

Article

On New Classes of Stancu-Kantorovich-Type Operators

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Abstract: The present paper introduces new classes of Stancu–Kantorovich operators constructed in the King sense. For these classes of operators, we establish some convergence results, error estimations theorems and graphical properties of approximation for the classes considered, namely, operators that preserve the test functions $e_0(x) = 1$ and $e_1(x) = x$, $e_0(x) = 1$ and $e_2(x) = x^2$, as well as $e_1(x) = x$ and $e_2(x) = x^2$. The class of operators that preserve the test functions $e_1(x) = x$ and $e_2(x) = x^2$ is a genuine generalization of the class introduced by Indrea et al. in their paper “A New Class of Kantorovich-Type Operators”, published in Constr. Math. Anal.

Keywords: Stancu operators; Kantorovich operators; Stancu–Kantorovich operators; King-type operators; approximation by positive linear operators

MSC: 41A36; 41A25



Citation: Vasian, B.I.; Garoiu, Ș.L.; Păcurar, C.M. On New Classes of Stancu-Kantorovich-Type Operators. *Mathematics* **2021**, *9*, 1235. <https://doi.org/10.3390/math9111235>

Academic Editor:
Panagiotis-Christos Vassiliou

Received: 14 April 2021

Accepted: 25 May 2021

Published: 28 May 2021

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1. Introduction

By $C[0, 1]$, we denote the space of continuous functions defined on $[0, 1]$, and by $L_1[0, 1]$, the space of all functions defined on $[0, 1]$, which are Lebesgue integrable. Let $\mathbb{N}$ be the set of all positive integers.

We consider $e_j(t) = t^j$ for $t \in [0, 1]$, $j \in \mathbb{N}$.

In order to prove Weierstrass's approximation theorem [1], S.N. Bernstein introduced the Bernstein operators in paper [2], which, for $f \in C[0, 1]$, are defined as:

$$B_n(f, x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, for every $m \in \mathbb{N}$, and $p_{m,k}(x) = 0$ if $k < 0$ or $k > m$.

Let $\alpha, \beta \geq 0$ and $\alpha \leq \beta$. D. D. Stancu introduced in paper [3] the following operators, which are a generalization of the well-known Bernstein operators (see [2]), $S_m^{(\alpha, \beta)} : C\left[\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}\right] \rightarrow C[0, 1]$ defined as:

$$S_m^{(\alpha, \beta)}(f, x) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \quad x \in [0, 1],$$

where $p_{m,k}(x)$, $m \in \mathbb{N}$, are defined as before, and $f \in C[0, 1]$.

The Bernstein operators have been intensively studied, and many generalizations were considered, one of which is the Kantorovich variant, due to L.V. Kantorovich, from 1930 (see [4]). The Kantorovich operators $K_m : L_1[0, 1] \rightarrow C[0, 1]$ are positive linear operators and can be defined as:

$$K_m(f, x) = (m+1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s) ds, \quad x \in [0, 1],$$

where $p_{m,k}(x)$, $m \in \mathbb{N}$, are the Bernstein basis, and $f \in L_1[0, 1]$.

Following the generalization of Bernstein operators proposed by Stancu, D. Bărbosu introduced in paper [5] the Kantorovich variant of Stancu–Bernstein operators, which, for $f \in L_1\left[\frac{\alpha}{m+\beta+1}, \frac{m+\alpha+1}{m+\beta+1}\right]$, is defined as:

$$K_m^{(\alpha, \beta)}(f, x) = (m + \beta + 1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s) ds, \quad x \in [0, 1],$$

with $0 \leq \alpha \leq \beta$, $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, for every $m \in \mathbb{N}$ and $f \in L_1[0, 1]$

The study of Kantorovich operators is still in the spotlight of many recent research papers (see [6–8]). Among the numerous generalizations of the Kantorovich–Bernstein type operators, we mention the one by Indrea et al., (see [9]), which introduces a new general class which preserves the test functions $e_1(x)$ and $e_2(x)$.

In [10], J.P. King introduced a new class of positive linear Bernstein-type operators which reproduce constant functions ($e_1(x)$) and $e_2(x)$). These operators are a generalization of the Bernstein operators, but they are not polynomial-type operators. With the results introduced by King, a new direction of research was initiated, which concerns the construction of new operators with better approximation properties, obtained by modifying existing sequences of linear positive operators. This subject has been one of great interest. Gonska and Pişul (see [11]) studied estimates in terms of the first and second moduli of continuity for the operators introduced by King. Among the first generalizations of King's result, we mention those of Agratini (see [12]), Cardena-Morales et al., (see [13]), Duman and Özarslan (see [14,15]) and Gonska et al., (see [16]). The subject is still of interest. Among the more recent studies, we mention the one by Popa (see [17]) where Voronovskaja-type theorems for King operators are studied. A recent extensive review of King type operators is that of Acar et al. (see [18]).

Based on the results in [3,9,19,20], we introduce three new classes of King-type approximation operators. The aim of our paper is to obtain convergence properties of the uniform approximation of continuous functions using a Korovkin-type theorem (see [21]). Our results are a generalization of previous results on the topic.

The article is structured as follows. Section 2 presents some known results and notions that are to be used throughout the paper. In Section 3, we introduce the general form of our operators and some properties they satisfy. In Sections 4–6, we introduced three new classes of operators, which preserve exactly two of the test functions e_i , $i = 0, 1, 2$. These operators are particular variants of the operator considered in Section 3.

2. Preliminaries

In the following, we present the notions and results that will be used to prove the main results of the paper.

We will denote by $\mathcal{F}(I)$ the set of all functions defined on $I \subset \mathbb{R}$.

