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Multiple Diamond-Alpha Integral in General Form and Their Properties, Applications

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Abstract: In this paper, we introduce the concept of n -dimensional Diamond-Alpha integral on time scales. In particular, it transforms into multiple Delta, Nabla and mixed integrals by taking different values of alpha. Some of its properties are explored, and the relationship between it and the multiple mixed integral is provided. As an application, we establish some weighted Ostrowski type inequalities through the new integral. These new inequalities expand some known inequalities in the monographs and papers, and in addition, furnish some other interesting inequalities. Examples of Ostrowski type inequalities are posed in detail at the end of the paper.

Keywords: multiple Diamond-Alpha integral; Ostrowski type inequalities; Delta integral; Nabla integral

MSC: Primary 26E70; 26E25

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1. Introduction

Since Hilger [1] initiated the theory of time scales in 1988, it has been used widely by many branches of sciences. Moreover, many studies of the topic of time scales have been published, such as [2–15].

In [2], Bohner and Georgiev defined the following multiple Delta integral by using Riemann's definition of the integral,

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n. \quad (1)$$

Moreover, Bohner and Georgiev pointed out that multiple Nabla integral

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n, \quad (2)$$

and mixed integrals that contain part of “ Δ ” and part of “ ∇ ” can be defined in the same way.

As a linear combination of Δ and ∇ dynamic derivatives, researchers introduced the Diamond-Alpha differentiation on time scales. More details about $\diamond_\alpha$ dynamic derivatives can be found in [16–19].

In [20], Tian, Zhu, and Cheung proposed the definition of multiple Diamond-Alpha integral

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \diamond_\alpha v_1 \diamond_\alpha v_2 \cdots \diamond_\alpha v_n, \quad (3)$$

by employing antiderivatives of single-variable functions.

In this paper we will consider the Diamond-Alpha integral of different alpha values, which is a more general form of (3), namely,

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n, \quad (4)$$

we call it multiple Diamond- α_i integral for convenience.

Ostrowski type inequality on time scales has received a lot attention [9,21–24]. In particular, the authors provided several Diamond-Alpha weighted Ostrowski type inequalities [21] as follows.

Theorem 1. If ω is a positive and continuous function, $\phi_k (k = 1, 2, 3) \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ are bounded and satisfy

$$\|\phi_k^\Delta\|_\infty = \sup_{x \in [a, b]} |\phi_k^\Delta(x)| < \infty, \quad \|\phi_k^\nabla\|_\infty = \sup_{x \in [a, b]} |\phi_k^\nabla(x)| < \infty, \quad k = 1, 2, 3,$$

then the following inequalities

$$\begin{aligned} & \left| \prod_{k=1}^3 \phi_k(x) - \frac{1}{3 \int_a^b \omega(v) \diamond_\alpha v} \sum_{k=1}^3 \left(\prod_{j \neq k} \phi_j(x) \int_a^b \omega(v) \phi_k(v) \diamond_\alpha v \right) \right| \\ & \leq \frac{1}{3 \int_a^b \omega(v) \diamond_\alpha v} \sum_{k=1}^3 \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \right) \int_a^b \omega(v) |x - v| \diamond_\alpha v, \end{aligned}$$

and

$$\begin{aligned} & \left| \int_a^b \omega(x) \prod_{k=1}^3 \phi_k(x) \diamond_\alpha x \right. \\ & \left. - \frac{1}{3 \int_a^b \omega(v) \phi_k(v) \diamond_\alpha v} \sum_{k=1}^3 \left(\int_a^b \omega(x) \prod_{j \neq k} \phi_j(x) \diamond_\alpha x \int_a^b \omega(x) \phi_k(x) \diamond_\alpha x \right) \right| \\ & \leq \frac{1}{3 \int_a^b \omega(v) \phi_k(v) \diamond_\alpha v} \times \\ & \int_a^b \omega(x) \sum_{k=1}^3 \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \right) \left(\int_a^b \omega(v) |x - v| \diamond_\alpha v \right) \diamond_\alpha x, \end{aligned}$$

hold for all $x \in [a, b]$.

Theorem 2. If ω is a positive and continuous function, $\phi_k (k = 1, 2) \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ are bounded and satisfy

$$\|\phi_k^\Delta\|_\infty = \sup_{x \in [a, b]} |\phi_k^\Delta(x)| < \infty, \quad \|\phi_k^\nabla\|_\infty = \sup_{x \in [a, b]} |\phi_k^\nabla(x)| < \infty, \quad k = 1, 2,$$

then the following inequalities

$$\begin{aligned} & \left| \int_a^b \omega(v) \left(\phi_1(x) \phi_2(x) - \phi_1(x) \phi_2(v) - \phi_1(v) \phi_2(x) + \phi_1(v) \phi_2(v) \right) \diamond_\alpha v \right| \\ & \leq \prod_{k=1}^2 (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \int_a^b \omega(v) |x - v|^2 \diamond_\alpha v, \end{aligned}$$

