{(m+1)^{\rho\gamma} - m^{\rho\gamma}}{\rho}.$$

We consider the following integral mean value of $\Phi_1(\gamma, x_v, T)$,

$$D(\gamma_0, x_v, T) = \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Phi_1(\gamma, x_v, T)|^2 d\gamma \quad (26)$$

with $\gamma_0 \in \left[\frac{1}{(\sqrt{5}+3)/8-\varepsilon}, \frac{1}{\varepsilon} \right]$.

From Lemma 6, we know that the upper bound of S depends on the range of T , so we have the following three cases.

Case 1. When $T \geq x_v^{4/5}$.

In this case, we have $x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < T$ and use (9) to get

$$\begin{aligned} D(\gamma_0, x_v, T_v) &\ll x_v^{2/\gamma_0} T^{-3} \left(x_v^{1/2} T^{5/4+\varepsilon} \right)^2 = x_v^{1+2/\gamma_0} T^{-1/2+2\varepsilon} \\ &\ll x_v^{1+3/2\gamma_0+2\varepsilon/\gamma_0+(1/2-2\varepsilon)\delta_1(\varepsilon)}. \end{aligned}$$

Note that if $1/\gamma_0 \leq (\sqrt{5}+3)/8-\varepsilon$, we have

$$D(\gamma_0, x_v, T) \ll x_v^{2-\varepsilon}.$$

Case 2. When $x_v^{9/20} \leq T < x_v^{4/5}$.

In case of $x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < T$, we use (9) and get

$$\begin{aligned} D(\gamma_0, x_v, T_v) &\ll x_v^{2/\gamma_0} T^{-3} \left(x_v^{3/2} T^{5/2+\varepsilon} e^{-\sqrt{5 \log x_v \log T}} \right)^2 \\ &= x_v^{3+2/\gamma_0} T^{2+2\varepsilon} e^{-2\sqrt{5 \log x_v \log T}} \\ &= e^{\left[(2+2\varepsilon) \left(\sqrt{\frac{\log T}{\log x_v}} \right)^2 - 2\sqrt{5 \frac{\log T}{\log x_v}} + 3 + \frac{2}{\gamma_0} \right] \log x_v} \\ &= x_v^{(2+2\varepsilon) \left(\sqrt{\frac{\log T}{\log x_v}} \right)^2 - 2\sqrt{5 \frac{\log T}{\log x_v}} + 3 + 2/\gamma_0}, \end{aligned}$$

For convenience, we take $t = \sqrt{\frac{\log T}{\log x_v}}$ and consider a quadratic function $g(t) = (2+2\varepsilon)t^2 - 2\sqrt{5}t + 3 + 2/\gamma_0$.

If $x_v^{1/\gamma_0 - \delta_1(\varepsilon)} \leq x_v^{9/20} \leq T$, we have $t \in [\sqrt{9/20}, \sqrt{4/5})$ and $g(t)$ is monotonically decreasing in this interval. Hence,

$$g(t) \leq g(\sqrt{9/20}) = 9(1+\varepsilon)/10 + 2/\gamma_0 \leq 9/5 + 9/10\varepsilon + 2\delta_1(\varepsilon)$$

with $1/\gamma_0 \leq 9/20 + \delta_1(\varepsilon)$. Therefore,

$$D(\gamma_0, x_v, T) \ll x_v^{2-\varepsilon}.$$

If $x_v^{9/20} < x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < T < x_v^{4/5}$, we get analogously $t \in (\sqrt{1/\gamma_0 - \delta_1(\varepsilon)}, \sqrt{4/5})$ and

$$\begin{aligned} g(t) &< g\left(\sqrt{1/\gamma_0 - \delta_1(\varepsilon)}\right) = 3 + 4/\gamma_0 - 2\sqrt{5(1/\gamma_0 - \delta_1(\varepsilon))} + 2\varepsilon/\gamma_0 - (2 + 2\varepsilon)\delta_1(\varepsilon) \\ &\leq 3 + 4/\gamma_0 - 2\sqrt{5}/\sqrt{\gamma_0} + 2\varepsilon/\gamma_0 + (\sqrt{10\gamma_0} - 2 - 2\varepsilon)\delta_1(\varepsilon) \\ &\leq 2 - \varepsilon, \end{aligned}$$

where we choose $\delta_1(\varepsilon)$ sufficiently small and use the elementary inequality

$$\sqrt{1-x} \geq 1 - x/\sqrt{2}, \text{ for } x \in [0, 1/2].$$

Therefore,

$$D(\gamma_0, x_v, T) \ll x_v^{2-\varepsilon}.$$

In case of $T \leq x_v^{1/\gamma_0 - \delta_1(\varepsilon)}$, we use (10) and get