Definition 1. Let I and J be two intervals of $\mathbb{R}$, such that $I \cap J \neq \emptyset$. For $x \in I$, let $\mu_x : I \rightarrow \mathbb{R}$, $\mu_x(t) = t - x$, $t \in I$. For any $m \in \mathbb{N}$, we set the functions $\phi_{m,k} : J \rightarrow \mathbb{R}$, such that $\phi_{m,k}(x) \geq 0$ for every $x \in J$, $k \in \{0, 1, \dots, m\}$ and the positive linear functionals $\Phi_{m,k} : U(I) \subset \mathcal{F}(I) \rightarrow \mathbb{R}$, $k \in \{0, 1, \dots, m\}$, where $U(I)$ is a linear subspace. For $m \in \mathbb{N}$ we define the operator $L_m : U(I) \subset \mathcal{F}(I) \rightarrow V(J) \subset \mathcal{F}(J)$ as

$$L_m(f, x) = \sum_{k=0}^m \phi_{m,k}(x) \Phi_{m,k}(f), \quad x \in J, \quad f \in U(I). \quad (1)$$

Remark 1. The operators $(L_m)_{m \in \mathbb{N}}$ defined above are linear and positive on $U(I)$.

Definition 2. For $i \in \mathbb{N}$ we denote by $(\Gamma_i L_m)$ the moments of order i of the operators defined in (1):

$$(\Gamma_i L_m)(x) = L_m\left((e_1 - x)^i, x\right) = \sum_{k=0}^m \phi_{m,k}(x) \Phi_{m,k}\left((e_1 - x)^i\right). \quad (2)$$

Definition 3. Let $I \subset \mathbb{R}$ be a compact interval and f be continuous function on I . The modulus of continuity is a function $\omega(f, \cdot) : [0, \infty) \rightarrow \mathbb{R}$ defined for any $h \geq 0$,

$$\omega(f, h) = \sup\{|f(x) - f(t)| : x, t \in I, |x - t| \leq h\}.$$

Now, let us recall the well-known result by Shisha and Mond (see [22]).

Theorem 1. Let L be a linear positive operator on I , a compact interval. If f is a continuous function on I , then for every $x \in I$ and every $h > 0$, one has

$$|L(f, x) - f(x)| \leq |f(x)| \cdot |L(e_0, x) - 1| + \left(L(e_0, x) + \frac{1}{h} \sqrt{L(e_0, x) \cdot L((e_0 - x)^2, x)}\right) \omega(f, h).$$

Other recent evaluations with moduli of continuity can be seen in [23].

3. A General Method for Constructing New Classes of Stancu-Type Operators

In this section, we consider a general method of constructing new types of Stancu operators, namely Stancu–Kantorovich operators with King modification. In the following sections, we will construct three new classes of such operators and we will study some properties of approximation for these operators, taking into account their expressions on the test functions $e_i(t) = t^i$, $i = 0, 1, 2$, and imposing that the operators preserve two of the test functions e_i and e_j , $i \neq j$, $i, j \in \{0, 1, 2\}$. A motivation for this type of modification of the operators is, as pointed out, for instance, by Acar et al., in [18], finding better properties of approximation and improved error estimates. Our approach is influenced by the modifications of Bernstein–Kantorovich operators considered in [6,9]. With this in mind, let us introduce the following operator.

Definition 4. Let I be a compact interval and $c_m, d_m : I \rightarrow \mathbb{R}$ be some functions that satisfy $c_m(x) \geq 0$, $d_m(x) \geq 0$ for all $x \in I$, $0 \leq \alpha \leq \beta$ and $m \in \mathbb{N}$. We define the following Stancu–Kantorovich type operators:

$$S_m^{(\alpha, \beta)*}(f, x) = (m + \beta + 1) \sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t) dt \quad (3)$$

for any $x \in I$, $m \in \mathbb{N}$ and $f \in L_1\left[\frac{\alpha}{m+\beta+1}, \frac{m+\alpha+1}{m+\beta+1}\right]$.

Lemma 1. The operator proposed in relation (3) has the following properties

$$S_m^{(\alpha, \beta)*}(e_0, x) = (c_m(x) + d_m(x))^m, \quad (4)$$

$$S_m^{(\alpha, \beta)*}(e_1, x) = \frac{m}{m + \beta + 1} c_m(x) (c_m(x) + d_m(x))^{m-1} + \frac{2\alpha + 1}{2(m + \beta + 1)} (c_m(x) + d_m(x))^m, \quad (5)$$

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_2, x) &= \frac{m(m-1)}{(m+\beta+1)^2} c_m^2(x) (c_m(x) + d_m(x))^{m-2} \\
&+ \frac{2m(1+\alpha)}{(m+\beta+1)^2} c_m(x) (c_m(x) + d_m(x))^{m-1} \\
&+ \frac{3\alpha(\alpha+1)+1}{3(m+\beta+1)^2} (c_m(x) + d_m(x))^m
\end{aligned} \quad (6)$$

for any $x \in I$, $m \in \mathbb{N}$, where $e_i(t) = t^i, i \in \{0, 1, 2\}$.

Proof. For $e_0(x) = 1$, we have:

$$\begin{aligned}
&S_m^{(\alpha, \beta)*}(e_0, x) = \\
&(m+\beta+1) \sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{m+\beta+1},
\end{aligned}$$

which, from the binomial theorem, yields:

$$S_m^{(\alpha, \beta)*}(e_0, x) = (c_m(x) + d_m(x))^m.$$

Now, let us evaluate $S_m^{(\alpha, \beta)*}$ for $e_1(x) = x$:

$$\begin{aligned}
&S_m^{(\alpha, \beta)*}(e_1, x) = \\
&(m+\beta+1) \sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{2} \left[\left(\frac{k+\alpha+1}{m+\beta+1} \right)^2 - \left(\frac{k+\alpha}{m+\beta+1} \right)^2 \right],
\end{aligned}$$

which is

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_1, x) &= \frac{1}{m+\beta+1} \sum_{k=0}^m \binom{m}{k} k (c_m(x))^k (d_m(x))^{m-k} \\
&+ \frac{2\alpha+1}{2(m+\beta+1)} (c_m(x) + d_m(x))^m.
\end{aligned}$$

Denoting $k-1 = l$ in the sum from the right hand side in the above relation, we get:

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_1, x) &= \frac{m}{m+\beta+1} c_m(x) \sum_{l=0}^{m-1} \binom{m-1}{l} (c_m(x))^l (d_m(x))^{m-l-1} \\
&+ \frac{2\alpha+1}{m+\beta+1} (c_m(x) + d_m(x))^m
\end{aligned}$$

and, again, by the binomial theorem, we get:

