mathematics

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Studying the Effect of Noise on Pricing and Marketing Decisions of New Products under Co-op Advertising Strategy in Supply Chains: Game Theoretical Approaches

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Abstract: The success of launching new products is the main challenge of companies since it is one of the key factors of competition. Thus, success in today's high rival markets depends on the presentation of new products with new options, which must be compatible with customers' desires. This research aims to analyze the psychological effect of the noise of a new product on the total profit of the chain and the optimal pricing and marketing decisions of the chain's members. Additionally, a cooperative (co-op) advertising strategy as a coordination mechanism is considered among the partners such that it helps them to obtain their target markets. Commonly, under co-op advertising, the manufacturer pays a percentage of the retailer's advertising costs. In this chain, the manufacturer and the retailer agree to share the retailer's advertising costs. Afterwards, four different relations between the manufacturer and retailer are studied and analyzed including three non-cooperative games with symmetrical distribution of market power and one asymmetrical distribution of it. So, four game models and their closed-form solutions are illustrated with a numerical example. It was found that the noise effect affects the total profit of the manufacturer and the retailer, as well as the supply chain by influencing the partners' advertising policies. In other word, increasing the noise effect of the product indicates to the manufacturer and the retailer to globally and locally advertise more, respectively. In turn, their profits increase, although also increasing the advertising costs. Finally, a complete sensitivity analysis is conducted and reported.

Keywords: vertical cooperative advertising; pricing; noise effect; game theory; supply chain

MSC: 2010:90B05

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Citation: Taleizadeh, A.A.; Cheraghi, Z.; Cárdenas-Barrón, L.E.; Noori-Daryan, M. Studying the Effect of Noise on Pricing and Marketing Decisions of New Products under Co-op Advertising Strategy in Supply Chains: Game Theoretical Approaches. *Mathematics* **2021**, *9*, 1222. https://doi.org/10.3390/math9111222

Academic Editor: Chia-Nan Wang

Received: 24 February 2021
Accepted: 27 April 2021
Published: 27 May 2021

1. Introduction and Literature Review

Nowadays, firms motivate customers' demand by launching new products considering the interests of customers. However, the success of presenting new products is a challenge that companies face. So, they commonly apply marketing tools such as optimal pricing and advertising strategies as the most efficient marketing policies to attract the market demand for the new products. These strategies are employed to model several supply chain settings in which there exists a considerable amount of research where the pricing and advertising decisions are jointly considered. To name a few works, Bergen and John [1], Kim and Staelin [2], Swami and Khainar [3], Karray and Zaccour [4], Yenipazarli [5], and He et al. [6].

In the literature, there exist several papers that have dealt with how the co-op advertising strategy affects the optimal decisions of supply chains' partners. As mentioned earlier, under the co-op advertising strategy, the manufacturer pays either a portion or all of retailer's advertising costs, where the retailer is responsible for the preparation and organization of the local advertising following some basic guidelines established by the manufacturer. It is well-known that the first mathematical model of co-op advertising was proposed by Berger [7]. He considers a situation in which the manufacturer gives an advertising payment to its retailer.

Jørgensen and Zaccour [8] considered a differential game model for a two-level supply chain with cooperation and non-cooperation settings where demand function is influenced by the retail price and advertising goodwill. Later, Jørgensen et al. [9] also used a demand function that is affected by the retail price and advertising goodwill to study the leading role in a supply chain where each member of the chain controls its advertising and margins. Generally speaking, there exist two types of advertising (i.e., global and local advertising strategies) in the manufacturer–retailer relationship to stimulate the consumer demand.

One of the most efficient marketing policies applied by the firms is called vertical cooperative advertising, also known as co-op advertising. Under this strategy, the retailers' advertising costs are shared with the companies. Indeed, the manufacturer can cooperate with the local retailers to pay a portion of its retailers' advertising costs in order to decrease the retailer's costs and subsequently increase the market share. An online advertising cooperative is one of this kind of strategy in which the companies gain a full-page advertisement while these are responsible for half the price. For instance, advertisement of Zomato (www.zomato.com) on a Facebook page is an application for searching restaurants on cell phones which is accessible by clicking on a Facebook link and redirecting to Zomato. Thereupon, with financial support, the retailer could increase its advertising and consequently raise its sales. For the sake of simplicity, in this paper, the term co-op advertising is used. Some researchers have employed co-op advertising strategy in their research, such as Yue et al. [10] and Szmerekovsky and Zhang [11] extended the work of Huang et al. [12].

Afterwards, Xie and Neyret [13] and Xie and Wei [14] derived optimal pricing and co-op advertising strategies for different relations between a manufacturer and a retailer. Wang et al. [15] and SeyedEsfahani et al. [16] studied co-op advertising for a supply chain under four decision models to obtain the optimal co-op advertising policies.

Later, Aust and Buscher [17] utilized SeyedEsfahani et al. [16]'s price-sensitive demand function and incorporated pricing policies and co-op advertising in a two-echelon supply chain. In addition, they studied four strategies using game theory and compared cooperation and non-cooperation policies. Other recent interesting researches that considered the co-op advertising strategy are Ahmadi-Javid and Hoseinpour [18], Yang et al. [19], Yue et al. [20]. Aust and Busher [21,22] and Jørgensen and Zaccour [23] provided a complete and comprehensive review on co-op advertising.

Game theory is a useful tool that allows us to model and analyze the interactions between the members of a supply chain [24–31]. In other word, game theory is an appropriate approach to study the behaviors of supply chains' members against each other's reactions. It is evident that all the partners intend to achieve their own desired goals, which can be vindicated by applying the coordination mechanisms, of which co-op advertising strategy is one. Indeed, these mechanisms are employed to coordinate the supply chains' decisions and close them to the optimal ones. As a result, a win–win relationship nearly is established among the members of the chains.

In this direction, Huang and Li [32] applied game theory to study co-op advertising policy in a two-echelon supply chain comprised of one manufacturer and one retailer. Their study concluded that the manufacturer always prefers the Stackelberg game compared with other games, simultaneously. Later, Xie and Ai [33] extended the models of Huang and Li [32] and Li et al. [34] to the situation when the manufacturer's marginal profit is not large enough. Equivalent approaches with a little change in the demand functions are presented in the research works of Li et al. [34], Huang et al. [12], Huang and Li [35] and He et al. [36].

The existing literature shows that the pricing and advertising policies are employed to coordinate the decisions of the chain's members in several investigations. However, an interesting and also sensible issue, which is not considered by them, is the effect of noise. It is known that the noise effect is considered for new products, which are launched to the competitive markets, to analyze the psychological effect of the customers' satisfaction or dissatisfaction on the sales. So, this issue is an important factor that influences the sales. Recently, the noise effect was modeled as a random component in an additive way for a supply chain to study pricing and inventory decisions under demand uncertainty by Chen et al. [37]. Additionally, some related research can be found in works [38–47].

In this research, in addition to co-op advertising and price policies, the noise effect as a psychological impact of the product is considered in the customers' demand function. Indeed, the noise effect of a product, which shows the end-users' satisfaction or dissatisfaction measure, is primarily a random element representing the effect of the word of mouth on the sale of the product. The principal aim of this research is to study the optimal pricing and advertising decisions of a one-manufacturer–one-retailer supply chain under different game-theoretic approaches. Here, two different models in terms of demand functions are considered, which are randomly dependent on the noise effect and also dependent on the price and advertising in the product's market acceptance. Using cooperative and non-cooperative game theory, the following four classical scenarios are considered: (1) Nash game, (2) Manufacturer-Stackelberg game, (3) Retailer-Stackelberg game, and (4) cooperative game. In both models, the demand is a function of the retailer's local advertising, the manufacturer's national advertising, the price, and the noise effect.

Mainly, the contribution of this paper is threefold. The first one is to introduce two uncertain pricing and advertising models in order to study the optimal decisions of the supply chain's members in the presence of the noise effect in order to maximize the total profit of the chain. The second one is to study four game-theoretic approaches among the partners to analyze their behaviors under different market powers and choose the best one. The third one is to derive the closed-form solutions of the decision variables where the concavity of the objective functions for both models under different scenarios is evidenced.

The rest of the paper is organized as follows. Section 2 defines the on-hand problem and notation used. Section 3 develops the optimal policies for four game models considering two types of demand functions. Section 4 shows the applicability of the proposed models with a numerical example. Section 5 provides a complete sensitivity analysis. Finally, Section 6 gives some conclusions and future research directions.

2. Problem Definition

This paper considers a supply chain comprised of one manufacturer and one retailer. The manufacturer sells his/her new product to the retailer; the retailer vends only the product to his/her own customers. In this supply chain, the manufacturer globally advertises to introduce the new product under its brand name and the retailer locally advertises to inform and also attract the customers. Here, a co-op advertising strategy as a coordinating mechanism is established between the manufacturer and the retailer to share the retailer's advertising costs. In other words, a portion of the retailer's advertising cost is supported by the manufacturer.

It is assumed that the good news from customers, who have bought and used the new product for the first time, encourages new customers to purchase the product, which is titled the noise effect. The noise effect represents the psychological effect of the word of mouth among the customers which results from the measures of the customers' satisfaction/dissatisfaction. So, the effect of noise is an important and drastic factor in the market because it makes an uncertainty in demand of each new product in presence of other products. Moreover, four game-theoretic approaches such as Nash, Manufacturer-Stackelberg, Retailer-Stackelberg, and cooperative games are considered to analyze the optimal pricing and marketing decisions under different market powers.

The manufacturer decides on the wholesale price, w, the cost of global advertising, A, and the participation rate, t, in local advertising a; the retailer decides on retail price, p, and the cost of the local advertising a. Therefore, the customers' demand with the price, advertising, and the noise effect considerations in the market is given as follows:

$$D(p, a, A) = D_0 . \, n(x) \, . \, g(p) \, . \, h(a, A) \tag{1}$$

where $g(p)$ is the effect of retail price on demand, $h(a, A)$ shows the advertising effect on demand, and $n(x)$ is the noise impact of the market acceptance. In order to model the problem, the following parameters and variables are defined. Notice that some symbols are the same as they were used previously in other research works.

The principal objective of this research is to optimize the total profit of the supply chain under both models with respect to the decision variables. It is important to mention that Szmerekovsky and Zhang [11] used the following functions $g_1(p) = p^{-e} \ ; \ \ e > 1$ and $h_1(a, A) = \alpha_2 - \beta_2 a^{-\gamma} A^{\delta}$; where $\alpha_2, \beta_2, \gamma, \delta > 0$. On the other hand, Xie and Wei [14] considered $g_2(p) = \alpha_1 - \beta_1 p \ ; \ \alpha_1, \beta_1 > 0$ and $h_2(a, A) = k_1\sqrt{a} + k_2\sqrt{A}\,; \ \ k_1, k_2 > 0$. This research incorporates the noise effect of a new product into the above-mentioned functions, which is described as follows:

$$n(x) = e_1^x; \ x > 0 \tag{2}$$

3. Modeling

This section develops two models for a two-echelon supply chain where the noise effect of the product is considered. According to the assumptions, demand is uncertain due to the essence of the noise effect. Using the expressions of g_1, h_1, g_2, h_2 and Equation (2), the demand functions for the first and the second models are given by Equations (3) and (4), respectively.

$$D_1(p, a, A) = D_0 \, e_1^x (p^{-e})(\alpha_2 - \beta_2 a^{-\gamma} A^{\delta}) \tag{3}$$

$$D_2(p, a, A) = D_0 \, e_1^x \, (\alpha_1 - \beta_1 p)(k_1\sqrt{a} + k_2\sqrt{A}) \tag{4}$$

Since x is a random variable, the expected profit functions for the manufacturer, the retailer, and the whole supply chain are as follows:

Model 1:

$$E[\Pi_m^1(w, A, t)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x (w - c)(p^{-e})(\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) - A - ta \right] f(x)dx \tag{5}$$

$$E[\Pi_r^1(p, a)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x (p - w - d)(p^{-e})(\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) - (1 - t)a \right] f(x)dx \tag{6}$$

$$E[\Pi_{sc}^1(p, a, A)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x (p - c - d)(p^{-e})(\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) - A - a \right] f(x)dx \tag{7}$$