and

$$\left| \int_a^b \omega(v) \diamond_{\alpha} v \int_a^b \omega(v) \phi_1(v) \phi_2(v) \diamond_{\alpha} v - \left(\int_a^b \omega(v) \phi_1(v) \diamond_{\alpha} v \right) \left(\int_a^b \omega(v) \phi_2(v) \diamond_{\alpha} v \right) \right|$$

$$\leq \prod_{k=1}^2 (\alpha \|\phi_k^{\Delta}\|_{\infty} + (1-\alpha) \|\phi_k^{\nabla}\|_{\infty}) \int_a^b \omega(x) \left(\int_a^b \omega(v) |x-v|^2 \diamond_{\alpha} v \right) \diamond_{\alpha} x,$$

hold for all $x \in [a, b]$.

The structure of the rest of this paper is as follows: we provide some preliminaries in Section 2 and consider the definition of multiple Diamond- α_i integral and its properties in Section 3. In Section 4, we generalize Theorems 1 and 2 to the inequalities involving m functions; the inequalities involving m functions and n variables are given. Some examples and conclusions are presented in Sections 5 and 6, respectively.

2. Preliminaries

In this section, we list some common definitions and properties related to time scales; they can be found in [2].

Time scales are an arbitrary nonempty closed subset of the real number including set of real numbers $\mathbb{R}$, positive integer numbers $\mathbb{N}$, and denoted as $\mathbb{T}$. For more comprehensive knowledge about time scales, readers can consult [2,4,6,8,11,18,25–28]. In what follows, $[a_i, b_i]_{\mathbb{T}_i}$ bespeak $[a_i, b_i] \cap \mathbb{T}_i$ and we default $a_i, b_i \in \mathbb{T}_i$ for all $i = 1, 2, \dots, n$ always.

Definition 1. For $h \in \mathbb{T}$, forward and backward jump operator $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\sigma(h) = \inf\{l \in \mathbb{T} : l > h\}, \text{ and } \rho(h) = \sup\{l \in \mathbb{T} : l < h\},$$

respectively.

Clearly, we have $\sigma(h) \geq h \geq \rho(h)$.

Definition 2. If $\phi : \mathbb{T} \rightarrow \mathbb{R}$ and for all given $\epsilon > 0$ there exists a neighborhood U and $\phi^{\Delta}(h), \phi^{\nabla}(h)$ of h such that

$$\left| \phi(\sigma(h)) - \phi(l) - \phi^{\Delta}(h)(\sigma(h) - l) \right| \leq \epsilon |\sigma(h) - l|,$$

and

$$\left| \phi(\rho(h)) - \phi(l) - \phi^{\nabla}(h)(\rho(h) - l) \right| \leq \epsilon |\rho(h) - l|,$$

for all $l \in U$, then ϕ is respectively called Delta and Nabla differentiable on $\mathbb{T}^k$ and $\mathbb{T}_k$, where $\mathbb{T}_k$, $\mathbb{T}^k$ is defined as follows:

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T})), & \text{if } \inf \mathbb{T} > -\infty, \\ \mathbb{T}, & \text{if } \inf \mathbb{T} = -\infty. \end{cases}$$

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 3. Let $\Phi, \phi : \mathbb{T} \rightarrow \mathbb{R}$. If $\Phi^{\Delta}(h) = \phi(h)$ holds for any $h \in \mathbb{T}^k$. Then Φ is called a Δ -antiderivative of ϕ . Moreover

$$\int_{\alpha}^h \phi(\tau) \Delta \tau = \Phi(h) - \Phi(\alpha).$$

If for any $h \in \mathbb{T}_k$ satisfies $\Phi^\nabla(h) = \phi(h)$. Then Φ is called a ∇ -antiderivative of ϕ . Moreover

$$\int_{\alpha}^h \phi(\tau) \nabla \tau = \Phi(h) - \Phi(\alpha).$$

Obviously, Delta, Nabla derivatives and integrals satisfy linearity.

Diamond-alpha derivative and integral are respectively defined as linear combination of Δ , ∇ dynamic derivatives and integrals.

Definition 4. If $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is Delta and Nabla differentiable, then

$$\phi^{\diamond_{\alpha}}(h) = \alpha \phi^{\Delta}(h) + (1 - \alpha) \phi^{\nabla}(h).$$

Definition 5. If $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is continuous on $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} \phi(h) \diamond_{\alpha} h = \alpha \int_{\alpha}^{\beta} \phi(h) \Delta h + (1 - \alpha) \int_{\alpha}^{\beta} \phi(h) \nabla h.$$

Diamond-Alpha derivatives and integrals have the following qualities.

