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Existence and Symmetry of Solutions for a Class of Fractional Schrödinger–Poisson Systems

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Abstract: In this paper, we investigate a class of Schrödinger–Poisson systems with critical growth. By the principle of concentration compactness and variational methods, we prove that the system has radially symmetric solutions, which improve the related results on this topic.

Keywords: fractional Laplacian; fractional Schrödinger–Poisson systems; weak solution; variational methods



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1. Introduction

In recent years, fractional equations or systems have been studied extensively by researchers due to their various applications in various fields, such as obstacle problems, electrical circuits, quantum mechanics, and phase transitions; see [1–7] and their references. It is particularly important to mention that Laskin in [7] established the following time-dependent Schrödinger equation involving a fractional Laplacian when he expanded the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths

$$i \frac{\partial \psi}{\partial t} = (-\Delta)^\alpha \psi + (V(x) + \kappa) \psi - g(x, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \quad (1)$$

where $\kappa > 0$ is a constant and $(-\Delta)^\alpha = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} u)$ is the fractional Laplacian of order α , and \mathcal{F} denotes the usual Fourier transform in \mathbb{R}^3 .

The fractional Schrödinger equation is a fundamental equation in the fractional quantum mechanics when investigating the quantum particles on stochastic fields modeled, and it has been getting a lot of attention from researchers; see [8–13] and their references. For example, Li et al. in [13] studied the following form of fractional Schrödinger equations

$$(-\Delta)^\alpha u + V(x)u = \lambda K(x)f(u) + |u|^{2_\alpha^*-2}u, \quad (2)$$

where $\alpha \in (0, 1)$, $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the fractional critical Sobolev exponent, $\lambda > 0$ is a parameter, $V(x)$ is potential function with $\lim_{|x| \rightarrow +\infty} V(x) = 0$, i.e., $V(x)$ is vanishing at infinity. Using

the variational method, they obtained the existence of a positive solution for (2).

When dealing with the quantum particle in three-dimensional space interacting with an unknown electromagnetic field, Benci and Fortunato in [14] first introduced the following classical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \quad (3)$$

The authors obtained a sequence solution of (3) by variational methods. Such a system (3), also called the Schrödinger–Maxwell system, arises in an interesting physical context. According to a classical model, the interaction of a charge particle with an electromagnetic

field can be described by coupling the nonlinear Schrödinger–Poisson system. Recently, system (3) has been widely investigated because it has a strong physical meaning; see [15–17] and their references. For example, Azzollini and Pomponio in [15] considered system (3) by variational methods; they established the existence of a ground state solution when potential $V(x)$ is a positive constant or non-constant.

Fractional Schrödinger–Poisson systems have received lots of attention in recent years, and many of the works have studied the existence of solutions of it; see [18–22] and their references. As far as we know, few studies have considered the existence of solutions for the fractional Schrödinger–Poisson system with critical growth. Gu et al. in [19] only studied the existence of a positive solution by variational methods, and there are no relevant articles that consider the existence of radially symmetric solutions of the fractional Schrödinger–Poisson system with critical growth. We tried to deal with this problem and obtained novel existence results by using new analytical methods, which are different from the related conclusions on this topic.

Motivated by above results, in this paper, we mainly study the following fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^\alpha u + V(x)u + \phi u = K(x)f(u) + \lambda|u|^{2_\alpha^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \quad (4)$$

where $0 < \alpha \leq t < 1$, $4\alpha + 2t > 3$, $2_\alpha^* = \frac{6}{3-2\alpha}$. We assume $V(x)$ and $K(x)$ are radially symmetric, i.e., $V(x) = V(|x|)$ and $K(x) = K(|x|)$ for any $x \in \mathbb{R}^3$, and satisfies the following assumptions

(V1) $V \in C^1(\mathbb{R}^3, \mathbb{R})$, $\nabla V(x) \cdot x \leq 0$ for any $x \in \mathbb{R}^3$;

(V2) Positive constants V_1 and V_2 exist such that $V_1 \leq V(x) \leq V_2$ for any $x \in \mathbb{R}^3$;

(K1) $K \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists a constant K_0 such that $0 \leq \nabla K(x) \cdot x \leq K_0$ for any $x \in \mathbb{R}^3$;

(K2) Positive constants K_1 and K_2 exist such that $K_1 \leq K(x) \leq K_2$ for any $x \in \mathbb{R}^3$;

We make the following assumptions for the function f .

(f1) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$;

(f2) There exists $C > 0$ such that $|f(t)| \leq C(1 + |t|^{p-1})$ for all $t \in \mathbb{R}$, where $p \in (2, 2_\alpha^*)$.

(f3) There exists $\theta > 4$ such that $0 < \theta F(t) \leq tf(t)$ for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(\eta) d\eta$.

(f4) $f(t) = f(-t)$ for all $t \in \mathbb{R}$.

Now we state the main results of this paper.

