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The Existence and Uniquenes Solution of Nonlinear Integral Equations via Common Fixed Point Theorems

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Abstract: In this paper, we prove some common fixed-point theorems on complex partial metric space. The presented results generalize and expand some of the well-known results in the literature. We also explore some of the applications of our key results.

Keywords: integral equations; complex partial metric space; common fixed point

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1. Introduction

Azam et al. [1] introduced the concept of complex-valued metric spaces and studied some fixed point theorems for mappings satisfying a rational inequality.

Two years later, in [2], Rao et al. discussed for the first time the idea of complex-valued b-metric spaces.

In 2017, Dhivya and Marudai [3] introduced the concept of complex partial metric space and suggested a plan to expand the results, as well as proving common fixed-point theorems under the rational expression contraction condition. This idea has been followed by Gunaseelan [4], who introduced the concept of complex partial b-metric spaces and discussed some results of fixed-point theory for self-mappings in these new spaces.

In [5], Prakasam and Gunaseelan proved the existence and uniqueness of a common fixed-point (with an illustrative example) theorem using CLR and E.A. properties in complex partial b-metric spaces. Their proved results generalize and extend some of the well-known results in the literature.

In [6], Gunaseelan et al. proved a fixed-point theorem in complex partial b-metric spaces under a contraction mapping. They also gave some applications of their main results.

In this paper, we prove some common fixed-point theorems on complex partial metric space.

2. Preliminaries

Let $\mathfrak{C}$ be the set of complex numbers and $\tau_1, \tau_2, \tau_3 \in \mathfrak{C}$. Define a partial order $\preceq$ on $\mathfrak{C}$ as follows:

$\tau_1 \preceq \tau_2$ if and only if $\mathcal{R}(\tau_1) \leq \mathcal{R}(\tau_2), \mathcal{I}(\tau_1) \leq \mathcal{I}(\tau_2)$.

Consequently, one can infer that $\tau_1 \preceq \tau_2$ if one of the following conditions is satisfied:

(i) $\mathcal{R}(\tau_1) = \mathcal{R}(\tau_2), \mathcal{I}(\tau_1) < \mathcal{I}(\tau_2)$,
(ii) $\mathcal{R}(\tau_1) < \mathcal{R}(\tau_2), \mathcal{I}(\tau_1) = \mathcal{I}(\tau_2)$,
(iii) $\mathcal{R}(\tau_1) < \mathcal{R}(\tau_2), \mathcal{I}(\tau_1) < \mathcal{I}(\tau_2)$,

(iv) $\mathcal{R}(\tau_1) = \mathcal{R}(\tau_2), \mathcal{I}(\tau_1) = \mathcal{I}(\tau_2)$.

In particular, we write $\tau_1 \precnsim \tau_2$ if $\tau_1 \neq \tau_2$, and one of (i), (ii) and (iii) is satisfied and we write $\tau_1 \prec \tau_2$ if only (iii) is satisfied. Notice that

(a) If $0 \preceq \tau_1 \precnsim \tau_2$, then $|\tau_1| < |\tau_2|$,
(b) If $\tau_1 \preceq \tau_2$ and $\tau_2 \prec \tau_3$, then $\tau_1 \prec \tau_3$,
(c) If $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma$, then $\eta\tau_1 \preceq \gamma\tau_1$ for all $0 \preceq \tau_1 \in \mathfrak{C}$.

Here $\mathfrak{C}_+(= \{(\aleph, \mathfrak{y}) | \aleph, \mathfrak{y} \in \mathbb{R}_+\})$ and $\mathbb{R}_+(= \{\aleph \in \mathbb{R} | \aleph \geq 0\})$ denote the set of non-negative complex numbers and the set of non negative real numbers, respectively.

Now, let us recall some basic concepts and notations that will be used below.

Definition 1 ([3])**.** *A complex partial metric on a non-void set G is a function $\varrho_{cb} : G \times G \to \mathbb{C}^+$ such that for all $\theta, \omega, \vartheta \in G$:*

(i) $0 \preceq \varrho_{cb}(\theta, \theta) \preceq \varrho_{cb}(\theta, \omega)$*(small self-distances)*
(ii) $\varrho_{cb}(\theta, \omega) = \varrho_{cb}(\omega, \theta)$*(symmetry)*
(iii) $\varrho_{cb}(\theta, \theta) = \varrho_{cb}(\theta, \omega) = \varrho_{cb}(\omega, \omega)$ *if and only if* $\theta = \omega$*(equality)*
(iv) $\varrho_{cb}(\theta, \omega) \preceq \varrho_{cb}(\theta, \vartheta) + \varrho_{cb}(\vartheta, \omega) - \varrho_{cb}(\vartheta, \vartheta)$*(triangularity).*

A complex partial metric space is a pair (G, ϱ_{cb}) such that G is a non-void set and ϱ_{cb} is the complex partial metric on G.