Proposition 1. If $\phi_1, \phi_2 : \mathbb{T} \rightarrow \mathbb{R}$ are Delta and Nabla differentiable (so called Diamond-Alpha differentiable) and continuous functions, $k_1, k_2 \in \mathbb{R}$, then

$$(k_1 \phi_1(h) + k_2 \phi_2(h))^{\diamond_{\alpha}} = k_1 \phi_1^{\diamond_{\alpha}}(h) + k_2 \phi_2^{\diamond_{\alpha}}(h),$$

and

$$\int_{\alpha}^{\beta} (k_1 \phi_1(h) + k_2 \phi_2(h)) \diamond_{\alpha} h = k_1 \int_{\alpha}^{\beta} \phi_1(h) \diamond_{\alpha} h + k_2 \int_{\alpha}^{\beta} \phi_2(h) \diamond_{\alpha} h.$$

3. Definition and Simple Properties

For more comprehensive knowledge about time scales, readers can consult [2,4,6,8,11,18,25–28].

Basically consistent with the definition process from [20], we suppose $D = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ and $\inf \mathbb{T}_i = a_i, \sup \mathbb{T}_i = b_i (i = 1, 2, \dots, n)$. In what follows, $[a_i, b_i]_{\mathbb{T}_i}$ bespeak $[a_i, b_i] \cap \mathbb{T}_i$ and by default we always have $a_i, b_i \in \mathbb{T}_i$ for all $i = 1, 2, \dots, n$. Every $\alpha_i (i = 1, 2, \dots, n) \in [0, 1]$.

We denote $\int_{a_k}^{b_k} Q_k(v_1, v_2, \dots, v_n) \diamond_{\alpha_k} v_k$ as diamond- α_k integral with respect to v_k ($k = 1, 2, \dots, n$). If there exist functions $Q_1, Q_2, \dots, Q_n$ defined on D such that

$$\begin{aligned} & Q_{k+1}(v_1, v_2, \dots, v_{k-1}, b_{k+1}, v_{k+1}, \dots, v_n) - Q_{k+1}(v_1, v_2, \dots, v_{k-1}, a_{k+1}, v_{k+1}, \dots, v_n) \\ &= \int_{a_{k+1}}^{b_{k+1}} Q_k(v_1, v_2, \dots, v_n) \diamond_{\alpha_{k+1}} v_{k+1} \\ &= \alpha_{k+1} \int_{a_{k+1}}^{b_{k+1}} Q_k(v_1, v_2, \dots, v_n) \Delta_{k+1} v_{k+1} + (1 - \alpha_{k+1}) \int_{a_{k+1}}^{b_{k+1}} Q_k(v_1, v_2, \dots, v_n) \nabla_{k+1} v_{k+1}, \end{aligned}$$

for all $k = 0, 1, 2, \dots, n - 1$, where we write $h(v_1, v_2, \dots, v_n)$ as $Q_0(v_1, v_2, \dots, v_n)$, then we define the multiple Diamond- α_i integral as follows

$$\begin{aligned}
& \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
&= \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_2}^{b_2} Q_1(b_1, v_2, \dots, v_n) - Q_1(a_1, v_2, \dots, v_n) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \cdots \diamond_{\alpha_n} v_n \\
&= \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_3}^{b_3} Q_2(b_1, b_2, \dots, v_n) - Q_2(b_1, a_2, \dots, v_n) \\
&\quad - Q_2(a_1, b_2, \dots, v_n) + Q_2(a_1, a_2, \dots, v_n) \diamond_{\alpha_3} v_3 \diamond_{\alpha_4} v_4 \cdots \diamond_{\alpha_n} v_n \\
&= \cdots \\
&= l_{n,2^n-1} Q_n(b_1, b_2, \dots, b_{n-1}, b_n) + l_{n,2^n-2} Q_n(b_1, b_2, \dots, b_{n-1}, a_n) \\
&\quad + l_{n,2^n-3} Q_n(b_1, b_2, \dots, a_{n-1}, b_n) + l_{n,2^n-4} Q_n(b_1, b_2, \dots, a_{n-1}, a_n) \\
&\quad + \cdots \\
&\quad + l_{n,3} Q_n(a_1, a_2, \dots, b_{n-1}, b_n) + l_{n,2} Q_n(a_1, a_2, \dots, b_{n-1}, a_n) \\
&\quad + l_{n,1} (Q_n(a_1, a_2, \dots, a_{n-1}, b_n) + l_{n,0} Q_n(a_1, a_2, \dots, a_{n-1}, a_n)), \tag{5}
\end{aligned}$$

where $l_{i,j}$ is defined by recursive formula, $l_{0,0} = 1$ and for all $m = 0, 1, 2, \dots, n$

$$l_{m+1,v} = \begin{cases} l_{m,v}, & \text{if } v = 0, 1, 2, \dots, 2^m - 1, \\ 1 - l_{m,v}, & \text{if } v = 2^m, 2^m + 1, \dots, 2^{m+1} - 1. \end{cases} \tag{6}$$