Theorem 1. Assume that $V(x), K(x)$ fulfills (V1), (V2), (K1), (K2) and f satisfies (f1)–(f3). There exists $\lambda_1 > 0$ such that for any $0 < \lambda < \lambda_1$, problem (4) has a nontrivial radially symmetric solution.

Theorem 2. Assume that $V(x), K(x)$ fulfills (V1), (V2), (K1), (K2) and f satisfies (f1)–(f4). Then, problem (4) has infinitely many radially symmetric solutions.

This paper is organized as follows. In Section 2, we introduce the preliminaries of fractional Sobolev space and some lemmas. In Section 3, we give the proofs of our main result.

2. Preliminaries

In this section, we present a short review of fractional Sobolev spaces and some lemmas, which we used to prove our main result. The complete introduction of fractional Sobolev spaces can be found in [23]. For the properties of the function ϕ_u^s , see [18]. Throughout this paper, we denote $\|\cdot\|_p$ as the usual norm of $L^p(\mathbb{R}^3)$, $C_i (i = 1, 2, \dots)$ or C denotes the positive constants.

For $\alpha \in (0, 1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\}.$$

and the norm is

$$\|u\|_{H^\alpha} := \left(\int_{\mathbb{R}^3} |u(\xi)|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

$H^\alpha(\mathbb{R}^3)$ is a Hilbert space with the inner product for any $u, v \in H^\alpha(\mathbb{R}^3)$

$$\langle u, v \rangle := \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v dx + \int_{\mathbb{R}^3} u v dx,$$

and norm

$$\|u\| := \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^3} u^2 dx \right)^{\frac{1}{2}}.$$

For the second equation in problem (4), it has a unique solution ϕ_u^t (see [18]). Substituting ϕ_u^t in (4), we have the following fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u + \phi_u^t u = K(x)f(u) + \lambda|u|^{2_\alpha^*-2}u, \quad x \in \mathbb{R}^3, \quad (5)$$

whose solutions can be obtained by seeking critical points of the energy functional $I_\lambda : H^\alpha(\mathbb{R}^3) \rightarrow \mathbb{R}$

$$\begin{aligned} I_\lambda(u) := & \frac{1}{2} \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)u^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx \\ & - \int_{\mathbb{R}^3} K(x)F(u) dx - \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx. \end{aligned}$$

which is well defined in $H^\alpha(\mathbb{R}^3)$ and $I \in C^1(H^\alpha(\mathbb{R}^3), \mathbb{R})$, and for all $u, v \in H^\alpha(\mathbb{R}^3)$,

$$\begin{aligned} \langle I'_\lambda(u), v \rangle := & \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v dx + \int_{\mathbb{R}^3} V(x)u v dx + \int_{\mathbb{R}^3} \phi_u^t u v dx \\ & - \int_{\mathbb{R}^3} K(x)f(u)v dx - \lambda \int_{\mathbb{R}^3} |u|^{2_\alpha^*-2} u v dx. \end{aligned}$$

Definition 1 ([24]). Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R}^1)$. A sequence $\{u_n\}$ in X is a (PS)-sequence if $|\varphi(u_n)| \leq c$ uniformly in n , while $\|\varphi'(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 1 ([23]). Let $\alpha \in (0, 1)$ and $N \geq 1$ satisfies $N > 2\alpha$. Then there exists $C = C(N, \alpha) > 0$ such that

$$\|u\|_{L^{2_\alpha^*}(\mathbb{R}^3)} \leq C \|u\|_{H^\alpha(\mathbb{R}^3)}$$

for every $u \in H^\alpha(\mathbb{R}^3)$, where $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the fractional critical exponent. Moreover, the embedding $H^\alpha(\mathbb{R}^3) \hookrightarrow L^\gamma(\mathbb{R}^3)$ is continuous for each $\gamma \in [2, 2_\alpha^*]$ and is locally compact for $\gamma \in [2, 2_\alpha^*)$.

In the following, the symmetric mountain pass lemma is presented and used to prove our main result.

Lemma 2 ([25]). Let X be an infinite-dimensional Banach space with $X = M \oplus N$, where M is a finite-dimensional subspace of X . Assume that $I \in C^1(X, \mathbb{R})$ is a functional that satisfies the (PS)-condition and the following properties:

(A1) I is even for all $u \in X$ and $I(0) = 0$.

(A2) There exist two constant $\rho, \sigma > 0$ such that $I(u) \geq \sigma$.

(A3) There exists constant $\xi = \xi(\Omega) > 0$ such that $0 \geq I(u)$ for any $u \in X \setminus B_\xi(\Omega)$, where Ω is an arbitrary finite-dimensional subspace of X and $B_\xi(\Omega) := \{u \in \Omega : \|u\| \leq \xi\}$. Then the functional I has an unbounded sequence of critical values.

In the following, we prove that the functional I_λ satisfies the mountain pass geometry.