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_1, x) &= \frac{m}{m+\beta+1} c_m(x) (c_m(x) + d_m(x))^{m-1} \\
&+ \frac{2\alpha+1}{m+\beta+1} (c_m(x) + d_m(x))^m.
\end{aligned}$$

Lastly, we shall compute $S_m^{(\alpha, \beta)*}$ for $e_2(x) = x^2$:

$$\begin{aligned}
&S_m^{(\alpha, \beta)*}(e_2, x) = \\
&(m+\beta+1) \sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{3} \left[\left(\frac{k+\alpha+1}{m+\beta+1} \right)^3 - \left(\frac{k+\alpha}{m+\beta+1} \right)^3 \right]
\end{aligned}$$

Now, by doing the calculations in the square brackets, we get:

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_2, x) &= \frac{1}{(m + \beta + 1)^2} \sum_{k=0}^m \binom{m}{k} k^2 (c_m(x))^k (d_m(x))^{m-k} \\
&+ \frac{2\alpha + 1}{(m + \beta + 1)^2} \sum_{k=0}^m \binom{m}{k} k (c_m(x))^k (d_m(x))^{m-k} \\
&+ \frac{3\alpha(\alpha + 1) + 1}{3(m + \beta + 1)^2} (c_m(x) + d_m(x))^m.
\end{aligned}$$

By denoting $k - 2 = l$ in the first sum from the right hand side in the relation from above and by the binomial theorem, we will have:

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_2, x) &= \frac{m(m-1)}{(m + \beta + 1)^2} c_m^2(x) (c_m(x) + d_m(x))^{m-2} \\
&+ \frac{2m(\alpha + 1)}{(m + \beta + 1)^2} c_m(x) (c_m(x) + d_m(x))^{m-1} \\
&+ \frac{3\alpha(\alpha + 1) + 1}{3(m + \beta + 1)^2} (c_m(x) + d_m(x))^m,
\end{aligned}$$

which completes the proof. $\square$

4. Stancu–Kantorovich Type Operators Which Preserve the Functions e_0 and e_1

In this section, we shall construct an operator of Stancu–Kantorovich type as in (3), that preserves the test functions e_0 and e_1 , i.e., an operator that satisfies

$$\begin{aligned}
S_m^{(\alpha, \beta)*}(e_0, x) &= 1 \\
S_m^{(\alpha, \beta)*}(e_1, x) &= x \\
S_m^{(\alpha, \beta)*}(e_2, x) &\xrightarrow{u} x^2.
\end{aligned} \tag{7}$$

Now, from the conditions in (7) and relations (4), (5) we get

$$c_m(x) = \frac{m + \beta + 1}{m} x - \frac{2\alpha + 1}{m}, \tag{8}$$

and

$$d_m(x) = \frac{-(m + \beta + 1)x}{m} + \frac{m + 2\alpha + 1}{m}, \tag{9}$$

for any $m \in \mathbb{N}$ and $x \in I$.

In order to have a positive operator, we shall assume that the functions c_m and d_m are positive. This condition yields the following inequality

$$\frac{2\alpha + 1}{m + \beta + 1} \leq x \leq \frac{m + 2\alpha + 1}{m + \beta + 1}, \forall x \in I, \forall m \in \mathbb{N}.$$

Lemma 2. For $0 \leq 2\alpha \leq \beta$ and any integers $m_0 < m$, we have

$$\left[\frac{2\alpha + 1}{m_0 + \beta + 1}; \frac{m_0 + 2\alpha + 1}{m_0 + \beta + 1} \right] \subset \left[\frac{2\alpha + 1}{m + \beta + 1}; \frac{m + 2\alpha + 1}{m + \beta + 1} \right].$$

Proof. Let us consider the sequences $(x_m)_{m \geq 1}$,

$$x_m = \frac{2\alpha + 1}{m + \beta + 1}$$

and $(y_m)_{m \geq 1}$,

$$y_m = \frac{m + 2\alpha + 1}{m + \beta + 1}.$$

Imposing the condition $0 \leq 2\alpha \leq \beta$ we have that $(x_m)_{m \geq 1}$ is a decreasing sequence, and $(y_m)_{m \geq 1}$ is an increasing sequence, thus implying that our inclusion holds for any $m \geq m_0$. $\square$

Remark 2. From now on, we will consider $0 \leq 2\alpha \leq \beta$.

Remark 3. Since on the interval $\left[\frac{2\alpha+1}{m_0+\beta+1}; \frac{m_0+2\alpha+1}{m_0+\beta+1}\right]$ we have that $c_m, d_m \geq 0$, for every $m \in \mathbb{N}$, we will consider $I = \left[\frac{2\alpha+1}{m_0+\beta+1}; \frac{m_0+2\alpha+1}{m_0+\beta+1}\right]$, where m_0 is a positive integer which is arbitrarily fixed. Note that for any $\varepsilon > 0$, if we make m_0 sufficiently large, then $[\varepsilon, 1 - \varepsilon] \subset I$.

Now, taking into account the sequences c_m and d_m obtained in (8) and (9), the operator in (3) will be

$$S_{1,m}^{(\alpha,\beta)*}(f, x) = (m + \beta + 1) \sum_{k=0}^m \binom{m}{k} \left(\frac{m + \beta + 1}{m} x - \frac{2\alpha + 1}{m} \right)^k \times \left(\frac{-(m + \beta + 1)x}{m} + \frac{m + 2\alpha + 1}{m} \right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t) dt, \quad (10)$$

for any $x \in I$.

Lemma 3. The operator $S_{1,m}^{(\alpha,\beta)*}$ from (10) satisfies

$$\begin{aligned} S_{1,m}^{(\alpha,\beta)*}(e_0, x) &= 1; \\ S_{1,m}^{(\alpha,\beta)*}(e_1, x) &= x; \\ S_{1,m}^{(\alpha,\beta)*}(e_2, x) &= \frac{m-1}{m} x^2 + \frac{m+2\alpha-1}{m(m+\beta+1)} x + \frac{12\alpha(\alpha+1) - m(5-36\alpha) - 3}{12m(m+\beta+1)^2} \end{aligned} \quad (11)$$

for $x \in I$.

