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Navier–Stokes Cauchy Problem with $|v_0(x)|^2$ Lying in the Kato Class K_3

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Abstract: We investigate the 3D Navier–Stokes Cauchy problem. We assume the initial datum v_0 is weakly divergence free, $\sup_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_0(y)|^2}{|x-y|}dy<\infty$ and $|v_0(y)|^2\in K_3$, where K_3 denotes the Kato class. The existence is local for arbitrary data and global if $\sup_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_0(y)|^2}{|x-y|}dy$ is small. Regularity and uniqueness also hold.

Keywords: Navier–Stokes equations; existence; regular solutions

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1. Introduction

We consider the Navier–Stokes Cauchy problem:

$$\begin{aligned} &v_t+v\cdot\nabla v+\nabla\pi_v=\Delta v,\ \nabla\cdot v=0,\ \text{in } (0,T)\times\mathbb{R}^3,\\ &v=v_0 \text{ on } \{0\}\times\mathbb{R}^3. \end{aligned}\tag{1}$$

In System (1) v is the fluid velocity field,π_v is the pressure field of an incompressible viscous fluid, $v_t:=\frac{\partial}{\partial t}v$ and $v\cdot\nabla v:=v_k\frac{\partial}{\partial x_k}v$.

We set

$$\begin{aligned} L^2_{wt}&:=\{u:\sup_x\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}dy<\infty\},\\ K_3&:=\{U(y):\lim_{\rho\to0}\sup_x\int_{B(x,\rho)}\frac{|U(y)|}{|x-y|}dy=0\}. \end{aligned}$$

In a different context, the set K_3 was introduced by Kato in [1]. In [2], Simon studies and develops properties related to the elements of K_3 (see also [3]). For $\rho>0$, we set

$$\begin{aligned} &\|u\|^2_{K^\rho}:=\sup_x\int_{B(x,\rho)}\frac{|u(y)|^2}{|x-y|}dy,\ \text{ and }\ \|u\|^2_{wt}:=\sup_x\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}dy,\\ &|||u|||_{(t,\rho)}:=\sup_{(0,t)}\tau^{\frac12}\|u(\tau)\|_\infty+\sup_{(0,t)}\|u(\tau)\|_{K^\rho}+\frac{t^{\frac12}}{\rho}\sup_{(0,t)}\|u(\tau)\|_{wt}. \end{aligned}\tag{2}$$

By the symbol $\mathbb{K}$ we mean the set

$$\mathbb{K}:=\{u\in L^2_{wt},\ \text{weakly divergence free and } |u|^2\in K_3\}.$$

For all $\rho>0$ and $t>0$, we put

$$h(t,\rho):=e^{-\frac{\rho^2}{4t}}\left[\rho^2t^{-1}+2\right]^{\frac12}/4\pi\,.\tag{3}$$

We set $T(v_0) := \sup\limits_{\rho>0} t(\rho)$, where $t(\rho)$ is such that

$$1 - 4c\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big] > 0\,, \tag{4}$$

where c is an absolute constant. The definition of $T(v_0)$ is well posed for all $v_0 \in \mathbb{K}$. Indeed, observing that $|v_0|^2 \in K_3$ and taking into account that, for fixed ρ, $h(t,\rho) \to 0$ as $t \to 0$ (see Remark 1), one can choose ρ and subsequently t in such a way that (4) is satisfied.

We are interested in the following result.

Theorem 1. *For all $v_0 \in \mathbb{K}$ there exists a solution (v, π_v) to problem (1) on $(0, T(v_0)) \times \mathbb{R}^3$ enjoying the properties*

$$\begin{aligned}
&\text{for all } \eta > 0,\ t \in (\eta, T(v_0)),\ \theta \in [0,1),\\
&v \in C^{2,\theta}(\mathbb{R}^3) \cap \mathbb{K} \text{ and } v_t,\ D^2 v \in C^{0,\frac{\theta}{2}}((\eta, T(v_0)) \times \mathbb{R}^3),\\
&|\!|\!|v|\!|\!|_{(t,\rho)} \le \frac{2\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]}{1 + \big(1 - 4c\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]\big)^{\frac{1}{2}}},
\end{aligned} \tag{5}$$

for all $t \in (0, T(v_0))$, with

$$\begin{aligned}
&t^{\frac{1}{2}}\|\pi_v(t)\|_{wt} \le \bar{c}|\!|\!|v|\!|\!|^2_{(t,\rho)}\,,\ \pi_v \in C^{1,\theta}(\mathbb{R}^3) \text{ for all } t \in (0, T(v_0))\,,\\
&\lim_{t\to 0} t^{\frac{1}{2}}\|v(t)\|_\infty = 0 \text{ and, for all } x \in \mathbb{R}^3,\quad \lim_{t\to 0}\int_{\mathbb{R}^3} \frac{|v(t,y) - v_0(y)|^2}{|x-y|}dy = 0\,,
\end{aligned} \tag{6}$$

where $\bar{c}$ is an absolute constant. If the norm $\|v_0\|_{wt}$ is suitably small, then the result holds for all $t > 0$.

Proposition 1 (Weighted energy relation). *Let v be a solution to (1) enjoying properties (5) and (6). Then the following weighted energy relation holds:*

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{|v(t,y)|^2}{|x-y|}dy + 2\int_s^t\int_{\mathbb{R}^3} \frac{|\nabla v(t,y)|^2}{|x-y|}dyd\tau + 4\pi\int_s^t |v(\tau,x)|^2 d\tau\\
&\qquad = \int_{\mathbb{R}^3} \frac{|v(s,y)|^2}{|x-y|}dy + \int_s^t\int_{\mathbb{R}^3} (v^2 + 2\pi_v)\, v \cdot \nabla_y \frac{1}{|x-y|}dyd\tau\,,
\end{aligned} \tag{7}$$

for all $s < t$ in $(0, T(v_0))$.

Proposition 2 (Uniqueness). *For all $v_0 \in \mathbb{K}$ a solution of Theorem 1 is unique.*

To better explain the aim of the previous theorem, we recall a result by Caffarelli, Kohn, and Nirenberg in [4] and another, based on [4], achieved in [5,6]. In [4], Proposition 1 is a criterion of regularity for a suitable weak solution. As a consequence, one understands that if a suitable weak solution $v(t,x)$, corresponding to an initial datum $v_0 \in L^2(\mathbb{R}^3)$, admits a possible singularity in (t_0, x_0), then in a neighborhood of the point (t_0, x_0) there is the behavior (see (1.18) in [4])

$$|v(t,x)| \ge c(|x-x_0|^2 + |t-t_0|)^{-\frac{1}{2}}. \tag{8}$$

Instead, in [5,6], recognizing that, via the Hardy–Littlewood–Sobolev inequality, $v(t,y) \in L^2(\mathbb{R}^3)$ ensures that the Newtonian potential $\psi(t,x) := \int\limits_{\mathbb{R}^3} \frac{|v(t,y)|^2}{|x-y|}\,dy$ is finite almost everywhere in $x \in \mathbb{R}^3$, by means of the quoted criterion of regularity the authors prove that such points (t,x) are the center of suitable parabolic cylinders of regularity $Q(t,x)$ for a suitable weak solution v (we stress that in [7], under a suitable assumption of smallness, the parabolic cylinder $Q(t,x)$ of the partial regularity is of the kind $(0,\infty) \times \mathbb{R}^3 - B(0,R)$, see also [4]). Therefore, the Kato class K_3 appears of some interest since, in connection with (8), the class K_3 has the potential to preserve the solution from the singularity of the kind (8), and in connection with results of [5,6], the properties of K_3 make the potential $\psi(t,x)$ a continuous function of $x \in \mathbb{R}^3$.