Definition 2 ([3])**.** *Let $(G, \wp_{cb})$ be a complex partial metric space. Let $\{\theta_n\}$ be any sequence in G. Then*

(i) *The sequence $\{\theta_n\}$ is said to converge to θ, if $\lim_{n\to\infty} \wp_{cb}(\theta_n, \theta) = \wp_{cb}(\theta, \theta)$.*
(ii) *The sequence $\{\theta_n\}$ is said to be a Cauchy sequence in $(G, \wp_{cb})$ if $\lim_{n,m\to\infty} \wp_{cb}(\theta_n, \theta_m)$ exists and is finite.*
(iii) *$(G, \wp_{cb})$ is said to be a complete complex partial metric space if for every Cauchy sequence $\{\theta_n\}$ in G there exists $\theta \in G$ such that $\lim_{n,m\to\infty} \wp_{cb}(\theta_n, \theta_m) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta) = \wp_{cb}(\theta, \theta)$.*
(iv) *A mapping $\Pi : G \to G$ is said to be continuous at $\theta_0 \in G$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\Pi(B_{\wp_{cb}}(\theta_0, \delta)) \subset B_{\wp_{cb}}(\Pi(\theta_0, \epsilon))$.*

Definition 3 ([3])**.** *Let Π and Ψ be self-mappings of non-void set G. A point $\aleph \in G$ is called a common fixed point of Π and Ψ if $\aleph = \Pi\aleph = \Psi\aleph$.*

Theorem 1 ([3])**.** *Let $(G, \preceq)$ be a partially ordered set and suppose that there exists a complex partial metric ϱ_{cb} in G such that (G, ϱ_{cb}) is a complete complex partial metric space. Let $\Pi, \Psi : G \to G$ be a pair of weakly increasing mappings, and suppose that for every comparable $\aleph, \mathfrak{y} \in G$ we have either*

$$\varrho_{cb}(\Pi\aleph, \Psi\mathfrak{y}) \preceq a\frac{\varrho_{cb}(\aleph, \Pi\aleph)\varrho_{cb}(\mathfrak{y}, \Psi\mathfrak{y})}{\varrho_{cb}(\aleph, \mathfrak{y})} + b\varrho_{cb}(\aleph, \mathfrak{y})$$

for $\varrho_{cb}(\aleph, \mathfrak{y}) \neq 0$ with $a \geq 0, b \geq 0$, $a + b < 1$, or

$$\varrho_{cb}(\Pi\aleph, \Psi\mathfrak{y}) = 0 \text{ if } \varrho_{cb}(\aleph, \mathfrak{y}) = 0.$$

If Π or Ψ is continuous, then Π and Ψ have a common fixed point $\propto \in G$ and $\varrho_{cb}(\propto, \propto) = 0$.

Inspired by Theorem 1, here we prove some common fixed-point theorems on complex partial metric space with an application. For complex partial metric space, we will use the CPMS notation.

3. Main Results

Theorem 2. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi\colon G \to G$ be two continuous mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\}, \tag{1}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Psi\omega) \neq 0$. Then, the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Let θ_0 be arbitrary point in G and define a sequence $\{\theta_n\}$ as follows:

$$\theta_{2n+1} = \Pi\theta_{2n} \quad \textit{and} \quad \theta_{2n+2} = \Psi\theta_{2n+1}, n = 0, 1, 2, \ldots \tag{2}$$

Then by (1) and (2), we obtain

$$\begin{aligned}
\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) &= \wp_{cb}(\Pi\theta_{2n}, \Psi\theta_{2n+1}) \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n}, \Pi\theta_{2n}), \wp_{cb}(\theta_{2n+1}, \Psi\theta_{2n+1}), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n}, \Psi\theta_{2n+1}) + \wp_{cb}(\theta_{2n+1}, \Pi\theta_{2n}))\} \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n}, \theta_{2n+2}) + \wp_{cb}(\theta_{2n+1}, \theta_{2n+1}))\} \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n}, \theta_{2n+1}) + \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) - \wp_{cb}(\theta_{2n+1}, \theta_{2n+1}) \\
&\quad + \wp_{cb}(\theta_{2n+1}, \theta_{2n+1}))\} \\
&= \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n}, \theta_{2n+1}) + \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}))\}
\end{aligned}$$

Case I:

If $\max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \frac{1}{2}(\wp_{cb}(\theta_{2n}, \theta_{2n+1}) + \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}))\} = \wp_{cb}$ $(\theta_{2n+1}, \theta_{2n+2})$, then we have

$$\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) \preceq \curlywedge \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}).$$

This implies $\curlywedge \geq 1$, which is a contradiction.

Case II:

If $\max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \frac{1}{2}(\wp_{cb}(\theta_{2n}, \theta_{2n+1}) + \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}))\} = \wp_{cb}$ $(\theta_{2n}, \theta_{2n+1})$, then we have

$$\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) \preceq \curlywedge \wp_{cb}(\theta_{2n}, \theta_{2n+1}). \tag{3}$$

From the next step, we have

$$\wp_{cb}(\theta_{2n+2}, \theta_{2n+3}) \preceq \curlywedge \max\{\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}), \wp_{cb}(\theta_{2n+2}, \theta_{2n+3}), \frac{1}{2}(\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) + \wp_{cb}(\theta_{2n+2}, \theta_{2n+3}))\}.$$

The following three cases arise.

Case IIa:

$$\wp_{cb}(\theta_{2n+2}, \theta_{2n+3}) \preceq \curlywedge \wp_{cb}(\theta_{2n+2}, \theta_{2n+3}),$$

which implies $\curlywedge \geq 1$, which is a contradiction.

Case IIb:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \text{人}\frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) + \wp_{cb}(\theta_{2n+2},\theta_{2n+3})).$$

This implies that

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \frac{\text{人}}{(2-\text{人})}\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{4}$$

Since $a := \dfrac{\text{人}}{2-\text{人}} < 1$, we get $\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq a\wp_{cb}(\theta_n,\theta_{n+1})$. Therefore $\{\theta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in G.