Proof. The first two relations from (11) are obvious, and the third follows by applying relations (5), (8) and (9) and after some computations. $\square$

Lemma 4. The following relations hold

$$\left(\Gamma_0 S_{1,m}^{(\alpha,\beta)*} \right)(x) = 1, \quad (12)$$

$$\left(\Gamma_1 S_{1,m}^{(\alpha,\beta)*} \right)(x) = 0, \quad (13)$$

$$\begin{aligned} \left(\Gamma_2 S_{1,m}^{(\alpha,\beta)*} \right)(x) &= \frac{12m(x - x^2) + (-24x^2(\beta + 1) + 12x(2\alpha + \beta) + 36\alpha - 5)}{12(m + \beta + 1)^2} \\ &+ \frac{-12x^2(\beta + 1)^2 + 12x(2\alpha\beta + 2\alpha - \beta - 1) + 3(4\alpha^2 + 4\alpha - 1)}{12m(m + \beta + 1)^2}. \end{aligned} \quad (14)$$

Proof. Using the previous lemma and relation (2), we get

$$\begin{aligned} \left(\Gamma_0 S_{1,m}^{(\alpha,\beta)*} \right)(x) &= 1, \\ \left(\Gamma_1 S_{1,m}^{(\alpha,\beta)*} \right)(x) &= \left[S_{1,m}^{(\alpha,\beta)*}(e_1, x) - x S_{1,m}^{(\alpha,\beta)*}(e_0, x) \right] = 0 \end{aligned}$$

and

$$\begin{aligned} (\Gamma_2 S_{1,m}^{(\alpha,\beta)*})(x) &= S_{1,m}^{(\alpha,\beta)*}(e_2, x) - 2x S_{1,m}^{(\alpha,\beta)*}(e_1, x) + x^2 S_{1,m}^{(\alpha,\beta)*}(e_0, x) \\ &= \frac{m-1}{m} x^2 + \frac{m+2\alpha-1}{m(m+\beta+1)} x + \frac{12\alpha(\alpha+1) - m(5-36\alpha) - 3}{12m(m+\beta+1)^2} - x^2. \end{aligned}$$

which, after some calculations, yields (14). $\square$

Lemma 5. We have

$$\lim_{m \rightarrow \infty} m (\Gamma_2 S_{1,m}^{(\alpha,\beta)*})(x) = x(1-x) \quad (15)$$

uniformly with respect to $x \in I$. Consequently, for any $\varepsilon > 0$, there exists an integer $m_\varepsilon \geq m_0$, sufficiently large, such that

$$(\Gamma_2 S_{1,m}^{(\alpha,\beta)*})(x) \leq \frac{\frac{1}{4} + \varepsilon}{m}, \quad (16)$$

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_\varepsilon$.

Proof. The relation (15) follows from (12) and (14). The existence of m_ε follows from the definition of the limit of a function and the inequality (16) follows from (15) by applying the inequality $x(1-x) \leq \frac{1}{4}$, which is true for every $x \in [0, 1]$. $\square$

Theorem 2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function on $[0, 1]$. Then, we have

$$\lim_{m \rightarrow \infty} S_{1,m}^{(\alpha,\beta)*} f = f$$

uniformly on I and for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that

$$|S_{1,m}^{(\alpha,\beta)*}(f, x) - f(x)| \leq \left(1 + \sqrt{\frac{1}{4} + \varepsilon}\right) \omega\left(f; \frac{1}{\sqrt{m}}\right),$$

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_\varepsilon$.

Proof. The Theorem from above follows from Theorem 1 by taking $h = \frac{1}{\sqrt{m}}$. $\square$

Graphic Properties of Approximation

Remark 4. It can be seen in both Figures 1 and 2 that our operators approximate the given functions for α and β chosen such that $\alpha \leq 2\beta$.

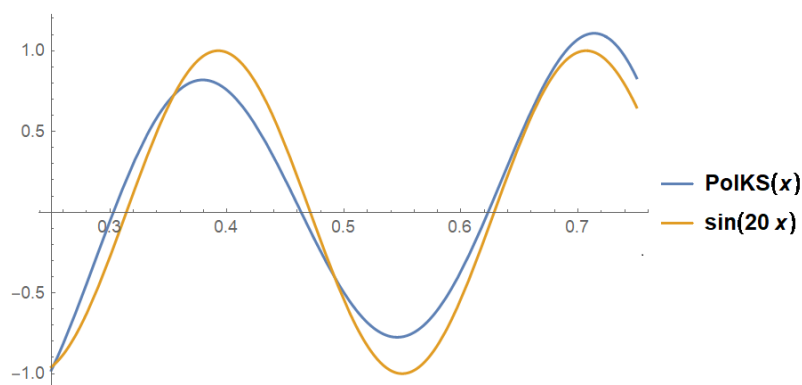


Figure 1. $\alpha = 0.1$, $\beta = 0.2$, $m = 25$ iterations.

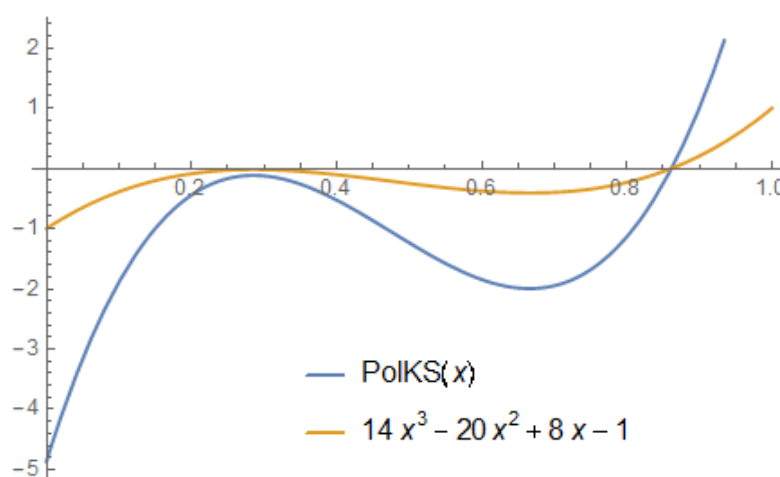


Figure 2. $\alpha = 0.1, \beta = 0.5, m = 500$ iterations.