We point out that, from (4), the existence interval $(0, T(v_0))$ in Theorem 1is determined by means of properties of the elements of $\mathtt{K}$. We can iterate the arguments, achieving a sequence $\{(T(v_0^{m-1}), T(v_0^m))\}$ of existence intervals that by the construction collapse to the empty set. The collapse is due to the fact that $\rho_m \to 0$. Hence, in the limit on m one loses the opportunity to take advantage of the smallness related to $\|v_0^m\|_{K^\rho}$. Concerning estimate $(6)_1$for the pressure field, it is actually stronger, in the sense that one obtains $\|\pi_v(t)\|_{wt} \leq \bar{c}\|v(t)\|_\infty \|v(t)\|_{wt}$ for all $t \in (0, T(v_0))$.

We point out that in the case of small data in L^2_{wt} the existence can be given relaxing the assumptions on the initial datum to v_0 divergence free and belonging to L^2_{wt}. The result is weaker. In fact, the properties $(6)_{3,4}$ do not hold.The initial datum is assumed in weak form, the uniqueness can be discussed following the duality approach furnished in [8], but, as made in [6], with $v \in L^2_w(0,T;L^\infty) \cap L^2(\eta, L^\infty)$, for all $\eta > 0$.

We remark that in the hypothesis $v_0 \in \mathtt{K} \cap L^2$, the results of existence and regularity of Theorem 1, coupled with the uniqueness theorems proved in [6], can be employed to deduce a structure theorem for suitable weak solutions.

The set L^2_{wt} is enclosed in BMO^{-1}, space detected in [9] as the widest scale-invariant space where problem (1) is well posed ($\dot{H}^{\frac12} \hookrightarrow L^3 \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty} \hookrightarrow BMO^{-1}$ holds, $p < \infty$). We also have $\mathtt{K} \subset BMO^{-1}_R$, cf. [9], or BMO^{-1}_0 (for this space see [10]). In fact, $w_0 \in BMO^{-1}_0$ means $w_0 \in BMO^{-1}$ and

$$\lim_{T\to 0}\Big[\sup_{(0,T)} t^{\frac12}\|w(t)\|_\infty + \sup_x \sup_{(0,T)} t^{-\frac34}\Big[\int_0^t \int_{B(x,\sqrt{t})} |w(\tau,y)|^2 dy d\tau\Big]^{\frac12}\Big] = 0\,,$$

where w is the heat solution corresponding to the distribution w_0. For $w_0 \in \mathtt{K}$, by virtue of Lemmas 4 and 6 in Section 3, we get

$$t^{\frac12}\|w(t)\|_\infty \leq c\|w_0\|_{K^\rho} + h(t,\rho)\|w_0\|_{wt} \ \text{ and } \|w(t)\|_{K^\rho} \leq c\|w_0\|_{K^\rho}\,,$$

where for all $\rho > 0$, we have $h(t,\rho) \to 0$ letting $t \to 0$. Hence, we easily get

$$\lim_{\rho\to 0}\lim_{T\to 0}\sup_{(0,T)} t^{\frac12}\|w(t)\|_\infty = 0\,,$$

and

$$
\begin{aligned}
&\lim_{T\to 0}\sup_x \sup_{(0,T)} t^{-\frac{3}{2}}\int_0^t \int_{B(x,\sqrt{t})} |w(\tau,y)|^2 dy d\tau \\
&\qquad \le \lim_{T\to 0}\sup_x \sup_{(0,T)} t^{-1}\int_0^t \int_{B(x,\sqrt{t})} \frac{|w(\tau,y)|^2}{|x-y|} dy d\tau \\
&\qquad\qquad \le c \lim_{T\to 0}\sup_x \sup_{(0,T)} \|w_0\|^2_{K\sqrt{t}} \le c\lim_{T\to 0}\sup_x \|w_0\|^2_{K\sqrt{T}} = 0\,.
\end{aligned}
$$

Hence, in connection with these spaces our result does not add a new statement. On the other hand, in the case of scale-invariant norms, to date, a functional dependence between the dimensionless size of the initial datum and the dimensionless size of the existence interval $(0,T)$ is not known and, to the best of our knowledge, the one exhibited in (4) is the first. Actually, in the setting of the scaling of the Navier–Stokes equations, one can consider the dimensionless ratio $t^{\frac{1}{2}}/\rho$, as we make in (4). In (4) this ratio *a priori* cannot be constant. The ratio changes by means of the size of the other quantities, which are all dimensionless, and they realize the size of the existence interval. In the proof of global existence for small data, being possible a constant ratio $t^{\frac{1}{2}}/\rho$ (which follows from the smallness of $\|v_0\|_{wt}$), we achieve $t^{\frac{1}{2}} \to \infty$ choosing proportionally ρ . So, *a priori*, one cannot compare the interval of existence given in [9] and the one given in Theorem 1. However, as remarked by the authors of [9], one of the interesting aspects of these results is the strict connection between the metrics employed in the existence theorems and the regularity criteria, such as the ones given in [4], which could be useful for an improvement of the partial regularity.

It is natural to inquire about the connection between $\mathtt{K}$ and the scale-invariant spaces $L^3, \dot{B}_{p,\infty}^{-1+\frac{3}{p}}$. In fact, no comparison is possible. This is a consequence of the fact that L^3 is not included in $\mathtt{K}$ and $\mathtt{K}$ is not included in $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$ (in this connection see Remark 5 in [6]). The Lorentz space $L(3,2)$ is enclosed in L_{wt}.

We conclude claiming that an existence theorem of weak solutions corresponding to a datum $v_0 \in \mathtt{K}$ holds. Roughly speaking, for this goal it is enough to follow the argument lines given in [11,12]. That is, to look for a weak solution v to problem (1) as the sum of two fields u and w. The field w is a smooth solution to a linear problem. Instead, in the ordinary L^2 setting, u is a weak solution to a suitable perturbed Navier–Stokes system. If the goal is the well posedness for the Navier–Stokes Cauchy problem, such an existence result of weak solutions appears of little interest since, by introducing the field u, one loses the advantage of $v_0 \in\mathtt{K}$. This is why we do not give the proof. The question is different in the context of *stability of motions*, such as in [13], where thanks to the result in [11] it is possible to show a transition from a stationary regime in Finn's class to a non-stationary one in $L(3,\infty)$ a.e. in t, or to show the converse transition.