Model 2:

$$E[\Pi_m^2(w, A, t)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x \, (w - c)(\alpha_1 - \beta_1 p)(k_1\sqrt{a} + k_2\sqrt{A}) - A - ta \right] f(x)dx \tag{8}$$

$$E[\Pi_r^2(p, a)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x (p - w - d)(\alpha_1 - \beta_1 p)(k_1\sqrt{a} + k_2\sqrt{A}) - (1 - t)a \right] f(x)dx \tag{9}$$

$$E[\Pi_{sc}^2(p, a, A)] = \int_{-\infty}^{+\infty} \left[D_0 \, e_1^x (p - c - d)(\alpha_1 - \beta_1 p)(k_1\sqrt{a} + k_2\sqrt{A}) - A - a \right] f(x)dx \tag{10}$$

where Model 1 is based on the function of Szmerekovsky and Zhang [11] and Model 2 is based on the function of Xie and Wei [14]. The subscripts m, r, and SC represent the manufacturer, retailer and whole supply chain system, respectively. The demand function must be positive; hence, the condition $p < \frac{\alpha_1}{\beta_1}$ must be established and satisfied. To avoid negative profit functions, the following conditions $\Pi_m > 0 \Rightarrow w > c$, $\Pi_r > 0 \Rightarrow p > w + d$, $\Pi_{sc} > 0 \Rightarrow p > c + d$, and $\alpha_1 - \beta_1(c + d) > 0$ are stated. In order to simplify the calculations, we apply an appropriate change in the variables as follows: $\alpha_1{}' = \alpha_1 - \beta_1(c + d)$,

$p' = \frac{\beta_1}{\alpha_1{}'}(p-(c+d))$, $w' = \frac{\beta_1}{\alpha_1{}'}(w-c)$, $k_1{}' = D_0\frac{\alpha_1{}'^2}{\beta_1}k_1$, and $k_2{}' = D_0\frac{\alpha_1{}'^2}{\beta_1}k_2$. Furthermore, for the sake of simplicity, the superscript ($'$) is removed and the new profit functions for the second model are stated as below (See [14])

Model 2:

$$E[\Pi_m^2(w,A,t)] = \int_{-\infty}^{+\infty}\left[e_1^x w(1-p)(k_1\sqrt{a}+k_2\sqrt{A})-A-ta\right] f(x)dx \tag{11}$$

$$E[\Pi_r^2(p,a)] = \int_{-\infty}^{+\infty}\left[e_1^x (p-w)(1-p)(k_1\sqrt{a}+k_2\sqrt{A})-(1-t)a\right] f(x)dx \tag{12}$$

$$E[\Pi_{sc}^2(p,a,A)] = \int_{-\infty}^{+\infty}\left[e_1^x p(1-p)(k_1\sqrt{a}+k_2\sqrt{A})-a-A\right] f(x)dx \tag{13}$$

In the next section, for both models, the optimal values of decision variables under the well-known Nash, Manufacturer-Stackelberg, Retailer-Stackelberg and cooperative games are determined.

3.1. Nash Game (NG)

In a Nash equilibrium game, the players with equal market power act independently and simultaneously. So, here, there is no cooperation and the manufacturer and the retailer make decisions individually about their own decision variables. Thus, in both models, the following optimization problems for the manufacturer are solved:

Model 1:

$$\begin{gathered} \underset{w,A,t}{Max} E[\Pi_m^1(w,A,t)] = D_0 E[e_1^x](w-c)(p^{-e})(\alpha_2-\beta_2 a^{-\gamma}A^{-\delta})-A-ta \\ s.t.\ \ 0\le A,\ \ c\le w\le 1,\ \text{ and } 0\le t\le 1 \end{gathered} \tag{14}$$

Model 2:

$$\begin{gathered} \underset{w,A,t}{Max} E[\Pi_m^2(w,A,t)] = E[e_1^x]\, w\,(1-p)(k_1\sqrt{a}+k_2\sqrt{A})-A-ta \\ s.t.\ \ 0\le w\le 1,\ \ 0\le A \text{ and } 0\le t\le 1 \end{gathered} \tag{15}$$

Under this game, whereas the partners of the chain play independently, it is obvious that the manufacturer is not interested in participating in local advertising strategy because it results in increasing its profit. Consequently, the optimum value of t will be zero because it has a negative coefficient on the manufacturer's objective function.

According to Jørgensen and Zaccour [8], Xie and Neyret [13] and SeyedEsfahani et al. [16], the retailer sells the product if they get at least a minimum unit margin. Hence, to solve the problem, the manufacturer's profit margin is considered as a minimum level. In turn, it is known that $p-w\ge w \Rightarrow w\le \frac{p}{2}$. Thus, the optimum value for w is $\frac{p}{2}$. In this case, the objective functions of the retailer in both models are given by:

Model 1:

$$\begin{gathered} \underset{p,a}{Max} E[\Pi_r^1(p,a)] = D_0 E[e_1^x](p-w-d)(p^{-e})(\alpha_2-\beta_2 a^{-\gamma}A^{-\delta})-(1-t)a \\ s.t.\ \ \ w+d\le p\ \text{ and } 0\le a \end{gathered} \tag{16}$$

Model 2:

$$\begin{gathered} Max E[\Pi_r^2(p,a)] = E[e_1^x](p-w)(1-p)(k_1\sqrt{a}+k_2\sqrt{A})-(1-t)a \\ s.t.\ \ w\le p\le 1\ \text{ and } 0\le a \end{gathered} \tag{17}$$

Theorem 1. *The optimal values of the decision variables in the first model under a Nash game are given in the second column of Table 1.*

Table 1. Optimal values for the decision variables under the four strategies for the first model.

CG	*SG-RL*	*SG-ML*	*NG*	Decision Variable
$\frac{e(c+d)}{(e-1)}$	$\frac{c+d}{e-1}+c+d$	$\frac{e(w+d)}{e-1}$	$\frac{2ed}{e-2}$	p
-	$\frac{1}{2}(\frac{c+d}{e-1})+\frac{c+d}{2}$	Use Equation (A21) from Appendix C	$\frac{ed}{e-2}$	w
$\frac{\delta}{\gamma}a^{co}$	$\left(\begin{array}{c} D_0E[e_1^x]\delta\beta_2\frac{e^{-e}}{2} \\ \times(\frac{c+d}{e-1})^{1-e}a^{-\gamma} \end{array}\right)^{\frac{1}{\delta+1}}$	$\delta^{\frac{1+\gamma}{1+\gamma+\delta}}D_0^{\frac{1}{1+\gamma+\delta}}E[e_1^x]^{\frac{1}{1+\gamma+\delta}}(e-1)^{\frac{e-1}{1+\gamma+\delta}}$ $\times e^{\frac{-e}{1+\gamma+\delta}}(1-t)^{\frac{-1}{1+\gamma+\delta}}(w+d)^{\frac{-\gamma-e}{1+\gamma+\delta}}$ $\times\beta_2^{\frac{1}{1+\gamma+\delta}}\gamma^{\frac{-\gamma}{1+\gamma+\delta}}(\gamma+1)^{\frac{-(1+\gamma)}{1+\gamma+\delta}}$ $\times\left(\begin{array}{c}\gamma(w+d)t \\ +(w-c)(1-t)(e-1)\end{array}\right)^{\frac{1+\gamma}{1+\gamma+\delta}}$	$(\frac{\delta}{\gamma})\frac{e(d-c)+2c}{(2d)}a^N$	A
$(\frac{\gamma}{\delta})^{\frac{\delta}{\delta+\gamma+1}}$ $\times\left(\begin{array}{c} D_0E[e_1^x]\beta_2e^{-e} \\ \times(\frac{c+d}{e-1})^{1-e}\end{array}\right)^{\frac{1}{\delta+\gamma+1}}$	$\left(\gamma\frac{\beta_2^{\frac{1}{\delta+1}}\delta^{\frac{-\delta}{\delta+1}}}{\delta+1}\right)^{\frac{\delta+1}{\delta+\gamma+1}}$ $\times\left(\begin{array}{c} D_0E[e_1^x]\frac{e^{-e}}{2} \\ \times(\frac{c+d}{e-1})^{1-e}\end{array}\right)^{\frac{1}{\delta+\gamma+1}}$	$\left(\frac{\gamma D_0\beta_2A^{-\delta}e^{-e}(\frac{w+d}{e-1})^{1-e}}{1-t}\right)^{\frac{1}{\gamma+1}}$	$\left(\gamma D_0E[e_1^x]\beta_2e^{-e}(\frac{2d}{e-2})^{1-e}\right)$ $\times\left((\frac{\delta}{\gamma})\frac{e(d-c)+2c}{(2d)}\right)^{-\delta}$	a
-	0	0	0	t

Proof. See Appendix A.

Theorem 2. *The optimal values of the decision variables in the second model under a Nash game are given in the second column of Table 2.*

Table 2. Optimal values for the decision variables under the four strategies for the second model.

CG	*SG-RL*	*SG-ML*	*NG*	Decision Variable
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{(\sqrt{16k^2+16k+9}-4k)+1}{2(\sqrt{16k^2+16k+9}-4k)}$	$\frac{2}{3}$	p
-	$\frac{1}{4}$	$\frac{1}{\sqrt{16k^2+16k+9}-4k}$	$\frac{1}{3}$	w
$\left(\frac{1}{8}E[e_1^x]k_2\right)^2$	$E[e_1^x]^2\frac{k_2^2}{16^2}$	$\left(\frac{E[e_1^x]k_2}{4}\right)^2\left(\frac{(\sqrt{16k^2+16k+9}-4k)-1}{(\sqrt{16k^2+16k+9}-4k)^2}\right)^2$	$\frac{1}{324}E[e_1^x]^2k_2^2$	A
$\left(\frac{1}{8}E[e_1^x]k_1\right)^2$	$E[e_1^x]^2\frac{k_1^2}{8^2}$	$\left(\frac{E[e_1^x]k_1}{16}\right)^2\left(\frac{(\sqrt{16k^2+16k+9}-4k)-1}{\sqrt{16k^2+16k+9}-4k}\right)^2$ $\times\left(\frac{3+(\sqrt{16k^2+16k+9}-4k)}{\sqrt{16k^2+16k+9}-4k}\right)^2$	$\frac{1}{324}E[e_1^x]^2k_1^2$	a
-	0	$\frac{5+4k-\sqrt{16k^2+16k+9}}{3-4k+\sqrt{16k^2+16k+9}}$	0	t

Proof. See Appendix B.

3.2. Stackelberg Game—The Manufacturer Is a Leader (SG-ML)

Under this non-cooperative game, the manufacturer as a powerful member of the chain, in terms of its reputation and popularity, is considered as a leader of the market while the retailer plays a follower role. In this game, the best answers of the retailer, as a follower, should be determined first; the leader's decision problem is solved based on the follower's responses. Hence, the retailer's best responses are as follows:

Model 1:

$$p_1^{SM^*}=\frac{e(w+d)}{e-1} \tag{18}$$

$$a_1^{SM^*} = \left(\frac{\gamma D_0 E[e_1^x] \beta_2 A^{-\delta} e^{-e} (\frac{w+d}{e-1})^{1-e}}{1-t} \right)^{\frac{1}{\gamma+1}} \tag{19}$$

Model 2:

$$p_2^{SM^*} = \frac{1+w}{2} \tag{20}$$

$$a_2^{SM^*} = \left(\frac{E[e_1^x] k_1}{2(1-t)} (\frac{1-w}{2})^2 \right)^2 \tag{21}$$

To solve the manufacturer's decision problem, the optimal values of p^{SM^*} and a^{SM^*} are substituted in the manufacturer's profit function. Then, the partial derivatives of the manufacturer's profit function regarding A^{SM} and t^{SM} are taken. The value of t will be equal to $t = 1 + \frac{(w+d)}{\gamma(w+d)-(e-1)(w-c)}$; the same as when there is no noise effect. Szmerekovsky and Zhang [11] have also shown that the optimal value of $t = 1 + \frac{(w+d)}{\gamma(w+d)-(e-1)(w-c)}$ is always equal to zero, and in this case (with noise effect), using the same manner, it can be shown that the optimal value of t^* is zero, too (see Szmerekovsky and Zhang [11]).

Theorem 3. *The optimal values of the decision variables in the first model under a Stackelberg game when the manufacturer is the leader are shown in the third column of Table 1.*

Proof. See Appendix C.

Theorem 4. *The optimal values of the decision variables in the second model under a Stackelberg game when the manufacturer is the leader are shown in the third column of Table 2.*

Proof. See Appendix D.