5. Stancu–Kantorovich-Type Operators Which Preserve the Functions e_0 and e_2

In this section, we shall construct an operator of Stancu–Kantorovich-type, as in (3), that preserves the test functions e_0 and e_2 , i.e., an operator that satisfies

$$\begin{aligned} S_m^{(\alpha, \beta)*}(e_0, x) &= 1 \\ S_m^{(\alpha, \beta)*}(e_2, x) &= x^2 \\ S_m^{(\alpha, \beta)*}(e_1, x) &\xrightarrow{u} x. \end{aligned} \quad (17)$$

Now, imposing the condition (17) and the Equations (4) and (6), we get:

$$c_m(x) + d_m(x) = 1, \quad \forall x \in I, \quad m \in \mathbb{N}, \quad (18)$$

and the following quadratic equation, in $c_m(x)$:

$$m(m-1)c_m^2(x) + 2m(1+\alpha)c_m(x) + \alpha(\alpha+1) + \frac{1}{3} = x^2(m+\beta+1)^2, \quad \forall x \in I, \quad m \in \mathbb{N}. \quad (19)$$

Note that for $\alpha \geq 0, \beta \geq 0$, the discriminant

$$\delta_m(x) = 4m \left[m \left(\frac{2}{3} + \alpha \right) + \alpha^2 + \alpha + \frac{1}{3} + x^2(m-1)(m+\beta+1)^2 \right] \quad (20)$$

of the quadratic Equation (19) is positive.

We make the following notation:

$$\Delta_m(x) = \frac{\delta_m(x)}{4}.$$

Now, solving Equation (19) we obtain, for $m \geq 2$:

$$c_m(x) = \frac{-m(1+\alpha) + \sqrt{\Delta_m(x)}}{m(m-1)} \quad (21)$$

and, from relation (18) we get

$$d_m(x) = \frac{m(m+\alpha) - \sqrt{\Delta_m(x)}}{m(m-1)}. \quad (22)$$

In order to have positive linear operators, we shall impose that the functions c_m and d_m from (21) and (22) are positive. In this case, we obtain the following inequalities:

$$\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \leq x \leq \frac{\sqrt{m(m + 2\alpha + 1) + \alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1}.$$

Lemma 6. Let $0 < \epsilon' < \frac{1}{2}$ be fixed. There is an integer $m_0 \in \mathbb{N}$ such that

$$[\epsilon', 1 - \epsilon'] \subset \left[\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1}; \frac{\sqrt{m(m + 2\alpha + 1) + \alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \right], \quad (23)$$

for every $m \in \mathbb{N}$ such that $m \geq m_\epsilon$ and α, β , satisfying $\alpha \leq \beta$.

Proof. We have that

$$\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \rightarrow 0$$

and

$$\frac{\sqrt{m(m + 2\alpha + 1) + \alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \rightarrow 1,$$

therefore, for all $\epsilon > 0$, the inclusion (23) holds. $\square$

Remark 5. Since the functions c_m and d_m are positive on the interval considered in (23), from now on, we will consider $I = [\epsilon', 1 - \epsilon']$, for all $\epsilon' > 0$ and $m \geq m_0$.

Now, taking into account the sequences c_m and d_m obtained in (21) and (22), the operator in (3) will be

$$S_{2,m}^{(\alpha,\beta)*}(f, x) = \frac{m + \beta + 1}{(m(m - 1))^m} \sum_{k=0}^m \binom{m}{k} \left(-m(1 + \alpha) + \sqrt{\Delta_m(x)} \right)^k \times \left(m(m + \alpha) - \sqrt{\Delta_m(x)} \right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t) dt, \quad (24)$$

for any $x \in I$ and $m \geq m_0$.

Lemma 7. The operator $S_{2,m}^{(\alpha,\beta)*}$ from (24) satisfies

$$\begin{aligned} S_{2,m}^{(\alpha,\beta)*}(e_0, x) &= 1; \\ S_{2,m}^{(\alpha,\beta)*}(e_1, x) &= \frac{-(m + 2\alpha + 1) + 2\sqrt{\Delta_m(x)}}{2(m + \beta + 1)(m - 1)}; \\ S_{2,m}^{(\alpha,\beta)*}(e_2, x) &= x^2. \end{aligned} \quad (25)$$

for $x \in I$ and $m \geq m_0$.

Proof. The first and last relation from (25) are obvious, and the second follows by applying relations (5) and (21). $\square$

Now, we can obtain the following result.

Lemma 8. The following relations hold

$$\left(\Gamma_0 S_{2,m}^{(\alpha,\beta)*}\right)(x) = 1, \quad (26)$$

$$\left(\Gamma_1 S_{2,m}^{(\alpha,\beta)*}\right)(x) = x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)}, \quad (27)$$

$$\left(\Gamma_2 S_{2,m}^{(\alpha,\beta)*}\right)(x) = 2x \left(x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)} \right), \quad (28)$$

for any $x \in I$ and $m \in \mathbb{N}$.

Proof. Using the previous lemma and the definition of the operator Γ_i from (2), we get the results after some calculations. $\square$

Lemma 9. We have:

$$\lim_{m \rightarrow \infty} m \left(\Gamma_1 S_{2,m}^{(\alpha,\beta)*} \right)(x) = \frac{1}{2}(1-x), \quad (29)$$

$$\lim_{m \rightarrow \infty} m \left(\Gamma_2 S_{2,m}^{(\alpha,\beta)*} \right)(x) = x(1-x), \quad (30)$$

uniformly with regard to $x \in I$. For any $\varepsilon > 0$, there exists $m_\varepsilon > m_0$ such that

$$\left(\Gamma_2 S_{2,m}^{(\alpha,\beta)*}\right)(x) \leq \frac{1}{m} \left(\frac{1}{4} + \varepsilon \right), \quad (31)$$

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_\varepsilon$.

