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Article

On the Paired-Domination Subdivision Number of Trees

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Abstract: A paired-dominating set of a graph G without isolated vertices is a dominating set of vertices whose induced subgraph has perfect matching. The minimum cardinality of a paired-dominating set of G is called the paired-domination number $\gamma_{pr}(G)$ of G. The paired-domination subdivision number $\mathrm{sd}_{\gamma_{pr}}(G)$ of G is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the paired-domination number. Here, we show that, for each tree $T \neq P_5$ of order $n \geq 3$ and each edge $e \notin E(T)$, $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$.

Keywords: paired-domination number; paired-domination subdivision number

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = V(G)$ and edge set $E(G) = E$ and let $n = |V|$. For any vertex $u \in V(G)$, the *open neighborhood* of u is the set $N(u) = N_G(u) = \{v \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of u is the set $N[u] = N_G[u] = \{u\} \cup N_G(u)$. The *degree* of a vertex u is $\deg(u) = \deg_G(u) = |N_G(u)|$. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*. A stem is said to be *strong* if it is adjacent to at least two leaves.

Throughout this paper, when an edge $e = uv$ is subdivided, $w_e = w_{uv}$ denotes the subdivision vertex for e. For a set $F \subseteq E(G)$, G_F denotes the graph obtained from G by subdividing every edge in F (note that we have $w_e \neq w_f$ for any $e, f \in F$ with $e \neq f$). The length of a shortest (u, v)-path in a graph G is the distance between u and v, and is written $d_G(u, v)$ or simply $d(u, v)$ if G is clear from context. The maximum distance among all pairs of vertices in G is called the diameter of G, written $\mathrm{diam}(G)$. A *diametral path* of G is a path of G with the length $\mathrm{diam}(G)$.

A subset S of V is a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex in S. A *paired-dominating set* (PD-set) of G is a subset S of V if S is a dominating set and the subgraph induced by S contains a perfect matching. The minimum cardinality of a PD-set of G is the *paired-domination number* $\gamma_{pr}(G)$. If S is a PD-set with a perfect matching M and $uv \in M$, then u and v are said to be *partners (or paired)* in S. We call a PD-set of minimum cardinality a *$\gamma_{pr}(G)$-set*. Since the end vertices of any maximal matching in G form a PD-set, every graph G without isolated vertices has a PD-set. Haynes and Slater [1] introduced the Paired-domination, which has been studied, for example, in [2–6]. For more details on paired-domination, we refer the reader to [7].

As good models of many practical problems, graphs sometimes have to be changed to adapt the changes in reality. Thus, we must pay attention to the change of the graph parameters under graph modifications, such as the deletion of vertices, deletion or addition of edges, and subdivision of edges. Velammal [8] was the first to study the domination subdivision number of a graph G defined to be the minimum number of edges that must be subdivided (each edge in G is subdivided at most once) to increase the domination number. Since then, subdivision parameters have been studied by several authors (see [9–15]).

In this paper, we study the *paired-domination subdivision number* of trees, which was introduced by Favaron et al. in [16] and has been further studied in [17–21]. The *paired-domination subdivision number* $\mathrm{sd}_{\gamma_{pr}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired-domination number of G.

If G is a connected graph of order at least 3, Favaron et al. [16] asked whether it is true that for any edge $e \notin E(G)$, $\mathrm{sd}_{\gamma_{pr}}(G+e) \leq \mathrm{sd}_{\gamma_{pr}}(G)$. Egawa et al. [18] gave a negative answer to this question. However, they proved the question in the affirmative if the following additional condition is added: $\gamma_{pr}(G+e) < \gamma_{pr}(G)$ for every $e \notin E(G)$. Recently, Hao et al. [19] showed that for any graph G without isolated vertices and different from mK_2, and for any edge $e \notin E(G)$, $\mathrm{sd}_{\gamma_{pr}}(G+e) \leq \mathrm{sd}_{\gamma_{pr}}(G) + 2\Delta(G)$.

Our aim in this paper is to further study paired-domination subdivision number and show that for each tree $T \neq P_5$ of order $n \geq 3$ and each edge $e \notin E(T)$, $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(\mathrm{T+e}) \leq n+2$.

We close this section by recalling some useful results.

Proposition 1 ([16])**.** *Let G be a connected graph of order $n \geq 3$ and let G' be obtained from G by subdividing the edge $e = uv \in E(G)$. Then, $\gamma_{pr}(G') \geq \gamma_{pr}(G)$.*

Proposition 2 ([20])**.** *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq n-1$.*

Proposition 3 ([18])**.** *For any graph G with no isolated vertex and any $uv \notin E(G)$, $\gamma_{pr}(G) = \gamma_{pr}(G+uv)$ or $\gamma_{pr}(G) = \gamma_{pr}(G+uv)+2$.*

Proposition 4 ([18])**.** *For any connected graph G of order at least 3 and $uv \notin E(G)$ with $\gamma_{pr}(G+uv) < \gamma_{pr}(G)$, $\mathrm{sd}_{\gamma_{pr}}(G+uv) \leq \mathrm{sd}_{\gamma_{pr}}(G)$.*

Proposition 5 ([16])**.** *If G contains either adjacent stems or a strong stem, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2$.*

Proposition 6 ([16])**.** *For any connected graph G containing a path $v_1v_2v_3v_4v_5$ such that $\deg(v_2) = \deg(v_3) = \deg(v_4) = 2$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq 4$.*