3.3. Stackelberg Game—The Retailer Is a Leader (SG-RL)

As in the previous section, here, we model the relation between the manufacturer and the retailer as a consecutive non-cooperative Stackelberg game. Now, it is important to remark that the retailer is a powerful member. In other words, the retailer is a leader and the manufacturer is a follower. Obviously, the first step is to find the manufacturer's best responses as a follower.

The manufacturer's profit must always be positive (i.e., $\Pi_m > 0$). Hence, $w > c \Rightarrow w - c > 0$ should be satisfied. On the other hand, in the whole chain, $p - (c+d) > 0$ should be established. Now, we define $w' = w - c$ and $p' = p - (c+d)$. So, the retailer's profit function in Equation (16) is rewritten as follows.

$$\underset{p,a}{Max} E[\Pi_r^1(p,a)] = D_0 E[e_1^x](p' - w')(p' + (c+d))^{-e} \times (\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) - (1-t)a \\ s.t. \;\; w' \leq p' \;\text{ and } 0 \leq a \tag{22}$$

According to Jørgensen and Zaccour [8], Xie and Neyret [13] and SeyedEsfahani et al. [16], the wholesale price of the first model can be written as $p' - w' \geq w' \Rightarrow w' \leq \frac{p'}{2}$. Thus, the optimal value of w' is $\frac{p'}{2}$ and the optimal value of A is equal to $\left(\delta D_0 E[e_1^x] \beta_2 (w-c)(p')^{-e} a^{-\gamma}\right)^{\frac{1}{\delta+1}}$. This is the result of the manufacturer's problem under a Nash game. Moreover, from the manufacturer's point of view, the optimal value of the participation rate t is zero. Hence, in this game, the optimal decision variables of the retailer in the first model are calculated by substituting the optimal values of t, w', and A into Equation (22). Similarly for the second model, we have $t^{SR*} = 0$, $w^{SR*} = \frac{p}{2}$, and $A^{SR*} = \left(\frac{1}{2} E[e_1^x] k_2 w(1-p)\right)^2$. Then by replacing

t^{SR*}, w^{SR*}, and A^{SR*} into Equation (17), the optimal decisions of the retailer can be easily obtained.

Theorem 5. *The optimal values of the decision variables for the first model under a Stackelberg game when the retailer is the leader are given in the fourth column of Table 1.*

Proof. See Appendix E.

Theorem 6. *The optimal values of the decision variables for the second model under a Stackelberg game when the retailer is the leader are given in the fourth column of Table 2.*

Proof. See Appendix F.

3.4. Cooperative Game (CG)

In a cooperative game, the manufacturer and the retailer cooperate, and they are willing to increase the whole system's profit to promote their profits more than non-cooperative games. Therefore, the total profit function of the chain for both models is optimized in order to obtain the optimal values of the decision variables.

Model 1:

$$\begin{gathered} \underset{p,a,A}{Max} E[\textstyle\prod_{sc}^{1}(p,a,A)] = D_0 E[e_1^x](p-c-d)(p^{-e})(\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) \ - a - A \\ s.t. \ \ c + d \leq p \ \text{ and } 0 \leq a, A \end{gathered} \tag{23}$$

Model 2:

$$\begin{gathered} \underset{p,a,A}{Max} E[\textstyle\prod_{sc}^{2}(p,a,A)] = E[e_1^x] p(1-p)(k_1\sqrt{a} + k_2\sqrt{A}) - a - A \\ s.t. \ 0 \leq p \leq 1 \text{ and } 0 \leq a, A \end{gathered} \tag{24}$$

Under this approach, the partners of the chain only make a decision on p^{CO}, A^{CO}, and a^{CO} as the decision variables. Conversely, the variables w and t do not affect the total profit since these are inner variables of the supply chain.

Theorem 7. *The optimal values of the decision variables for the first model under a cooperative game are shown in the last column of Table 1.*

Proof. See Appendix G.

Theorem 8. *The optimal values of the decision variables for the second model under a cooperative game are shown in the last column of Table 2.*

Proof. See Appendix H.

To measure the efficiency of a supply chain, the most important criterion is the total profit of the whole chain. According to Xie and Neyret [13], SeyedEsfahani et al. [16], and Aust and Buscher [17], the members of the chain agree to cooperate only when their profit is higher than those under the non-cooperative games. This means that:

$$\Delta\Pi_m = \Pi_m^c - \Pi_m^{\max} \geq 0 \tag{25}$$

$$\Delta\Pi_r = \Pi_r^c - \Pi_r^{\max} \geq 0 \tag{26}$$

where Π_m^c and Π_r^c are the profit of the manufacturer and the retailer in the cooperative game, respectively. Π_m^{max} and $\Pi_r^{\max}$ are the largest profit of the manufacturer and the retailer in every non-cooperative game, respectively. In turn, if inequalities (25) and (26) are satisfied, the cooperation is possible. In other words, the manufacturer and the retailer agree to cooperate with each other when they gain higher profit than with other non-cooperative attitudes. Thus, the following inequality for the whole supply chain is proposed:

$$\Delta\Pi_{m+r} = \Delta\Pi_m + \Delta\Pi_r = \Pi^c_{m+r} - \Pi^{\max}_m - \Pi^{\max}_r \geq 0 \tag{27}$$

Note that to find $\Pi^{\max}_m$ and $\Pi^{\max}_r$, it is necessary to compare the results of the games given in Sections 3.1–3.3.

4. Numerical Example

In this section, two numerical examples are presented to clarify the validation of the proposed models.

Example 1. The following parameter values are considered: $D_0 = 5$, $c = 1$, $d = 1$, $\alpha_2 = 1000$, $\beta_2 = 500$, $\delta = 1$, $\gamma = 1$, $e_1 = 3$, $k_1 = 2$, and $k_2 = 3$. It is assumed that the random element to model noise effect (x) has a normal distribution with parameters ($\mu = 0, \sigma = 1$) then $E[e_1^x] = e_1^{\mu+\frac{\sigma}{2}} = \sqrt{e_1} = 1.6487$. The optimal values of the decision variables in both models under the four different strategies are summarized in Tables 3 and 4, respectively.

Table 3. The results for the numerical example 1 in the first model.

CG	*SG-RL*	*SG-ML*	*NG*	**Decision Variable**
3	2.3333	4.4669	6	p_1
-	1.1667	1.9780	3	w_1
5.3445	5.3445	1.9510	38.1644	A_1
5.3445	2.6722	5.9408	38.1644	a_1
-	0	0	0	t_1
-	99.0453	84.6017	38.1382	$E[\Pi^1_m]$
-	101.6550	125.8264	38.1382	$E[\Pi^1_r]$
289.2814	200.7004	210.4282	76.2764	$E[\Pi^1_{sc}]$

Table 4. The results for the numerical example 1 in the second model.

CG	*SG-RL*	*SG-ML*	*NG*	**Decision Variable**
0.5000	0.5000	0.7168	0.6667	p_2
-	0.2500	0.4335	0.3333	w_2
0.3822	0.0956	0.0922	0.0755	A_2
0.1699	0.1699	0.0042	0.0336	a_2
-	0	0.5075	0	t_2
-	0.2655	0.1163	0.1424	$E[\Pi^2_m]$
-	0.1913	0.1356	0.1847	$E[\Pi^2_r]$
0.5521	0.4568	0.2519	0.3271	$E[\Pi^2_{sc}]$

As it was mentioned in the previous section, both players agree to cooperate only when their profits are higher than under non-cooperative games. So, according to the assumptions, both players have the minimum benefits $\Pi_m = 99.0453$ and $\Pi_r = 125.8264$ in the first model and $\Pi_m = 0.2655$ and $\Pi_r = 0.1913$ in the second model. The minimum benefits that both sides claim together are:

Model 1:

$$\Pi_m^{SR} + \Pi_r^{SR} = 99.0453 + 125.8264 = 224.8717 < \Pi_{m+r}^c = 289.2814 \tag{28}$$

Model 2:

$$\Pi_m^{SR} + \Pi_r^{SR} = 0.2655 + 0.1913 = 0.4568 < \Pi_{m+r}^c = 0.5521 \tag{29}$$

Obviously, the minimum benefits that both sides claim is lower than that of in the cooperative game. Therefore, there exists an incentive for cooperation between the manufacturer and the retailer. Notice that the non-cooperative settings are not beneficial to any of the players because in the non-cooperative settings, both players have lower profits.

Example 2. Here, the parameters are as follows: $D_0 = 5$, c = 1, d = 1, $\alpha_2 = 1400$, $\beta_2 = 700$, $\delta = 1.4$, $\gamma = 1$, $e_1 = 3$, $k_1 = 3$, and $k_2 = 4$. It is considered that the random element to model noise effect (x) has a normal distribution with parameters ($\mu = 0, \sigma = 1$) then $E[e_1^x] = e_1^{\mu+\frac{\sigma}{2}} = \sqrt{e_1} = 1.6487$. The optimal values for the decision variables in both models under the different game-theoretic approaches are given in Tables 5 and 6, respectively.

Table 5. The results for the numerical example 2 in the first model.

CG	*SG-RL*	*SG-ML*	*NG*	**Decision Variables**
3	2.3333	4.4682	6	p_1
-	1.1667	1.9791	3	w_1
5.5582	5.4994	2	45.0836	A_1
5.0489	2.4477	5.3966	50.9113	a_1
-	0	0	0	t_1
-	113.3375	96.6574	41.3799	$E[\Pi_m^1]$
-	116.3238	144.1467	40.6934	$E[\Pi_r^1]$
330.3593	229.6614	240.7929	76.9857	$E[\Pi_{sc}^1]$

Table 6. The results for the numerical example 2 in the second model.

CG	*SG-RL*	*SG-ML*	*NG*	**Decision Variable**
0.5000	0.5000	0.7170	0.6667	p_2
-	0.2500	0.4334	0.3333	w_2
0.5656	0.1414	0.1368	0.1117	A_2
0.2514	0.2514	0.0067	0.0502	a_2
-	0	0.4948	0	t_2
-	0.3928	0.1738	0.2111	$E[\Pi_m^2]$
-	0.2828	0.1997	0.2707	$E[\Pi_r^2]$
0.8171	0.6758	0.3735	0.4842	$E[\Pi_{sc}^2]$

Similarly, it is found that the players prefer to collaborate with each other due to higher profit.

5. Sensitivity Analysis

This section provides some sensitivity analyses for different values of parameters to investigate their influence on the decision variables. Tables 7–14 show the sensitivity analyses for all four games in both models. Tables 7–10 present the results of the first model and Tables 11–14 show the results of the second model.

In the Nash game, based on the results shown in Table 7, it is observed that:

- The p_1^N, w_1^N, A_1^N, a_1^N, and t_1^N are not sensitive regarding α_2 changes while $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are highly sensitive such that increasing α_2 causes increasing $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$, and vice versa.
- Additionally, p_1^N, w_1^N, and t_1^N are not sensitive regarding β_2 and δ changes while A_1^N, a_1^N, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive as when β_2 increases, a_1^N and A_1^N increase and $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ decrease. Moreover, decreasing δ decrease a_1^N and A_1^N and $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ increase.
- Furthermore, p_1^N, w_1^N, a_1^N, and t_1^N are not sensitive regarding γ changes while $E[\Pi_r^1]$ is slightly sensitive and A_1^N, $E[\Pi_m^1]$, and $E[\Pi_{sc}^1]$ are sensitive so that when γ increases A_1^N, $E[\Pi_r^1]$ increase and $E[\Pi_m^1]$ and $E[\Pi_{sc}^1]$ decrease, and vice versa.
- Moreover, p_1^N, w_1^N, A_1^N, a_1^N, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are highly and t_1^N is not sensitive regarding e_1 changes. Nonetheless, when e_1 increases all of them decrease and vice versa.
- Likewise, p_1^N, w_1^N and t_1^N are not sensitive regarding $E[e_1^x]$ changes while A_1^N, a_1^N, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive in order that increasing $E[e_1^x]$, increases A_1^N, a_1^N, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$, and vice versa.

Table 7. The results of the sensitivity analysis in the Nash game for the first model.