Proof. We have:

$$\begin{aligned} & \lim_{m \rightarrow \infty} m \left(x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)} \right) = \\ & \lim_{m \rightarrow \infty} \left(-\frac{2\sqrt{\Delta_m(x)} - 2(m+\beta+1)(m-1)x}{m} + \frac{m^2}{2(m+\beta+1)(m-1)} + \frac{m(m+2\alpha+1)}{2(m+\beta+1)(m-1)} \right) = \\ & \frac{1}{2} - \lim_{m \rightarrow \infty} \frac{\sqrt{\Delta_m(x)} - (m+\beta+1)(m-1)x}{m}, \end{aligned}$$

and after some calculations, we get:

$$\lim_{m \rightarrow \infty} m \left(x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)} \right) = \frac{1}{2}(1-x). \quad (32)$$

Now, replacing the right hand side term in (27) and (28) with (32), we will get the convergences in (29) and (30). Using the definition of the limit of a function and the inequality $x(1-x) \leq \frac{1}{4}$, $\forall x \in [0, 1]$, we have that for every $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that the inequality (31) holds, for every $m \geq m_\varepsilon$. $\square$

Now, using the above results we obtain the following theorem.

Theorem 3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function on $[0, 1]$. Then, we have

$$\lim_{m \rightarrow \infty} \left(S_{2,m}^{(\alpha,\beta)*} f \right)(x) = f(x)$$

uniformly on I and for every $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that

$$\left| \left(S_{2,m}^{(\alpha,\beta)*} f \right) (x) - f(x) \right| \leq \left(1 + \sqrt{\frac{1}{4} + \varepsilon} \right) \omega \left(f; \frac{1}{\sqrt{m}} \right),$$

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_\varepsilon$.

Proof. The Theorem follows from relation (31) and from Theorem 1 by taking $h = \frac{1}{\sqrt{m}}$. $\square$

Graphic Properties of Approximation

Remark 6. Furthermore, in this case, it can be seen in Figures 3 and 4 that our operators approximate the given functions.

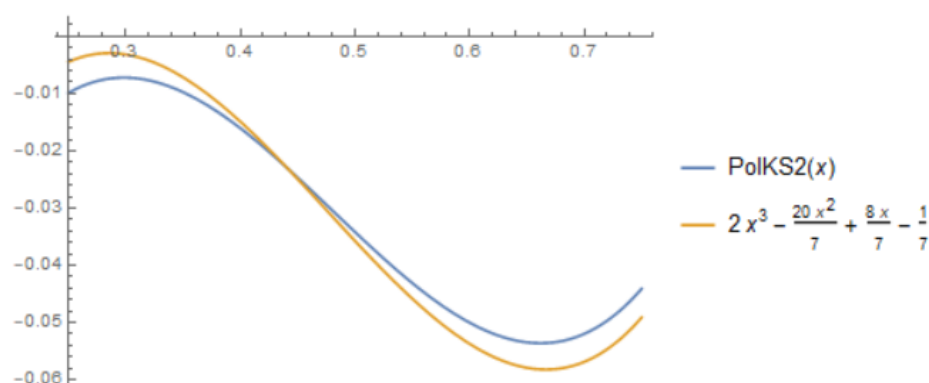


Figure 3. $\alpha = 0.1$, $\beta = 0.65$, $m = 50$ iterations.

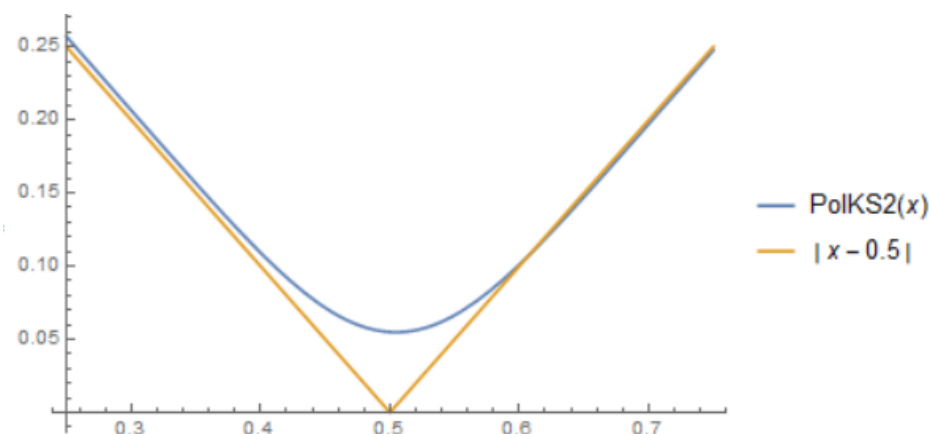


Figure 4. $\alpha = 0.1$, $\beta = 0.65$, $m = 50$ iterations.

6. Stancu–Kantorovich Type Operators Which Preserve the Functions e_1 and e_2

In this section, we shall construct an operator of Stancu–Kantorovich type as in (3), that preserves the test functions e_1 and e_2 , i.e., an operator that satisfies

$$\begin{aligned} S_m^{(\alpha,\beta)*}(e_0, x) &\xrightarrow{u} 1 \\ S_m^{(\alpha,\beta)*}(e_1, x) &= x \\ S_m^{(\alpha,\beta)*}(e_2, x) &= x^2. \end{aligned} \tag{33}$$

In order to obtain the main results of this section, we shall consider the following notation

$$S_m^{(\alpha, \beta)*}(e_0, x) = 1 + w_m(x), \quad (34)$$

where $x \in I$, $m \in \mathbb{N}$ and $w_m : I \rightarrow \mathbb{R}$.

With the previous notation, we have the following remark.