This paper is organized as follows. In Section 2 we recall some preliminary results. In Section 3 we furnish a priori estimates in the functionals that define $|||\cdot|||_{(t,\rho)}$ in (2). In Section 4 we achieve the proof of Theorem 1, Propositions 1 and 2.

2. Preliminaries

We set

$$H * a(t,x) := \int_{\mathbb{R}^3} H(t,x-y)a(y)dy,$$

and, $i = 1, 2, 3$,

$$\nabla_x E_i * (a \otimes b)(t, x) := \int_0^t \int_{\mathbb{R}^3} D_{x_j} E_{ih}(t - \tau, x - y)(a_h b_j)(y, \tau) dy d\tau.$$

We look for a solution to the integral equation

$$v(t, x) = H * v_0(t, x) - \nabla_x E * (v \otimes v)(t, x), \text{ for all } (t, x) \in (0, T) \times \mathbb{R}^3, \tag{9}$$

where $H(t, z) := (4\pi t)^{-\frac{3}{2}} \exp[-|z|^2/4t]$ is the fundamental solution of the heat equation and $E(s, z)$ is the Oseen tensor, fundamental solution of the Stokes system, with components

$$E_{ij}(s, z) := -H(s, z)\delta_{ij} + D_{z_i z_j}\phi(s, z),$$

$$\phi(s, z) := \mathcal{E}(z) s^{-\frac{3}{2}} \int_0^{|z|} \exp[-a^2/4s^{\frac{3}{2}}] da,$$

where $\mathcal{E}$ is the fundamental solution of the Laplace equation. For the Oseen tensor the following estimates hold (cf. [14], estimates (VI) on page 215 and (VIII) on page 216, or [15]):

$$|D_s^k D_z^h E(s, z)| \leq c(|z| + s^{\frac{1}{2}})^{-3-h-k}, \text{ for all } s > 0 \text{ and } z \in \mathbb{R}^3, \tag{10}$$

for all $\theta \in (0, 1)$, uniformly in (s, z)

$$\begin{aligned}
\big|D_z^h E(s, z) - D_z^h E(s, \overline{z})\big| &\leq c|z - \overline{z}|^\theta \big[(|z| + s^{\frac{1}{2}})^{-(3+h+1)\theta} + (|\overline{z}| + s^{\frac{1}{2}})^{-(3+h+1)\theta}\big] \\
&\quad \times \big[(|z| + s^{\frac{1}{2}})^{-(3+h)(1-\theta)} + (|\overline{z}| + s^{\frac{1}{2}})^{-(3+h)(1-\theta)}\big], \qquad (11)\\
\big|D_t^k E(s, z) - D_s^k E(\overline{s}, z)\big| &\leq c|s - \overline{s}|^{\frac{\theta}{2}} \big[(|z| + s^{\frac{1}{2}})^{-(3+h+1)\theta} + (|z| + \overline{s}^{\frac{1}{2}})^{-(3+k+1)\theta}\big] \\
&\quad \times \big[(|z| + s^{\frac{1}{2}})^{-(3+k)(1-\theta)} + (|z| + \overline{s}^{\frac{1}{2}})^{-(3+h)(1-\theta)}\big],
\end{aligned}$$

where D_z^h is the symbol of the partial derivatives with respect to z_i-variable α_i times, $i = 1, 2, 3$, and $h = \alpha_1 + \alpha_2 + \alpha_3$.

We use the method of successive approximations. The lemmas of this and the following section ensure boundedness and convergence of the approximating sequence of velocity fields $\{v^m\}$. Finally the pressure with the corresponding estimates are recovered by solving a suitable Poisson equation and applying Lemma 3.

Following [16] we state:

Lemma 1. *Let $\xi_0 > 0$ and $c > 0$. Assume $1 - 4c\xi_0 > 0$. Let $\{\xi_m\}$ be a non-negative sequence of real numbers such that*

$$\xi_m \leq \xi_0 + c\xi_{m-1}^2.$$

Then $\xi_{m-1} \leq \xi$ for all $m \in \mathbb{N}$, where ξ is the minimum solution of the algebraic equation $c\xi^2 - \xi + \xi_0 = 0$.

Proof. The proof is immediate. □

Lemma 2. *There exist constants c independent of u such that*

i. *If $u \in L^\infty(\mathbb{R}^3) \cap L^2_{wt}(\mathbb{R}^3)$, then the following holds:*

$$\int_{\mathbb{R}^3} \frac{|u(y)|^3}{|x - y|^2} dy \leq c\|u\|_\infty^2 \Big[\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} dy\Big]^{\frac{1}{2}}, \text{ for all } x \in \mathbb{R}^3. \tag{12}$$

ii. *If $|x-y|^{-\frac{1}{2}}u,\ |x-y|^{-\frac{1}{2}}\nabla u \in L^2(\mathbb{R}^3)$, then the following holds:*

$$\int\limits_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^2}\,dy \le c\Big[\int\limits_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}dy\Big]^{\frac{1}{2}}\Big[\int\limits_{\mathbb{R}^3} \frac{|\nabla u(y)|^2}{|x-y|}\Big]^{\frac{1}{2}}, \textit{ for all } x \in \mathbb{R}^3. \tag{13}$$

iii. *If $u \in L^\infty(\mathbb{R}^3)$ and $|x-y|^{-\frac{1}{2}}u,\ |x-y|^{-\frac{1}{2}}\nabla u \in L^2(\mathbb{R}^3)$, then the following holds:*

$$\int\limits_{\mathbb{R}^3} \frac{|u(y)|^3}{|x-y|^2}\,dy \le c\|u\|_\infty\Big[\int\limits_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}dy\Big]^{\frac{1}{2}}\Big[\int\limits_{\mathbb{R}^3} \frac{|\nabla u(y)|^2}{|x-y|}\Big]^{\frac{1}{2}}, \textit{ for all } x \in \mathbb{R}^3. \tag{14}$$

Proof. For all $R > \overline{R} > 0$, applying Hölder's inequality, we get

$$\begin{aligned}\int\limits_{B(x,R)} \frac{|u(y)|^3}{|x-y|^2}dy &= \int\limits_{B(x,R)-B(x,\overline{R})} \frac{|u(y)|^3}{|x-y|^2}dy + \int\limits_{B(x,\overline{R})} \frac{|u(y)|^3}{|x-y|^2}dy \\ &\le \frac{\|u\|_\infty}{\overline{R}}\int\limits_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\,dy + c\|u\|_\infty^3\overline{R}.\end{aligned}$$