$E[\Pi_{sc}^1]$	$E[\Pi_r^1]$	$E[\Pi_m^1]$	t_1^N	a_1^N	A_1^N	w_1^N	p_1^N	% Changes	Parameters
80.1	80.1	80.1	0	0	0	0	0	+40	$\alpha_2 = 1000$
40.0	40.0	40.0	0	0	0	0	0	+20	
−40.0	−40.0	−40.0	0	0	0	0	0	−20	
−80.1	−80.1	−80.1	0	0	0	0	0	−40	
−40.0	−40.0	−40.0	0	40.0	40.0	0	0	+40	$\beta_2 = 500$
−20.0	−20.0	−20.0	0	20.0	20.0	0	0	+20	
20.0	20.0	20.0	0	−20.0	−20.0	0	0	−20	
40.0	40.0	40.0	0	−40.0	−40.0	0	0	−40	
−20.0	0.1	−40.0	0	0	40.0	0	0	+40	$\gamma = 1$
−10.0	−0.04	−20	0	0	20.0	0	0	+20	
9.9	−0.1	19.9	0	0	−20.0	0	0	−20	
19.6	−0.4	39.6	0	0	−40.0	0	0	−40	
−37.3	−60.2	−14.4	0	60.2	14.4	0	0	+40	$\delta = 1$
−14.1	−24.4	−3.7	0	24.5	3.7	0	0	+20	
5.8	16.3	4.7	0	−16.3	−4.6	0	0	−20	
1.6	26.1	23.0	0	−26.4	−22.7	0	0	−40	
−64.8	−64.8	−64.8	0	−64.7	−64.7	−36.4	−36.4	+40	$e = 3$
−40.0	−40.0	−40.0	0	−39.9	−39.9	−25.0	−25.0	+20	
38.8	38.8	38.8	0	38.8	38.8	100.0	100.0	−20	
Infeasible								−40	
40.0	40.0	40.0	0	40.0	40.0	0	0	+40	$E[e_1^x] = 1.6487$
20.0	20.0	20.0	0	20.0	20.0	0	0	+20	
−20.0	−20.0	−20.0	0	−20.0	−20.0	0	0	−20	
−40.1	−40.1	−40.1	0	−40.0	−40.0	0	0	−40	

Table 8. The results of the sensitivity analysis in the SG-ML for the first model.

$E[\Pi_{sc}^1]$	$E[\Pi_r^1]$	$E[\Pi_m^1]$	t_1^{SM}	a_1^{SM}	A_1^{SM}	w_1^{SM}	p_1^{SM}	% Changes	Parameters
43.0	43.2	42.8	0	−0.3	0.1	0.3	0.2	+40	$\alpha_2 = 1000$
21.5	21.6	21.4	0	−0.2	0.1	0.2	0.1	+20	
−21.5	−21.6	−21.4	0	0.3	−0.1	−0.3	−0.2	−20	
−43.0	−43.2	−42.8	0	0.7	−0.4	−0.8	−0.5	−40	
−0.9	−0.9	−0.8	0	12.0	11.8	−0.1	−0.1	+40	$\beta_2 = 500$
−0.5	−0.5	−0.4	0	6.3	6.2	−0.1	−0.047	+20	
0.5	0.6	0.5	0	−7.2	−7.1	0.1	0.1	−20	
1.2	1.2	1.1	0	−15.8	−15.6	0.2	0.1	−40	
1.7	1.3	2.1	0	9.02	−4.0	0	0	+40	$\gamma = 1$
0.9	0.7	1.1	0	5.7	−2.6	0	0	+20	
−8.0	−0.6	−0.9	0	−7.9	4.6	0	0	−20	
−1.8	−1.31	−2.4	0	−19.8	12.8	0	0	−40	
1.2	1.6	0.75	0	−39.2	−3.2	0	0	+40	$\delta = 1$
0.6	0.7	0.41	0	−20.5	−2.8	0	0	+20	
−0.8	−1.01	−0.49	0	25.9	6.2	0	0	−20	
−1.4	−1.9	−1.15	0	61.2	16.5	0	0	−40	
−69.2	−69.3	−69.4	0	−35.6	−31.9	−4.3	−3.9	+40	$e = 3$
−45.7	−45.5	−45.8	0	−20.1	−20.2	−2.6	−2.5	+20	
88.6	89.1	88.1	0	27.3	24.2	3.4	4	−20	
293.8	290.9	296.4	0	67.4	62.3	8.9	8.8	−40	
42.1	42.2	41.9	0	11.6	12.0	0.2	0.2	+40	$E[e_1^x] = 1.6487$
21.0	21.1	20.9	0	6.1	6.3	0.1	0.1	+20	
−21.0	−21.0	−20.9	0	−7.0	−7.2	−0.2	−0.1	−20	
−41.8	−41.9	−41.7	0	−15.3	−15.8	−0.5	−0.3	−40	

Table 9. The results of the sensitivity analysis in the SG-RL for the first model.

$E[\Pi_{sc}^1]$	$E[\Pi_r^1]$	$E[\Pi_m^1]$	t_1^{SR}	a_1^{SR}	A_1^{SR}	w_1^{SR}	p_1^{SR}	% Changes	Parameters
43.1	42.5	43.7	0	0	0	0	0	+40	$\alpha_2 = 1000$
21.6	21.3	21.8	0	0	0	0	0	+20	
−21.6	−21.3	−21.8	0	0	0	0	0	−20	
−43.1	−42.5	−43.7	0	0	0	0	0	−40	
−0.9	−0.7	−1.1	0	11.9	11.9	0	0	+40	$\beta_2 = 500$
−0.5	−0.4	−0.6	0	6.3	6.3	0	0	+20	
0.6	0.5	0.7	0	−7.2	−7.2	0	0	−20	
1.2	1.0	1.4	0	−15.7	−15.7	0	0	−40	
1.4	1.2	1.6	0	8.6	−4.0	0	0	+40	$\gamma = 1$
0.8	0.6	0.9	0	5.4	−2.6	0	0	+20	
−1.0	−0.7	−1.2	0	−8.5	4.6	0	0	−20	
−2.0	−1.4	−2.7	0	−21.5	12.8	0	0	−40	
1.1	1.5	0.7	0	−37.1	−3.2	0	0	+40	$\delta = 1$
0.6	0.8	0.4	0	−21.2	−2.8	0	0	+20	
−0.7	−0.9	−0.5	0	28.6	6.2	0	0	−20	
−1.6	−2.0	−1.2	0	67.0	16.5	0	0	−40	
−72.1	−71.6	−72.7	0	−33.6	−33.6	−4.1	−4.1	+40	$e = 3$
−47.5	−47.1	−47.9	0	−19.0	−19.0	−2.4	−2.4	+20	
96.3	95.5	97.2	0	26.1	26.1	3.6	3.6	−20	
309.5	306.4	312.7	0	65.1	65.1	9.5	9.5	−40	
42.2	41.8	42.6	0	11.9	11.9	0	0	+40	$E[e_1^x] = 1.6487$
21.1	20.9	21.3	0	6.3	6.3	0	0	+20	
−21.0	−20.8	−21.2	0	−7.2	−7.2	0	0	−20	
−41.9	−41.5	−42.2	0	−15.7	−15.9	0	0	−40	

Table 10. The results of the sensitivity analysis with a cooperative game for the first model.

$E[\Pi^1_{sc}]$	a^{co}_1	A^{co}_1	p^{co}_1	% Changes	Parameters
42.2	0	0	0	+40	$\alpha_2 = 1000$
21.1	0	0	0	+20	
−21.1	0	0	0	−20	
−42.2	0	0	0	−40	
−0.7	11.9	11.9	0	+40	$\beta_2 = 500$
−0.3	6.3	6.3	0	+20	
0.4	−7.2	−7.2	0	−20	
0.9	−15.7	−15.7	0	−40	
1.0	−9.4	−35.3	0	+40	$\gamma = 1$
0.6	−4.7	−20.6	0	+20	
−0.7	4.1	30.1	0	−20	
−1.7	6.3	77.2	0	−40	
1.1	−28.5	0.1	0	+40	$\delta = 1$
0.6	−15.9	0.9	0	+20	
−0.7	20.1	−3.9	0	−20	
−1.5	45.6	−12.6	0	−40	
−72.8	−33.6	−33.6	−12.5	+40	$e = 3$
−48.5	−19.0	−19.0	−7.7	+20	
104.6	26.1	26.1	14.3	−20	
366.1	65.1	65.1	50.0	−40	
41.6	11.9	11.9	0	+40	$E[e^x_1] = 1.6487$
20.8	6.3	6.3	0	+20	
−20.7	−7.2	−7.2	0	−20	
−20.8	−15.7	−15.7	0	−40	

Table 11. The results of the sensitivity analysis with a Nash game for the second model.

$E[\Pi^2_{sc}]$	$E[\Pi^2_r]$	$E[\Pi^2_m]$	t^N_2	a^N_2	A^N_2	w^N_2	p^N_2	% Changes	Parameters
29.6	17.4	45.4	0	95.8	0	0	0	+40	$k_1 = 2$
13.5	8.0	20.9	0	43.8	0	0	0	+20	
−11.1	−6.6	−16.9	0	−36.0	0	0	0	−20	
−19.7	−11.7	−30.0	0	−64.0	0	0	0	−40	
66.5	78.5	51.1	0	0	96.0	0	0	+40	$k_2 = 3$
30.5	35.9	23.5	0	0	44.0	0	0	+20	
−24.9	−29.5	−19.0	0	0	−36.0	0	0	−20	
−44.3	−52.4	−33.8	0	0	−64.0	0	0	−40	
96.1	95.9	96.3	0	95.8	96.0	0	0	+40	$E[e^x_1] = 1.6487$
44.0	43.9	44.2	0	43.8	44.0	0	0	+20	
−36.0	−36.1	−35.9	0	−36.0	−36.0	0	0	−20	
−64.0	−64.0	−64.0	0	−64.0	−64.0	0	0	−40	

Table 12. The results of the sensitivity analysis with a SG-ML for the second model.

$E[\Pi_{sc}^2]$	$E[\Pi_r^2]$	$E[\Pi_m^2]$	t_2^{SM}	a_2^{SM}	A_2^{SM}	w_2^{SM}	p_2^{SM}	% Changes	Parameters
19.6	17.6	22.0	−4.4	138.1	−1.7	−3.3	−1.0	+40	$k_1 = 2$
9.0	8.2	10.0	−2.3	61.9	−0.9	−1.8	−0.5	+20	
−7.1	−7.0	−7.3	2.7	−42.9	1.0	2.1	0.6	−20	
−12.8	−13.1	−12.6	5.7	−71.4	1.8	4.5	1.4	−40	
77.0	77.0	76.9	3.9	−14.3	98.6	3.1	0.9	+40	$k_2 = 3$
35.2	35.5	34.8	2.2	−9.5	45.1	1.7	0.5	+20	
−28.7	−29.4	−27.9	−2.9	14.3	−36.7	−2.2	−0.7	−20	
−50.6	−52.3	−48.7	−6.7	35.7	−65.0	−5.1	−1.5	−40	
96.1	96.0	96.1	0	97.6	96.0	0	0	+40	$E[e_1^x] = 1.6487$
44.1	44.0	44.1	0	45.2	44.0	0	0	+20	
−36.0	−36.0	−35.9	0	−35.7	−36.0	0	0	−20	
−64.0	−64.0	−64.1	0	−64.3	−64.0	0	0	−40	

Table 13. The results of the sensitivity analysis in the SG-RL for the second model.

$E[\Pi_{sc}^2]$	$E[\Pi_r^2]$	$E[\Pi_m^2]$	t_2^{SR}	a_2^{SR}	A_2^{SR}	w_2^{SR}	p_2^{SR}	% Changes	Parameters
35.7	−0.1	61.4	0	96.0	0	0	0	+40	$k_1 = 2$
16.4	−0.1	28.1	0	44.0	0	0	0	+20	
−13.4	−0.1	−23.1	0	−36.0	0	0	0	−20	
−23.8	−0.1	−41.0	0	−64.0	0	0	0	−40	
60.2	95.8	34.5	0	0	95.9	0	0	+40	$k_2 = 3$
27.6	43.9	15.8	0	0	43.9	0	0	+20	
−22.6	−36.0	−13.0	0	0	−36.0	0	0	−20	
−40.2	−64.0	−23.1	0	0	−64.0	0	0	−40	
77.7	161.4	17.3	0	−51.0	96.0	0	0	+40	$E[e_1^x] = 1.6487$
30.6	92.1	−13.8	0	−64.0	43.9	0	0	+20	
−42.0	−14.6	−61.7	0	−84.0	−36.0	0	0	−20	
−67.4	−52.0	−78.5	0	−91.0	−64.0	0	0	−40	

Table 14. The results of the sensitivity analysis in the cooperative game for the second model.