If we default the following notations,

$$\begin{aligned}
I_a^b \{Q_n\}(2^n - 1) &= Q_n(b_1, b_2, \dots, b_{n-1}, b_n), & I_a^b \{Q_n\}(2^n - 2) &= Q_n(b_1, b_2, \dots, b_{n-1}, a_n), \\
I_a^b \{Q_n\}(2^n - 3) &= Q_n(b_1, b_2, \dots, a_{n-1}, b_n), & I_a^b \{Q_n\}(2^n - 4) &= Q_n(b_1, b_2, \dots, a_{n-1}, a_n), \\
&\cdots \\
I_a^b \{Q_n\}(3) &= Q_n(a_1, a_2, \dots, b_{n-1}, b_n), & I_a^b \{Q_n\}(2) &= Q_n(a_1, a_2, \dots, b_{n-1}, a_n), \\
I_a^b \{Q_n\}(1) &= Q_n(a_1, a_2, \dots, a_{n-1}, b_n), & I_a^b \{Q_n\}(0) &= Q_n(a_1, a_2, \dots, a_{n-1}, a_n), \tag{7}
\end{aligned}$$

then (5) transforms to

$$\begin{aligned}
& \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
&= \sum_{k=0}^{2^n-1} l_{n,k} I_a^b \{Q_n\}(k).
\end{aligned}$$

Next we give an example to show the new integral.

Example 1. If $h(v_1, v_2, \dots, v_n) = v_1 v_2 \cdots v_n$, $\mathbb{T}_i = \mathbb{Z} \cap [a_i, b_i]$ ($i = 1, 2, \dots, n$) and $a_i, b_i \in \mathbb{Z}$ ($i = 1, 2, \dots, n$). Then

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} v_1 v_2 \cdots v_n \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n = \prod_{k=1}^n \frac{(b_k + a_k - 2\alpha_k + 1)(b_k - a_k)}{2}.$$

Proof. According to the definition of Diamond-Alpha integral, for all $c \in \mathbb{R}$, we have

$$\begin{aligned}
\int_a^b c x \diamond_{\alpha} x &= c \left(\alpha \int_a^b x \Delta x + (1 - \alpha) \int_a^b x \nabla x \right) \\
&= c \left(\alpha \left(\frac{b^2 - b}{2} - \frac{a^2 - a}{2} \right) + (1 - \alpha) \left(\frac{b^2 + b}{2} - \frac{a^2 + a}{2} \right) \right) \\
&= \frac{c(b + a - 2\alpha + 1)(b - a)}{2}.
\end{aligned}$$

Using the above conclusion repeatedly, it leads to the desired equality.

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} v_1 v_2 \cdots v_n \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_2}^{b_2} \frac{(b_1 + a_1 - 2\alpha_1 + 1)(b_1 - a_1)}{2} v_2 v_3 \cdots v_n \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \cdots \diamond_{\alpha_n} v_n \\
 = & \cdots \\
 = & \prod_{k=1}^n \frac{(b_k + a_k - 2\alpha_k + 1)(b_k - a_k)}{2}.
 \end{aligned}$$

□

In particular, if we choose $\mathbb{T}_i = \mathbb{R} \cap [a_i, b_i] (i = 1, 2, \dots, n)$, then (4) changes into

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) dv_1 dv_2 \cdots dv_n. \quad (8)$$

If we let every $\alpha_i (i = 1, 2, \dots, n)$ to be equal to 0 or 1, then (4) changes into (1) or (2). If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, then (4) becomes (3). This shows (4) is a generalization of (1), (2) and (3).

Another powerful consequence of (4) is that it can represent multiple mixed integrals effectively. If every $\alpha_i = \omega_i = 0$ or 1, then the multiple mixed integral can be written as

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} h(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n.$$

For example, the following multiple mixed integral

$$\int_{a_4}^{b_4} \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1, v_2, v_3, v_4) \Delta_1 v_1 \nabla_2 v_2 \nabla_3 v_3 \Delta_4 v_4,$$

can be changed into

$$\int_{a_4}^{b_4} \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1, v_2, v_3, v_4) \diamond_1 v_1 \diamond_0 v_2 \diamond_0 v_3 \diamond_1 v_4.$$

For convenience, we define functions $\psi_i : \{0, 1\} \rightarrow [0, 1] (i = 1, 2, \dots, n)$ as follows,

$$\psi_i(x) = \begin{cases} \alpha_i, & \text{if } x = 0, \\ 1 - \alpha_i, & \text{if } x = 1. \end{cases} \quad (9)$$

The next theorem provides the link between multiple Diamond- α_i integral and multiple mixed integrals. It shows that the multiple Diamond- α_i integral is also a linear combination of multiple mixed integrals, similar to the Diamond- α integral of one variable.