Lemma 3. Suppose $V(x), K(x)$ satisfies (V1), (V2), (K1), (K2) and f satisfies (f1)–(f3), then the functional I_λ satisfies

- (i) There exists $\beta, \rho > 0$ such that $I_\lambda(u) \geq \beta$ for $\|u\| = \rho$.
- (ii) There exists $e \in H^\alpha(\mathbb{R}^3)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.

Proof. (i) If $V(X), K(X)$ satisfies (V1), (V2), (K1) and (K2), by (f1) and (f2), for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad (6)$$

and

$$|F(t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^p. \quad (7)$$

Therefore, by Lemma 1, (7) and Hölder's inequality, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} K(x)F(u)dx - \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} K(x)F(u)dx - \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} K(x) \left[\frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^p \right] dx - \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2} \int_{\mathbb{R}^3} K(x)|t|^2 dx - \frac{C_\varepsilon}{p} \int_{\mathbb{R}^3} K(x)|t|^p dx - \frac{\lambda C_{2_\alpha^*}}{2_\alpha^*} \|u\|^{2_\alpha^*} \\ &\geq \frac{1 - \varepsilon d_2}{2} \|u\|^2 - \frac{C_\varepsilon d_p}{p} \|u\|^p - \frac{\lambda C_{2_\alpha^*}}{2_\alpha^*} \|u\|^{2_\alpha^*}, \end{aligned}$$

which implies that there exist two positive constants $\beta, \rho > 0$ such that $I_\lambda(u) \geq \beta$ for $\|u\| = \rho$.

(ii) If $V(X), K(X)$ satisfies (V1), (V2), (K1) and (K2), from (f3), C_1 exists such that

$$F(u) \geq C_1|u|^\theta \quad (8)$$

for $|u| > M_1$, where $M_1 \rightarrow +\infty$. By the properties of the function ϕ_u^s (see Lemma 2.3-(2) in [18]) and (8), for $t > 0$ and any $u \in H^\alpha(\mathbb{R}^3)$, we have

$$\begin{aligned} I_\lambda(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} K(x)F(tu)dx - \frac{\lambda t^{2_\alpha^*}}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &\leq \frac{t^2}{2}\|u\|^2 + \frac{C_2 t^4}{4} \|u\|^4 - C_1 t^\theta \|u\|^\theta - \frac{\lambda C_{2_\alpha^*}}{2_\alpha^*} \|u\|^{2_\alpha^*}. \end{aligned}$$

Therefore, $I_\lambda(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$, which means that there exists a constant $M_2 > 0$ such that $t_0 > M_2$ such that $I_\lambda(t_0 u) \leq 0$. Hence, we can choose $e := t_0 u \in H^\alpha(\mathbb{R}^3)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. \square

Lemma 4. Assume that (f1)–(f4) are satisfied. Then, any (PS)-sequence for I_λ is bounded.

Proof. Let $\{u_n\} \subset$ be a (PS)-sequence. By (f3), for $\theta > 4$, we have

$$\begin{aligned} 1 + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \\ &\quad + \int_{\mathbb{R}^3} K(x) \left(\frac{1}{\theta} f(u_n) u_n - F(u_n)\right) dx + \lambda \left(\frac{1}{\theta} - \frac{1}{2_\alpha^*}\right) \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2 + \lambda \left(\frac{1}{\theta} - \frac{1}{2_\alpha^*}\right) C_{2_\alpha^*} \|u_n\|^{2_\alpha^*}, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$. \square

Lemma 5 ([26]). Let $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^3)$ be a sequence and $u_n \rightharpoonup u$ weakly as $n \rightarrow \infty$ and such that $|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \rightharpoonup \mu$ and $|u_n|^{2_\alpha^*} \rightharpoonup \nu$ weakly-* in $\mathcal{M}(\mathbb{R}^3)$. Then, either $u_n \rightarrow u$ in $L_{loc}^{2_\alpha^*}(\mathbb{R}^3)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j \in J} \subset \bar{\Omega}$ and positive numbers $\{v_j\}_{j \in J}$ such that $\nu = |u|^{2_\alpha^*} + \sum_{j \in J} v_j \delta_{x_j}$, where $\Omega \subseteq \mathbb{R}^3$ is an open subset. If, Ω is bounded, then there exists a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^3)$ with $\text{supp} \tilde{\mu} \subset \bar{\Omega}$ and positive numbers $\{\mu_j\}_{j \in J}$ such that $\mu = |(-\Delta)^{\frac{\alpha}{2}} u|^2 + \tilde{\mu} + \sum_{j \in J} \mu_j \delta_{x_j}$.