Proposition 7 ([16])**.** *For $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{pr}}(P_n) = \mathrm{sd}_{\gamma_{pr}}(C_n) = \begin{cases} 1, & \text{if} \quad n \equiv 0 \pmod 4, \\ 4, & \text{if} \quad n \equiv 1 \pmod 4, \\ 3, & \text{if} \quad n \equiv 2 \pmod 4, \\ 2, & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

Proposition 8 ([18])**.** *If a tree T contains a path $v_1v_2v_3v_4$ in which $\deg(v_1) = 1$ and $\deg(v_i) = 2$ for $i = 2, 3$, then $\mathrm{sd}_{\gamma_{pr}}(T) \leq 4$.*

Proposition 9 ([19])**.** *For any isolated-free graph G different from mK_2 and any $uv \notin E(G)$ satisfying that u or v is a stem,*

$$\mathrm{sd}_{\gamma_{pr}}(G+uv) \leq \mathrm{sd}_{\gamma_{pr}}(G) + 2.$$

For any positive integer $m \geq 1$, let S_m be the *healthy spider* obtained from the complete bipartite graph $K_{1,m}$ by subdividing every edge. Therefore $V(S_m) = \{u_1^i, u_2^i \mid 1 \leq i \leq m\} \cup \{x\}$ and $E(S_m) = \{xu_2^i, u_2^iu_1^i \mid 1 \leq i \leq m\}$ (see Figure 1). The vertex x is called the center of S_m. Let $T_{1,m}$ be the tree obtained from the disjoint union of two copies of the healthy spider S_m centered at x, y, by joining x and y (see Figure 2). Observe that $n(T_{1,m}) = 4m+2$, $\gamma_{pr}(T_{1,m}) = 4m$ and $\mathrm{sd}_{\gamma_{pr}}(T_{1,m}) = 2m+1$.

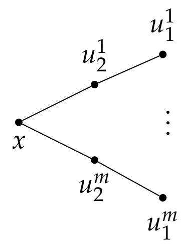

Figure 1. A healthy spider S_m.

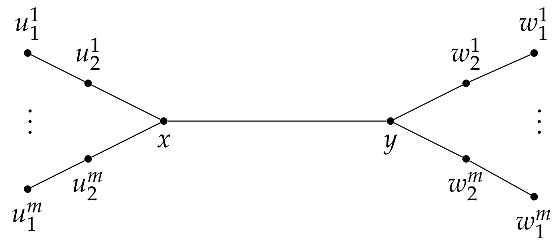

Figure 2. A tree $T_{1,m}$ with $\mathrm{sd}_{\gamma_{pr}}(T_{1,m}) = n(T_{1,m})/2$.

Let $T_{2,m}$ be the tree obtained from $T_{1,m}$ by subdividing the edge xy with a subdivision vertex u and adding a new vertex v and a new edge uv (Figure 3). Observe that $n(T_{2,m}) = 4m + 4$, $\gamma_{pr}(T_{2,m}) = 4m + 2$, and $\mathrm{sd}_{\gamma_{pr}}(T_{2,m}) = 2m + 2$.

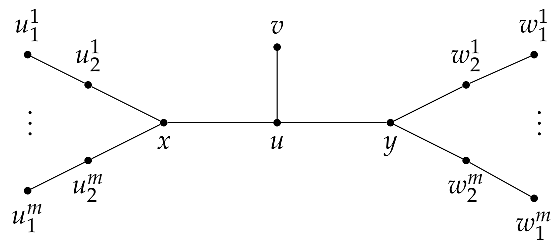

Figure 3. A tree $T_{2,m}$ with $\mathrm{sd}_{\gamma_{pr}}(T_{2,m}) = n(T_{2,m})/2$.

Now let $\mathcal{T} = \{K_{1,3}\} \cup \{S_m, T_{1,m}, T_{2,m} \mid m \geq 1\}$.

Proposition 10 ([18]). *For any tree T of order $n \geq 3$, $\mathrm{sd}_{\gamma_{pr}}(T) \geq n/2$ if and only if $T \in \mathcal{T}$.*

Proposition 11 ([20]). *For any tree T of order $n \geq 4$ different from a healthy spider, $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n}{2}$.*

Combining Propositions 10 and 11, we have the following corollary.

Corollary 1. *Let T be a tree of order $n \geq 4$ different from a healthy spider. Then, $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n}{2}$ with equality if and only if $T \in \{K_{1,3}, T_{1,m}, T_{2,m} \mid m \geq 1\}$.*

2. Main Result

Our aim in this section is to prove that, for each tree $T \neq P_5$ with $n \geq 3$ vertices and any edge $e \notin E(T)$, $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \leq n + 2$. First, we consider trees with diameter four and five.

Lemma 1. *Let T be a tree of order $n \geq 6$ with $\mathrm{diam}(T) = 4$ and let $e = xy \notin E(G)$. Then,*

$$\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \leq n + 2.$$

Proof. If T has a strong stem or adjacent stems, then by Propositions 2 and 5 we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \leq n + 1$. Suppose next that T has no adjacent stems and no strong stem. Then, T is a healthy spider. Let $V(T) = \{v, v_i, u_i \mid 1 \leq i \leq t\}$ and let $E(T) = \{vv_i, v_iu_i \mid 1 \leq i \leq t\}$. Since $n \geq 6$, we have $t \geq 3$. We consider the four cases.

Case 1. Both x and y are leaves.