Theorem 3. If $\phi(v_1, v_2, \dots, v_n)$ is Diamond- $\alpha_i (i = 1, 2, \dots, n)$ integrable on D , then

$$\begin{aligned}
 & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \phi(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \phi(v_1, \dots, v_n) \diamond_{\omega_1} v_1 \cdots \diamond_{\omega_n} v_n,
 \end{aligned}$$

where $\omega_1, \dots, \omega_n = 0$ or 1 means all non-repetitive possibilities in binary and $\psi_i (i = 1, 2, \dots, n)$ are defined by (9), that means there are 2^n total items in the right-hand side.

Proof. According to the definition of Diamond- α_i ($i = 1, 2, \dots, n$) integral, we have

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \alpha_1 \phi(v_1, v_2, \dots, v_n) \Delta_1 v_1 \right. \\
 + & \left. \int_{a_1}^{b_1} (1 - \alpha_1) \phi(v_1, v_2, \dots, v_n) \nabla_1 v_1 \right) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \cdots \diamond_{\alpha_n} v_n \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} \alpha_1 \alpha_2 \phi(v_1, v_2, \dots, v_n) \Delta_1 v_1 \Delta_2 v_2 \right. \\
 + & \int_{a_2}^{b_2} \int_{a_1}^{b_1} \alpha_1 (1 - \alpha_2) \phi(v_1, v_2, \dots, v_n) \Delta_1 v_1 \nabla_2 v_2 \\
 + & \int_{a_2}^{b_2} \int_{a_1}^{b_1} (1 - \alpha_1) \alpha_2 \phi(v_1, v_2, \dots, v_n) \nabla_1 v_1 \Delta_2 v_2 \\
 + & \left. \int_{a_2}^{b_2} \int_{a_1}^{b_1} (1 - \alpha_1) (1 - \alpha_2) \phi(v_1, v_2, \dots, v_n) \nabla_1 v_1 \nabla_2 v_2 \right) \diamond_{\alpha_3} v_3 \diamond_{\alpha_4} v_4 \cdots \diamond_{\alpha_n} v_n \\
 = & \cdots \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \alpha_1 \alpha_2 \cdots \alpha_n \phi(v_1, v_2, \dots, v_n) \Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n \\
 + & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \alpha_1 \alpha_2 \cdots (1 - \alpha_n) \phi(v_1, v_2, \dots, v_n) \Delta_1 v_1 \Delta_2 v_2 \cdots \nabla_n v_n \\
 + & \cdots \\
 + & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} (1 - \alpha_1) (1 - \alpha_2) \cdots \alpha_n \phi(v_1, v_2, \dots, v_n) \nabla_1 v_1 \nabla_2 v_2 \cdots \Delta_n v_n \\
 + & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} (1 - \alpha_1) (1 - \alpha_2) \cdots (1 - \alpha_n) \phi(v_1, v_2, \dots, v_n) \nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n \\
 = & \sum_{\omega_1, \dots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \phi(v_1, \dots, v_n) \diamond_{\omega_1} v_1 \cdots \diamond_{\omega_n} v_n.
 \end{aligned}$$

□

From the notations above, in [2], the following properties are introduced. These propositions show that multiple mixed integrals satisfy linearity and isotonicity.

Proposition 2 ([2], Theorem 7.50). Suppose ϕ_1, ϕ_2 is bounded, Delta and Nabla integrable on D , then the following formula holds

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) + \phi_2(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\
 + & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n,
 \end{aligned}$$

where $\omega_1, \dots, \omega_n = 0$ or 1 .

Proposition 3 (See [2], Theorem 7.49). Suppose that the bounded function ϕ is Delta and Nabla integrable on D , then for all $k \in \mathbb{R}$

$$\begin{aligned} & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} k \phi(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\ &= k \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n, \end{aligned}$$

where $\omega_1, \dots, \omega_n = 0$ or 1 .

Corollary 1 ([2], Corollary 7.51). If $\phi_1, \phi_2, \dots, \phi_m$ are bounded Delta and Nabla integrable over D , and $k_1, k_2, \dots, k_m \in \mathbb{R}$, then

$$\begin{aligned} & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \sum_{j=1}^m k_j \phi_j(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\ &= \sum_{j=1}^m \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} k_j \phi_j(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n, \end{aligned}$$

where $\omega_1, \dots, \omega_n = 0$ or 1 .

Proposition 4 ([2], Theorem 7.52). Suppose ϕ_1, ϕ_2 is bounded Delta and Nabla integrable on D , and $\phi_1(v_1, v_2, \dots, v_n) \geq \phi_2(v_1, v_2, \dots, v_n)$ holds on for all points in D , then

$$\begin{aligned} & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\ &\geq \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n, \end{aligned}$$

where $\omega_1, \dots, \omega_n = 0$ or 1 .

Remark 1. In fact, [2] just narrated the properties of multiple Delta integrals. However, since in the end of Chapter 7, the authors mentioned that multiple Nabla integrals and multiple mixed integrals should have the same properties, we give the general forms of the theorems in [2].