Setting

$$\overline{R} = \Big[\int\limits_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}dy\Big]^{\frac{1}{2}}\Big/\|u\|_\infty,$$

by virtue of the Beppo-Levi monotone convergence theorem, we arrive at (12). Inequality (13) is a particular case of a general weighted inequality proved in Lemma 7.1 of [4]. Inequality (14) is an immediate consequence of Hölder's inequality and (13). □

Let us consider the equation

$$\Delta\widetilde{\pi} = -\nabla a \cdot \nabla u^T \ \text{ in } \mathbb{R}^3. \tag{15}$$

For problem (15) we recall the following result:

Lemma 3. *Let a and u be divergence free in (15). For a solution $\widetilde{\pi}$ to problem (15) there exist constants c independent of u such that*

i. *If $|x-y|^{-\frac{2}{3}}a, |x-y|^{-\frac{2}{3}}u \in L^3(\mathbb{R}^3)$, then the following holds:*

$$\||x-y|^{-\frac{4}{3}}|\widetilde{\pi}|\|_{\frac{3}{2}} \le c\||x-y|^{-\frac{2}{3}}|a(y)|\|_3\||x-y|^{-\frac{2}{3}}|u(y)|\|_3. \tag{16}$$

ii. *If $a \in L^\infty(\mathbb{R}^3)$ and $|x-y|^{-\frac{1}{2}}\nabla u \in L^2(\mathbb{R}^3)$, then the following holds:*

$$\||x-y|^{-\frac{1}{2}}|\nabla\widetilde{\pi}|\|_2 \le c\|a\|_\infty\||x-y|^{-\frac{1}{2}}|\nabla u(y)|\|_2. \tag{17}$$

iii. *If, for $\theta \in (0,1)$, $a, u \in C^{1,\theta}(\mathbb{R}^3)$, then we get $\pi \in C^{1,\theta}(\mathbb{R}^3)$.*

Proof. We note that under our assumptions the following identity holds: $\nabla a \cdot \nabla u^T = \nabla\cdot\nabla\cdot(a\otimes u)$. Hence, from the representation formula for solutions of (15) and the theory of singular integrals (in particular [17] with weight), one deduces the result. □

3. Lemmas Related to the Functionals Defined on L^∞, K_3, L^2_{wt}

Lemma 4. *Let $a \in L^2_{wt}$. Then for the convolution product $H * a$ we get*

$$t^{\frac{1}{2}}\|H * a(t)\|_\infty \le h_0(t,\rho)\|a\|_{K^\rho} + h(t,\rho)\|a\|_{wt}\,,\ \text{for all}\ \rho > 0\ \text{and}\ t > 0\,,$$
$$\text{with}\ \ h_0(t,\rho) = \big[2 - e^{-\frac{\rho^2}{2t}}(\rho^2 t^{-1} + 2)\big]^{\frac{1}{2}}/4\pi\ \text{and}\ h(t,\rho) = e^{-\frac{\rho^2}{4t}}\big[\rho^2 t^{-1} + 2\big]^{\frac{1}{2}}/4\pi. \tag{18}$$

Proof. By the definition of the heat kernel and applying Hölder's inequality, we get

$$\begin{aligned}|H * a(t,x)| &\le \int\limits_{B(x,\rho)} |x-y|^{\frac{1}{2}} H(t,x-y)\frac{|a(y)|}{|x-y|^{\frac{1}{2}}}dy + \int\limits_{|x-y|>\rho} |x-y|^{\frac{1}{2}} H(t,x-y)\frac{|a(y)|}{|x-y|^{\frac{1}{2}}}dy \\ &\le \Big[\int\limits_{B(0,\rho)} |z|H^2(t,z)dz\Big]^{\frac{1}{2}}\|a\|_{K^\rho} + \Big[\int\limits_{|z|>\rho}|z|H^2(t,z)dz\Big]^{\frac{1}{2}}\|a\|_{wt} \\ &= t^{-\frac{1}{2}}\big[h_0(t,\rho)\|a\|_{K^\rho} + h(t,\rho)\|a\|_{wt}\big]\,,\end{aligned}$$

where for functions h_0 and h one easily proves the values claimed in (18). $\square$

Remark 1. Function $h_0(t,\rho)$ belongs to $(0, 1/(2^{\frac{3}{2}}\pi)]$ for all $(t,\rho) \in (0,\infty)\times(0,\infty)$. Instead, for all $\rho > 0$, function $h(t,\rho)$ is monotonically increasing in $t > 0$, with $\lim\limits_{t\to 0} h(t,\rho) = 0$.

Lemma 5. *Let $\sup_{(0,T)}[t^{\frac{1}{2}}\|a(t)\|_\infty + t^{\frac{1}{2}}\|b(t)\|_\infty] < \infty$ and $\sup_{(0,T)}[\|a(t)\|_{wt} + \|b(t)\|_{wt}] < \infty$. Then there exists a constant c independent of a and b such that*

$$\begin{aligned}t^{\frac{1}{2}}\|\nabla E * (a\otimes b)(t)\|_\infty \le c\Big[&\sup_{(0,t)} \tau\||a(\tau)||b(\tau)|\|_\infty \\ &+ \sup_{(0,t)}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{K^\rho} + t\rho^{-2}\sup_{(0,t)}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{wt}\Big],\end{aligned} \tag{19}$$
$$\text{for all}\ \rho > 0\ \text{for all}\ t \in (0,T)\,.$$

Proof. Via formulae (10) we get

$$\begin{aligned}|\nabla E * (a\otimes b)(t,x)| &\le \int\limits_{\frac{t}{2}}^{t}\int\limits_{\mathbb{R}^3}\frac{|a(\tau,y)||b(\tau,y)|}{(|x-y|^2 + t - \tau)^2}dy + \int\limits_{0}^{\frac{t}{2}}\int\limits_{\mathbb{R}^3}\frac{|a(\tau,y)||b(\tau,y)|}{(|x-y|^2+t-\tau)^2}dy \\ &=: I_1(t) + I_2(t)\,.\end{aligned}$$

By our hypotheses we get

$$I_1(t) \le c\int\limits_{\frac{t}{2}}^{t}\frac{1}{\tau}\sup \tau\||a(\tau)||b(\tau)|\|_\infty\int\limits_{\mathbb{R}^3}(|z|^2 + t - \tau)^{-2}dzd\tau \le ct^{-\frac{1}{2}}\sup_{(\frac{t}{2},t)}\tau\||a(\tau)||b(\tau)|\|_\infty\,,$$

and

$$\begin{aligned}
I_2(t) &\le c\int\limits_0^{\frac{t}{2}}\int\limits_{B(x,\rho)}\frac{|a(\tau,y)||b(\tau,y)|}{(|x-y|^2+t-\tau)^2}dzd\tau+\int\limits_0^{\frac{t}{2}}\int\limits_{|x-y|>\rho}\frac{|a(\tau,y)||b(\tau,y)|}{(|x-y|^2+t-\tau)^2}dzd\tau\\
&\le c\int\limits_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{2}}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{K^\rho}+c\rho^{-2}\int\limits_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{wt}\\
&\le ct^{-\frac{1}{2}}\sup_{(0,t)}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{K^\rho}+ct^{\frac{1}{2}}\rho^{-2}\sup_{(0,t)}\||a(\tau)|^{\frac{1}{2}}|b(\tau)|^{\frac{1}{2}}\|_{wt}.
\end{aligned}$$