Lemma 6 ([27]). Define

$$\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^3: |x| > R\}} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx,$$

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^3: |x| > R\}} |u_n|^{2_\alpha^*} dx.$$

Then the quantities μ_∞ and ν_∞ are well defined and satisfy

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx = \int_{\mathbb{R}^3} d\mu + \mu_\infty,$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty.$$

Lemma 7 ([27]). Let $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly-* in $\dot{H}^\alpha(\mathbb{R}^3)$, $|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \rightharpoonup \mu$ and $|u_n|^{2_\alpha^*} \rightharpoonup \nu$ weakly-* in $\mathcal{M}(\mathbb{R}^3)$ as $n \rightarrow \infty$. Then, $v_i \leq (S_\alpha^{-1} \mu\{x_i\})^{\frac{2_\alpha^*}{2}}$ for $i \in J$ and $\nu_\infty \leq (S_\alpha^{-1} \mu_\infty)^{\frac{2_\alpha^*}{2}}$, where x_i, v_i are from Lemma 5 and μ_∞, ν_∞ are given in Lemma 6, S_α is the best Sobolev constant of $\dot{H}^\alpha(\mathbb{R}^3) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^3)$, i.e.,

$$S_\alpha = \inf_{u \in \dot{H}^\alpha(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{\|u\|_{L^{2_\alpha^*}(\mathbb{R}^3)}^2}. \quad (9)$$

In the following lemma, we show that the functional I_λ satisfies the (PS)-condition.

Lemma 8. There exists $\lambda_* > 0$, in which each bounded (PS) sequence for I_λ contains a convergent subsequence for any $\lambda \in (0, \lambda_*)$.

Proof. Let $\{u_n\} \subset H^\alpha(\mathbb{R}^3)$ be a bounded (PS) sequence, i.e.,

$$I_\lambda(u_n) \leq C_3 \text{ and } I'_\lambda(u_n) \rightarrow 0 \text{ in } E, \text{ as } n \rightarrow \infty.$$

Passing to a subsequence, still denoted by $\{u_n\}$. Set $u_n \rightharpoonup u_0$ weakly in E . According to Lemma 1, we have $u_n \rightarrow u_0$ in $L^p(\mathbb{R}^3)$ and $u_n \rightarrow u_0$ a.e. in \mathbb{R}^3 as $n \rightarrow \infty$. Therefore, by Prokhorov's Theorem [28], there exists $\mu, \nu \in \mathcal{M}(\mathbb{R}^3)$ such that

$$|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \rightarrow \mu \text{ and } |u_n|^{2^*} \rightarrow \nu \text{ weakly-}^* \text{ in } \mathcal{M}(\mathbb{R}^3) \text{ as } n \rightarrow \infty.$$

By Lemma 5, we have $u_n \rightarrow u_0$ in $L_{loc}^{2^*}(\mathbb{R}^3)$ or $\nu = |u_0|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}$ as $n \rightarrow \infty$. For any $\phi \in H^\alpha(\mathbb{R}^3)$, we have

$$\begin{aligned} & \langle I'_\lambda(u_n), \phi \rangle - \langle I'_\lambda(u_0), \phi \rangle \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} (u_n - u_0) (-\Delta)^{\frac{\alpha}{2}} \phi dx + \int_{\mathbb{R}^3} V(x) (u_n - u_0) \phi dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t (u_n)^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_0}^t (u_0)^2 dx - \int_{\mathbb{R}^3} K(x) (f(u_n) - f(u_0)) \phi dx \\ &- \lambda \int_{\mathbb{R}^3} (|u_n|^{2^*-2} u_n - |u_0|^{2^*-2} u_0) \phi dx. \end{aligned}$$

Since $u_n \rightharpoonup u_0$ weakly in $H^\alpha(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} (u_n - u_0) (-\Delta)^{\frac{\alpha}{2}} \phi dx + \int_{\mathbb{R}^3} V(x) (u_n - u_0) \phi dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since

$$\{|u_n|^{2^*-2} u_n - |u_0|^{2^*-2} u_0\}_n \text{ is bounded in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^3)$$

and

$$|u_n|^{2^*-2} u_n - |u_0|^{2^*-2} u_0 \rightarrow 0 \text{ a.e. in } \mathbb{R}^3 \text{ as } n \rightarrow \infty,$$

then

$$|u_n|^{2^*-2} u_n - |u_0|^{2^*-2} u_0 \rightarrow 0 \text{ weakly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^3) \text{ as } n \rightarrow \infty,$$

which implies that

$$\int_{\mathbb{R}^3} (|u_n|^{2^*-2} u_n - |u_0|^{2^*-2} u_0) \phi dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By the properties of the function ϕ_u^s (see Lemma 2.3-(6) in [18])

$$\int_{\mathbb{R}^3} \phi_{u_n}^t (u_n)^2 dx - \int_{\mathbb{R}^3} \phi_{u_0}^t (u_0)^2 dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the following, we prove that

$$\int_{\mathbb{R}^3} K(x) (f(u_n) - f(u_0)) \phi dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As $\langle I'_\lambda(u_n), \phi \rangle \rightarrow 0$, then $\langle I'_\lambda(u_0), \phi \rangle = 0$, i.e., $I'_\lambda(u_0) = 0$. Thus,

$$\begin{aligned} & \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx + \int_{\mathbb{R}^3} V(x) u_0^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_0}^t (u_0)^2 dx \\ &= \int_{\mathbb{R}^3} K(x) f(u_0) u_0 dx + \frac{\lambda}{2^*} \int_{\mathbb{R}^3} |u_0|^{2^*} dx. \end{aligned}$$