Without a loss of generality, assume that $x = u_1$ and $y = u_2$. Observe that $\gamma_{pr}(T+e) = 2t-2$. Let T' be obtained from $T+e$ by subdividing the edges vv_1, vv_2, v_1u_1 with the vertices w_1, w_2, and w_3, respectively. We show that $\gamma_{pr}(T') \geq 2t$. Let D be a $\gamma_{pr}(T')$-set. If $v \notin D$, then we must have $v_i, u_i \in D$ for each $3 \leq i \leq t$ and $|D \cap \{v_1, v_2, u_1, u_2, w_1, w_2, w_3\}| \geq 4$, which leads to $\gamma_{pr}(T') = |D| \geq 2t$. Assume now that $v \in D$. If v is paired with w_1 or w_2, then we must have $v_i, u_i \in D$ for each $3 \leq i \leq t$ and $|D \cap \{v, v_1, v_2, u_1, u_2, w_1, w_2, w_3\}| \geq 4$ implying that $\gamma_{pr}(T') = |D| \geq 2t$. Assume next that v is not paired with w_1 or w_2. Without a loss of generality, assume that v is paired with v_3. Thus, $v_i, u_i \in D$ for each $4 \leq i \leq t$ and $|D \cap \{v_1, v_2, u_1, u_2, w_1, w_2, w_3\}| \geq 4$ yielding $\gamma_{pr}(T') = |D| \geq 2t$. Thus, $\mathrm{sd}_{\gamma_{pr}}(T+e) \leq 3$ and, by Proposition 2, we obtain $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$.

Case 2. x is a leaf and y is a stem.

Without a loss of generality, assume that $x = u_1$ and $y = v_2$. Then, it is clear that $\gamma_{pr}(T+e) = 2t-2$. Let T' be obtained from $T+e$ by subdividing the edges v_2u_2 and e with the vertices w_1 and w_2, respectively. As in Case 1, we can see that $\gamma_{pr}(T') \geq 2t$. Hence, $\mathrm{sd}_{\gamma_{pr}}(T+e) \leq 2$ and, by Proposition 2, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+1$.

Case 3. x is a leaf and $y = v$.

Without a loss of generality, assume that $x = u_1$. Clearly $\gamma_{pr}(T+e) = 2t-2$. Let T' be obtained from $T+e$ by subdividing the edges u_1v_1 with w_1. We show that $\gamma_{pr}(T') \geq 2t$. Let D be a $\gamma_{pr}(T')$-set. To paired-dominate $u_2, \ldots, u_t$, we must have $|D \cap \{v, u_i, v_i \mid 2 \leq i \leq t\}| \geq 2t-2$ and to paired-dominate u_1 we must have $|D \cap \{u_1, v_1, w_1\}| \geq 1$. Since $|D|$ is even, we obtain $\gamma_{pr}(T') \geq 2t$. Hence, $\mathrm{sd}_{\gamma_{pr}}(T+e) = 1$ and, by Proposition 2, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n$.

Case 4. Both x and y are stems.

In this case, T' has adjacent stems, and we deduce from Propositions 2 and 5 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+1$. $\square$

Lemma 2. *Let T be a tree of order n with $\mathrm{diam}(T) = 5$ and let $e = xy \notin E(G)$. Then,*

$$\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2.$$

Proof. As in the proof of Lemma 1, we may assume that T has no strong stem and no adjacent stems. Let $u_1w_1v_1v_2x_1y_1$ be a diametral path in T. By the assumption, the components of $T - v_1v_2$ are P_3 or a healthy spider. It is easy to see that $\gamma_{pr}(T) = n-2$. Let T_1 and T_2 be the components of $T - v_1v_2$ containing v_1 and v_2, respectively. If $T = P_6$, then we deduce from Propositions 2 and 7 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$. Assume that $T \neq P_6$. Suppose that $\deg(v_2) \geq 3$. If $\deg(v_1) = 2$, then it is easy to see that subdividing the edges u_1w_1, w_1v_1, v_1v_2 increases the paired-domination number of T and so $\mathrm{sd}_{\gamma_{pr}}(T) \leq 3$ and the result follows by Proposition 2. Hence, we assume that $\deg(v_1) \geq 3$. Let $V(T_1) = \{v_1, w_i, u_i \mid 1 \leq i \leq t\}$, $E(T_1) = \{v_1w_i, w_iu_i \mid 1 \leq i \leq t\}$, $V(T_2) = \{v_2, x_i, y_i \mid 1 \leq i \leq s\}$ and let $E(T_2) = \{v_2x_i, x_iy_i \mid 1 \leq i \leq s\}$. We consider the four cases.

Case 1. Both x and y are leaves.

If x and y are the leaves of T_1 (resp., T_2), then, clearly, $\gamma_{pr}(T+xy) < \gamma_{pr}(T)$ and by Propositions 4 and 11, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n$. If x is a leaf of T_1 and y is a leaf of T_2, then, we deduce from Propositions 6 and 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n/2 + 4 < n+2$.

Case 2. x is a leaf and y is a stem.