Case IIc:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \text{人}\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{5}$$

From (3) and (5), $\forall n = 0,1,2,\ldots$, we get

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \text{人}\wp_{cb}(\theta_n,\theta_{n+1}) \preceq \ldots \preceq \text{人}^{n+1}\wp_{cb}(\theta_0,\theta_1).$$

For $m,n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\wp_{cb}(\theta_n,\theta_m) &\preceq \wp_{cb}(\theta_n,\theta_{n+1}) + \wp_{cb}(\theta_{n+1},\theta_m) - \wp_{cb}(\theta_{n+1},\theta_{n+1}) \\
&\preceq \wp_{cb}(\theta_n,\theta_{n+1}) + \wp_{cb}(\theta_{n+1},\theta_m) \\
&\preceq \wp_{cb}(\theta_n,\theta_{n+1}) + \wp_{cb}(\theta_{n+1},\theta_{n+2}) + \wp_{cb}(\theta_{n+2},\theta_m) \\
&\quad - \wp_{cb}(\theta_{n+2},\theta_{n+2}) \\
&\preceq \wp_{cb}(\theta_n,\theta_{n+1}) + \wp_{cb}(\theta_{n+1},\theta_{n+2}) + \wp_{cb}(\theta_{n+2},\theta_m) \\
&\preceq \wp_{cb}(\theta_n,\theta_{n+1}) + \wp_{cb}(\theta_{n+1},\theta_{n+2}) + \wp_{cb}(\theta_{n+2},\theta_{n+3}) \\
&\quad + \ldots + \wp_{cb}(\theta_{m-2},\theta_{m-1}) + \wp_{cb}(\theta_{m-1},\theta_m).
\end{aligned}$$

Moreover, by using (5), we get

$$\begin{aligned}
\wp_{cb}(\theta_n,\theta_m) &\preceq \text{人}^n\wp_{cb}(\theta_0,\theta_1) + \text{人}^{n+1}\wp_{cb}(\theta_0,\theta_1) + \text{人}^{n+2}\wp_{cb}(\theta_0,\theta_1) \\
&\quad + \ldots + \text{人}^{m-2}\wp_{cb}(\theta_0,\theta_1) + \text{人}^{m-1}\wp_{cb}(\theta_0,\theta_1) \\
&= \sum_{i=1}^{m-n} \text{人}^{i+n-1}\wp_{cb}(\theta_0,\theta_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
|\wp_{cb}(\theta_n,\theta_m)| &\leq \sum_{i=1}^{m-n} \text{人}^{i+n-1}|\wp_{cb}(\theta_0,\theta_1)| = \sum_{t=n}^{m-1} \text{人}^t|\wp_{cb}(\theta_0,\theta_1)| \\
&\leq \sum_{i=n}^{\infty} |\wp_{cb}(\theta_0,\theta_1)| \\
&= \frac{\text{人}^n}{1-\text{人}}|\wp_{cb}(\theta_0,\theta_1)|.
\end{aligned}$$

Then, we have

$$|\wp_{cb}(\theta_n,\theta_m)| \leq \frac{\text{人}^n}{1-\text{人}}|\wp_{cb}(\theta_0,\theta_1)| \to 0 \quad as \quad n \to \infty.$$

Hence, $\{\theta_n\}$ is a Cauchy sequence in G.

Case III:

If $\max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\ \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))\} = \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))$.

Then, we have

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \frac{\curlywedge}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))$$

Hence,

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \frac{\curlywedge}{2-\curlywedge}\wp_{cb}(\theta_{2n},\theta_{2n+1}). \tag{6}$$

For the next step, we have

$$\begin{aligned}\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\max\{&\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),\\ &\frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2})+\wp_{cb}(\theta_{2n+2},\theta_{2n+3}))\}.\end{aligned}$$

Then, we have the following three cases:

Case IIIa:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),$$

which implies $\curlywedge \geq 1$, which is a contradiction.

Case IIIb:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{7}$$

Then by (6) and (7), we get $\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \gamma\wp_{cb}(\theta_n,\theta_{n+1})$, where $\gamma = \max\left\{\curlywedge, \frac{\curlywedge}{2-\curlywedge}\right\} < 1$. Hence $\{\theta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in G.

Case IIIc:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2})+\wp_{cb}(\theta_{2n+2},\theta_{2n+3})).$$

Hence, we obtain

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \frac{\curlywedge}{(2-\curlywedge)}\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{8}$$

Using (6) and (8) yields

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \wr\wp_{cb}(\theta_n,\theta_{n+1}), \tag{9}$$

where $0 \leq \wr = \frac{\curlywedge}{2-\curlywedge} < 1$.

Then, $\forall n = 0,1,2,\ldots$, and we get

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \wr\wp_{cb}(\theta_n,\theta_{n+1}) \preceq \ldots \preceq \wr^{n+1}\wp_{cb}(\theta_0,\theta_1).$$

For $m, n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) - \wp_{cb}(\theta_{n+1}, \theta_{n+1}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\quad - \wp_{cb}(\theta_{n+2}, \theta_{n+2}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \ldots + \wp_{cb}(\theta_{m-2}, \theta_{m-1})) + \wp_{cb}(\theta_{m-1}, \theta_m).
\end{aligned}$$