By (6) and the Young inequality, one has

$$\begin{aligned} & |[f(u_n) - f(u_0)]\phi| \\ & \leq |u_n||\phi| + C_4|u_n|^{p-1}|\phi| + |u_0||\phi| + C_5|u_0|^{p-1}|\phi| \\ & \leq |u_n - u_0||\phi| + 2|u_0||\phi| + C_4|u_n - u_0|^{p-1}|\phi| + C_5|u_n - u_0|^{p-1}|\phi| \\ & \leq \varepsilon|u_n - u_0|^2 + C_\varepsilon|\phi|^2 + 2|u_0||\phi| + \varepsilon|u_n - u_0|^p + C_4C_\varepsilon|\phi|^p + C_5|u_0|^{p-1}|\phi|. \end{aligned}$$

Set

$$G_{\varepsilon,n}(x) = \max\{|K(x)(f(u_n) - f(u_0))\phi| - \varepsilon|u_n - u_0|^2 - \varepsilon|u_n - u_0|^p, 0\}.$$

Then

$$0 \leq G_{\varepsilon,n}(x) \leq C_\varepsilon|\phi|^2 + 2|u_0||\phi| + C_4C_\varepsilon|\phi|^p + C_5|u_0|^{p-1}|\phi| \in L^1(\mathbb{R}^3),$$

and $G_{\varepsilon,n}(x) \rightarrow 0$ a.e. on \mathbb{R}^3 . By the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^3} G_{\varepsilon,n}(x)dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^3} K(x)(f(u_n) - f(u_0))\phi dx \right| \\ & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} G_{\varepsilon,n}(x)dx + \varepsilon \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n - u_0|^2 dx + \varepsilon \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n - u_0|^p dx \\ & \leq C_3\varepsilon. \end{aligned}$$

By the arbitrariness of ε , we have

$$\int_{\mathbb{R}^3} K(x)(f(u_n) - f(u_0))\phi dx \rightarrow 0.$$

Next we will verify that $u_n \rightarrow u_0$ in $L^{2^*}_\alpha(\mathbb{R}^3)$. We show that there exists $\lambda_* > 0$ such that $v_i = 0$ and $v_\infty = 0$ for $0 < \lambda < \lambda_*$ and $i \in J$. We argue by contradiction. Assume that $i_0 \in J$ exists such that $v_{i_0} > 0$ or $v_\infty > 0$, by Lemma 7, we have

$$v_{i_0} \leq (S_\alpha^{-1}\mu(x_{i_0}))^{\frac{2^*}{2}}. \quad (10)$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ satisfy $\varphi \in (0, 1)$, $\varphi = 1$ in $B(0, 1)$ and $\varphi = 0$ in $\mathbb{R}^3 \setminus B(0, 2)$. For any $\varepsilon > 0$, we define $\varphi_\varepsilon(x) = \varphi(\frac{x-x_{i_0}}{\varepsilon})$, where $i_0 \in J$. Using (9), one has

$$\int_{\mathbb{R}^3} |u_n \varphi_\varepsilon|^{2^*}_\alpha dx \leq \left(S_\alpha^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)\varphi_\varepsilon(x) - u_n(y)\varphi_\varepsilon(y)|^2}{|x-y|^{N+2\alpha}} dx dy \right)^{\frac{2^*}{2}},$$

which means that

$$\int_{\mathbb{R}^3} |u_n \varphi_\varepsilon|^{2^*}_\alpha dx \rightarrow \int_{\mathbb{R}^3} \varphi_\varepsilon^{2^*}_\alpha dv \text{ as } n \rightarrow \infty, \quad (11)$$

and

$$\int_{\mathbb{R}^3} \varphi_\varepsilon^{2^*}_\alpha dv \rightarrow v(\{x_{i_0}\}) = v_{i_0}, \text{ as } \varepsilon \rightarrow 0. \quad (12)$$

By (f3), we obtain

$$\begin{aligned}
& \theta I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n \rangle \\
&= \frac{\theta}{2} \|u_n\|^2 + \frac{\theta}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t(u_n)^2 dx - \theta \int_{\mathbb{R}^3} K(x) F(u_n) dx - \frac{\lambda \theta}{2_\alpha^*} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \\
&\quad - \|u_n\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t(u_n)^2 dx + \int_{\mathbb{R}^3} K(x) f(u_n) u_n dx + \lambda \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \\
&= \left(\frac{\theta-2}{2} \right) \|u_n\|^2 + \left(\frac{\theta-1}{4} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t(u_n)^2 dx + \int_{\mathbb{R}^3} K(x) [f(u_n) u_n - \theta F(u_n)] dx \\
&\quad + \lambda \left(1 - \frac{\theta}{2_\alpha^*} \right) \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \\
&\geq \lambda \left(1 - \frac{\theta}{2_\alpha^*} \right) \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \\
&\geq \lambda \left(\frac{2_\alpha^* - \theta}{2_\alpha^*} \right) \int_{\mathbb{R}^3} |u_n \varphi_\varepsilon|^{2_\alpha^*} dx.
\end{aligned}$$