Via estimates for I_1 and I_2 we arrive at (19). □

Lemma 6. *Let $\|a\|_{K^\rho}<\infty$. Then we get*

$$\|H*a(t)\|_{K^\rho}\le\|a\|_{K^\rho}\,,\ \textit{for all } t>0\,. \tag{20}$$

Proof. Employing Minkowski's inequality, we get

$$\begin{aligned}
\int\limits_{B(x,\rho)}|x-y|^{-1}\Big[\int\limits_{\mathbb{R}^3}H(t,y-z)a(z)dz\Big]^2dy &= \int\limits_{B(x,\rho)}\Big[\int\limits_{\mathbb{R}^3}H(t,\xi)\frac{a(y-\xi)}{|x-y|^{\frac{1}{2}}}\,d\xi\Big]^2dy\\
&\le\Big[\int\limits_{\mathbb{R}^3}H(t,\xi)\Big[\int\limits_{B(x,\rho)}\frac{|a(y-\xi)|^2}{|x-y|}dy\Big]^{\frac{1}{2}}d\xi\Big]^2.
\end{aligned}$$

Hence one easily arrives at (20). □

Lemma 7. *Let $a(x)\in L^2_{wt}$. Then we get*

$$\|H*a(t)\|_{wt}\le\|a\|_{wt}\,,\ \textit{for all } t>0\,. \tag{21}$$

Proof. The proof is analogous to the one of the previous lemma. Hence it is omitted. □

Lemma 8. *Let $\sup_{(0,T)}\big[t^{\frac{1}{2}}\|a(t)\|_\infty+\|b(t)\|_{K^\rho}\big]<\infty$. Then there exists a constant c independent of $a(t,x)$ and $b(t,x)$ such that*

$$\|\nabla E*(a\otimes b)(t)\|_{K^\rho}\le c\sup_{(0,t)}\tau^{\frac{1}{2}}\|a(\tau)\|_\infty\|b(\tau)\|_{K^\rho},\ \textit{for all } \rho>0 \textit{ and } t\in(0,T)\,. \tag{22}$$

Proof. Setting $y-z=\xi$ in the convolution product, we have to estimate the integral

$$\int\limits_{B(x,\rho)}|x-y|^{-1}\Big[\int\limits_0^t\int\limits_{\mathbb{R}^3}\nabla E(t-\tau,\xi)\cdot(a(\tau,y-\xi)\otimes b(\tau,y-\xi))d\xi d\tau\Big]^2dy=:I(t,x)$$

Employing the Minkowski inequality, we get

$$I(t,x)\le\Big[\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla E(t-\tau,\xi)|\Big[\int\limits_{B(x,\rho)}\frac{|a(\tau,y-\xi)|^2|b(\tau,y-\xi)|^2}{|x-y|}dy\Big]^{\frac{1}{2}}d\xi d\tau\Big]^2.$$

By virtue of our hypotheses and estimate (10) for the Oseen tensor, we find

$$I(t,x)\le c\sup_{(0,t)}\tau^{\frac{1}{2}}\|a(\tau)\|_\infty\|b(\tau)\|_{K^\rho}\int\limits_0^t(t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\,.$$

Hence one easily arrives at (22). □

Lemma 9. *Let* $\sup_{(0,T)}\left[t^{\frac{1}{2}}\|a(t)\|_\infty+\|b(t)\|_{wt}\right]<\infty$. *Then there exists a constant c independent of* $a(t,x)$ *and* $b(t,x)$ *such that*

$$\|\nabla E*(a\otimes b)(t)\|_{wt}\le c\sup_{(0,t)}\tau^{\frac{1}{2}}\|a(\tau)\|_\infty\|b(\tau)\|_{wt},\ \textit{for all } t\in(0,T)\,. \tag{23}$$

Proof. The proof is analogous to the one of the previous lemma. Hence it is omitted. □

Lemma 10. *In the hypotheses of Lemmas 4 and 5 the convolution products* $H*a$ *and* $\nabla E*(a\otimes b)$ *are Hölder continuous functions, with exponent* $\theta\in[0,1)$, *in* $(t,x)\in((\eta,T)\times\mathbb{R}^3)$, $\eta>0$. *In particular we get*

$$\begin{aligned}\frac{|H*a(t,x)-H*a(\bar t,\bar x)|}{\left[|x-\bar x|+|t-\bar t|^{\frac{1}{2}}\right]^\theta}&\le\frac{c}{\eta^{\frac{1+\theta}{2}}}\|a\|_{wt}\\ \frac{|\nabla E*(a\otimes b)(t,x)-\nabla E*(a\otimes b)(\bar t,\bar x)|}{\left[|x-\bar x|+|t-\bar t|^{\frac{1}{2}}\right]^\theta}&\le\frac{c}{\eta^{\frac{1+\theta}{2}}}\Big[\sup_{(0,t)}\tau\|a(\tau)b(\tau)\|_\infty+\sup_{(0,t)}\||a(\tau)b(\tau)|^{\frac{1}{2}}\|_{wt}\Big].\end{aligned} \tag{24}$$

Proof. Following the classical proof of the Hölder property for solutions to the heat equation, and the estimate given in Lemma 4, one obtains $(24)_1$. We omit further details, as the computations are similar to the following ones for ∇E.

Taking estimates (11) into account, applying Hölder's inequality, for the term $\nabla E*(a\otimes b)(t,x)$ we easily get the following estimate, $\theta\in[0,1)$,

$$\frac{\big|\nabla E*(a\otimes b)(t,x)-\nabla E*(a\otimes b)(t,\bar x)\big|}{|x-\bar x|^\theta}\le$$
$$c\int_0^t\int_{\mathbb{R}^3}\Big[\frac{|a(\tau,y)||b(\tau,y)|}{(|x-y|+(t-\tau)^{\frac{1}{2}})^{4+\theta}}+\frac{|a(\tau,y)||b(\tau,y)|}{(|\bar x-y|+(t-\tau)^{\frac{1}{2}})^{4+\theta}}\Big]dyd\tau\,.$$