Next, we explore properties of (4) based on the Properties 2–4 and Theorem 3. We find that our new integrals also satisfy linearity and isotonicity.

Proposition 5. Suppose ϕ_1, ϕ_2 is bounded Diamond- α_i ($i = 1, 2, \dots, n$) integrable on D , then we have

$$\begin{aligned} & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} (\phi_1(v_1, v_2, \dots, v_n) + \phi_2(v_1, v_2, \dots, v_n)) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\ &= \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\ &+ \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n. \end{aligned} \quad (10)$$

Proof. According to Proposition 2 and Theorem 3, we obtain

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) + \phi_2(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \\
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, \dots, v_n) + \phi_2(v_1, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\
 = & \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \\
 & \left(\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right. \\
 & \left. + \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right) \\
 = & \left(\sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \right. \\
 & \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right) \\
 + & \left(\sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \right. \\
 & \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right) \\
 = & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 + & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n.
 \end{aligned}$$

Therefore, the proof is complete. $\square$

Proposition 6. If $\phi(v_1, v_2, \dots, v_n)$ is bounded Diamond- α_i ($i = 1, 2, \dots, n$) integrable on D and $k \in \mathbb{R}$, then

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} k \phi(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & k \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n. \quad (11)
 \end{aligned}$$

The proof of Proposition 6 is similar to the proof of Proposition 5, we will not repeat it. Combining (10) and (11), we have the following corollary.

Corollary 2. If $\phi_1, \phi_2, \dots, \phi_m$ is bounded Diamond- α_i ($i = 1, 2, \dots, n$) integrable on D and $k_i \in \mathbb{R}$ ($i = 1, 2, \dots, m$), then

$$\begin{aligned}
 & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \sum_{j=1}^m k_j \phi_j(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\
 = & \sum_{j=1}^m k_j \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_j(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n.
 \end{aligned}$$

Proposition 7. If ϕ_1, ϕ_2 is bounded Diamond- α_i ($i = 1, 2, \dots, n$) integrable on D , and $\phi_1(v_1, v_2, \dots, v_n) \geq \phi_2(v_1, v_2, \dots, v_n)$ holds for all points in D , then

$$\begin{aligned} & \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_1(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\ & \geq \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi_2(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n. \end{aligned}$$

In particular, if $\phi_2(v_1, v_2, \dots, v_n) \equiv 0$, then we have the following corollary.

Corollary 3. If ϕ is bounded Diamond- α_i ($i = 1, 2, \dots, n$) integrable on D , and $\phi(v_1, v_2, \dots, v_n) \geq 0$ holds for all points in D , then

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \phi(v_1, v_2, \dots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \geq 0.$$

4. Ostroski Type Inequalities on Multiple Diamond- α_i Integrals

In this section, we establish some Ostrowski type inequalities through multiple Diamond- α_i integral. Specifically, we consider Ostrowski type inequalities with more functions and more variables. Theorem 1 just considered three functions, in fact, we found it is also holds for the case with m functions. Thus, we have the following theorem.

Theorem 4. If λ is a positive and continuous function, ϕ_k ($k = 1, 2, \dots, m$) $\in C([a, b]_{\mathbb{T}}, \mathbb{R})$ are bounded, nonzero and satisfy

$$\|\phi_k^\Delta\|_\infty := \sup_{x \in [a, b]} |\phi_k^\Delta(x)| < \infty, \quad \|\phi_k^\nabla\|_\infty := \sup_{x \in [a, b]} |\phi_k^\nabla(x)| < \infty, \quad k = 1, 2, \dots, m,$$

for given $m \geq 2$, then the following inequalities

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x) - \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x) \int_a^b \lambda(y) \phi_k(y) \diamond_\alpha y \right) \right| \\ & \leq \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \right) \int_a^b \lambda(y) |x - y| \diamond_\alpha y, \quad (12) \end{aligned}$$

and

$$\begin{aligned} & \left| m \int_a^b \lambda(y) \phi_k(y) \diamond_\alpha y \int_a^b \lambda(x) \prod_{k=1}^m \phi_k(x) \diamond_\alpha x \right. \\ & \left. - \sum_{k=1}^m \left(\int_a^b \lambda(x) \prod_{j \neq k} \phi_j(x) \diamond_\alpha x \int_a^b \lambda(x) \phi_k(x) \diamond_\alpha x \right) \right| \\ & \leq \int_a^b \lambda(x) \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \right) \left(\int_a^b \lambda(y) |x - y| \diamond_\alpha y \right) \diamond_\alpha x, \quad (13) \end{aligned}$$

hold for all $x \in [a, b]$.