Using (9), we get

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wr^n \wp_{cb}(\theta_0, \theta_1) + \wr^{n+1} \wp_{cb}(\theta_0, \theta_1) + \wr^{n+2} \wp_{cb}(\theta_0, \theta_1) \\
&\quad + \ldots + \wr^{m-2} \wp_{cb}(\theta_0, \theta_1) + \wr^{m-1} \wp_{cb}(\theta_0, \theta_1) \\
&= \sum_{i=1}^{m-n} \wr^{i+n-1} \wp_{cb}(\theta_0, \theta_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\wp_{cb}(\theta_n, \theta_m)| &\leq \sum_{i=1}^{m-n} \wr^{i+n-1} |\wp_{cb}(\theta_0, \theta_1)| = \sum_{t=n}^{m-1} \wr^t |\wp_{cb}(\theta_0, \theta_1)| \\
&\leq \sum_{i=n}^{\infty} \wr^t |\wp_{cb}(\theta_0, \theta_1)| \\
&= \frac{\wr^n}{1 - \wr} |\wp_{cb}(\theta_0, \theta_1)|.
\end{aligned}$$

Hence, we have

$$|\wp_{cb}(\theta_n, \theta_m)| \leq \frac{\wr^n}{1 - \wr} |\wp_{cb}(\theta_0, \theta_1)| \to 0 \quad as \quad n \to \infty.$$

Hence, $\{\theta_n\}$ is a Cauchy sequence in G. In all cases above discussed, we get the sequence $\{\theta_n\}_{n \in \mathbb{N}}$, which is a Cauchy sequence. Since G is complete, there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n \to \infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n \to \infty} \wp_{cb}(\theta_n, \theta_n) = 0$$

By the continuity of Π, it follows that $\theta_{2n+1} = \Pi\theta_{2n} \to \Pi\theta^*$ as $n \to \infty$.

$$i.e., \ \wp_{cb}(\Pi\theta^*, \Pi\theta^*) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta^*, \Pi\theta_{2n}) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}).$$

However,

$$\wp_{cb}(\Pi\theta^*, \Pi\theta^*) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}) = \lim_{n \to \infty} \wp_{cb}(\theta_{2n+1}, \theta_{2n+1}) = 0.$$

Next, we have to prove that θ^* is a fixed point of Π.

$$\wp_{cb}(\Pi\theta^*, \theta^*) \preceq \wp_{cb}(\Pi\theta^*, \Pi\theta_{2n}) + \wp_{cb}(\Pi\theta_{2n}, \theta^*) - \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}).$$

As $n \to \infty$, we obtain $|\wp_{cb}(\Pi\theta^*, \theta^*)| \leq 0$. Thus, $\wp_{cp}(\Pi\theta^*, \theta^*) = 0$. Hence $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Pi\theta^*) = \wp_{cb}(\Pi\theta^*, \Pi\theta^*) = 0$ and $\Pi\theta^* = \theta^*$. In the same way, we have $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n\to\infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta_n) = 0$$

By the continuity of Π, it follows $\theta_{2n+2} = \Psi\theta_{2n+1} \to \Psi\theta^*$ as $n \to \infty$.

$$i.e., \; \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta^*, \Psi\theta_{2n+1}) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta_{2n+1}, \Psi\theta_{2n+1}).$$

However,

$$\wp_{cb}(\Psi\theta^*, \Psi\theta^*) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta_{2n+1}, \Psi\theta_{2n+1}) = \lim_{n\to\infty} \wp_{cb}(\theta_{2n+2}, \theta_{2n+2}) = 0.$$

Next we have to prove that θ^* is a fixed point of Ψ.

$$\wp_{cb}(\Psi\theta^*, \theta^*) \preceq \wp_{cb}(\Psi\theta^*, \Psi\theta_{2n+1}) + \wp_{cb}(\Psi\theta_{2n+1}, \theta^*) - \wp_{cb}(\Psi\theta_{2n+1}, \Pi\theta_{2n+1}).$$

As $n \to \infty$, we obtain $|\wp_{cb}(\Psi\theta^*, \theta^*)| \leq 0$. Thus, $\wp_{cp}(\Psi\theta^*, \theta^*) = 0$. Hence, $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Psi\theta^*) = \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = 0$ and $\Psi\theta^* = \theta^*$. Therefore, θ^* is a common fixed point of the pair (Π, Ψ).

To prove uniqueness, let us consider $\omega^* \in G$ is another common fixed point for the pair (Π, Ψ). Then

$$\begin{aligned}
\wp_{cb}(\theta^*, \omega^*) &= \wp_{cb}(\Pi\theta^*, \Psi\omega^*) \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \wp_{cb}(\theta^*, \Pi\theta^*), \wp_{cb}(\omega^*, \Psi\omega^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta^*, \Psi\omega^*) + \wp_{cb}(\omega^*, \Pi\theta^*))\} \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \wp_{cb}(\theta^*, \theta^*), \wp_{cb}(\omega^*, \omega^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta^*, \omega^*) + \wp_{cb}(\omega^*, \theta^*))\} \\
&\preceq \curlywedge \wp_{cb}(\theta^*, \omega^*).
\end{aligned}$$

This implies that $\theta^* = \omega^*$. □

In the absence of the continuity condition for the mappings Π and Ψ, we get the the following theorem.

Theorem 3. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi\colon G \to G$ be two mappings such that*

$$\begin{aligned}
\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\
&\frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},
\end{aligned} \tag{10}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Psi\omega) \neq 0$. Then, the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Following from Theorem 2, we get that the sequence $\{\theta_n\}$ is a Cauchy sequence. Since G is complete, there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$.
Since Π and Ψ are not continuous, we have $\wp_{cb}(\theta^*, \Pi\theta^*) = \vartheta > 0$.