Assume first that x, y are the vertices of T_1. We may assume that $x = u_1$ and $y = w_2$. Then, the set $D = \{v_1, w_2\} \cup \{w_i, u_i \mid 3 \le i \le t\} \cup \{x_j, y_j \mid 1 \le j \le s\}$ is a paired-dominating set of $T + e$ of size $n - 4$ and so $\gamma_{pr}(T + e) < n - 2 = \gamma_{pr}(T)$. If x and y are the vertices of T_2, then similarly we have $\gamma_{pr}(T + e) < \gamma_{pr}(T)$. Assume second that $x \in V(T_1)$ and $y \in V(T_2)$. Without a loss of generality, we can suppose that $x = u_1$ and $y = x_1$. Then, the set $\{v_1, v_2, x_1, w_2\} \cup \{w_i, u_i \mid 3 \le i \le t\} \cup \{x_j, y_j \mid 2 \le j \le s\}$ is a paired-dominating set of $T + e$ with cardinality $n - 4$, and thus $\gamma_{pr}(T + e) < n - 2 = \gamma_{pr}(T)$. Applying Propositions 4 and 11 we obtain $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n$.

Case 3. x is a leaf and $y \in \{v_1, v_2\}$.

Without loss of generality, we may assume that $x = u_1$. If $y = v_1$, then the set $\{v_1, w_2\} \cup \{w_i, u_i \mid 3 \le i \le t\} \cup \{x_j, y_j \mid 1 \le j \le s\}$ is a paired-dominating set of $T + e$ of size $n - 4$ and so $\gamma_{pr}(T + e) < n - 2 = \gamma_{pr}(T)$. If $y = v_2$, then the set $\{v_1, v_2, w_2, x_1\} \cup \{w_i, u_i \mid 3 \le i \le t\} \cup \{x_j, y_j \mid 2 \le j \le s\}$ is a paired-dominating set of $T + e$ of size $n - 4$, and thus $\gamma_{pr}(T + e) < n - 2 = \gamma_{pr}(T)$. Using Propositions 4 and 11, we obtain $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n$.

Case 4. Both x and y are stems.

In this case, $T' + xy$ has adjacent stems, and we deduce from Propositions 2 and 5 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n + 1$. $\square$

Theorem 1. *Let T be a tree different from P_5 of order $n \ge 3$ and let $e = xy \notin E(G)$. Then,*

$$\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n + 2.$$

Proof. If T has a strong stem or adjacent stems, then, by Propositions 2 and 5, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n + 1$. Hence, we assume that T has no strong stem and no adjacent stems. It follows that $\mathrm{diam}(T) \ge 4$. According to Lemmas 1 and 2, we may assume that $\mathrm{diam}(T) \ge 6$. If x or y is a stem, then by Propositions 9 and 11, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n + 2$. Hence, we assume that neither x nor y is a stem. If x (resp., y) is a leaf with support vertex x' (resp., y') of degree 2, respectively, then for $x'' \in N(x') - \{x\}$, $x''x'xyy'$ is a path in T with $\deg(x') = \deg(x) = \deg(y) = 2$, and therefore by Propositions 6 and 11, we have $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) \le n/2 + 4 < n + 2$. Thus, we may assume that, if both x and y are leaves, then the stem of x or y is of at least degree 3.

Let $v_1 v_2 \ldots v_k$ be a diametral path such that $\deg(v_3)$ is minimized. By assumption, the trees T_1 and T_2 (the components of $T - \{v_3v_4, v_{k-2}v_{k-3}\}$ containing v_3 and v_{k-2}, respectively) are P_3 or a healthy spider. Let $N(v_3) = \{v_4\} \cup \{u_1 = v_2, u_2, \ldots, u_t\}$ and let u'_i be the leaf adjacent to u_i for each i. Root T is at v_4. We consider three cases.

Case 1. $x, y \notin V(T_1)$.

We distinguish the following subcases.

Subcase 1.1. v_4 is a stem.

By our assumption, T has no strong stem. Let w be a unique leaf adjacent to v_4. First, let $w \notin \{x, y\}$. Let T' be obtained from $T + e$ by subdividing the edges v_3u_i and $u_iu'_i$ with vertices x_i and y_i, respectively for each i and the edges v_3v_4 and v_4w with vertices r and s, respectively. We show that $\gamma_{pr}(T') > \gamma_{pr}(T + e)$. Let P be a $\gamma_{pr}(T')$-set. To paired-dominate u'_i, we may assume that $u_i, y_i \in P$ for each i. On the other hand, to paired-dominate w and v_3, we must have $|P \cap \{v_3, v_4, w, r, s, x_1, x_2, \ldots, x_t\}| \ge 4$.

If $v_4 \notin P$ or $v_4 \in P$ and v_4 is paired with r or s, then the set

$$(P - \{v_3, v_4, w, r, s, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{v_3, v_4, u'_1, u'_2, \ldots, u'_t\}$$

is a PD-set of $T + e$ with cardinality of less than P. Assume that $v_4 \in P$ and v_4 is paired with a vertex not in $\{r, s\}$. Then, we have $|P \cap \{w, r, s, v_3, x_1, x_2, \ldots, x_t\}| \ge 4$, and clearly the set $(P - \{v_3, w, r, s, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{u'_1, u'_2, \ldots, u'_t\}$ is a PD-set of $T+e$ with cardinality of less than P. Since the number of subdivided edges is at most $n/2$, we deduce from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \le n$.

Now, let $w \in \{x, y\}$. Let $N(v_4) = \{z_1 = v_3, z_2 = v_5, z_3 = w, z_4, \ldots, z_\ell\}$. Note that each component of $(T+e) - \{v_4 z_i \mid 1 \le j \le \ell\}$ has at least two vertices. Let T' be obtained from $T+e$ by subdividing $v_3 u_i$ and $u_i u'_i$ with vertices x_i and y_i, respectively for each i, the edge $v_4 z_j$ with vertex z'_j for each j, and the edge e with vertex q. By our assumptions, it is not difficult to verify that the number of subdivided edges is at most $n/2 + 2$. We next show that $\gamma_{pr}(T') > \gamma_{pr}(T+e)$. Let P be a $\gamma_{pr}(T')$-set, and let F be the set of all edges in $\{e, v_4 z_j \mid 2 \le j \le \ell\}$ whose subdivision vertices are in P. To paired-dominate u'_i, we may assume that $u_i, y_i \in P$ for each i.