Proof. It is easy to check

$$\begin{aligned} \phi_k(x) - \phi_k(y) &= \int_x^y \phi_k^\Delta(v) \Delta v = \int_x^y \phi_k^\nabla(v) \nabla v \\ &= \alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v, \quad k = 1, 2, \dots, m. \end{aligned}$$

Then we have

$$\begin{aligned} & \phi_k(x) \prod_{j \neq k} \phi_j(x) - \phi_k(y) \prod_{j \neq k} \phi_j(y) = \prod_{k=1}^m \phi_k(x) - \phi_k(y) \prod_{j \neq k} \phi_j(x) \\ &= \prod_{j \neq k} \phi_j(x) \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right), \quad k = 1, 2, \dots, m. \end{aligned}$$

By adding up from $k = 1$ to m , we can obtain

$$\begin{aligned} & m \prod_{k=1}^m \phi_k(x) - \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \phi_k(y) \\ &= \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right), \end{aligned}$$

namely,

$$\begin{aligned} & m \prod_{k=1}^m \phi_k(x) \lambda(y) - \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \phi_k(y) \lambda(y) \\ &= \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y). \end{aligned}$$

Diamond-Alpha integrating on both side from a to b with respect to y and rewriting, we get

$$\begin{aligned} & \prod_{k=1}^m \phi_k(x) - \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \phi_k(y) \lambda(y) \diamond_\alpha y \\ &= \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y) \diamond_\alpha y. \end{aligned} \quad (14)$$

Then we can get the desired inequality (12) by

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x) - \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \phi_k(y) \lambda(y) \diamond_\alpha y \right| \\ &= \left| \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y) \diamond_\alpha y \right| \\ &= \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} |\phi_j(x)| \left| \int_a^b \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y) \diamond_\alpha y \right| \\ &\leq \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} |\phi_j(x)| \int_a^b \left| \alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right| \lambda(y) \diamond_\alpha y \\ &\leq \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} |\phi_j(x)| \int_a^b \left(\alpha \int_x^y |\phi_k^\Delta(v)| \Delta v + (1 - \alpha) \int_x^y |\phi_k^\nabla(v)| \nabla v \right) \lambda(y) \diamond_\alpha y \\ &\leq \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \prod_{j \neq k} |\phi_j(x)| \int_a^b \left(\alpha \int_x^y \|\phi_k^\Delta\| \Delta v + (1 - \alpha) \int_x^y \|\phi_k^\nabla\| \nabla v \right) \lambda(y) \diamond_\alpha y \\ &\leq \frac{1}{m \int_a^b \lambda(y) \diamond_\alpha y} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \right) \int_a^b \lambda(y) |x - y| \diamond_\alpha y. \end{aligned}$$

From Inequality (14), we also have

$$\begin{aligned} & m \int_a^b \lambda(y) \diamond_\alpha y \prod_{k=1}^m \phi_k(x) \lambda(x) - \lambda(x) \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \phi_k(y) \lambda(y) \diamond_\alpha y \\ &= \lambda(x) \sum_{k=1}^m \prod_{j \neq k} \phi_j(x) \int_a^b \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1-\alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y) \diamond_\alpha y. \end{aligned}$$

Diamond-Alpha integrating with respect to x , the inequality leads to

$$\begin{aligned} & \left| m \int_a^b \lambda(y) \diamond_\alpha y \int_a^b \prod_{k=1}^m \phi_k(x) \lambda(x) \diamond_\alpha x - \sum_{k=1}^m \left(\int_a^b \lambda(x) \prod_{j \neq k} \phi_j(x) \diamond_\alpha x \int_a^b \phi_k(x) \lambda(x) \diamond_\alpha x \right) \right| \\ &= \left| \sum_{k=1}^m \int_a^b \left(\lambda(x) \prod_{j \neq k} \phi_j(x) \int_a^b \left(\alpha \int_x^y \phi_k^\Delta(v) \Delta v + (1-\alpha) \int_x^y \phi_k^\nabla(v) \nabla v \right) \lambda(y) \diamond_\alpha y \right) \diamond_\alpha x \right| \\ &\leq \sum_{k=1}^m \int_a^b \left(\lambda(x) \left(\prod_{j \neq k} |\phi_j(x)| (\alpha \|\phi_k^\Delta\|_\infty + (1-\alpha) \|\phi_k^\nabla\|_\infty) \right) \int_a^b \lambda(y) |x-y| \diamond_\alpha y \right) \diamond_\alpha x \\ &= \int_a^b \lambda(x) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x) (\alpha \|\phi_k^\Delta\|_\infty + (1-\alpha) \|\phi_k^\nabla\|_\infty) \right) \left(\int_a^b \lambda(y) |x-y| \diamond_\alpha y \right) \diamond_\alpha x, \end{aligned}$$

which is the desired Inequality (13). Therefore, we complete the proof. $\square$

If we take $\mathbb{T} = \mathbb{R}$, then the inequalities in Theorem 4 change into