Then, we estimate

$$\begin{aligned}
\vartheta &= \wp_{cb}(\theta^*, \Pi\theta^*) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*) - \wp_{cb}(\theta_{2i+2}, \theta_{2i+2}) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Psi\theta_{2i+1}, \Pi\theta^*) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\{\wp_{cb}(\theta_{2i+1}, \theta^*), \wp_{cb}(\theta_{2i+1}, \Psi\theta_{2i+1}), \wp_{cb}(\theta^*, \Pi\theta^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2i+1}, \Pi\theta^*) + \wp_{cb}(\theta^*, \Psi\theta_{2i+1}))\} \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\{\wp_{cb}(\theta_{2i+1}, \theta^*), \wp_{cb}(\theta_{2i+1}, \theta_{2i+2}), \wp_{cb}(\theta^*, \Pi\theta^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2i+1}, \Pi\theta^*) + \wp_{cb}(\theta^*, \theta_{2i+2}))\} \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \wp_{cb}(\theta^*, \Pi\theta^*) \\
&\preceq s\wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \vartheta.
\end{aligned}$$

This yields

$$|\vartheta| \leq |\wp_{cb}(\theta^*, \theta_{2i+2})| + \curlywedge |\vartheta|.$$

By definition,

$$\lim_i \wp_{cb}(\theta^*, \theta_{2i+2}) = \wp_{cb}(\theta^*, \theta^*)$$

From the Cauchy property of (θ_n), the above limit is zero, and then,

$$|\vartheta| \leq \curlywedge |\vartheta|.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then $\theta^* = \Pi\theta^*$. In the same way, we obtain $\theta^* = \Psi\theta^*$. Hence θ^* is a common fixed point for the pair (Π, Ψ) and $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Psi\theta^*) = \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = 0$. Uniqueness of the common fixed point θ^* follows from Theorem 2. □

For $\Pi = \Psi$, we get the following fixed points results on CPMS.

Theorem 4. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$\begin{aligned}
\wp_{cb}(\Pi\theta, \Pi\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Pi\omega), \\
&\frac{1}{2}(\wp_{cb}(\theta, \Pi\omega) + \wp_{cb}(\omega, \Pi\theta))\},
\end{aligned} \tag{11}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Pi\omega) \neq 0$. Then the pair Π has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Remark 1. *Similarly, we get a fixed point result in the absence of continuity condition for the mapping Π.*

Corollary 1. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Psi\colon G \to G$ be a continuous mapping such that*

$$\begin{aligned}
\wp_{cb}(\Psi^n\theta, \Psi^n\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Psi^n\theta), \wp_{cb}(\omega, \Psi^n\omega), \\
&\frac{1}{2}(\wp_{cb}(\theta, \Psi^n\omega) + \wp_{cb}(\omega, \Psi^n\theta))\},
\end{aligned}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$, $\wp_{cb}(\Psi^n\theta, \Psi^n\omega) \neq 0$ and $n \in \mathbb{N}$. Then, Ψ has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. By Theorem 2, we get $\theta^* \in G$ such that $\Psi^n\theta^* = \theta^*$ and $\wp_{cb}(\theta^*, \theta^*) = 0$. Then, we get

$$\begin{aligned}\wp_{cb}(\Psi\theta^*, \theta^*) &= \wp_{cb}(\Psi\Psi^n\theta^*, \Psi^n\theta^*) = \wp_{cb}(\Psi^n\Psi\theta^*, \Psi^n\theta^*)\\ &\preceq \curlywedge \max\{\wp_{cb}(\Psi\theta^*, \theta^*), \wp_{cb}(\Psi\theta^*, \Psi^n\Psi\theta^*), \wp_{cb}(\theta^*, \Psi^n\theta^*),\\ &\quad \frac{1}{2}(\wp_{cb}(\Psi\theta^*, \Psi^n\theta^*) + \wp_{cb}(\theta^*, \Psi^n\Psi\theta^*))\}\\ &\preceq \curlywedge \max\{\wp_{cb}(\Psi\theta^*, \theta^*), \wp_{cb}(\Psi\theta^*, \Psi\theta^*), \wp_{cb}(\theta^*, \theta^*),\\ &\quad \frac{1}{2}(\wp_{cb}(\Psi\theta^*, \theta^*) + \wp_{cb}(\theta^*, \Psi\theta^*))\}\\ &= \curlywedge \wp_{cb}(\Psi\theta^*, \theta^*).\end{aligned}$$

Hence $\Psi^n\theta^* = \Psi\theta^* = \theta^*$. Then Ψ has a unique fixed point. □

Remark 2. *From the above Corollary 1, similarly, we get a fixed-point result in the absence of continuity condition for the mapping Ψ.*

Next, we present a new generalization of a common fixed point theorem on CPMS.