$$\begin{aligned} D(\gamma_0, x_v, T) &\ll x_v^{3+2/\gamma_0} T^{2+2\varepsilon} e^{-2\sqrt{5 \log T \log x_v}} \left(T x_v^{-1/\gamma_0}\right)^2 \\ &= x_v^3 T^{4+2\varepsilon} e^{-2\sqrt{5 \log T \log x_v}} \\ &= x_v^{(4+2\varepsilon) \left(\sqrt{\frac{\log T}{\log x_v}}\right)^2 - 2\sqrt{5 \frac{\log T}{\log x_v}} + 3}, \end{aligned}$$

so we consider a new quadratic function $g_1(t) = (4 + 2\varepsilon)t^2 - 2\sqrt{5}t + 3$.

If $x_v^{9/20} \leq T \leq x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < x_v^{4/5}$, we get $t \in [\sqrt{9/20}, \sqrt{1/\gamma_0 - \delta_1(\varepsilon)}]$ and

$$\begin{aligned} g_1(t) &\leq g_1(1/\gamma_0 - \delta_1(\varepsilon)) = 4(1/\gamma_0 - \delta_1(\varepsilon)) - 2\sqrt{5} \cdot \sqrt{1/\gamma_0 - \delta_1(\varepsilon)} + 3 + 2\varepsilon/\gamma_0 - 2\varepsilon\delta_1(\varepsilon) \\ &\leq 4/\gamma_0 - 2\sqrt{5}/\sqrt{\gamma_0} + 3 + 2\varepsilon/\gamma_0 + (\sqrt{10\gamma_0} - 4 - 2\varepsilon)\delta_1(\varepsilon) \\ &\leq 2 - \varepsilon. \end{aligned}$$

Therefore,

$$D(\gamma_0, x_v, T) \ll x_v^{2-\varepsilon}.$$

Case 3. When $T < x_v^{9/20}$.

If $x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < T < x_v^{9/20}$, we use (9) to get

$$x_v^{2/\gamma_0} T^{-3} \left(x_v^{3/4} T^{5/6+\varepsilon}\right)^2 = x_v^{3/2+2/\gamma_0} T^{-4/3+2\varepsilon} \ll x_v^{2-\varepsilon}$$

and

$$x_v^{2/\gamma_0} T^{-3} \left(x_v^{1-\beta} T^{3\beta+\varepsilon}\right)^2 = x_v^{2-2\beta+2/\gamma_0} T^{-3+6\beta+2\varepsilon} \ll x_v^{2-\varepsilon}.$$

Therefore,

$$D(\gamma_0, x_v, T) \ll x_v^{2/\gamma_0} T^{-3} \left(x_v^{3/4} T^{5/6+\varepsilon} + x_v^{1-\beta} T^{3\beta+\varepsilon}\right)^2 \ll x_v^{2-\varepsilon}.$$

If $T \leq x_v^{1/\gamma_0 - \delta_1(\varepsilon)} < x_v^{9/20}$ or $T < x_v^{9/20} \leq x_v^{1/\gamma_0 - \delta_1(\varepsilon)}$, we use (10) to get

$$x_v^{2/\gamma_0} T^{-3} \left(x_v^{3/4} T^{5/6+\varepsilon}\right)^2 \left(T x_v^{-1/\gamma_0}\right)^2 \ll x_v^{3/2} T^{2/3+2\varepsilon} \ll x_v^{2-\varepsilon}$$

and

$$x_v^{2/\gamma_0} T^{-3} \left(x_v^{1-\beta} T^{3\beta+\varepsilon}\right)^2 \left(T x_v^{-1/\gamma_0}\right)^2 \ll x_v^{2-2\beta} T^{-1+6\beta+2\varepsilon} \ll x_v^{2-\varepsilon}.$$

Therefore,

$$D(\gamma_0, x_v, T) \ll x_v^{2/\gamma_0} T^{-3} \left(x_v^{3/4} T^{5/6+\varepsilon} + x_v^{1-\beta} T^{3\beta+\varepsilon} \right)^2 \left(T x_v^{-1/\gamma_0} \right)^2 \ll x_v^{2-\varepsilon}.$$

Combining all the above cases, we obtain

$$D(\gamma_0, x_v, T) = \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Phi_1(\gamma, x_v, T)|^2 d\gamma \ll x_v^{2-\varepsilon}.$$