Employing the arguments developed in the proof of Lemma 5, one easily arrives to the estimate

$$\begin{aligned}&\big|\nabla E*(a\otimes b)(t,x)-\nabla E*(a\otimes b)(t,\bar x)\big|\\&\qquad\le ct^{-\frac{1}{2}-\frac{\theta}{2}}|x-\bar x|^\theta\big[\sup_{(0,t)}\tau\|a(\tau)b(\tau)\|_\infty+\sup_{(0,t)}\||a(\tau)b(\tau)|^{\frac{1}{2}}\|_{wt}\big]\,.\end{aligned} \tag{25}$$

Analogously, one proves the Hölder property with respect to time for the term $\nabla E*(a\otimes b)$. Hence, for all $\eta>0$, in $(\eta,T)\times\mathbb{R}^3$, we have $(24)_2$. □

We study the integral relation

$$v^m(t,x)=H*v_0(t,x)-\nabla_xE*(v^{m-1}\otimes v^{m-1})(t,x)\,. \tag{26}$$

In (2), for $t>0$ and $\rho>0$, we set

$$|||u|||_{(t,\rho)}:=\sup_{(0,t)}\tau^{\frac{1}{2}}\|u(\tau)\|_\infty+\sup_{(0,t)}\|u(\tau)\|_{K^\rho}+\frac{t^{\frac{1}{2}}}{\rho}\sup_{(0,t)}\|u(\tau)\|_{wt}\,.$$

Lemma 11. *Let $v_0 \in \mathbb{K}$. Set $v^0(t,x) := H * v_0$. Then there exists a constant c, independent of v_0 and $m \in \mathbb{N}$, such that for the sequence (26) we get*

$$\|\!|v^m|\!\|_{(t,\rho)} \leq (1/(2^{\frac{3}{2}}\pi)+1)\|v_0\|_{K^\rho} + (h(t,\rho)+\tfrac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt} + c\|\!|v^{m-1}|\!\|^2_{(t,\rho)}, \quad \text{for all } t>0 \text{ and } \rho>0\,. \tag{27}$$

Proof. From definition (26), by virtue of Lemmas 4–9, for all $t>0$ and $\rho>0$, we get

$$\begin{aligned}
s^{\frac{1}{2}}\|v^1(s)\|_\infty &\leq h_0(s,\rho)\|v_0\|_{K^\rho} + h(s,\rho)\|v_0\|_{wt} \\
&\quad + c\Big[\sup_{(0,s)}\tau^{\frac{1}{2}}\|v^0(\tau)\|_\infty + \sup_{(0,s)}\|v^0(\tau)\|_{K^\rho} + \frac{s^{\frac{1}{2}}}{\rho}\sup_{(0,s)}\|v^0(\tau)\|_{wt}\Big]^2, \\
\|v^1(s)\|_{K^\rho} &\leq \|v_0\|_{K^\rho} + c\sup_{(0,s)}\tau^{\frac{1}{2}}\|v^0(\tau)\|_\infty\|v^0(\tau)\|_{K^\rho} \\
&\leq \|v_0\|_{K^\rho} + c\Big[\sup_{(0,s)}\tau^{\frac{1}{2}}\|v^0(\tau)\|_\infty + \sup_{(0,s)}\|v^0(\tau)\|_{K^\rho} + \frac{t^{\frac{1}{2}}}{\rho}\sup_{(0,t)}\|v^0(\tau)\|_{wt}\Big]^2, \\
\|v^1(s)\|_{wt} &\leq \|v_0\|_{wt} + c\sup_{(0,s)}\tau^{\frac{1}{2}}\|v^0(\tau)\|_\infty\|v^0(\tau)\|_{wt}\,,
\end{aligned} \tag{28}$$

where c is a constant independent of t, ρ. Multiplying $(28)_3$ for $\frac{t^{\frac{1}{2}}}{\rho}$, we get

$$\frac{t^{\frac{1}{2}}}{\rho}\|v^1(s)\|_{wt} \leq \frac{t^{\frac{1}{2}}}{\rho}\|v_0\|_{wt} + c\Big[\sup_{(0,s)}\tau^{\frac{1}{2}}\|v^0(\tau)\|_\infty + \sup_{(0,s)}\|v^0(\tau)\|_{K^\rho} + \frac{t^{\frac{1}{2}}}{\rho}\sup_{(0,s)}\|v^0(\tau)\|_{wt}\Big]^2.$$

Taking $\sup_{(0,t)}$ in each of (28), then summing $(28)_1$, $(28)_2$, and the last inequality, recalling the definition of the functional $\|\!|\cdot|\!\|_{(t,\rho)}$ and taking Remark 1 into account, we arrive at

$$\|\!|v^1|\!\|_{(t,\rho)} \leq (1/(2^{\frac{3}{2}}\pi)+1)\|v_0\|_{K^\rho} + (h(t,\rho)+\frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt} + 3c\|\!|v^0|\!\|^2_{(t,\rho)}, \quad \text{for all } \rho>0 \text{ and } t>0,$$

with a constant c independent of the datum v_0. So, for $m=1$, (26) is well defined and estimate (27) is true. Then by induction one proves the estimate for all $m \in \mathbb{N}$. □

Lemma 12. *Let $\{v^m\}$ be the sequence defined in (26) corresponding to $v_0 \in \mathbb{K}$. Then, there exists a $T(v_0)>0$ such that, for all η, the sequence strongly converges in $C^{0,\theta}((\eta,T(v_0))\times\mathbb{R}^3)$, $\theta\in[0,1)$ to a solution v to (9), and for all $t\in(0,T(v_0))$, the sequence converges to v in L^2_{wt}. In particular we get, for all $t\in[0,T(v_0))$,*

$$\|\!|v|\!\|_{(t,\rho)} \leq \frac{2\big[(1/(2^{\frac{3}{2}}\pi)+1)\|v_0\|_{K^\rho} + (h(t,\rho)+\frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]}{1+\big(1-4c\big[(1/(2^{\frac{3}{2}}\pi)+1)\|v_0\|_{K^\rho} + (h(t,\rho)+\frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]\big)^{\frac{1}{2}}}, \tag{29}$$

and

$$\lim_{t\to 0} t^{\frac{1}{2}}\|v(t)\|_\infty = 0\,. \tag{30}$$

Finally, for all $t\in[0,T(v_0))$, the limit $v(t,x)$ belongs to $\mathbb{K}$.

Proof. Recalling that $|v_0|^2 \in K_3$ and Remark 1 for h, we choose ρ and subsequently t in such a way that

$$1 - 4c\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big] > 0\,. \tag{31}$$

We denote by $T(v_0)$ the supremum of $t(\rho)$ for which (31) holds. Then, by virtue of (27) and Lemma 1, $t \in [0, T(v_0))$ and uniformly in $m \in \mathbb{N}$, we get

$$|||v^m|||_{(t,\rho)} \le \frac{2\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]}{1 + \big(1 - 4c\big[(1/(2^{\frac{3}{2}}\pi) + 1)\|v_0\|_{K^\rho} + (h(t,\rho) + \frac{t^{\frac{1}{2}}}{\rho})\|v_0\|_{wt}\big]\big)^{\frac{1}{2}}} =: A(\rho,t). \tag{32}$$

Employing again the definition given in (3), we also have the following immediate property:

P: for any sequence $\{t_p\} \to 0$, one constructs a sequence $\{\rho_p\} \to 0$ such that for all $p \in \mathbb{N}$ (31) holds and the right-hand side of (32) tends to zero.