Theorem 5. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi\colon G \to G$ be two continuous mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\omega, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}, \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\Pi\theta, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}\right\}, \tag{12}$$

for all $\theta, \omega \in G$, where $0 \le \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Psi\omega) \neq 0$. Then, the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Let θ_0 be arbitrary point in G and define a sequence $\{\theta_n\}$ as follows:

$$\theta_{2n+1} = \Pi\theta_{2n} \quad \textit{and} \quad \theta_{2n+2} = \Psi\theta_{2n+1}, n = 0, 1, 2, \dots \tag{13}$$

Then, by (12) and (13), we obtain

$$\begin{aligned}\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) &= \wp_{cb}(\Pi\theta_{2n}, \Psi\theta_{2n+1})\\ &\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\Psi\theta_{2n+1}, \Pi\theta_{2n})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})},\\ &\quad \frac{\wp_{cb}(\theta_{2n}, \Pi\theta_{2n},)\wp_{cb}(\Pi\theta_{2n}, \Psi\theta_{2n+1})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})}\}\\ &\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\theta_{2n+1}, \theta_{2n+2})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})},\\ &\quad \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\theta_{2n+1}, \theta_{2n+2})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})}\}\\ &\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2})\}.\end{aligned}$$

If $\max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2})\} = \wp_{cb}(\theta_{2n+1}, \theta_{2n+2})$, then

$$\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) \preceq \curlywedge \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}).$$

This shows that $\curlywedge \ge 1$, which is a contradiction. Therefore

$$\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) \preceq \curlywedge \wp_{cb}(\theta_{2n}, \theta_{2n+1}). \tag{14}$$

Similarly, we obtain

$$\wp_{cb}(\theta_{2n+2}, \theta_{2n+3}) \preceq \curlywedge \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}). \tag{15}$$

From (14) and (15), $\forall n = 0, 1, 2, \ldots$, we get

$$\wp_{cb}(\theta_{n+1}, \theta_{n+2}) \preceq \curlywedge \wp_{cb}(\theta_n, \theta_{n+1}) \preceq \ldots \preceq \curlywedge^{n+1} \wp_{cb}(\theta_0, \theta_1). \tag{16}$$

For $m, n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) - \wp_{cb}(\theta_{n+1}, \theta_{n+1}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\quad - \wp_{cb}(\theta_{n+2}, \theta_{n+2}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \ldots + \wp_{cb}(\theta_{m-2}, \theta_{m-1}) + s^{m-n} \wp_{cb}(\theta_{m-1}, \theta_m).
\end{aligned}$$

By using (16), we get

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \curlywedge^n \wp_{cb}(\theta_0, \theta_1) + \curlywedge^{n+1} \wp_{cb}(\theta_0, \theta_1) + \curlywedge^{n+2} \wp_{cb}(\theta_0, \theta_1) \\
&\quad + \ldots + \curlywedge^{m-2} \wp_{cb}(\theta_0, \theta_1) + \curlywedge^{m-1} \wp_{cb}(\theta_0, \theta_1) \\
&= \sum_{i=1}^{m-n} \curlywedge^{i+n-1} \wp_{cb}(\theta_0, \theta_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\wp_{cb}(\theta_n, \theta_m)| &\leq \sum_{i=1}^{m-n} \curlywedge^{i+n-1} |\wp_{cb}(\theta_0, \theta_1)| = \sum_{i=1}^{m-n} \curlywedge^{t} |\wp_{cb}(\theta_0, \theta_1)| \\
&\leq \sum_{i=n}^{\infty} \curlywedge^{t} |\wp_{cb}(\theta_0, \theta_1)| \\
&= \frac{\curlywedge^n}{1 - \curlywedge} |\wp_{cb}(\theta_0, \theta_1)|.
\end{aligned}$$

Hence, we have

$$|\wp_{cb}(\theta_n, \theta_m)| \leq \frac{\curlywedge^n}{1 - \curlywedge} |\wp_{cb}(\theta_0, \theta_1)| \to 0 \quad as \quad n \to \infty.$$

Hence, $\{\theta_n\}$ is a Cauchy sequence in G. Since G is complete, there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n \to \infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n \to \infty} \wp_{cb}(\theta_n, \theta_n) = 0.$$

Since Ψ is continuous, it yields

$$\theta^* = \lim_{n \to \infty} \theta_{2n+2} = \lim_{n \to \infty} \Psi \theta_{2n+1} = \Psi \lim_{n \to \infty} \theta_{2n+1} = \Psi \theta^*.$$

Similarly, by the continuity of Π, we get $\theta^* = \Pi \theta^*$. Then the pair (Π, Ψ) has a common fixed point. To prove uniqueness, let us consider that $\omega^* \in G$ is another common fixed point for the pair (Π, Ψ). Then

$$\begin{aligned}\wp_{cb}(\theta^*, \omega^*) &= \wp_{cb}(\Pi\theta^*, \Psi\omega^*) \\ &\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\omega^*, \Psi\omega^*)}{1+\wp_{cb}(\theta^*, \omega^*)}, \\ &\quad \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\Psi\omega^*, \Pi\theta^*)}{1+\wp_{cb}(\theta^*, \omega^*)}\} \\ &\preceq \curlywedge \wp_{cb}(\theta^*, \omega^*)\end{aligned}$$

This implies that $\theta^* = \omega^*$. □

In the absence of the continuity condition for the mapping Π and Ψ in the Theorem 5, we obtain the following result.

Theorem 6. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi \colon G \to G$ be two mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\omega, \Psi\omega)}{1+\wp_{cb}(\theta, \omega)}, \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\Pi\theta, \Psi\omega)}{1+\wp_{cb}(\theta, \omega)}\right\}, \tag{17}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Psi\omega) \neq 0$. Then the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Following from Theorem 5, we get that the sequence $\{\theta_n\}$ is a Cauchy sequence. Since G is complete, then there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n\to\infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta_n) = 0.$$