Inequality (20) gives us

$$\begin{aligned} \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Psi_{2,c}(x)|^2 d\gamma &\ll \log^4 x \cdot \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Delta \Psi_{2,c}(x_v)|^2 d\gamma \\ &\ll \log^4 x \cdot \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Phi(\gamma, x_v, T_v)|^2 d\gamma + x_v^{2c} \log^{4A+10} x \\ &\ll \log^6 x \cdot \max_{1 \leq T \leq T_v} \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_v} |\Phi_1(\gamma, x_v, T)|^2 d\gamma + x_v^{2c} \log^{4A+10} x \\ &\ll x^{2-\varepsilon}. \end{aligned}$$

Let $\gamma_0 := \frac{1}{(\sqrt{5}+3)/8-\varepsilon}$, $\gamma_{i+1} := \gamma_i + \frac{1}{\log^2 x_v}$ and $\mathcal{A}_i := \{\gamma \in [\gamma_i, \gamma_{i+1}] / |\Psi_{2,c}(x)| > x \log^{-A} x\}$. Then by Tscheytschev's inequality, we obtain

$$\begin{aligned} \mu(\mathcal{A}_i) &\ll x^{-2} \log^{2A} x \cdot \int_{\gamma_i}^{\gamma_{i+1}} |\Psi_{2,c}(x)|^2 d\gamma \\ &\ll x^{-\varepsilon} \log^{2A} x \ll x^{-\varepsilon}. \end{aligned}$$

Author Contributions: Conceptualization, X.H. and X.Y.; Methodology, D.Z.; Software, X.H.; Validation, X.H., X.Y. and D.Z.; Formal Analysis, X.H.; Investigation, X.H.; Resources, X.Y.; Data Curation, D.Z.; Writing—Original Draft Preparation, X.H.; Writing—Review & Editing, X.Y.; Visualization, D.Z.; Supervision, D.Z.; Project Administration, D.Z.; Funding Acquisition, X.Y. and D.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 11771256 and No. 11801327), and Natural Science Foundation of Shandong Province (Grant No. ZR201709280100).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the referees for their many useful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Deligne, P. La conjecture de Weil. I. *Inst. Hautes Etudes Sci. Publ. Math.* **1974**, *43*, 273–307. [\[CrossRef\]](#)
2. Deligne, P. La conjecture de Weil. II. *Inst. Hautes Etudes Sci. Publ. Math.* **1980**, *52*, 137–252. [\[CrossRef\]](#)
3. Chandrasekharan, K.; Narasimhan, R. Functional equations with multiple gamma factors and the average order of arithmetical functions. *Ann. Math.* **1962**, *76*, 93–136. [\[CrossRef\]](#)
4. Davenport, H. On certain exponential sums. *J. Reine Angew. Math.* **1933**, *169*, 158–176.
5. Hafner, J.L.; Ivić, A. On sums of Fourier coefficients of cusp forms. *Enseign. Math.* **1989**, *35*, 373–382.
6. Hecke, E. Theorie der Eisensteinsche Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik. *Abh. Math. Sem. Univ. Hamburg* **1927**, *5*, 199–224. [\[CrossRef\]](#)
7. Jiang, Y.J.; Lü, G.S.; Yan, X.F. Mean value theorem connected with Fourier coefficients of Hecke-Maass forms for $SL(m, \mathbb{Z})$. *Math. Proc. Camb. Philos. Soc.* **2016**, *161*, 339–356. [\[CrossRef\]](#)