$E[\Pi_{sc}^2]$	a_2^{co}	A_2^{co}	p_2^{co}	% Changes	Parameters
29.5	96.0	0	0	+40	$k_1 = 2$
13.5	44.0	0	0	+20	
−11.1	−36.0	0	0	−20	
−19.7	−64.0	0	0	−40	
66.5	0	96.0	0	+40	$k_2 = 3$
30.5	0	44.0	0	+20	
−24.9	0	−36.0	0	−20	
−44.3	0	−64.0	0	−40	
96.0	96.0	96.0	0	+40	$E[e_1^x] = 1.6487$
44.0	44.0	44.0	0	+20	
−36.0	−36.0	−36.0	0	−20	
−64.0	−64.0	−64.0	0	−40	

Hence, under the first model where the Nash game is established between the manufacturer and the retailer, it was found that all the decision variables are considerably sensitive regarding the noise effect changes. It means that the noise effect changes of a new product signals to the manufacturer and the retailer to change their advertising policies so that increasing the noise effect motivates the manufacturer and the retailer to advertise more, and although their costs increase, their profits are higher.

In the Stackelberg game, when the manufacturer is the leader (See Table 8), it is concluded that:

- The p_1^{SM}, w_1^{SM}, A_1^{SM}, a_1^{SM} are slightly sensitive concerning α_2 changes while $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive. Conversely, when α_2 increases, p_1^{SM}, w_1^{SM}, A_1^{SM}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ increase and a_1^{SM} decreases.
- According to Szmerekovsky and Zhang [11], $t_1^{SM^*}$ is zero; thus, t_1^{SM} is not sensitive regarding the parameter changes.
- Likewise, p_1^{SM}, w_1^{SM}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are slightly sensitive regarding β_2 changes while a_1^{SM} and A_1^{SM} are sensitive so that when β_2 increases, a_1^{SM} and A_1^{SM} increase and p_1^{SM}, w_1^{SM}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ decrease, and vice versa.
- Similarly, p_1^{SM}, w_1^{SM}, A_1^{SM}, a_1^{SM}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive regarding e_1 changes. Nonetheless, when e_1 increases all of them decrease and vice versa.
- Moreover, p_1^{SM} and w_1^{SM} are slightly sensitive regarding $E[e_1^x]$ changes while A_1^{SM}, a_1^{SM}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive. When $E[e_1^x]$ increases all of them increase and vice versa.

Therefore, under the Stackelberg manufacturer game, it was found that all the decision variables are significantly sensitive regarding the noise effect changes. It is stated that the noise effect is an important factor on the sales of a new product so that when the effect of noise increases, the chain's members are incentivized to advertise their product more than earlier, leading to increased popularity of the product, promoting their market share, and consequently enhancing the chain profit.

Table 9 shows the results of the Stackelberg game when the retailer is the leader. As it is shown:

- The α_2 and β_2 changes the influence on $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$. When α_2 increases and β_2 decreases, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$, increase and vice versa. Furthermore, A_1^{SR} and a_1^{SR} are sensitive regarding β_2 changes so that when β_2 increases, A_1^{SR} and a_1^{SR} increase.
- Equally, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are slightly sensitive regarding γ changes.
- Moreover, A_1^{SR} and a_1^{SR} are sensitive regarding γ and δ changes such that when γ increases, a_1^{SR}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ increase and A_1^{SR} decreases and vice versa. Furthermore, increasing δ increase, a_1^{SR} and A_1^{SR} increase and $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$, decrease and vice versa.
- Additionally, p_1^{SR} and w_1^{SR} are slightly sensitive regarding e_1 changes while a_1^{SR}, A_1^{SR}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are highly sensitive so that when e_1 increases, p_1^{SR}, w_1^{SR}, A_1^{SR}, a_1^{SR}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ decrease, and vice versa.
- Additionally, A_1^{SR}, a_1^{SR}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ are sensitive regarding $E[e_1^x]$ changes. When $E[e_1^x]$ increases, A_1^{SR}, a_1^{SR}, $E[\Pi_m^1]$, $E[\Pi_r^1]$, and $E[\Pi_{sc}^1]$ increase, and vice versa.

Then, under the Stackelberg retailer game, it was found that the noise effect is an efficient factor for a new product launched to the market due to the fact that all the decision variables are significantly sensitive regarding its changes.

Table 10 presents the results under the cooperative game. From the results given in Table 10, it is easy to see that:

- $E[\Pi_{sc}^1]$ is slightly sensitive regarding α_2 and δ changes while A_1^{co}, a_1^{co} and $E[\Pi_{sc}^1]$ are sensitive concerning β_2, γ and δ changes so that when γ increases, a_1^{co} and A_1^{co} decrease and $E[\Pi_{sc}^1]$ increases. However, when δ increases, A_1^{co} and $E[\Pi_{sc}^1]$ increase and a_1^{co} decreases and vice versa.
- Furthermore, p_1^{co}, a_1^{co} and A_1^{co} are sensitive regarding e_1 changes while $E[\Pi_{sc}^1]$ is highly sensitive. When e_1 increases, p_1^{co}, a_1^{co} and A_1^{co}, and $E[\Pi_{sc}^1]$ decrease, and vice versa. Also, A_1^{co}, a_1^{co} and $E[\Pi_{sc}^1]$ are sensitive regarding $E[e_1^x]$ changes.

Thus, under the cooperative game, similarly, the noise effect plays a remarkable role in the chain profit changes so that increasing the effect of noise in the market, which is originated from satisfaction or dissatisfaction of customers, promotes the total profit of the chain due to more advertising by the partners.

Under the second model, when the Nash game is established among the manufacturer and the retailer (See Table 11), one can conclude that:

- Similar to the first model, k_1 and k_2 directly affect $E[\Pi_m^2]$, $E[\Pi_r^2]$ and $E[\Pi_{sc}^2]$. Also a_2^N and A_2^N are highly sensitive regarding k_1 and k_2 changes, respectively. Additionally, A_2^N, a_2^N, $E[\Pi_m^2]$, $E[\Pi_r^2]$, and $E[\Pi_{sc}^2]$ are highly sensitive regarding $E[e_1^x]$ changes so that by increasing $E[e_1^x]$ then a_2^N, A_2^N, $E[\Pi_m^2]$, $E[\Pi_r^2]$ and $E[\Pi_{sc}^2]$ increase and vice versa.

Under the Stackelberg game, when the manufacturer is the leader, from Table 12, it is concluded that:

- As with the previous model, $E[\Pi_m^2]$, $E[\Pi_r^2]$ and $E[\Pi_{sc}^2]$ are sensitive regarding the changes of k_1 and k_2 while p_2^{SM}, w_2^{SM}, t_2^{SM} are slightly sensitive. Additionally a_2^{SM} and A_2^{SM} are highly sensitive regarding k_1 and k_2 changes, respectively. Furthermore, A_2^{SM}, a_2^{SM}, $E[\Pi_m^2]$, $E[\Pi_r^2]$, and $E[\Pi_{sc}^2]$ are highly sensitive regarding $E[e_1^x]$ changes so that by increasing $E[e_1^x]$ then A_2^{SM}, a_2^{SM}, $E[\Pi_m^2]$, $E[\Pi_r^2]$, and $E[\Pi_{sc}^2]$ increase, and vice versa.

In the Stackelberg game, when the retailer is the leader, it is easy to observe (See Table 13) that:

- Here, it is found that k_1 and k_2 highly influence $E[\Pi_m^2]$ and $E[\Pi_r^2]$, respectively, and subsequently affect $E[\Pi_{sc}^2]$. Additionally, a_2^{SR} and A_2^{SR} are highly sensitive regarding the changes of k_1 and k_2, respectively. Moreover, A_2^{SR}, a_2^{SR}, $E[\Pi_m^2]$, $E[\Pi_r^2]$, and $E[\Pi_{sc}^2]$ are highly sensitive regarding $E[e_1^x]$ changes so that by increasing $E[e_1^x]$ then a_2^{SR}, A_2^{SR}, $E[\Pi_m^2]$, $E[\Pi_r^2]$ and $E[\Pi_{sc}^2]$ increase, and vice versa.

Finally, in the cooperative game, whose results are given in Table 14, it was found that:

- In this case, a_2^{co} is highly sensitive and $E[\Pi_{sc}^2]$ is sensitive regarding k_1 changes. Conversely, A_2^{co} and $E[\Pi_{sc}^2]$ are highly sensitive so that increasing k_1 leads to an increase A_2^{co} and $E[\Pi_{sc}^2]$, and vice versa. Moreover, p_2^{co} is not sensitive concerning $E[e_1^x]$ changes while A_2^{co}, a_2^{co}, and $E[\Pi_{sc}^2]$ are highly sensitive.

Similar to the first model, it is concluded that the noise effect changes affect all the decision variables more than other parameters. So, this issue is correctly considered in the market demand of the new product and it has a considerable effect on the chain profit.

6. Conclusions

This research evaluates pricing and marketing decisions under a cooperative advertising strategy in a two-echelon supply chain comprised of one manufacturer and one retailer. Therefore, the pricing, advertising, and noise effect are proposed as the marketing policies into the market demand by considering two well-known different demand functions under four game-theoretic attitudes consisting of three non-cooperative games (i.e., Nash equilibrium, Stackelberg equilibrium when the manufacturer is the leader, and Stackelberg equilibrium when the retailer is the leader) and one cooperative game. Using a numerical example, it was found that under the non-cooperative approaches, the sum of the minimum benefits that both sides gain is lower than that of under the cooperative game.

In the first model, the price changes have a large impact on the demand. Under a non-cooperative environment, the retailer tries to increase his/her profits through increasing the local advertising because it causes an increase to their market share and consequently enhances their profit. In the second model, it is observed that the effects of national advertising on the manufacturer's, retailer's and whole supply chain's profits are higher than local advertising. Moreover, in both models, it was shown that both global and local advertising are sensitive regarding the noise effect changes such that its changes signal to the manufacturer and the retailer to advertise more due to increasing the new product popularity. So, the total profits of the manufacturer, the retailer, as well as the whole supply chain are sensitive regarding the noise effect changes and increasing with increasing the effect of noise of the new product.

Additionally, in the second model, it was indicated that the sensitivities of the advertising and profit function regarding the changes of the noise effect are higher than the sensitivities of the advertising and profit function, in the first model. Furthermore, in the second model, it was found that the most influential parameter on the profit function is the noise effect parameter. Thus, it can be claimed that the noise effect in the market demand of the new product is correctly considered and it has a considerable impact on the chain profit. However, in the non-cooperative games, the manufacturer tries to globally advertise less than the retailer. So, increasing the retailer's profit will cause a bigger selling price, and this makes higher profits for the retailer compared with the manufacturer's profit. Consequently, the manufacturer, in order to not be removed from the market, has to pay the co-op advertising costs and they prefer to cooperate in advertising.

There are several research directions that can be carried out, which can be outlined as follows:

- Consider other types of demand functions with even asymmetric or non-asymmetric information.
- Allow that parameters and variables be time-dependent (dynamic).
- Consider a two-echelon supply chain comprised of two manufacturers and one retailer where there exits competition between the manufacturers.
- Extend the present research work to a three-echelon supply chain consisting of a supplier, a manufacturer, and a retailer.