Since Π and Ψ are not continuous, we have $\wp_{cb}(\theta^*, \Pi\theta^*) = \vartheta > 0$.
Then, we estimate

$$\begin{aligned}\vartheta &= \wp_{cb}(\theta^*, \Pi\theta^*) \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*) - \wp_{cb}(\theta_{2i+2}, \theta_{2i+2}) \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Pi\theta^*, \theta_{2i+2}) \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Pi\theta^*, \Psi\theta_{2i+1}) \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\left\{\wp_{cb}(\theta^*, \theta_{2i+1}), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\theta_{2i+1}, \Psi\theta_{2i+1})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}, \right. \\ &\quad \left. \frac{\wp_{cb}(\theta^*, \Pi\theta_*)\wp_{cb}(\Pi\theta^*, \Psi\theta_{2i+1})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\} \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\left\{\wp_{cb}(\theta^*, \theta_{2i+1}), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\theta_{2i+1}, \theta_{2i+2})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\}, \\ &\quad \left. \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\Pi\theta^*, \theta_{2i+2})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\} \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \wp_{cb}(\theta^*, \Pi\theta^*)^2 \\ &\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \vartheta^2.\end{aligned}$$

This yields

$$|\vartheta| \leq |\wp_{cb}(\theta^*, \theta_{2i+2})| + \curlywedge |\vartheta|^2.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then $\theta^* = \Pi\theta^*$. In the same way, we obtain $\theta^* = \Psi\theta^*$. Hence, θ^* is a common fixed point for the pair (Π, Ψ). For uniqueness of the common fixed point, θ^* follows from Theorem 5. □

For $\Pi = \Psi$, we get the following fixed-points results on CPMS.

Theorem 7. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$\wp_{cb}(\Pi\theta, \Pi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta,\omega), \frac{\wp_{cb}(\theta,\Pi\theta)\wp_{cb}(\omega,\Pi\omega)}{1+\wp_{cb}(\theta,\omega)}, \frac{\wp_{cb}(\theta,\Pi\theta)\wp_{cb}(\Pi\theta,\Pi\omega)}{1+\wp_{cb}(\theta,\omega)}\right\},$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi\theta, \Pi\omega) \neq 0$. Then, Π has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Remark 3. *Similarly, in the absence of continuity condition, we can get a fixed point result on Π.*

Corollary 2. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$\wp_{cb}(\Pi^n\theta, \Pi^n\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta,\omega), \frac{\wp_{cb}(\theta,\Pi^n\theta)\wp_{cb}(\omega,\Pi^n\omega)}{1+\wp_{cb}(\theta,\omega)}, \frac{\wp_{cb}(\theta,\Pi^n\theta)\wp_{cb}(\Pi^n\theta,\Pi\omega)}{1+\wp_{cb}(\theta,\omega)}\right\},$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$ and $\wp_{cb}(\Pi^n\theta, \Pi^n\omega) \neq 0$. Then Π has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. By Theorem 5, we get $\theta^* \in G$ such that $\Pi^n\theta^* = \theta^*$ and $\wp_{cb}(\theta^*, \theta^*) = 0$. Then we get

$$\begin{aligned}
\wp_{cb}(\Pi\theta^*, \theta^*) &= \wp_{cb}(\Pi\Pi^n\theta^*, \Pi^n\theta^*) = \wp_{cb}(\Pi^n\Pi\theta^*, \Pi^n\theta^*) \\
&\preceq \curlywedge \max\left\{\wp_{cb}(\Pi\theta^*,\theta^*), \frac{\wp_{cb}(\Pi\theta^*,\Pi^n\Pi\theta^*)\wp_{cb}(\theta^*,\Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*,\theta^*)}, \frac{\wp_{cb}(\Pi\theta^*,\Pi^n\Pi\theta^*)\wp_{cb}(\Pi^n\Pi\theta^*,\Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*,\theta^*)}\right\} \\
&\preceq \curlywedge \max\left\{\wp_{cb}(\Pi\theta^*,\theta^*), \frac{\wp_{cb}(\Pi\theta^*,\Pi\Pi^n\theta^*)\wp_{cb}(\theta^*,\Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*,\theta^*)}, \frac{\wp_{cb}(\Pi\theta^*,\Pi\Pi^n\theta^*)\wp_{cb}(\Pi\Pi^n\theta^*,\Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*,\theta^*)}\right\} \\
&= \curlywedge \wp_{cb}(\Pi\theta^*, \theta^*).
\end{aligned}$$

Hence $\Pi^n\theta^* = \Pi\theta^* = \theta^*$. Then, Π has a unique fixed point. □

Remark 4. *From the above corollary 2, similarly, we get a fixed point result in the absence of continuity condition for the mapping Π.*

Example 1. *Let $G = \{1,2,3,4\}$ be endowed with the order $\theta \preceq \omega$ if and only if $\theta \leq \omega$. Then, $\preceq$ is a partial order in G. Define the complex partial metric space $\wp_{cb} : G \times G \to \mathbb{C}^+$ as follows:*

(θ, ω)	$\wp_{cb}(\theta, \omega)$
(1,1), (2,2)	0
(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,3)	e^{ix}
(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)	$3e^{ix}$

Obviously, $(G, \wp_{cb})$ is a complete CPMS for $x \in [0, \frac{\pi}{2}]$. Define $\Pi, \Psi : G \to G$ by $\Pi\theta = 1$,

$$\Psi(\theta) = \begin{cases} 1 & \text{if } \theta \in \{1,2,3\} \\ 2 & \text{if } \theta = 4. \end{cases}$$

Clearly Π and Ψ are continuous functions. Now, for $\curlywedge = \frac{1}{3}$, we consider the following cases:

(A) *If $\theta = 1$ and $\omega \in G - \{4\}$, then $\Pi(\theta) = \Psi(\omega) = 1$ and the conditions of Theorem 2 are satisfied.*