Let $n \rightarrow \infty$, then we have $\theta C_3 \geq \lambda \left(\frac{2_\alpha^* - \theta}{2_\alpha^*} \right) \int_{\mathbb{R}^3} \varphi_\varepsilon^{2_\alpha^*} dx$. By (11) and (12), we know that

$$2C_3 \geq \lambda \left(\frac{2_\alpha^* - \theta}{2_\alpha^*} \right) v_{i_0}. \quad (13)$$

Since $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$, we know that $\{u_n \varphi_\varepsilon\}$ is also bounded in $H^\alpha(\mathbb{R}^3)$. Thus

$$\langle I'_\lambda(u_n), u_n \varphi_\varepsilon \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

which means that

$$\begin{aligned}
& \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u_n \cdot (-\Delta)^{\frac{\alpha}{2}} (u_n \varphi_\varepsilon) dx \\
&= \int_{\mathbb{R}^3} \left[K(x) f(u_n) u_n + \lambda |u_n|^{2_\alpha^*} - V(x) |u_n|^2 \right] \varphi_\varepsilon dx + o(1)
\end{aligned} \quad (14)$$

By (6), we have

$$tf(t) \leq \varepsilon |t|^2 + C_\varepsilon |t|^p.$$

Then

$$\begin{aligned}
& \int_{\mathbb{R}^3} \left[K(x) f(u_n) u_n + \lambda |u_n|^{2_\alpha^*} - V(x) |u_n|^2 \right] \varphi_\varepsilon dx \\
&\leq \int_{\mathbb{R}^3} \left[\varepsilon K(x) |u_n|^2 + C_\varepsilon K(x) |u_n|^{2_\alpha^*} + \lambda |u_n|^{2_\alpha^*} \right] \varphi_\varepsilon dx.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^3} K(x) |u_n|^2 \varphi_\varepsilon dx = \int_{B(x_i, 2\varepsilon)} K(x) |u_n|^2 \varphi_\varepsilon dx \rightarrow \int_{B(x_i, 2\varepsilon)} K(x) |u_0|^2 \varphi_\varepsilon dx \text{ as } n \rightarrow \infty,$$

and

$$\int_{B(x_i, 2\varepsilon)} K(x) |u_0|^2 \varphi_\varepsilon dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

then

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left[K(x)f(u_n)u_n + \lambda|u_n|^{2_\alpha^*} - V(x)|u_n|^2 \right] \varphi_\varepsilon dx \\
& \leq (C_\varepsilon + \lambda) \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} \varphi_\varepsilon dx \\
& \leq (C_\varepsilon + \lambda) \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \varphi_\varepsilon^{2_\alpha^*} dx \\
& = (C_\varepsilon + \lambda) \nu_{i_0}.
\end{aligned}$$

Choose λ large enough to satisfy $\lambda > C_\varepsilon$, then

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left[K(x)f(u_n)u_n + \lambda|u_n|^{2_\alpha^*} - V(x)|u_n|^2 \right] \varphi_\varepsilon dx \leq 2\lambda \nu_{i_0}. \quad (15)$$

By the proposition 3.6 in [23], we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx. \quad (16)$$

Therefore, for any $v \in H^\alpha(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}} dx dy = \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x) dx. \quad (17)$$

By (17), we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u_n \cdot (-\Delta)^{\frac{\alpha}{2}} (u_n \varphi_\varepsilon) dx \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(u_n(x) \varphi_\varepsilon(x) - u_n(y) \varphi_\varepsilon(y))}{|x - y|^{N+2\alpha}} dx dy \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))^2 \varphi_\varepsilon(y)}{|x - y|^{N+2\alpha}} dx dy \\
& \quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y)) u_n(x)}{|x - y|^{N+2\alpha}} dx dy.
\end{aligned}$$

It is easy to know that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))^2 \varphi_\varepsilon(y)}{|x - y|^{N+2\alpha}} dx dy \rightarrow \int_{\mathbb{R}^3} \varphi_\varepsilon d\mu \text{ as } n \rightarrow \infty,$$

and

$$\int_{\mathbb{R}^3} \varphi_\varepsilon d\mu \rightarrow \mu(\{x_{i_0}\}) \text{ as } \varepsilon \rightarrow 0.$$

By the Hölder inequality, one has

$$\begin{aligned}
& \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y)) u_n(x)}{|x - y|^{N+2\alpha}} dx dy \\
& \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)| |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| |u_n(x)|}{|x - y|^{N+2\alpha}} dx dy \\
& \leq C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{\frac{1}{2}}.
\end{aligned}$$