Remark 7. In order to have positive operators $S_m^{(\alpha, \beta)*}$, $m \in \mathbb{N}$, $0 \leq \alpha \leq \beta$ and for the relation (34) to hold, we shall impose that $S_m^{(\alpha, \beta)*}(e_0, x) \geq 0$, which implies

$$1 + w_m(x) \geq 0, \quad \forall x \in I, m \in \mathbb{N} \quad (35)$$

From (34), we get

$$(c_m(x) + d_m(x))^m = 1 + w_m(x), \quad \forall x \in I, m \in \mathbb{N} \quad (36)$$

which implies

$$c_m(x) + d_m(x) = (1 + w_m(x))^{\frac{1}{m}}, \quad \forall x \in I, m \in \mathbb{N}. \quad (37)$$

Now, from the above considerations and imposing the conditions

$$S_m^{(\alpha, \beta)*}(e_1, x) = x, \quad \forall x \in I, m \in \mathbb{N} \quad (38)$$

and

$$S_m^{(\alpha, \beta)*}(e_2, x) = x^2, \quad \forall x \in I, m \in \mathbb{N}, \quad (39)$$

we will obtain the following lemma.

Lemma 10. We have

$$c_m(x) = \frac{m + \beta + 1}{m} \left[x - \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) \right] (1 + w_m(x))^{\frac{1-m}{m}} \quad (40)$$

and

$$d_m(x) = (1 + w_m(x))^{\frac{1}{m}} \times \left[1 - \frac{m + \beta + 1}{m} \cdot \frac{1}{1 + w_m(x)} \left(x - \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) \right) \right]. \quad (41)$$

Proof. Equaling relations (5) and (38), we get

$$\frac{m}{m + \beta + 1} c_m(x) (c_m(x) + d_m(x))^{m-1} + \frac{2\alpha + 1}{2(m + \beta + 1)} (c_m(x) + d_m(x))^m = x.$$

Now, using the Equation (37), we have

$$\frac{m}{m + \beta + 1} c_m(x) (1 + w_m(x))^{\frac{m-1}{m}} + \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) = x,$$

which gives relation (40).

Now, using relations (37) and (40), by direct computation, we will get Formula (41). $\square$

Imposing the condition (39), we will obtain the following quadratic equation in $w_m(x)$:

$$\begin{aligned} & w_m^2(x)[-5m-3-\alpha(1+m+\alpha)] + \\ & w_m(x)\{-12m[(m+1)^2+\beta(\beta+2m+2)]x^2 + \\ & 12[(m+1)^2+2\alpha(1+m)+\beta(1+m+2\alpha)]x - 2[5m+3+12\alpha(1+m+\alpha)]\} + \\ & +\{-12[(m+1)^2+\beta(\beta+2m+2)]x^2 + \\ & +12[(m+1)^2+2\alpha(1+m)+\beta(1+m+2\alpha)]x - [5m+3+12\alpha(1+m+\alpha)]\} = 0, \end{aligned}$$

which has the following solutions:

$$\begin{aligned} w_{m,1}(x) &= \frac{1}{2(-3-5m-\alpha-m\alpha-\alpha^2)} \left(x^2(12m+24m^2+12m^3+24\beta+24m\beta+12\beta^2) + \right. \\ & \quad + x(-144-144m-24\alpha-24m\alpha-12\beta-12m\beta-24\alpha\beta) + \\ & \quad \left. + \left(6+10m+24\alpha+24m\alpha+24\alpha^2 - \sqrt{t_1^2(x)+t_2(x)} \right) \right) \\ w_{m,2}(x) &= \frac{1}{2(-3-5m-\alpha-m\alpha-\alpha^2)} \left(x^2(12m+24m^2+12m^3+24\beta+24m\beta+12\beta^2) + \right. \\ & \quad + x(-144-144m-24\alpha-24m\alpha-12\beta-12m\beta-24\alpha\beta) + \\ & \quad \left. + \left(6+10m+24\alpha+24m\alpha+24\alpha^2 + \sqrt{t_1^2(x)+t_2(x)} \right) \right) \end{aligned}$$

where

$$\begin{aligned} t_1(x) &= -6-10m-24\alpha-24m\alpha-24\alpha^2 + \\ & \quad + x(144+144m+24\alpha+24m\alpha+12\beta+12m\beta+24\alpha\beta) + \\ & \quad + x^2(-12m-24m^2-12m^3-24\beta-24m\beta-12\beta^2); \\ t_2(x) &= 4(-3-5m-12\alpha-12m\alpha-12\alpha^2)(-3-5m-\alpha-m\alpha-\alpha^2) + \\ & \quad + 4x^2(-3-5m-\alpha-m\alpha-\alpha^2)(-144-144m-36\beta-24m\beta) + \\ & \quad + 4x(-3-5m-\alpha-m\alpha-\alpha^2)(144+144m+24\alpha+24m\alpha+12\beta+12m\beta+24\alpha\beta). \end{aligned}$$

Remark 8. It is easy to verify that $\lim_{m \rightarrow \infty} w_{m,2}(x) = -\infty$ and $\lim_{m \rightarrow \infty} w_{m,1}(x) = 0$, uniformly for $x \in (0,1)$.

From now on, in this section we will consider $w_m(x) = w_{m,1}(x)$.

In order to have a positive operator, the quantities $c_m(x)$ and $d_m(x)$ from relations (40) and (41) shall be positive. With that condition, we get the following inequalities:

$$x - \frac{2\alpha+1}{2(m+\beta+1)}(1+w_m(x)) \geq 0,$$

and

$$1 - \frac{m+\beta+1}{m} \cdot \frac{1}{1+w_m(x)} \left(x - \frac{2\alpha+1}{2(m+\beta+1)}(1+w_m(x)) \right) \geq 0,$$

for all $x \in I$, $m \in \mathbb{N}$ and $0 \leq \alpha \leq \beta$ which lead to

$$\frac{2(m+\beta+1)}{2m+2\alpha+1}x - 1 \leq w_m(x) \leq \frac{2(m+\beta+1)}{2\alpha+1}x - 1, \quad (42)$$

for all $x \in I$, $m \in \mathbb{N}$ and $0 \leq \alpha \leq \beta$.