Author Contributions: Conceptualization, A.A.T. and L.E.C.-B.; methodology, A.A.T. and L.E.C.-B.; software, Z.C., L.E.C.-B., and M.N.-D.; validation, Z.C., L.E.C.-B., and M.N.-D.; formal analysis, A.A.T., Z.C., L.E.C.-B., M.N.-D.; investigation, Z.C.; resources, A.A.T. and L.E.C.-B.; data curation, Z.C.; writing—original draft preparation, Z.C.; writing—review and editing, A.A.T. and L.E.C.-B.; visualization, Z.C., and M.N.-D; supervision, A.A.T. and L.E.C.-B.; project administration, A.A.T. and L.E.C.-B.; funding acquisition, Not applicable. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

Variables

p:	Retail price
w:	Wholesale price
A:	Global advertising expenditures
a:	Local advertising expenditures
t:	Advertising participation rate

Parameters

D_0:	Base demand rate
α_1:	Price demand potential
β_1:	Price sensitivity
α_2:	Market cap or sales saturate asymptote
β_2:	Impact of local advertisement on market demand
γ:	Quasi-local advertising elasticity
δ:	Quasi-global advertising elasticity
e:	Price-elasticity index
k_1:	Effectiveness of local advertising
k_2:	Effectiveness of global advertising
c:	Manufacturer's unit production cost
d:	Retailer's unit handling cost
e_1:	Noise sensitivity index
x:	Random element to model noise effect
Π:	Expected profit

Appendix A. Proof of Theorem 1

For the first model, we can claim that the manufacturer's profit function is a concave function regarding A because the second derivative of the manufacturer's profit function regarding A is negative. Therefore, we have:

$$\frac{\partial^2 E[\Pi_m^1(w,A,t)]}{\partial A^2} = -(\delta+1)\delta D_0 E[e_1^x]\beta_2(w-c)(p^{-e})a^{-\gamma}A^{-(\delta+2)} < 0 \tag{A1}$$

Consequently, the optimal value of A is now obtained by setting the first derivative regarding to A equal to zero, which is given by

$$\frac{\partial E[\Pi_m^1(w,A,t)]}{\partial A} = \delta D_0 E[{e_1}^x]\beta_2(w-c)(p^{-e})a^{-\gamma}A^{-(\delta+1)} - 1 = 0$$

So, we have:

$$t^* = 0 \tag{A2}$$

$$w^* = \frac{p}{2} \tag{A3}$$

$$A^* = \left(\delta D_0 E[e_1^x]\beta_2(w-c)(p^{-e})a^{-\gamma}\right)^{\frac{1}{\delta+1}} \tag{A4}$$

To solve the retailer's problem, z is defined as $z = (p-w-d)(p^{-e})\ ;\ w+d \le p$. Taking the partial derivative of z regarding p yields:

$$\frac{\partial z}{\partial p} = (p^{-e}) - ep^{-(e+1)}(p-w-d) = 0 \ \Rightarrow (p^{-e-1})(p - e(p-w-d)) = 0 \tag{A5}$$

From Equation (A5), it is easy to see that $p^{-e-1} > 0$. Thus, the value of $p = p_1$ is equal to $\frac{e(w+d)}{e-1}$. The range of z is determined so that $p = w+d \Rightarrow z = 0,\ p = p_1 \Rightarrow z = e^{-e}(\frac{w+d}{e-1})^{1-e} > 0$, and $p \to \infty \Rightarrow z = 0$. So, the maximum value of z is the yield when $p = p_1$ and its minimum value is zero. Hence, $0 \le z \le e^{-e}(\frac{w+d}{e-1})^{1-e}$. As a result, the retailer's objective function shown in Equation (16) is rewritten as follows:

$$\begin{gathered} \underset{p,a}{Max} E[\Pi_r^1(p,a)] = D_0 E[e_1^x] z(\alpha_2 - \beta_2 a^{-\gamma}A^{-\delta}) - (1-t)a \\ s.t.\ \ 0 \le z \le e^{-e}(\tfrac{w+d}{e-1})^{1-e},\ \ 0 \le a \end{gathered} \tag{A6}$$

Of course, the expected profit $E[\Pi_r^1(p,a)]$ regarding z is an increasing function because its partial derivative regarding z is a positive value ($\frac{\partial E[\Pi_r^1(p,a)]}{\partial z} = D_0 E[e_1^x](\alpha_2 -$

$\beta_2 a^{-\gamma} A^{-\delta}) > 0$). Therefore, in this case, the optimal value of z is equal to $e^{-e}(\frac{w+d}{e-1})^{1-e}$. Now, the second derivative of the retailer's profit function regarding a is equal to $\frac{\partial^2 E[\Pi_r^1(p,a)]}{\partial a^2}$ $= -(\gamma+1)\gamma D_0 E[e_1^x] z \beta_2 a^{-(\gamma+2)} A^{-\delta} < 0$. Thus, the optimum value of a is now obtained from the first derivative. As a result, the optimal solution of the retailer's problem is as follows:

$$p^* = \frac{e(w+d)}{e-1} \tag{A7}$$

$$z^* = e^{-e}(\frac{w+d}{e-1})^{1-e} \tag{A8}$$

$$a^* = \left(\frac{\gamma D_0 E[e_1^x] z \beta_2 A^{-\delta}}{1-t}\right)^{\frac{1}{\gamma+1}} \tag{A9}$$

Using the obtained solutions, the following equilibrium points $p_1^N = \frac{2ed}{e-2}$, $w_1^N = \frac{ed}{e-2}$, $t_1^N = 0$, $A_1^N = (\frac{\delta}{\gamma})\frac{e(d-c)+2c}{(2d)} a_1^N$ and $a_1^N = \left(\gamma D_0 E[e_1^x] \beta_2 e^{-e}(\frac{2d}{e-2})^{1-e}\right)\left((\frac{\delta}{\gamma})\frac{e(d-c)+2c}{(2d)}\right)^{-\delta}$ are derived. It is important to remark that we use superscript N to denote the Nash game.

Appendix B. Proof of Theorem 2

In the second model, we can claim that the manufacturer's profit function is a concave function regarding A because $\frac{\partial^2 E[\Pi_m^2(w,A,t)]}{\partial A^2} = -\frac{1}{4}E[e_1^x]k_2 w(1-p)A^{-\frac{3}{2}} < 0$. So, the optimal value of A is obtained by using $\frac{\partial E[\Pi_m^2(w,A,t)]}{\partial A} = \frac{1}{2}E[e_1^x]k_2 w(1-p)A^{-\frac{1}{2}} - 1 = 0$. Consequently, the optimal solution of the manufacturer's problem is given by $t^* = 0$, $w^* = \frac{p}{2}$, and $A^* = \left(\frac{1}{2}E[e_1^x]k_2 w(1-p)\right)^2$. Now, z is defined as $z = (p-w)(1-p)$; $w \leq p \leq 1$. Then, taking the partial derivative of z regarding p leads to $\frac{\partial z}{\partial p} = (1-p) - (p-w) = 0$. So, the value for $p = p_1$ is equal to $\frac{e(w+d)}{e-1}$. Additionally, the range of z is obtained when $p = 1 \Rightarrow z = 0$, $p = p_1 \Rightarrow z = (\frac{1-w}{2})^2 > 0$, and $p = w \Rightarrow z = 0$. Therefore, we can write the following optimization problem:

$$\begin{gathered} MaxE[\Pi_r^2(p,a)] = E[e_1^x] z (k_1\sqrt{a} + k_2\sqrt{A}) - (1-t)a \\ s.t.\ 0 \leq z \leq (\frac{1-w}{2})^2,\ 0 \leq a \end{gathered} \tag{A10}$$

Following the same procedure as the first model, the optimal value of z is equal to $(\frac{1-w}{2})^2$. Since $\frac{\partial^2 E[\prod_r]}{\partial a^2} = -\frac{1}{4}E[e_1^x] z k_1 a^{-\frac{3}{2}} < 0$, a^* is obtained using the first derivative $\frac{\partial E[\prod_r]}{\partial a} = \frac{1}{2}E[e_1^x] z k_1 a^{-\frac{1}{2}} - (1-t) = 0$. Thus, the optimal solutions of a^*, p^*, and z^* are $a^* = \left(\frac{E[e_1^x]k_1}{2(1-t)}(\frac{1-w}{2})^2\right)^2$, $p^* = \frac{1+w}{2}$ and $z^* = (\frac{1-w}{2})^2$. Hence, by using these solutions, the equilibrium points $p_2^N = \frac{2}{3}$, $w_2^N = \frac{1}{3}$, $t_2^N = 0$, $A_2^N = \frac{1}{324}E[e_1^x]^2 k_2^2$ and $a_2^N = \frac{1}{324}E[e_1^x]^2 k_1^2$ are easily derived.

Appendix C. Proof of Theorem 3

In this case, as mentioned earlier, the manufacturer and the retailer play a leader and a follower, respectively. So, under this approach, the best responses of the retailer should be substituted (i.e., Equations (18) and (19)) into the manufacturer's profit function. The manufacturer's profit function in the first model changes to:

$$\underset{w,A,t}{Max}E[\Pi_m^1(w,A,t)] = E[e_1^x]D_0(w-c)\times\left(\frac{e(w+d)}{e-1}\right)^{-e}\left(\alpha_2-\beta_2\left(\frac{\gamma E[e_1^x]D_0\beta_2A^{-\delta}e^{-e}\left(\frac{w+d}{e-1}\right)^{1-e}}{1-t}\right)^{\frac{-\gamma}{\gamma+1}}A^{-\delta}\right) - A - t\left(\frac{\gamma E[e_1^x]D_0\beta_2A^{-\delta}e^{-e}\left(\frac{w+d}{e-1}\right)^{1-e}}{1-t}\right)^{\frac{1}{\gamma+1}} \quad s.t.\ 0\le A,\ c\le w\le 1,\ \text{and } 0\le t\le 1 \tag{A11}$$

By simplifying the above equation, we have:

$$\begin{aligned}\underset{w,A,t}{Max}E[\Pi_m^1(w,A,t)] = {} & -A - E[e_1^x]^{\frac{1}{\gamma+1}}D_0^{\frac{1}{\gamma+1}}A^{\frac{-\delta}{\gamma+1}}t(1-t)^{\frac{-1}{\gamma+1}}(e-1)^{\frac{e-1}{\gamma+1}}e^{\frac{-e}{\gamma+1}}(w+d)^{\frac{1-e}{\gamma+1}}\beta_2^{\frac{1}{\gamma+1}}\gamma^{\frac{1}{\gamma+1}} \\ & -E[e_1^x]^{\frac{1}{\gamma+1}}D_0^{\frac{1}{\gamma+1}}A^{\frac{-\delta}{\gamma+1}}(1-t)^{\frac{\gamma}{\gamma+1}}(e-1)^{\frac{e+\gamma}{\gamma+1}}e^{\frac{-e}{\gamma+1}}(w+d)^{\frac{-(\gamma+e)}{\gamma+1}}(w-c)\beta_2^{\frac{1}{\gamma+1}}\gamma^{\frac{-\gamma}{\gamma+1}} \\ & +E[e_1^x]D_0\alpha_2(e-1)^e e^{-e}(w+d)^{-e}(w-c)\end{aligned} \tag{A12}$$

Equation (A12) can be rewritten as follows:

$$Max\ E[\Pi_m^1(w,A,t)] = -A + xA^{\frac{-\delta}{\gamma+1}}(1-t)^{\frac{-1}{\gamma+1}}t + yA^{\frac{-\delta}{\gamma+1}}(1-t)^{\frac{\gamma}{\gamma+1}} + z \tag{A13}$$

where x, y and z are given by:

$$x = -E[e_1^x]^{\frac{1}{\gamma+1}}D_0^{\frac{1}{\gamma+1}}(e-1)^{\frac{e-1}{\gamma+1}}e^{\frac{-e}{\gamma+1}}(w+d)^{\frac{1-e}{\gamma+1}}\beta_2^{\frac{1}{\gamma+1}}\gamma^{\frac{1}{\gamma+1}} \tag{A14}$$

$$y = -E[e_1^x]^{\frac{1}{\gamma+1}}D_0^{\frac{1}{\gamma+1}}(e-1)^{\frac{e+\gamma}{\gamma+1}}e^{\frac{-e}{\gamma+1}}(w+d)^{\frac{-(\gamma+e)}{\gamma+1}}(w-c)\beta_2^{\frac{1}{\gamma+1}}\gamma^{\frac{-\gamma}{\gamma+1}} \tag{A15}$$

$$z = E[e_1^x]D_0\alpha_2(e-1)^e e^{-e}(w+d)^{-e}(w-c) \tag{A15}$$

To determine the optimal value of t, first, take the first derivative of Π_m regarding t and then set it to zero. So, the procedure is as follows:

$$\begin{aligned}\frac{\partial E[\Pi_m^1(w,A,t)]}{\partial t} &= \left((1-t)^{\frac{-1}{\gamma+1}} + \frac{1}{\gamma+1}(1-t)^{\frac{-(\gamma+2)}{\gamma+1}}t\right)xA^{\frac{-\delta}{\gamma+1}} - \frac{\gamma}{\gamma+1}(1-t)^{\frac{-1}{\gamma+1}}yA^{\frac{-\delta}{\gamma+1}} \\ &= \frac{1}{\gamma+1}A^{\frac{-\delta}{\gamma+1}}(1-t)^{\frac{-(\gamma+2)}{\gamma+1}}(\gamma(1-t)(x-y)+x) = 0\end{aligned} \tag{A17}$$

For every $A > 0$ and $0 \le t \le 1$, we have $\frac{1}{\gamma+1}A^{\frac{-\delta}{\gamma+1}}(1-t)^{\frac{-(\gamma+2)}{\gamma+1}} \ge 0$. Therefore, the partial derivative is determined by using $\gamma(1-t)(x-y)+x$. As a result, we have $t = 1+\frac{x}{\gamma(x-y)}$. The optimum value of t as a function of w is now obtained by substituting x and y given in Equations (A14) and (A15), respectively. So, $t = 1+\frac{(w+d)}{\gamma(w+d)-(e-1)(w-c)}$ is the same as when there is no noise effect. However, according to the research of Szmerekovsky and Zhang [11] in which they have shown that the optimal value of $t = 1+\frac{(w+d)}{\gamma(w+d)-(e-1)(w-c)}$ is always equal to zero and in this case (with the noise effect) by using the same approach, it can be shown that the optimal value of t^* is zero, too (see Szmerekovsky and Zhang [11]).