By the following equality (see (3.7) in [27])

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x - y|^{N+2\alpha}} dx dy = 0,$$

(14) and (15), we obtain that

$$\mu(\{x_{i_0}\}) \leq 2\lambda v_{i_0} \text{ for any } i_0 \in J.$$

From (10) and (13), one has

$$2C_3 \geq \lambda \left(\frac{2_\alpha^* - \theta}{2_\alpha^*} \right) \left(\frac{\lambda^{-1} S_\alpha}{2} \right)^{\frac{2_\alpha^*}{2_\alpha^* - 2}} = \left(\frac{2_\alpha^* - \theta}{2_\alpha^*} \right) \left(\frac{S_\alpha}{2} \right)^{\frac{N}{2_\alpha}} \lambda^{\frac{2_\alpha - N}{2_\alpha}},$$

which implies

$$\lambda \geq \left(\frac{2_\alpha^* - \theta}{2C_3 2_\alpha^*} \right) \left(\frac{S_\alpha}{2} \right)^{\frac{2_\alpha^*}{2}} := \lambda_*,$$

which contradicts $0 < \lambda < \lambda_*$. Therefore, for any $i \in J$, $v_i = 0$ and $v_\infty = 0$. By Lemma 6, we know

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx = \int_{\mathbb{R}^3} |u_0|^{2_\alpha^*} dx.$$

Since $|u_n - u_0|^{2_\alpha^*} \leq 2^{2_\alpha^*} (|u_n|^{2_\alpha^*} + |u_0|^{2_\alpha^*})$, by Fatou's Lemma, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} 2^{2_\alpha^*+1} |u_0|^{2_\alpha^*} dx &= \int_{\mathbb{R}^3} \liminf_{n \rightarrow \infty} (2^{2_\alpha^*} |u_n|^{2_\alpha^*} + 2^{2_\alpha^*} |u_0|^{2_\alpha^*} - |u_n - u_0|^{2_\alpha^*}) dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} (2^{2_\alpha^*} |u_n|^{2_\alpha^*} + 2^{2_\alpha^*} |u_0|^{2_\alpha^*} - |u_n - u_0|^{2_\alpha^*}) dx \\ &\quad + 2^{2_\alpha^*+1} \int_{\mathbb{R}^3} |u_0|^{2_\alpha^*} dx - \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n - u_0|^{2_\alpha^*} dx, \end{aligned}$$

which implies that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n - u_0|^{2_\alpha^*} dx = 0,$$

then $u_n \rightarrow u_0$ in $L^{2_\alpha^*}(\mathbb{R}^3)$ as $n \rightarrow \infty$.

Note that $I'_\lambda(u_n) \rightarrow 0$, then

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \\ &= \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} K(x) f(u_n) u_n dx + \lambda \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx - \int_{\mathbb{R}^3} V(x) u_n^2 dx \right. \\ &\quad \left. - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t(u_n)^2 dx \right) \\ &\leq \int_{\mathbb{R}^3} K(x) f(u_0) u_0 dx + \lambda \int_{\mathbb{R}^3} |u_0|^{2_\alpha^*} dx - \int_{\mathbb{R}^3} V(x) u_0^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_0}^t(u_0)^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx, \end{aligned}$$

which means that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx. \quad (18)$$

Thus,

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(x) u_n^2 dx \\
 &= \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} K(x) f(u_n) u_n dx + \lambda \int_{\mathbb{R}^3} |u_n|^{2^*} dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \right. \\
 &\quad \left. - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t(u_n)^2 dx \right) \\
 &= \int_{\mathbb{R}^3} K(x) f(u_0) u_0 dx + \lambda \int_{\mathbb{R}^3} |u_0|^{2^*} dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_0|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_0}^t(u_0)^2 dx \\
 &= \int_{\mathbb{R}^3} V(x) u_0^2 dx.
 \end{aligned}$$

Since $u_n \rightharpoonup u_0$ weakly in $H^\alpha(\mathbb{R}^3)$, then by (18), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} (u_n - u_0)|^2 dx + \int_{\mathbb{R}^3} V(x) (u_n - u_0)^2 dx = 0,$$

which implies that $u_n \rightarrow u_0$ in $H^\alpha(\mathbb{R}^3)$. \square

Lemma 9. Let Ω be every finite subspace of $H^\alpha(\mathbb{R}^3)$. Then, for any $u \in \Omega$, it holds

$$I_\lambda(u) \rightarrow -\infty \text{ as } \|u\| \rightarrow \infty.$$

Proof. Arguing by contradiction, there exists $C_6 > 0$ such that $I(u_n) \geq -C_6$ for any $n \in \mathbb{N}$. Let $v_n := \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing a subsequence, still denote by v_n , we assume that $v_n \rightharpoonup v$ in $H^\alpha(\mathbb{R}^3)$, by Lemma 1, we have $v_n \rightarrow v$ in $L^r(\mathbb{R}^3)$ and $v_n \rightarrow v$ a.e. on \mathbb{R}^3 .