On the other hand, to paired-dominate v_3, we must have $|P \cap \{v_3, v_4, z'_1, x_1, x_2, \ldots, x_t\}| \ge 2$. If $v_4 \notin P$ or $v_4 \in P$ and v_4 is not paired with z'_1, then $|P \cap \{z'_1, v_3, x_1, x_2, \ldots, x_t\}| \ge 2$ and the set $(P - \{v_3, z'_1, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{u'_1, u'_2, \ldots, u'_t\}$ is a PD-set of G_1 obtained from $T+e$ by subdividing all edges in F with cardinality of less than P. If $v_4 \in P$ and v_4 is paired with z'_1, then the set $(P - \{v_4, z'_1, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{v_3, u'_2, u'_3, \ldots, u'_t\}$ is a PD-set of G_2 obtained from $T+e$ by subdividing the edges in F with cardinality of less than P. By Proposition 1, we have $\gamma_{pr}(T') > \gamma_{pr}(G_i) \ge \gamma_{pr}(T+e)$ for each $i \in \{1, 2\}$. Therefore, in either case, we have $\mathrm{sd}_{\gamma_{pr}}(T+e) \le n/2 + 2$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \le n+2$.

Subcase 1.2. v_4 is not a stem and there is a path $v_4 w_2 w_1$ in T such that $\deg(w_1) = 1$ and $\deg(w_2) = 2$.

If e is incident to w_2, then it follows from Propositions 9 and 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \le n+2$. Hence, we assume that $w_2 \notin \{x, y\}$. First, let $w_1 \notin \{x, y\}$. Let T' be obtained from $T+e$ by subdividing the edges $v_3 u_i$ and $u_i u'_i$ with the vertices x_i and y_i, respectively, for each i and the edges $v_3 v_4$ and $v_4 w_2$ with the vertices r and s, respectively. By our assumptions, it is not difficult to verify that the number of subdivided edges is at most $n/2$. As above, we can see that $\gamma_{pr}(T') > \gamma_{pr}(T+e)$, and thus $\mathrm{sd}_{\gamma_{pr}}(T+e) \le n/2$. We deduce from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \le n$.

Now, let $w_1 \in \{x, y\}$. We distinguish the following situations.

(1.2.1) $v_4 \notin \{x, y\}$.

Let $N(v_4) = \{z_1 = v_3, z_2 = v_5, z_3 = w_2, z_4, \ldots, z_\ell\}$. Note that each component of $(T+e) - \{v_4 z_i \mid 1 \le j \le \ell\}$ has order at least two. Let T' be obtained from $T+e$ by subdividing the edges $v_3 u_i$ and $u_i u'_i$ with the vertices x_i and y_i, respectively, for each i, the edge $v_4 z_j$ with vertex z'_j for each j, and the edges $w_2 w_1$ and e with the vertices q_1 and q_2, respectively. By our assumptions, it is not difficult to verify that the number of subdivided edges is at most $(n+3)/2$. We show that $\gamma_{pr}(T') > \gamma_{pr}(T+e)$. Let P be a $\gamma_{pr}(T')$-set, and let F be the set of all edges in $\{w_1 w_2, e, v_4 z_j \mid 2 \le j \le \ell\}$ whose subdivision vertices are in P. To paired-dominate u'_i, we may assume that $u_i, y_i \in P$ for each i. On the other hand, to paired-dominate v_3, we must have $|P \cap \{v_4, z'_1, v_3, x_1, x_2, \ldots, x_t\}| \ge 2$.

If $v_4 \notin P$ or $v_4 \in P$ and v_4 is not paired with z'_1, then $|P \cap \{v_3, z'_1, x_1, x_2, \ldots, x_t\}| \ge 2$, and the set $(P - \{v_3, z'_1, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{u'_1, u'_2, \ldots, u'_t\}$ is a PD-set of G_3 obtained from $T+e$ by subdividing the edges F with cardinality of less than P. If $v_4 \in P$ and v_4 is paired with z'_1, then the set $(P - \{v_4, z'_1, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}) \cup \{v_3, u'_2, u'_3, \ldots, u'_t\}$ is a PD-set of G_4 obtained from $T+e$ by subdividing the edges in F with cardinality of less than P. By Proposition 1, we have $\gamma_{pr}(T') > \gamma_{pr}(G_i) \ge \gamma_{pr}(T+e)$ for each $i \in \{3, 4\}$. Therefore, in either case, we have $\mathrm{sd}_{\gamma_{pr}}(T+e) \le (n+3)/2$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) < n+2$.

(1.2.2) $v_4 \in \{x, y\}$.

Then, $e = v_4 w_1$. Let $N(v_4) = \{z_1 = v_3, z_2 = v_5, z_3 = w_2, z_4, \ldots, z_\ell\}$, and let T' be obtained from $T+e$ by subdividing the edges $v_3 u_i$ and $u_i u'_i$ with the vertices x_i and y_i, respectively, for each i, the edge $v_4 z_j$ with vertex z'_j for each j, and the edges $w_2 w_1$

and e. By our assumptions, it is not difficult to verify that the number of subdivided edges is at most $n/2+2$. Using an argument similar to that described in (1.2.1), we can see that $\mathrm{sd}_{\gamma_{pr}}(T)+\mathrm{sd}_{\gamma_{pr}}(T+e)\leq n+2$.