Lemma 11. Let $0 < \varepsilon' < \frac{1}{2}$. Then, there exists $m_0 \in \mathbb{N}$ such that relation (42) holds for any $x \in [\varepsilon', 1 - \varepsilon']$ and $m \in \mathbb{N}$, $m \geq m_0$.

Proof. We have $w_m \rightarrow 0$, uniformly on $(0, 1)$, and

$$\lim_{m \rightarrow \infty} \left(\frac{2(m + \beta + 1)}{2m + 2\alpha + 1} x - 1 \right) = x - 1 \leq \varepsilon',$$

uniformly for $x \in I$. $\square$

From now on, we will consider $I = [\varepsilon', 1 - \varepsilon']$, with fixed $0 < \varepsilon' < \frac{1}{2}$.

We can write the operators in (3) as

$$\begin{aligned} S_{3,m}^{(\alpha,\beta)*}(f, x) &= (m + \beta + 1) \sum_{k=0}^m \binom{m}{k} (1 + w_m(x))^{1-k} \\ &\quad \times \left(\frac{m + \beta + 1}{m} \left(x - \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) \right) \right)^k \\ &\quad \times \left(1 - \frac{m + \beta + 1}{m(1 + w_m(x))} \left(x - \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) \right) \right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha}{m+\beta+1}} f(t) dt. \end{aligned}$$

Lemma 12. For $x \in I$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} (\Gamma_0 S_{3,m}^{(\alpha,\beta)*})(x) &= 1 + w_m(x), \\ (\Gamma_1 S_{3,m}^{(\alpha,\beta)*})(x) &= -x w_m(x), \\ (\Gamma_2 S_{3,m}^{(\alpha,\beta)*})(x) &= x^2 w_m(x). \end{aligned}$$

Proof. The proof follows immediately from relation (2) and from the conditions (38), (39). $\square$

Theorem 4. We have

$$\lim_{m \rightarrow \infty} S_{3,m}^{(\alpha,\beta)*}(f, x) = f(x)$$

uniformly on I for every $f \in C[0, 1]$.

Proof. By applying Theorem 1. $\square$

Remark 9. It was proved in [24] that there is no sequence of positive, linear and analytic operators $L : C[0, 1] \rightarrow C[0, 1]$ that preserves the test functions e_1 and e_2 . Therefore, we have operators $S_{3,m}^{(\alpha,\beta)*} : C[0, 1] \rightarrow C[\varepsilon', 1 - \varepsilon']$.

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As a first comparison, we considered the function $f(x) = \sin(20x)$, and we obtained the following graphics, Figure 5, where $PolKS(x)$ represents our operators that preserve e_1 and e_2 , and $P(x)$ is the operator obtained by Indrea et al. in [9], which is also a particular case of our operators considered in the third section for $\alpha = \beta = 0$.

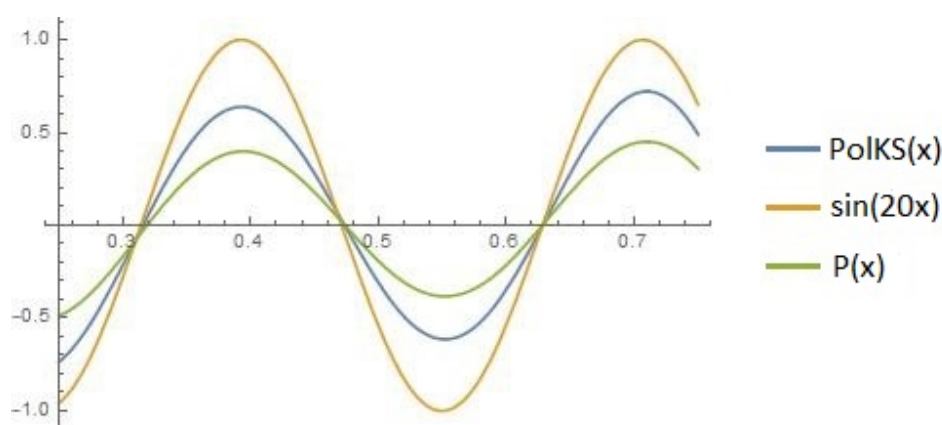


Figure 5. $f(x) = \sin(20x)$, $\alpha = 10$, $\beta = 20$, $m = 50$ iterations.

Now, we considered the function $f(x) = |x - 0.5|$ and we obtained Figure 6:

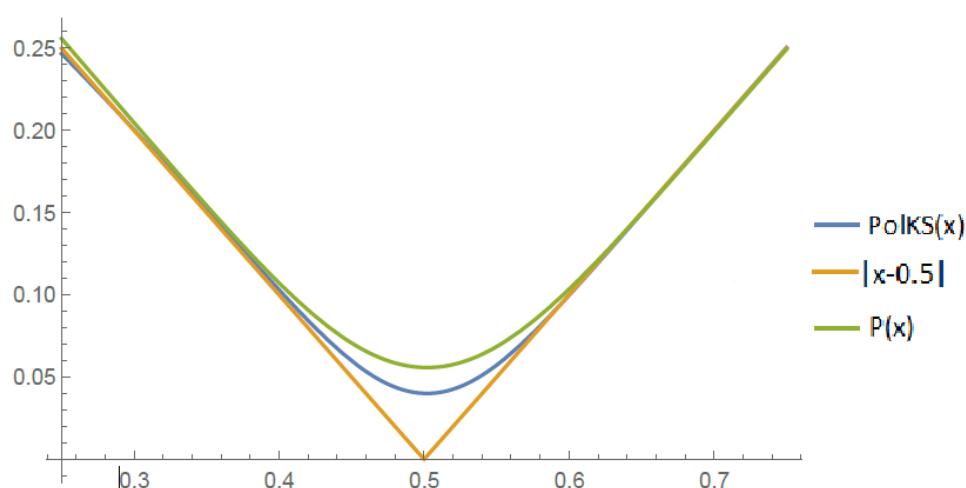


Figure 6. $f(x) = |x - 0.5|$, $\alpha = 10$, $\beta = 20$, $m = 50$ iterations.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The APC was funded by Transilvania University of Braşov.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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