Moreover, to determine the optimum value of A, we get the second partial derivative of Equation (A13) regarding A as follows:

$$\frac{\partial^2 E[\Pi_m^1(w,A,t)]}{\partial A^2} = \frac{\delta(\delta+\gamma+1)}{(\gamma+1)^2}A^{\frac{-(\delta+2\gamma+2)}{\gamma+1}}x(1-t)^{\frac{-1}{\gamma+1}}t + \frac{\delta(\delta+\gamma+1)}{(\gamma+1)^2}A^{\frac{-(\delta+2\gamma+2)}{\gamma+1}}y(1-t)^{\frac{\gamma}{\gamma+1}} \tag{A18}$$

According to the sign of x and y, for all $c < w$ and $0 \le t \le 1$, we have $\frac{\partial^2 E[\Pi_m^1(w,A,t)]}{\partial A^2} \le 0$. Thus, the optimal value of A is obtained as follows:

$$\frac{\partial E[\Pi_m^1(w,A,t)]}{\partial A} = -1 - \frac{\delta}{\gamma+1}A^{\frac{-(\delta+\gamma+1)}{\gamma+1}}x(1-t)^{\frac{-1}{\gamma+1}}t - \frac{\delta}{\gamma+1}A^{\frac{-(\delta+\gamma+1)}{\gamma+1}}y(1-t)^{\frac{\gamma}{\gamma+1}} = 0 \tag{A19}$$

So, we have:

$$A^{\frac{(\delta+\gamma+1)}{\gamma+1}} = -\frac{\delta}{\gamma+1}(1-t)^{\frac{-1}{\gamma+1}}(xt+y(1-t)) \tag{A20}$$

By replacing the values of x and y shown in Equations (A14) and (A15), respectively, the optimal value of A is as follows:

$$\begin{aligned} A = \; & \delta^{\frac{1+\gamma}{1+\gamma+\delta}} E[e_1^x]^{\frac{1}{1+\gamma+\delta}} D_0^{\frac{1}{1+\gamma+\delta}} (e-1)^{\frac{e-1}{1+\gamma+\delta}} e^{\frac{-e}{1+\gamma+\delta}} (1-t)^{\frac{-1}{1+\gamma+\delta}} (w+d)^{\frac{-\gamma-e}{1+\gamma+\delta}} \beta_2^{\frac{1}{1+\gamma+\delta}} \gamma^{\frac{-\gamma}{1+\gamma+\delta}} (\gamma+1)^{\frac{-(1+\gamma)}{1+\gamma+\delta}} \\ & \times(\gamma(w+d)t+(w-c)(1-t)(e-1))^{\frac{1+\gamma}{1+\gamma+\delta}} \end{aligned} \tag{A21}$$

To get the wholesale price w, we must determine the first derivative of $E[\Pi_m^1(w,A,t)]$ regarding w.

$$\begin{aligned} \frac{\partial E[\Pi_m^1(w,A,t)]}{\partial w} = \; & \alpha_2 E[e_1^x] D_0 (\tfrac{e}{e-1})^{-e}\left[-e(w+d)^{-e-1}(w-c)+(w+d)^{-e}\right] \\ & -E[e_1^x] D_0 (\tfrac{e}{e-1})^{-e}\left(\gamma^{-\gamma}\beta_2 E[e_1^x]^{-\gamma} D_0^{-\gamma} A^{-\delta} e^{e\gamma}(e-1)^{\gamma-e\gamma}(1-t)^{\gamma}\right)^{\frac{1}{\gamma+1}} \\ & \times\left[\tfrac{-(\gamma+e)}{\gamma+1}(w+d)^{\frac{-(2\gamma+e+1)}{\gamma+1}}(w-c)+(w+d)^{\frac{-(\gamma+e)}{\gamma+1}}\right] \\ & -t\left(\gamma\beta_2 E[e_1^x] D_0 A^{-\delta} e^{-e}(e-1)^{e-1}(1-t)^{-1}\right)^{\frac{1}{\gamma+1}}\left[\tfrac{1-e}{\gamma+1}(w+d)^{\frac{-(e+\gamma)}{\gamma+1}}\right] \end{aligned} \tag{A22}$$

Then, the optimum value for w is calculated by using Equation (A22). By using the optimal value of w, the other optimal values are obtained.

Appendix D. Proof of Theorem 4

In this case, similar to the previous one, the follower's responses should be substituted (i.e., Equations (20) and (21)) into the manufacturer's profit function ($E[\Pi_m^2(w,A,t)]$) as follows:

$$\begin{aligned} Max\ E[\Pi_m^2(w,A,t)] = E[e_1^x]\, w(\tfrac{1-w}{2})\left(\left(\tfrac{E[e_1^x]k_1^2}{2(1-t)}(\tfrac{1-w}{2})^2\right)+k_2\sqrt{A}\right) - A - \tfrac{E[e_1^x]^2 k_1^2 t}{4(1-t)^2}(\tfrac{1-w}{2})^4 \\ s.t.\ \ 0\le A,\ \ 0\le w\le 1,\ \ \text{and } 0\le t\le 1 \end{aligned} \tag{A23}$$

The optimal values of A, t, and w are determined using the first partial derivative of $E[\Pi_m^2(w,A,t)]$ regarding A, t, and w, respectively. From the first derivative of $E[\Pi_m^2(w,A,t)]$ regarding A, the optimal value of A is:

$$\frac{\partial E[\Pi_m^2(w,A,t)]}{\partial A} = \frac{E[e_1^x]k_2}{2\sqrt{A}}(\frac{1-w}{2})w - 1 = 0 \quad \Rightarrow A = \frac{E[e_1^x]^2 k_2^2}{16} w^2(1-w)^2 \tag{A24}$$

The first derivative regarding t is given by:

$$\begin{aligned} \frac{\partial E[\Pi_m^2(w,A,t)]}{\partial t} &= (\tfrac{E[e_1^x]^2 k_1^2}{2(1-t)^2})(\tfrac{1-w}{2})^3 w - \tfrac{E[e_1^x]^2 k_1^2}{4}(\tfrac{t+1}{(1-t)^3})(\tfrac{1-w}{2})^4 \\ &= \tfrac{E[e_1^x]^2(\frac{1-w}{2})^3 k_1^2}{2(1-t)^2}\left[w - \tfrac{t+1}{4(1-t)}(1-w)\right] = 0 \end{aligned} \tag{A25}$$

Hence, the optimal value of t is:

$$t = \frac{5w-1}{3w+1} \tag{A26}$$

Moreover, the first derivative regarding w is equal to:

$$\frac{\partial E[\Pi_m^2(w,A,t)]}{\partial w} = 8k_2(1-2w)(1-t)^2\sqrt{A} + E[e_1^x]k_1^2(1-4w)(1-w)^2(1-t) + E[e_1^x]k_1^2(1-w)^3 t = 0 \tag{A27}$$

Substituting the optimal values of A and t into Equation (A27) yields:

$$\begin{aligned}\frac{\partial E[\Pi_m^2(w,A,t)]}{\partial w} = & \ 8(1-2w)\left(\tfrac{2(1-w)}{3w+1}\right)^2 k_2\left(\tfrac{E[e_1^x]k_2}{4}w(1-w)\right) \\ & +E[e_1^x]k_1^2(1-4w)(1-w)^2\left(\tfrac{2(1-w)}{3w+1}\right)+E[e_1^x]k_1^2(1-w)^3\tfrac{5w-1}{3w+1}=0\end{aligned} \tag{A28}$$

Equation (A28) is expressed as a quadratic function of w and is shown as below:

$$w^2(16{k_2}^2+9k_1^2)-w(8{k_2}^2)-k_1^2=0 \tag{A29}$$

Since $0<w<1$ and using $k=\frac{{k_2}^2}{k_1^2}$, the optimal value of w is as follows:

$$w=\frac{4k+\sqrt{16k^2+16k+9}}{9+16k} \tag{A30}$$

Using Equation (A30), the optimal values for the decision variables are derived and they are given by $A_2^{SM}=\left(\frac{E[e_1^x]k_2}{4}\right)^2\times\left(\frac{(\sqrt{16k^2+16k+9}-4k)-1}{\left(\sqrt{16k^2+16k+9}-4k\right)^2}\right)^2$, $p_2^{SM}=\frac{\left(\sqrt{16k^2+16k+9}-4k\right)+1}{2\left(\sqrt{16k^2+16k+9}-4k\right)}$, $t_2^{SM}=\frac{5+4k-\sqrt{16k^2+16k+9}}{3-4k+\sqrt{16k^2+16k+9}}$, $w_2^{SM}=\frac{1}{\sqrt{16k^2+16k+9}-4k}$, and $a_2^{SM}=\left(\frac{E[e_1^x]k_1}{16}\right)^2\left(\frac{(\sqrt{16k^2+16k+9}-4k)-1}{\sqrt{16k^2+16k+9}-4k}\right)^2\left(\frac{3+(\sqrt{16k^2+16k+9}-4k)}{\sqrt{16k^2+16k+9}-4k}\right)^2$. It is important to remark that we use the superscript SM to refer to the Stackelberg game when the manufacturer is the leader.

Appendix E. Proof of Theorem 5

As in the previous section, $t=0$, $w'=\frac{p'}{2}$, and $A=\left(\delta D_0\beta_2(w-c)(p')^{-e}a^{-\gamma}\right)^{\frac{1}{\delta+1}}$ should be replaced into Equation (16); then, the retailer's objective function is:

$$\begin{aligned}\underset{p,a}{Max}E[\Pi_r^1(p,a)] &= E[e_1^x]D_0(p'-w')(p'+(c+d))^{-e}(\alpha_2-\beta_2a^{-\gamma}A^{-\delta})-(1-t)a \\ & s.t.\ \ w'\le p' \ \text{ and } 0\le a,\end{aligned} \tag{A31}$$

To solve this problem, we define $y=\frac{p'}{2}(p'+(c+d))^{-e}$. Then, by taking the partial derivative of y regarding p', we have:

$$\frac{\partial y}{\partial p'}=\frac{1}{2}(p'+(c+d))^{-e}-\frac{p'}{2}e(p'+(c+d))^{-e-1}=(p'+(c+d))^{-e}\left(\frac{1}{2}-\frac{p'}{2}e(p'+(c+d))^{-1}\right)=0 \tag{A32}$$

Since $(p'+(c+d))^{-e}>0$, from Equation (A32), the value of $p=p_1$ is equal to $p'=\frac{c+d}{e-1}$. The range for y is determined as $p'=0\Rightarrow y=0$, $p'=\frac{c+d}{e-1}\Rightarrow y=\frac{e^{-e}}{2}\left(\frac{c+d}{e-1}\right)^{1-e}>0$, and $p'\to\infty\Rightarrow y=0$. So, the maximum value of z is $p'=\frac{c+d}{e-1}$ and its minimum value is zero. Therefore, the range for y is $0\le y\le\frac{e^{-e}}{2}\left(\frac{c+d}{e-1}\right)^{1-e}$. Now, the retailer's problem in Equation (A31) is rewritten as follows:

$$\begin{aligned}\underset{p,a}{Max}E[\Pi_r^1(p,a)] &= E[e_1^x]D_0y(\alpha_2-\beta_2a^{-\gamma}(\delta\beta_2ya^{-\gamma})^{\frac{-\delta}{\delta+1}})-a \\ & s.t.\ \ 0\le y\le\tfrac{e^{-e}}{2}\left(\tfrac{c+d}{e-1}\right)^{1-e} \ \text{ and } 0\le a,\end{aligned} \tag{A33}$$