Subcase 1.3. All children of v_4 except v_5 have depth 3.

Considering the above cases and subcases and the choice of diametral path, we may assume that each component of $(T+e)-v_4$ has at least three vertices. Let $N(v_4)=\{z_1=v_3, z_2=v_5, z_3,\dots,z_\ell\}$, and let T' be obtained from $T+e$ by subdividing the edges v_3u_i and u_iu_i' with the vertices x_i and y_i, respectively, for each i and the edges v_4z_j with vertex z_j' for each j. By our assumptions, it is not difficult to verify that the number of subdivided edges is at most $n/2+1$. We show that $\gamma_{pr}(T')>\gamma_{pr}(T+e)$. Let P be a $\gamma_{pr}(T')$-set. To paired-dominate u_i', we may assume that $u_i, y_i\in P$ for each i. On the other hand, to paired-dominate v_3, we must have $|P\cap\{v_3, v_4, z_1', x_1,\dots,x_t\}|\geq 2$.

Let F be the set of subdivided edges incident to v_4 whose subdivision vertices belong to P. If $v_4\notin P$ or $v_4\in P$ and its partner is in $\{z_2',\dots,z_s'\}$, then we have $|P\cap\{v_3, z_1', x_1,\dots,x_t\}|\geq 2$, and the set $(P-(\{v_3, z_1'\}\cup\{x_i, y_i\mid 1\leq i\leq t\})\cup\{v_3, u_1\}\cup\{u_i, u_i'\mid 2\leq i\leq t\}$ is a PD-set of G_5 obtained from $T+e$ by subdividing the edges of F with cardinality of less than P.

If $v_4\in P$, and its partner is z_1', then the set $(P-(\{v_4, z_1'\}\cup\{y_i\mid 1\leq i\leq t\})\cup\{v_3, u_1\}\cup\{u_i, u_i'\mid 2\leq i\leq t\}$ is a PD-set of G_6 obtained from $T+e$ by subdividing the edges of F with cardinality of less than P. By Proposition 1, we have $\gamma_{pr}(T')>\gamma_{pr}(G_i)\geq\gamma_{pr}(T+e)$ for each $i\in\{5,6\}$. Therefore, in either case, we have $\mathrm{sd}_{\gamma_{pr}}(T+e)\leq n/2+1$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T)+\mathrm{sd}_{\gamma_{pr}}(T+e)\leq n+1$.

Case 2. $x, y\in V(T_1)$.

By our earlier assumptions, T has no strong stem, neither x nor y is a stem, and if both x and y are leaves, then the stem of x or y is of at least degree 3. Therefore, we may assume without a loss of generality that $x=v_3$ and $y=v_1$. Let T' be obtained from $T+e$ by subdividing the edges v_3v_2, v_2v_1, and e by the subdivision vertices x_1, x_2, and x_3, respectively. We show that $\gamma_{pr}(T')>\gamma_{pr}(T+e)$. Let P be a $\gamma_{pr}(T')$-set. If $v_3\notin P$, then we must have $|P\cap\{x_1, x_2, x_3, v_1, v_2\}|\geq 4$, and clearly $(P-\{x_1, x_2, x_3, v_1, v_2\})\cup\{v_1, v_2\}$ is a PD-set of $T+e$ with cardinality of less than P.

If $v_3\in P$ and its partner is not x_1 or x_3, then $|P\cap\{x_1, x_2, x_3, v_1, v_2\}|\geq 2$, and hence $P-\{x_1, x_2, x_3, v_1, v_2\}$ is a PD-set of $T+e$ with cardinality of less than P. If $v_3\in P$ and its partner is x_1 or x_3, then $|P\cap\{x_1, x_2, x_3, v_1, v_2\}|\geq 3$, and hence $(P-\{x_1, x_2, x_3, v_1, v_2\})\cup\{v_2\}$ is a PD-set of $T+e$ with cardinality of less than P. Therefore, we have $\mathrm{sd}_{\gamma_{pr}}(T+e)\leq 3$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T)+\mathrm{sd}_{\gamma_{pr}}(T+e)\leq n/2+3<n+2$.

Case 3. One of x and y belongs to $V(T_1)$, and the other does not belong to $V(T_1)$.

Assume, without a loss of generality, that $x\in V(T_1)$ and $y\notin V(T_1)$. By our earlier assumption, x is not a stem. We distinguish two subcases.

Subcase 3.1. x is a leaf of T_1.

Assume without a loss of generality that $x=v_1$. If $\deg_T(v_3)=2$ or y is a leaf, then it follows from Proposition 6 that $\mathrm{sd}_{\gamma_{pr}}(T+e)\leq 4$, and the result follows from Proposition 11. Let $\deg_T(v_3)\geq 3$, and y is not a leaf. By our earlier assumption, y is not a stem, and hence each component of $(T+e)-y$ has at least two vertices. Note that the component of $(T+e)-y$ containing v_3 has at least five vertices and, thus, $\deg_{T+e}(y)<n/2-1$. Let T' be obtained from $T+e$ by subdividing the edges v_3v_2, v_2v_1, and v_1y by the subdivision vertices x_1, x_2, and x_3, respectively, and all edges incident to y in T. We show that $\gamma_{pr}(T')>\gamma_{pr}(T+e)$. Let P be a $\gamma_{pr}(T')$-set, F' be the set of subdivided edges incident to y whose subdivision vertices are in P and let T_1 be obtained from $T+e$ by subdividing the edges in F'.