(B) *If* $\theta = 1$, $\omega = 4$, *then* $\Pi\theta = 1$, $\Psi\omega = 2$,

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, 0, 3e^{ix}, \frac{1}{2}(e^{ix} + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

(C) *If* $\theta = 2$, $\omega = 4$, *then* $\Pi\theta = 1$, $\Psi\omega = 2$,

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, e^{ix}, 3e^{ix}, \frac{1}{2}(0 + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

(D) *If* $\theta = 3$, $\omega = 4$, *then* $\Pi\theta = 1$, $\Psi\omega = 2$,

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, e^{ix}, 3e^{ix}, \frac{1}{2}(e^{ix} + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

(E) *If* $\theta = 4$, $\omega = 4$, *then* $\Pi\theta = 2$, $\Psi\omega = 2$,

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, 3e^{ix}, 3e^{ix}, \frac{1}{2}(3e^{ix} + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

Moreover, for $\curlywedge = \frac{1}{3}$, *with* $\curlywedge < 1$, *the conditions of Theorem 2 are satisfied. Therefore,* 1 *is the unique common fixed point of* Π *and* Ψ.

4. Application

Consider the following systems of nonlinear integral equations:

$$w(s) = \mathfrak{F}(s) + \int_a^b T_1(s, p, w(p))dp, \tag{18}$$

and

$$z(s) = \mathfrak{F}(s) + \int_a^b T_2(s, p, z(p))dp, \tag{19}$$

where

(i) $\mathfrak{F} : [a, b] \to \mathbb{R}^n$ is a continuous mapping and $\mathfrak{F}(s)$ is a given function in $(C([a, b]), \mathbb{R}^n)$,
(ii) $w(s)$ and $z(s)$ are unknown variables for each $s \in J = [a, b]$, $b > a \geq 0$,
(iii) $T_1(s, p)$ and $T_2(s, p)$ are deterministic kernels defined for $s, p \in J = [a, b]$.

In this section, we present an existence theorem for a common solution to (18) and (19) that belongs to $G = (C(J), \mathbb{R}^n)$ (the set of continuous functions defined on J) by using the obtained result in Theorem 2. We consider the continuous mappings $\Pi, \Psi : G \to G$ given by

$$\Pi w(s) = \mathfrak{F}(s) + \int_a^b T_1(s, p, w(p))dp, w \in G, s \in J,$$

and

$$\Psi z(s) = \mathfrak{F}(s) + \int_a^b T_2(s, p, z(p))dp, z \in G, s \in J,$$

Then, the existence of a common solution to the nonlinear integral Equations (18) and (19) is equivalent to the existence of a common fixed point of Π and Ψ. It is well known that G, endowed with the metric $\wp_{cb}$, defined by

$$\wp_{cb}(w, z) = \sup_{s \in J} |w(s) - z(s)| + 2,$$

for all $w, z \in G$, is a complete CPMS. G can also be equipped with the partial order $\preceq$ given by

$$w, z \in G,\ w \preceq z \text{ if and only } w(s) \geq z(s),\ \text{ for all}\, s \in J.$$

Further, let us consider a system of nonlinear integral equation as (18) and (19) under the following condition hold:

(A) $T_1, T_2 : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfying

$$|T_1(s, p, w(p)) - T_2(s, p, z(p))| \preceq \frac{S(w, z)}{(b - a)e^t} - \frac{2}{b - a}, \forall t > 0,$$

where

$$S(w, z) = \max\{\wp_{cb}(w, z), \wp_{cb}(w, \Pi w), \wp_{cb}(z, \Psi z), \frac{1}{2}(\wp_{cb}(w, \Psi z) + \wp_{cb}(z, \Pi w))\}.$$

Theorem 8. *Let $(C(J), \mathbb{R}^n, \wp_{cb})$ be a complete CPMS; then, the system (18) and (19) under condition (A) have a unique common solution.*

Proof. For $w, z \in (C(J), \mathbb{R}^n)$ and $s \in J$, we define the continuous mappings $\Pi, \Psi : G \to G$ by

$$\Pi w(s) = \mathfrak{F}(s) + \int_a^b T_1(s, p, w(p))dp,$$

and

$$\Psi z(s) = \mathfrak{F}(s) + \int_a^b T_2(s, p, z(p))dp.$$

Then, we have

$$\begin{aligned}
\wp_{cb}(\Pi w(s), \Psi z(s)) &= \sup_{s \in J} |\Pi w(s) - \Psi z(s)| + 2 \\
&\precsim \int_a^b |T_1(s,p,w(p)) - T_2(s,p,z(p))| dp + 2 \\
&\precsim \int_a^b \left(\frac{S(w,z)}{(b-a)e^t} - \frac{2}{b-a} \right) dp + 2 \\
&= \frac{S(w,z)}{e^t} \\
&= \curlywedge S(w,z) \\
&= \curlywedge \max\{\wp_{cb}(w,z), \wp_{cb}(w,\Pi w), \wp_{cb}(z,\Psi z), \\
&\quad \frac{1}{2}(\wp_{cb}(w,\Psi z) + \wp_{cb}(z,\Pi w))\}.
\end{aligned}$$

Hence, all the conditions of Theorem 2 are satisfied for $0 < \curlywedge = \frac{1}{e^t} < 1$ with $t > 0$. Therefore the system of nonlinear integral Equations (18) and (19) have a unique common solution. $\square$

5. Conclusions

In this paper, we proved some common fixed-point theorems on complex partial metric space. An illustrative example and application on complex partial metric space is given.

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