The optimal value of y is equal to $y^*=y_{\max}=\frac{e^{-e}}{2}\left(\frac{c+d}{e-1}\right)^{1-e}$. Since the second derivative of $E[\Pi_r^1(p,a)]$ regarding a is negative, the optimal value of a is obtained as below:

$$\frac{\partial^2E[\Pi_r^1(p,a)]}{\partial a^2}=-\left(\frac{\gamma}{\delta+1}\right)\left(\frac{\gamma+\delta+1}{\delta+1}\right)E[e_1^x]D_0y\beta_2{}^{\frac{1}{\delta+1}}(\delta yE[e_1^x]D_0)^{\frac{-\delta}{\delta+1}}a^{\frac{-\gamma}{\delta+1}-2}<0 \tag{A34}$$

So, we have:

$$\begin{aligned}\frac{\partial E[\Pi_r^1(p,a)]}{\partial a} &= \frac{\gamma}{\delta+1}E[e_1^x]D_0 y\beta_2^{\frac{1}{\delta+1}}\left(\delta y E[e_1^x]D_0\right)^{\frac{-\delta}{\delta+1}}a^{\frac{-(\gamma+\delta+1)}{\delta+1}}-1=0\\ &\Rightarrow \quad a=\left(\gamma\frac{\beta_2^{\frac{1}{\delta+1}}\delta^{\frac{-\delta}{\delta+1}}}{\delta+1}\right)^{\frac{\delta+1}{\delta+\gamma+1}}\left(yE[e_1^x]D_0\right)^{\frac{1}{\delta+\gamma+1}}\end{aligned} \tag{A35}$$

According to the Equations $p'=p-(c+d)$ and $p'=\frac{c+d}{e-1}$, we have $p_1^{SR}=\frac{c+d}{e-1}+c+d$. Hence, $w_1^{SR}=\frac{1}{2}(\frac{c+d}{e-1})+\frac{c+d}{2}$. Using Equation (A35) and $y^*=\frac{e^{-e}}{2}(\frac{c+d}{e-1})^{1-e}$, we have $a_1^{SR}=\left(\gamma\frac{\beta_2^{\frac{1}{\delta+1}}\delta^{\frac{-\delta}{\delta+1}}}{\delta+1}\right)^{\frac{\delta+1}{\delta+\gamma+1}}\left(E[e_1^x]D_0\frac{e^{-e}}{2}(\frac{c+d}{e-1})^{1-e}\right)^{\frac{1}{\delta+\gamma+1}}$. Additionally, we will have the optimum values of $t^{SR}=0$ and $A_1^{SR}=\left(\delta\beta_2 E[e_1^x]D_0\frac{e^{-e}}{2}(\frac{c+d}{e-1})^{1-e}a^{-\gamma}\right)^{\frac{1}{\delta+1}}$ for the Stackelberg-retailer equilibrium solution. Note that we use the superscript *SR* to refer to the Stackelberg game when the retailer is the leader.

Appendix F. Proof of Theorem 6

By substituting Equations $t^*=0$, $w^*=\frac{p}{2}$, $A^*=\left(\frac{1}{2}E[e_1^x]k_2 w(1-p)\right)^2$ into Equation (17), the retailer's objective function is:

$$\begin{aligned}\underset{p,a}{Max}\,E[\Pi_r^2(p,a)] &= E[e_1^x]\frac{p}{2}(1-p)\left(k_1\sqrt{a}+\frac{1}{4}D_0k_2^2p(1-p)\right)-a\\ &s.t.\ 0\le p\le 1 \ \text{ and } 0\le a,\end{aligned} \tag{A36}$$

We define $y=\frac{p}{2}(1-p)$. According to $\frac{\partial y}{\partial p}=\frac{(1-p)}{2}-\frac{p}{2}=0$, the value of $p=p_2$ is equal to $\frac{1}{2}$. The range for y is determined such that if $p=0\Rightarrow y=0$, if $p=p_1\Rightarrow y=\frac{1}{8}>0$, and if $p=1\Rightarrow y=0$, so $0\le y\le\frac{1}{8}$. Now, the retailer's profit function shown in Equation (A36) is re-expressed as follows:

$$\begin{aligned}\underset{p,a}{Max}\,E[\Pi_r^2(p,a)] &= E[e_1^x]y\left(k_1\sqrt{a}+\frac{1}{4}D_0{k_2}^2y\right)-a\\ &s.t.\ \ 0\le y\le\tfrac{1}{8}\ \text{ and }\ 0\le a\end{aligned} \tag{A37}$$

Similar to the first model, the optimal values of y and a are equal to $y^*=y_{\max}=\frac{1}{8}$ and $\frac{\partial E[\Pi_r^2(p,a)]}{\partial a}=\frac{1}{2}E[e_1^x]k_1 y a^{-\frac{1}{2}}-1=0$, which yields $a=(\frac{1}{2}E[e^x]k_1y)^2$. Therefore, the optimal values are $p_2^{SR}=\frac{1}{2}$, $w_2^{SR}=\frac{1}{4}$, $t_2^{SR}=0$, $a_2^{SR}=\left(E[e_1^x]\frac{k_1}{8}\right)^2$ and $A_2^{SR}=\left(E[e_1^x]\frac{k_2}{16}\right)^2$.

Appendix G. Proof of Theorem 7

To prove this theorem, z is defined as follows:

$$\begin{aligned}z &= (p-c-d)(p^{-e})\\ &s.t.\ \ c+d\le p\end{aligned} \tag{A38}$$

The first derivative of z regarding t is equal to $\frac{\partial z}{\partial p}=p^{-e}-ep^{-e-1}(p-c-d)=p^{-e}\left(1-ep^{-1}(p-c-d)\right)=0$. Since $p^{-e}>0$, $p=p_1=\frac{e(c+d)}{(e-1)}$, if $p=c+d\Rightarrow z=0$, if $p=p_1\Rightarrow z=e^{-e}(\frac{c+d}{e-1})^{1-e}>0$, and if $p\to\infty\Rightarrow z=0$. So, we have $0\le z\le e^{-e}(\frac{c+d}{e-1})^{1-e}$. The objective function of the supply chain problem shown in Equation (23) is expressed as follows:

$$\begin{aligned}\underset{p,a,A}{Max}\,E[\textstyle\prod_{sc}^1(p,a,A)] &= D_0E[e_1^x]z(\alpha_2-\beta_2a^{-\gamma}A^{-\delta})-a-A\\ &s.t.\ \ 0\le z\le e^{-e}(\tfrac{c+d}{e-1})^{1-e}\ \text{ and } 0\le a,A\end{aligned} \tag{A39}$$

Since $\frac{\partial E[\prod_{sc}^{1}(p,a,A)]}{\partial z} = D_0(\alpha_2 - \beta_2 a^{-\gamma} A^{-\delta}) > 0$, we have $z^* = z_{\max} = e^{-e}\left(\frac{c+d}{e-1}\right)^{1-e}$. To determine the optimal values for A and a, we calculate the second partial derivatives for the total supply chain profit in Equation (23) regarding A and a,

$$\frac{\partial E[\prod_{sc}^{1}(p,a,A)]}{\partial a} = \gamma D_0 E[e_1^x]\beta_2 a^{-\gamma-1} A^{-\delta} z - 1 \quad \Rightarrow \frac{\partial^2 E[\prod_{sc}^{1}(p,a,A)]}{\partial a^2} = \frac{-\gamma(\gamma+1) D_0 E[e_1^x]\beta_2 z}{a^{\gamma+2} A^{\delta}} < 0 \tag{A40}$$

$$\frac{\partial E[\prod_{sc}^{1}(p,a,A)]}{\partial A} = \delta D_0 E[e_1^x]\beta_2 A^{-\delta-1} a^{-\gamma} z - 1 \quad \Rightarrow \frac{\partial^2 E[\prod_{sc}^{1}(p,a,A)]}{\partial A^2} = \frac{-\delta(\delta+1) D_0 E[e_1^x]\beta_2 z}{a^{\gamma} A^{\delta+2}} < 0 \tag{A41}$$

Since the second derivatives are both negative, the optimal values of A and a are easily obtained from the first derivative:

$$a = \left(\gamma D_0 E[e_1^x]\beta_2 A^{-\delta} z\right)^{\frac{1}{\gamma+1}} \tag{A42}$$

$$A = \left(\delta D_0 E[e_1^x]\beta_2 a^{-\gamma} z\right)^{\frac{1}{\delta+1}} \tag{A43}$$

Hence, the optimal values for the decision variables in the cooperative game are $p_1^{co} = \frac{e(c+d)}{(e-1)}$, $A_1^{co} = \frac{\delta}{\gamma} a_1^{co}$ and $a_1^{co} = \left(\left(\frac{\gamma}{\delta}\right)^{\delta} D_0 E[e_1^x]\beta_2 e^{-e}\left(\frac{c+d}{e-1}\right)^{1-e}\right)^{\frac{1}{\delta+\gamma+1}}$. It is important to mention that we use the superscript co to refer to the cooperative game.

Appendix H. Proof of Theorem 8

According to the objective function of the supply chain given by Equation (24), z is defined as follows:

$$\begin{aligned} &z = p(1-p) \\ &s.t.\ \ 0 \le p \le 1 \end{aligned} \tag{A44}$$

Such that if $p = 0 \Rightarrow z = 0$, if $p = p_1 \Rightarrow z = \frac{1}{4} > 0$ and if $p = 1 \Rightarrow z = 0$. So, we have $0 \le z \le \frac{1}{4}$ and then we can rewrite the objective function as follows:

$$\begin{aligned} &Max\ E[\prod_{sc}^{2}(p,a,A)] = E[e_1^x] z (k_1\sqrt{a} + k_2\sqrt{A}) - a - A \\ &s.t.\ \ 0 \le z \le \tfrac{1}{4},\ \ 0 \le a, A \end{aligned} \tag{A45}$$

Since $\frac{\partial E[\prod_{sc}^{2}(p,a,A)]}{\partial z} = E[e_1^x](k_1\sqrt{a} + k_2\sqrt{A}) > 0$, we have $z^* = z_{\max} = \frac{1}{4}$. To obtain the values of a and A we use the Hessian matrix as below:

$$H = \begin{bmatrix} \frac{\partial^2 E[\prod_{sc}^{2}(p,a,A)]}{\partial a^2} & \frac{\partial^2 E[\prod_{sc}^{2}(p,a,A)]}{\partial a \partial A} \\ \frac{\partial^2 E[\prod_{sc}^{2}(p,a,A)]}{\partial a \partial A} & \frac{\partial^2 E[\prod_{sc}^{2}(p,a,A)]}{\partial A^2} \end{bmatrix} = \begin{bmatrix} \frac{-k_1 z}{4a\sqrt{a}} & 0 \\ 0 & \frac{-k_1 z}{4A\sqrt{A}} \end{bmatrix} \tag{A46}$$

The Hessian matrix is negative definite, so the optimal values are:

$$\frac{\partial E[\prod_{sc}^{2}(p,a,A)]}{\partial a} = \frac{1}{2} E[e_1^x] k_1 z a^{-\frac{1}{2}} - 1 = 0 \quad \Rightarrow a = \left(\frac{1}{2} E[e_1^x] k_1 z\right)^2 \tag{A47}$$

$$\frac{\partial E[\prod_{sc}^{2}(p,a,A)]}{\partial A} = \frac{1}{2} E[e_1^x] k_2 z A^{-\frac{1}{2}} - 1 = 0 \quad \Rightarrow A = \left(\frac{1}{2} E[e_1^x] k_2 z\right)^2 \tag{A48}$$

Therefore, the optimal values of the cooperative game in the second model are $p_2^{co} = \frac{1}{2}$, $A_2^{co} = \left(\frac{1}{8} E[e_1^x] k_2\right)^2$, and $a_2^{co} = \left(\frac{1}{8} E[e_1^x] k_1\right)^2$.

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