Estimate (32) ensures that, for all $t \in [0, T(v_0))$, the sequence $\{|||v^m|||_{(t,\rho)}\}$ is bounded. We set $w^m := v^m - v^{m-1}$. Hence from (26) we arrive at ($m \ge 0$ and $v^{-1} = 0$)

$$w^{m+1}(t,x) = -\nabla_x E * (w^m \otimes v^m)(t,x) - \nabla_x E * (v^{m-1} \otimes w^m)(t,x)\,.$$

Employing the arguments of Lemmas 5, 6, and 9, and recalling estimate (32), we easily arrive at the sequence of estimates

$$|||w^1|||_{(t,\rho)} \le cA^2(\rho,t), \ldots, |||w^m|||_{(t,\rho)} \le 2^{m-1}c^m A^{m+1}(\rho,t), \ldots\,. \tag{33}$$

Since (31) furnishes $A(\rho,t) < 1/2c < 1$ for all $t \in (0, T(v_0))$, we get the convergence of $\{v^m\}$ with respect to the functional $|||\cdot|||_{(t,\rho)}$. The uniform convergence of the sequence of continuous functions $\{v^m\}$ on $(\eta, T) \times \mathbb{R}^3$ ensures that the limit is a continuous function in $(t,x) \in C((\eta,T) \times \mathbb{R}^3)$. We denote by v the limit. Recalling that for all $\theta' < \theta$ the following holds

$$[v^m - v^p]^{\mathbb{R}^3}_{\theta'} \le 2^{\frac{\theta-\theta'}{\theta}}\big[\sup_{\mathbb{R}^3}|v^m - v^p|\big]^{\frac{\theta-\theta'}{\theta}}\big[[v^m - v^p]^{\mathbb{R}^3}_\theta\big]^{\frac{\theta'}{\theta}},$$

where $[\,\cdot\,]^{\mathbb{R}^3}_\lambda$ denotes the Hölder seminorm, thanks to the Hölder properties (24) and the pointwise convergence just proved, for the limit v we obtain the Hölder property with $\theta' \in (0,\theta)$. Moreover, by virtue of property P, we deduce (30). We conclude the proof by proving that $v(t,x) \in \mathrm{K}$ for all $t \in [0, T(v_0))$. In fact, we have to prove that

$$\lim_{\overline{\rho}\to 0}\sup_x \int\limits_{B(x,\overline{\rho})} \frac{|v(t,y)|^2}{|x-y|}\,dy = 0\,.$$

The convergence with respect to the functional $|||\cdot|||_{(t,\rho)}$ in particular ensures that the limit v satisfies the integral Equation (9). Actually, the field v enjoys the hypotheses of Lemma 5. Thus considering $u^{m-1} := v - v^{m-1}$, we have to estimate in L^∞ the quantity

$$\nabla E * (u^{m-1} \otimes v)(t,x) + \nabla E * (v^{m-1} \otimes u^{m-1})(t,x)\,.$$

By virtue of estimate (19) , applying the Schwartz inequality, we get

$$\begin{aligned}&\|\nabla E * (u^{m-1} \otimes v)(t,x) + \nabla E * (v^{m-1} \otimes u^{m-1})(t,x)\|_\infty \\ &\qquad\qquad \le c|||u^{m-1}|||_{(t,\rho)}\big(|||v|||_{(t,\rho)} + |||v^{m-1}|||_{(t,\rho)}\big)\,,\end{aligned}$$

which ensures the desired convergence, hence

$$v(t,x) = H * v_0(t,x) - \nabla_x E * (v \otimes v)(t,x)\,. \tag{34}$$

Hence, applying Lemmas 6 and 8, we deduce

$$\|v(t)\|_{K^{\overline{\rho}}} \leq \|v_0\|_{K^{\overline{\rho}}} + c \sup_{(0,t)} \tau^{\frac{1}{2}} \|v(\tau)\|_\infty \|v(\tau)\|_{K^{\overline{\rho}}} \text{ for all } \overline{\rho} > 0 \text{ and } t \in [0, T(v_0))\,.$$

Since $A(\rho, t) < 1/2c$ for all $t \in (0, T(v_0))$, we deduce that

$$\sup_{(0,t)} \|v(\tau)\|_{K_{\overline{\rho}}} < \|v_0\|_{K_{\overline{\rho}}} + \frac{1}{2} \sup_{(0,t)} \|v(\tau)\|_{K^{\overline{\rho}}}\,, \text{ for all } \overline{\rho} > 0 \text{ and } t \in [0, T(v_0))\,.$$

Since $v_0 \in K_3$, we deduce $v \in K_3$ for all $t > 0$. □

Remark 2. By the definitions, $T(v_0)$ in Lemma 12 and $T(v_0)$ introduced in (4) coincide.

4. Proof of the Main Results Stated in the Introduction

Proof of Existence. In the hypothesis of Theorem 1, by virtue of Lemmas 11 and 12, we establish a solution $v(t,x)$ divergence free to the integral Equation (9) such that

$$\begin{aligned} &\text{for all } t \in (\eta, T(v_0)),\ \theta \in [0,1),\ v \in C^{0,\theta}(\mathbb{R}^3), \\ &\text{for all } t \in [0, T(v_0)),\ v(t,x) \in \mathbb{K} \text{ with } \|v(t)\|_{wt} \in L^\infty((0, T(v_0))). \end{aligned} \tag{35}$$

Subsequently, by means of integral equation and thanks to the Hölder property, one proves that v admits $\nabla\nabla v(t,x), v_t(t,x)$ with the regularity stated in (5) (see, e.g., [14]). Then, we consider π_v solution to the Poisson equation $\Delta \pi_v = -\nabla v \cdot (\nabla v)^T$. By Lemma 3 one arrives at $(6)_{1,2}$. Since v is solution to the integral Equation (34), by the couple (v, π_v) one finds the desired solution to System (1) (cf. [14,15] or [18] Section 4.6). Concerning the initial condition v_0, we first observe that the limit property $(6)_4$ trivially holds for $v^0(t,x)$, and then, via the integral Equation (9) and Lemma 9 for $\nabla E * (v \otimes v)$, we get

$$\|v(t) - v^0(t)\|_{wt} \leq c \sup_{(0,t)} \tau^{\frac{1}{2}} \|v(\tau)\|_\infty \|v(\tau)\|_{wt} \text{ for all } t \in [0, T(v_0))\,.$$