First, we assume that $y\in P$ and y is paired with x_3. Then, clearly $|P\cap\{x_1, x_2, v_1, v_2\}\geq 2$. We may assume that $v_3\in P$, otherwise $u_2, u_2'\in P$, and hence we may consider $(P-\{u_2'\})\cup\{v_3\}$ as a $\gamma_{pr}(T')$-set. If v_3 is paired with a vertex other than x_1, then $P-$

$\{x_1, x_2, v_2, v_1\}$ is a PD-set of T_1 with cardinality of less than P. If v_3 is paired with x_1, then $|P \cap \{v_2, v_1, x_2\}| \geq 2$ and $(P - \{x_1, x_2, v_1, v_2\}) \cup \{v_2\}$ is also a PD-set of T_1 with cardinality of less than P.

Second, assume that $y \in P$ and y is paired with a subdivision vertex other than x_3. Then, $|P \cap \{x_1, x_2, v_1, v_2, x_3\}| \geq 2$. As above, we may assume that $v_3 \in P$. If v_3 is paired with a vertex other than x_1, then $P - \{x_1, x_2, v_1, v_2, x_3\}$ is a PD-set of T_1 with cardinality of less than P. If v_3 is paired with x_1, then $u_2, u_2' \in P$ and hence $P - \{u_2', x_1, x_2, v_1, v_2, x_3\}$ is a PD-set of T_2 obtained from T by subdividing the edges in $F' - \{e\}$, with cardinality of less than P.

Finally, we assume that $y \notin P$. Then, y must be dominated by a subdivision vertex. As above, we may assume that $v_3 \in P$. We consider the following.

(3.1.1) $x_3 \in P$.
Then, x_3 and v_1 are partners and to paired-dominate v_2, we must have $|P \cap \{x_1, x_2, v_2, v_3\}| \geq 2$. If v_3 is paired with x_1, then $u_2, u_2' \in P$, and hence $(P - \{u_2', x_1, x_2, x_3, v_1, v_2\}) \cup \{v_1, v_2\}$ is a PD-set of T_1 with cardinality of less than P. If v_3 is paired with a vertex other than x_1, then $|P \cap \{x_1, x_2, v_2\}| \geq 2$, and hence $(P - \{x_1, x_2, x_3, v_1, v_2\}) \cup \{v_1, v_2\}$ is a PD-set of T_2 with cardinality of less than P.

(3.1.2) $x_3 \notin P$.
Then, we must have $v_1, x_2 \in P$. Let $w_{yz} \in P$ be a subdivision vertex that dominates y, and let T_3 be obtained from $T + e$ by subdividing the edges $F' \setminus \{yz\}$. If v_3 is paired with x_1, then $u_2, u_2' \in P$, and hence $(P - \{u_2', x_1, x_2, x_3, v_1, v_2, w_{yz}\}) \cup \{y\}$ is a PD-set of T_2 with cardinality of less than P. If v_3 is paired with a vertex other than x_1, then $(P - \{x_1, x_2, x_3, v_1, v_2, w_{yz}\}) \cup \{y\}$ is also a PD-set of T_3 with cardinality of less than P.

In all cases, it follows from Proposition 1 that $\gamma_{pr}(T') > \gamma_{pr}(T_i) \geq \gamma_{pr}(T + e)$ for each $i \in \{1, 2, 3\}$. Therefore, we have $\mathrm{sd}_{\gamma_{pr}}(T + e) \leq \deg_{T+e}(y) + 2 < n/2 + 1$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T + e) < n + 1$.

Subcase 3.2. $x = v_3$.

By assumption, y is not a support vertex. We consider the following situations.

(3.2.1) $\deg(y) = 1$.
Let w_1 be the stem of y and w_2 be the parent of w_1. Let T' be obtained from $T + e$ by subdividing the edges v_3v_4, v_3y, yw_1, and w_1w_2 with the vertices y_1, y_2, y_3, y_4, respectively, and the edge v_3u_i with vertex x_i for each i. Let F be the set of all subdivided edges. Note that by the choice of the diametral path $v_1v_2 \cdots v_k$, it is not difficult to check that $|F| \leq n/2 + 2$. We show that $\gamma_{pr}(T') > \gamma_{pr}(T + e)$. Let P be a $\gamma_{pr}(T')$-set. To paired-dominate v_1, we must have $|P \cap \{x_1, v_1, v_2\}| \geq 2$.

If $v_3 \in P$ and v_3 is the partner of x_1, then $(P - \{x_1, v_1, v_2\}) \cup \{v_2\}$ is a PD-set of T_3 obtained from $T + e$ by subdividing the edges in $F \setminus \{v_2v_3\}$ with cardinality of less than P. If $v_3 \in P$ and v_3 is the partner of w_{v_3z} $(\neq x_1)$, then $(P - \{x_1, v_1, v_2, w_{v_3z}\}) \cup \{v_2\}$ is a PD-set of T_4 obtained from $T + e$ by subdividing the edge in $F \setminus \{v_3v_2, v_3z\}$ with cardinality of less than P.