8. Kloosterman, H.D. Asymptotische Formeln für die Fourier-koeffizienten ganzer Modulformen. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*; Springer: Berlin/Heidelberg, Germany, 1927; Volume 5, pp. 337–353.
9. Lao, H.X. The cancellation of Fourier coefficients of cusp forms over different sparse sequences. *Acta Math. Sin. (Engl. Ser.)* **2013**, *29*, 1963–1972. [[CrossRef](#)]
10. Lao, H.X. On comparing Hecke eigenvalues of cusp forms. *Acta Math. Hungar.* **2020**, *160*, 58–71. [[CrossRef](#)]
11. Liu, H.F.; Li, S.; Zhang, D.Y. Power moments of automorphic L-function attached to Maass forms. *Int. J. Number Theory* **2016**, *12*, 427–443. [[CrossRef](#)]
12. Liu, H.F.; Zhang, R. Some problems involving Hecke eigenvalues. *Acta Math. Hungar.* **2019**, *159*, 287–298. [[CrossRef](#)]
13. Rankin, R.A. *Sums of Cusp Form Coefficients*; Automorphic forms and analytic number theory (Montreal, PQ, 1989); University Montreal: Montreal, QC, Canada, 1990; pp. 115–121.
14. Saljé, H. Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen. *Math. Z.* **1933**, *36*, 263–278.
15. Song, P.; Zhai, W.G.; Zhang, D.Y. Power moments of Hecke eigenvalues for congruence group. *J. Number Theory* **2019**, *198*, 139–158. [[CrossRef](#)]
16. Walfisz, A. Über die Koeffizientensummen einiger Moduformen. *Math. Ann.* **1933**, *108*, 75–90. [[CrossRef](#)]
17. Weil, A. On some exponential sums. *Proc. Nat. Acad. Sci. USA* **1948**, *34*, 204–207. [[CrossRef](#)]
18. Wilton, J.R. A note on Ramanujans arithmetical function $\tau(n)$. *Proc. Cambridge Philos. Soc.* **1929**, *25*, 121–129. [[CrossRef](#)]
19. Wu, J. Power sums of Hecke eigenvalues and applications. *Acta Arith.* **2009**, *137*, 333–344. [[CrossRef](#)]
20. Zhang, D.Y.; Lau, Y-K.; Wang, Y.N. Remark on the paper “On products of Fourier coefficients of cusp forms”. *Arch. Math. (Basel)* **2017**, *108*, 263–269. [[CrossRef](#)]
21. Zhang, D.Y.; Wang, Y.N. Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$. *Ramanujan J.* **2018**, *47*, 685–700. [[CrossRef](#)]
22. Zhang, D.Y.; Wang, Y.N. Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form. *J. Number Theory* **2017**, *176*, 211–225. [[CrossRef](#)]
23. Iwaniec, H.; Kowalski, E. *Analytic Number Theory*; American Mathematical Society Colloquium Publications; American Mathematical Society: Providence, RI, USA, 2004; Volume 53.
24. Baier, S.; Zhao, L.Y. On Hecke eigenvalues at Piatetski-Shapiro primes. *J. Lond. Math. Soc.* **2010**, *81*, 175–201. [[CrossRef](#)]
25. Balog, A. On a variant of the Pjateckij-Šapiro prime number problem. In *Groupe de Travail en Théorie Analytique et Élémentaire des Nombres, 1987–1988*; Publ. Math. Orsay, 89-01; Université Paris-Saclay: Orsay, France, 1989; pp. 3–11.
26. Dufner, G. Pjateckij-Shapiro prime twins. *J. Number Theory* **1993**, *43*, 370–378. [[CrossRef](#)]
27. Zhang, D.Y.; Zhai, W.G. On the distribution of Hecke eigenvalues over Piatetski-Shapiro prime twins. *Acta Math. Sin. (Engl. Ser.)* **2021**. [[CrossRef](#)]
28. Fomenko, O.M. Identities Involving Coefficients of Automorphic L-Functions. *J. Math. Sci.* **2006**, *133*, 1749–1755. [[CrossRef](#)]
29. Huang, J.; Liu, H.F.; Xu, F.X. Two-Dimensional Divisor Problems Related to Symmetric L-Functions, *Symmetry* **2021**, *13*, 359. [[CrossRef](#)]
30. Ichihara, Y. Estimates of a certain sum involving coefficients of cusp forms in weight and level aspects. *Lith. Math. J.* **2008**, *48*, 188–202. [[CrossRef](#)]
31. Jiang, Y.J.; Lü, G.S. Uniform estimates for sums of coefficients of symmetric square L-function. *J. Number Theory* **2015**, *148*, 220–234. [[CrossRef](#)]
32. Lao, H.X. On the fourth moment of coefficients of symmetric square L-function. *Chin. Ann. Math. Ser. B* **2012**, *33*, 877–888. [[CrossRef](#)]
33. Lü, G.S. Uniform estimates for sums of Fourier coefficients of cusp forms. *Acta Math. Hungar.* **2009**, *124*, 83–97. [[CrossRef](#)]
34. Sankaranarayanan, A. On a sum involving Fourier coefficients of cusp forms. *Lithuanian Math. J.* **2006**, *46*, 459–474. [[CrossRef](#)]
35. Tang, H.C. A note on the Fourier coefficients of Hecke Maass forms. *J. Number Theory* **2013**, *133*, 1156–1167. [[CrossRef](#)]
36. Zhang, R.; Han, X.; Zhang, D.Y. Power moments of the Riesz mean error term of symmetric square L-function in short intervals. *Symmetry* **2020**, *12*, 2036. [[CrossRef](#)]
37. Gallagher, P.X. A large sieve density estimate near $\sigma = 1$. *Invent. Math.* **1970**, *11*, 329–339. [[CrossRef](#)]
38. Ye, Y.B.; Zhang, D.Y. Zero density for automorphic L-functions. *J. Number Theory* **2013**, *133*, 3877–3901. [[CrossRef](#)]
39. Dong, L.L.; Liu, H.F.; Zhang, D.Y. Zero density estimates for automorphic L-functions of $SL(2, \mathbb{Z})$. *Acta Math. Hungar.* **2016**, *148*, 191–210. [[CrossRef](#)]