By condition (f3), we know $F(u_n) > 0$ for every $u_n \in H^\alpha(\mathbb{R}^3)$, then $F(u_n) \geq -C_7 u_n^2$. Define $\Omega_n(0, r_1) := \{x \in \mathbb{R}^3 : 0 < |u_n| < r_1\}$, where $r_1 > 0$ is a constant. For $u_n \in E$, we have

$$\begin{aligned}
 \frac{1}{\|u_n\|^4} \int_{\Omega_n(0, r_1)} K(x) F(u) dx &\geq -\frac{C_7}{\|u_n\|^4} \int_{\Omega_n(0, r_1)} K(x) u^2 dx \\
 &\geq -\frac{C_7 \|u_n\|^2}{\|u_n\|^4}.
 \end{aligned} \tag{19}$$

Using the properties of the function ϕ_u^s (Lemma 2.3-(2) in [18]), (19) and Fatou's Lemma, we have

$$\begin{aligned}
 0 &= \lim_{n \rightarrow \infty} \frac{I_\lambda(u_n)}{\|u_n\|^4} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{1}{4\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n}^t u^2 dx - \frac{1}{\|u_n\|^4} \int_{\mathbb{R}^3} K(x) F(u) dx - \frac{\lambda}{2^* \|u_n\|^4} \int_{\mathbb{R}^3} |u|^{2^*} dx \right] \\
 &\leq C_8 + \lim_{n \rightarrow \infty} \left[-\frac{1}{\|u_n\|^4} \int_{\Omega_n(0, r_1)} K(x) F(u) dx - \frac{1}{\|u_n\|^4} \int_{\Omega_n(r_1, \infty)} K(x) F(u) dx \right] \\
 &\leq C_8 + \limsup_{n \rightarrow \infty} \left[\frac{\|v_n\|_{L_K^2}^2}{\|u_n\|^2} - \int_{\Omega_n(r_1, \infty)} \frac{K(x) F(u)}{u_n^4} v_n^4 dx \right] \\
 &\leq C_8 - \liminf_{n \rightarrow \infty} \left[\int_{\Omega_n(r_1, \infty)} \frac{K(x) F(u)}{u_n^4} v_n^4 dx \right] \\
 &\leq C_8 - \int_{\mathbb{R}^3} \liminf_{n \rightarrow \infty} \frac{K(x) F(u)}{u_n^4} [\chi_{\Omega_n(r_1, \infty)}] v_n^4 dx = -\infty,
 \end{aligned}$$

where $\chi_{\Omega_n(r_1, \infty)}$ is the characteristic function of $\Omega_n(r_1, \infty)$. This is a contradiction and for any $u \in \Omega$, we obtain $I(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$. \square

Proof of Theorem 1. Let $\lambda_1 := \min\{\lambda_0, \lambda_*\}$, where λ_0, λ_* are given in Lemma 3 and Lemma 8. By mountain pass theorem and Lemma 3, there exists a (PS) sequence $\{u_n\} \subset E$ for I_λ in $H^\alpha(\mathbb{R}^3)$ for any $0 < \lambda < \lambda_1$. According to Lemma 8, there exists a subsequence of $\{u_n\} \subset H^\alpha(\mathbb{R}^3)$, still denoted by $\{u_n\}$, and $\{u_{**}\} \subset E$ such that $u_n \rightarrow u_{**}$ in $H^\alpha(\mathbb{R}^3)$ as $n \rightarrow \infty$. Moreover, $I'_\lambda(u_{**}) = 0$ and $I_\lambda(u_{**}) \geq \beta > 0$. Hence, u_{**} is a nontrivial radially symmetric solution of problem (4). \square

Proof of Theorem 2. From the definition of I , we have $I_\lambda(0) = 0$. By the condition (f4), we easily know I_λ is an even functional, i.e., $I_\lambda(-u) = I_\lambda(u)$. Hence, (A1) of Lemma 2 holds. According to Lemma 9, there exists $\xi = \xi(\Omega) > 0$, for any $u \in \Omega$ with $\|u\| \geq \xi$, there is $I_\lambda(u) \leq 0$. Therefore, (A3) of Lemma 2 is satisfied. By Lemma 3-(i), we know that (A2) of Lemma 2 is satisfied. Consequently, by Lemma 2, problem (4) has infinitely nontrivial radially symmetric solutions. \square

3. Conclusions

In this paper, we study a class of fractional Schrödinger–Poisson systems with critical growth (4). Problem (4) comes from the interaction of a charge particle with an electromagnetic field in three-dimensional space. By using the principle of concentration compactness, we established a compactness result about functional I_λ . Applying the mountain pass theorem, a nontrivial radially symmetric solution was obtained. Infinitely nontrivial radially symmetric solutions are presented by the symmetric mountain pass lemma. Therefore, our results improve the related conclusions on this topic.

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