Thus, since $\lim_{t\to 0} \sup_{(0,t)} \tau^{\frac{1}{2}} \|v(\tau)\|_\infty = 0$, we arrive at the limit property (6). Concerning the global existence, we remark that $\|v_0\|_{K^\rho} \leq \|v_0\|_{wt}$ holds for all $\rho > 0$. Hence, considering $t^{\frac{1}{2}}/\rho$ in constant ratio, we can satisfy (31) by just requiring $\|v_0\|_{wt}$ to be sufficiently small. Since we can consider arbitrary ρ, the same holds for t, which means global existence. □

Proof of Proposition 1. We denote by φ_R a smooth nonegative cutoff function with value $\varphi_R(y) = 1$ for $|y| \leq R$ and $\varphi_R(y) = 0$ for $|y| \geq 2R$ and $R|\nabla \varphi_R| + |\nabla\nabla\varphi_R| \leq cR^{-2}$ for all $y \in \mathbb{R}^3$. In order to prove (7) we multiply Equation (1) by $|x-y|^{-1} v(t,y)\varphi_R(y)$. Integrating by parts on $(s,t) \times \mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} \varphi_R \frac{|v(t,y)|^2}{|x-y|} dy + 2 \int_s^t \int_{\mathbb{R}^3} \varphi_R \frac{|\nabla v(t,y)|^2}{|x-y|} dy d\tau \leq \sum_{i=1}^3 I_i(t)\,, \tag{36}$$

where we set

$$I_1(t) := \int_s^t \int_{\mathbb{R}^3} |v(\tau, y)|^2 |\Delta \frac{\varphi_R}{|x-y|}| dy d\tau$$
$$I_2(t) := \int_s^t \int_{\mathbb{R}^3} \frac{c|v|^3}{R|x-y|} + \frac{|v|^3}{|x-y|^2} dy d\tau$$
$$I_3(t) := 2 \int_s^t \int_{\mathbb{R}^3} \frac{|\pi_v||v|}{|x-y|^2} + \frac{c|\pi_v||v|}{R|x-y|} dy d\tau\,.$$

By virtue of the regularity of v, applying Hölder's inequality and employing Lemmas 2 and 3, we get the right and sides of I_i, $i = 1, 2, 3$ uniformly bounded in R. Hence applying the Beppo Levi monotonic convergence theorem, for all $s, t \in (0, T(v_0))$, we deduce that $|x-y|^{-\frac{1}{2}} |\nabla v(t, y)| \in L^2(s, t; L^2(\mathbb{R}^3))$. This last integrability property and the regularity of v allow us to arrive at (7). The result is proved. $\square$

Proof of Proposition 2. Finally, we prove the uniqueness in the class of existence. Consider two solutions v and $\overline{v}$ enjoying the properties indicated in Theorem 1; then, as it is known, for both solutions one writes the integral Equation (9). Hence by difference we arrive at

$$u(t, x) = -\nabla_x E * (u \otimes v) - \nabla_x E * (\overline{v} \otimes u)\,,$$

where $u := \overline{v} - v$ and $t \in [0, T(v)) \cap [0, T(\overline{v}))$. By virtue of Lemma 9 we get

$$\|u(t)\|_{wt} \le c \sup_{(0,t)} \tau^{\frac{1}{2}} (\|v(\tau)\|_\infty + \|\overline{v}(\tau)\|_\infty) \sup_{(0,t)} \|u(\tau)\|_{wt}\,.$$

Since the limit property (6) holds, one easily deduce the uniqueness on some interval $(0, \delta]$. In order to complete the uniqueness for $t \in [\delta, T(v)) \cap [\delta, T(\overline{v}))$, one employs the weighted energy inequality. Multiplying equation of u, that is $u_t - \Delta u + v \cdot \nabla u + u \cdot \nabla \overline{v} + \nabla \pi_u = 0$, by $u/|x-y|$ and integrating on $(\delta, t) \times \mathbb{R}^3$, one obtains

$$\int_{\mathbb{R}^3} \frac{|u(t, y)|^2}{|x-y|} dy + 2 \int_\delta^t \int_{\mathbb{R}^3} \frac{|\nabla u(t, y)|^2}{|x-y|} dy d\tau \le 2 \sum_{i=1}^3 I_i(t)\,, \tag{37}$$

where we set

$$I_1(t) := \int_\delta^t \int_{\mathbb{R}^3} \frac{|v \cdot \nabla u \cdot u| + |u \cdot \nabla u \cdot \overline{v}|}{|x-y|} dy$$
$$I_2(t) := \int_\delta^t \int_{\mathbb{R}^3} \frac{|u||u \cdot \overline{v}|}{|x-y|^2} dy$$
$$I_3(t) := \int_\delta^t \int_{\mathbb{R}^3} \frac{|\nabla \pi_u||u|}{|x-y|} dy d\tau\,.$$

Since $v, \overline{v} \in L^\infty((\delta, T(v)) \cap (\delta, T(\overline{v})) \times \mathbb{R}^3)$, by virtue of Lemma 2, we obtain the estimates:

$$I_1(t) \le c \int_\delta^t [\|v(\tau)\|_\infty + \|\overline{v}(t)\|_\infty] \Big[\int_{\mathbb{R}^3} \frac{|u|^2}{|x-y|} dy \Big]^{\frac{1}{2}} \Big[\int_{\mathbb{R}^3} \frac{|\nabla u|^2}{|x-y|} dy \Big]^{\frac{1}{2}} d\tau$$
$$I_2(t) \le c \int_\delta^t \|\overline{v}(\tau)\|_\infty \Big[\int_{\mathbb{R}^3} \frac{|u|^2}{|x-y|} dy \Big]^{\frac{1}{2}} \Big[\int_{\mathbb{R}^3} \frac{|\nabla u|^2}{|x-y|} dy \Big]^{\frac{1}{2}} d\tau\,.$$

Since $\Delta\pi_u = -\nabla\cdot\nabla\cdot(u\otimes v + \overline{v}\otimes u)$, employing Lemma 3, then we get the estimate

$$\int_{\mathbb{R}^3} \frac{|\nabla\pi_u|^2}{|x-y|} \le c\big[\|v\|_\infty + \|\overline{v}\|_\infty\big]^2 \int_{\mathbb{R}^3} \frac{|\nabla u|^2}{|x-y|}dy\,.$$

Hence for I_3 we obtain

$$I_3(t) \le c\int_\delta^t \big[\|v(\tau)\|_\infty + \|\overline{v}(\tau)\|_\infty\big]\Big[\int_{\mathbb{R}^3} \frac{|u|^2}{|x-y|}dy\Big]^{\frac12}\Big[\int_{\mathbb{R}^3} \frac{|\nabla u|^2}{|x-y|}dy\Big]^{\frac12}d\tau\,.$$

Increasing the right-hand side of (37) with the previous estimates we deduce an integral inequality for the uniqueness. The theorem is completely proved. □

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