In the following, we may assume that $v_3 \notin P$. To paired-dominate v_1, we may assume that $x_1, v_2 \in P$, and, to paired-dominate y_2, we must have $y \in P$. First, we assume that y and y_2 are partners. If $y_3 \in P$, then $(P - \{y_2, y, x_1\}) \cup \{v_3\}$ is a PD-set of T_5 obtained from $T + e$ by subdividing the edges in $F \setminus \{v_3y, v_2v_3\}$ with cardinality of less than P.

If $y_3 \notin P$, then $(P - \{y_2, y, x_1\}) \cup \{v_3\}$ is a PD-set of T_6 obtained from $T + e$ by subdividing the edges in $F \setminus \{v_3y, v_2v_3, yw_1\}$ with cardinality of less than P. Second, we assume that y and y_3 are partners. If $y_4 \notin P$, then $w_2 \in P$ and $(P - \{y, y_3, x_1\}) \cup \{v_3\}$, is a PD-set of T_7 obtained from $T + e$ by subdividing the edges in $F \setminus \{yw_1, v_3y, v_3v_2\}$ with cardinality of less than P. If $y_4 \in P$, then $(P - \{y, y_3, x_1\}) \cup \{v_3\}$ is also a PD-set of T_7 with cardinality of less than P.

In all cases, it follows from Proposition 1 that $\gamma_{pr}(T') > \gamma_{pr}(T_i) \geq \gamma_{pr}(T+e)$ for each $i \in \{3,4,\ldots,7\}$. Therefore, we have $\mathrm{sd}_{\gamma_{pr}}(T+e) \leq |F| \leq n/2+2$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$.

(3.2.2) There is a path yy_1y_2 in T satisfying that $\deg(y_2) = 1$ and $\deg(y_1) = 2$.
Let T' be obtained from $T+e$ by subdividing the edge v_3u_i with vertex x_i for each i, the edges v_3v_4, e, and yy_1 with the vertices z_1, z_2, and z_3, respectively, and the other edges incident to y. Let F be the set of all subdivided edges. We note that $|F| \leq n/2+2$. Let P be a $\gamma_{pr}(T')$-set. To paired-dominate v_1 and y_2, we must have $|P \cap \{x_1, v_1, v_2\}| \geq 2$ and $|P \cap \{y_1, y_2, z_3\}| \geq 2$. To dominate z_2, we must have $y \in P$ or $v_3 \in P$.

First, assume that $y \in P$. If y is paired with z_3, then $y_1, y_2 \in P$ and $P - \{y_2, z_3\}$ is a PD-set of T_8 obtained from $T+e$ by subdividing the edges $F - \{yy_1\}$ with cardinality of less than P. If y is paired with z_2, then $(P - \{y_1, y_2, z_2, z_3\}) \cup \{y_1\}$ is a PD-set of T_9 obtained from $T+e$ by subdividing the edges $F - \{yy_1, e\}$ with cardinality of less than P. If y is partner with a subdivision vertex $w_{yz} \notin \{z_2, z_3\}$, then $(P - \{y_1, y_2, z_3, w_{yz}\}) \cup \{y_1\}$ is a PD-set of T_{10} obtained from $T+e$ by subdividing the edges $F - \{yz, yy_1\}$ with cardinality of less than P.

Second, assume that $v_3 \in P$. If v_3 is paired with a subdivision vertex $w_{v_3z} \neq x_1$, then $(P - \{v_1, v_2, x_1, w_{v_3z}\}) \cup \{v_2\}$ is a PD-set of T_{11} obtained from $T+e$ by subdividing the edges $F - \{v_3z, v_3v_2\}$ with cardinality of less than P. If v_3 is paired with x_1, then $(P - \{x_1, v_1, v_2\}) \cup \{v_2\}$ is a PD-set of T_{12} obtained from $T+e$ by subdividing the edges $F - \{v_3v_2\}$ with cardinality of less than P.

In all cases, it follows from Proposition 1 that $\gamma_{pr}(T') > \gamma_{pr}(T_i) \geq \gamma_{pr}(T+e)$ for each $i \in \{8,9,\ldots,12\}$. Therefore, we have $\mathrm{sd}_{\gamma_{pr}}(T+e) \leq |F| \leq n/2+2$. It follows from Proposition 11 that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$.

(3.2.3) There is no path yy_1y_2 in T with $\deg(y_2) = 1$ and $\deg(y_1) = 2$.
Let $w_1w_2\cdots w_m$ be the shortest (w_1, w_m)-path in T such that w_1 is a leaf of T and $w_m = y$. By our assumption, T has no strong stem. Thus, we have $m \geq 4$. Let T_1' be the component of $T - \{w_3w_4\}$ containing w_1. Note that $x, y \notin V(T_1')$. Now, using the argument applied in Case 1, we can see that $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$.

This completes the proof.

□

3. Conclusions

The main objective of this paper was to study the paired-domination subdivision number $\mathrm{sd}_{\gamma_{pr}}(G)$ of a graph G defined to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired-domination number of G. We focused on trees and proved that, for any tree $T \neq P_5$ of order $n \geq 3$ and any edge $e \notin E(T)$, $\mathrm{sd}_{\gamma_{pr}}(T) + \mathrm{sd}_{\gamma_{pr}}(T+e) \leq n+2$. As a consequence of this study, we pose the following conjecture.

Conjecture 1. *Let G be a connected graph of order $n \geq 6$. Then,* $\mathrm{sd}_{\gamma_{pr}}(G) + \mathrm{sd}_{\gamma_{pr}}(G+e) \leq n+2$.

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