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x) - \frac{1}{m \int_a^b \lambda(y) dy} \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x) \int_a^b \lambda(y) \phi_k(y) dy \right) \right| \\ &\leq \frac{1}{m \int_a^b \lambda(y) dy} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x)| \|\phi_k'\|_\infty \int_a^b \lambda(y) |x-y| dy, \right. \end{aligned}$$

and

$$\begin{aligned} & \left| m \int_a^b \lambda(y) \phi_k(y) dy \int_a^b \lambda(x) \prod_{k=1}^m \phi_k(x) dx - \sum_{k=1}^m \left(\int_a^b \lambda(x) \prod_{j \neq k} \phi_j(x) dx \int_a^b \lambda(x) \phi_k(x) dx \right) \right| \\ &\leq \int_a^b \lambda(x) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x) \|\phi_k'\|_\infty + \left(\int_a^b \lambda(y) |x-y| dy \right) \right) dx. \end{aligned}$$

In the same way, the inequalities in Theorem 2 are hold for the case with more functions. The theorem shows there is a upper bound of $\int_a^b \lambda(y) \prod_{k=1}^n (\phi_k(x) - \phi_k(y)) \diamond_\alpha y$ and the upper bound is depend on ϕ_k and λ .

Theorem 5. If λ is a positive and continuous function, $\phi_k (k = 1, 2, \dots, m) \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ are bounded, nonzero and satisfy

$$\|\phi_k^\Delta\|_\infty = \sup_{x \in [a, b]} |\phi_k^\Delta(x)| < \infty, \quad \|\phi_k^\nabla\|_\infty = \sup_{x \in [a, b]} |\phi_k^\nabla(x)| < \infty, \quad k = 1, 2, \dots, m,$$

then the following inequality

$$\left| \int_a^b \lambda(y) \prod_{k=1}^n (\phi_k(x) - \phi_k(y)) \diamond_\alpha y \right| \leq \prod_{k=1}^m (\alpha \|\phi_k^\Delta\|_\infty + (1-\alpha) \|\phi_k^\nabla\|_\infty) \int_a^b \lambda(y) |x-y|^m \diamond_\alpha y, \quad (15)$$

holds for all $x \in [a, b]$.

Proof. Clearly, we have

$$\phi_k(y) - \phi_k(x) = \int_x^y \phi^\Delta(v) \Delta v = \int_x^y \phi^\nabla(v) \nabla v = \alpha \int_x^y \phi^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi^\nabla(v) \nabla v.$$

Multiplying from $k = 1$ to m , we obtain

$$\prod_{k=1}^m (\phi_k(y) - \phi_k(x)) = \prod_{k=1}^m \left(\alpha \int_x^y \phi^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi^\nabla(v) \nabla v \right),$$

namely,

$$\lambda(y) \prod_{k=1}^m (\phi_k(y) - \phi_k(x)) = \lambda(y) \prod_{k=1}^m \left(\alpha \int_x^y \phi^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi^\nabla(v) \nabla v \right).$$

Diamond-Alpha integrating with respect to y leads to

$$\begin{aligned} & \left| \int_a^b \lambda(y) \prod_{k=1}^m (\phi_k(y) - \phi_k(x)) \diamond_\alpha y \right| \\ &= \left| \int_a^b \lambda(y) \prod_{k=1}^m \left(\alpha \int_x^y \phi^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi^\nabla(v) \nabla v \right) \diamond_\alpha y \right| \\ &\leq \int_a^b \lambda(y) \prod_{k=1}^m \left| \alpha \int_x^y \phi^\Delta(v) \Delta v + (1 - \alpha) \int_x^y \phi^\nabla(v) \nabla v \right| \diamond_\alpha y \\ &\leq \prod_{k=1}^m (\alpha \|\phi_k^\Delta\|_\infty + (1 - \alpha) \|\phi_k^\nabla\|_\infty) \int_a^b \lambda(y) |x - y|^m \diamond_\alpha y, \end{aligned}$$

which is the desired Inequality (15). $\square$

If we chose $\mathbb{T} = \mathbb{R}$, then Inequality (15) transforms to

$$\left| \int_a^b \lambda(y) \prod_{k=1}^m (\phi_k(y) - \phi_k(x)) dy \right| \leq \prod_{k=1}^m \|\phi_k'\|_\infty \int_a^b \lambda(y) |x - y|^m dy.$$

Next we consider the inequalities involving m functions and n variables based on the multiple Diamond- α_i integral in Section 2. We denote $D^* = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Theorem 6. If λ is a positive and continuous function with n variables, $\phi_k (k = 1, 2, \dots, m) \in C([a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n}, \mathbb{R})$ are bounded, nonzero and satisfy

$$\left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2} \cdots \diamond_{\omega_n}} \right\|_\infty := \sup_{(x_1, x_2, \dots, x_n) \in D^*} \left| \frac{\partial^n \phi_k(x_1, x_2, \dots, x_n)}{\diamond_{\omega_1} x_1 \diamond_{\omega_2} x_2 \cdots \diamond_{\omega_n} x_n} \right| < \infty, \quad k = 1, 2, \dots, m,$$

where $\omega_i = 0$ or 1 for all $i = 1, 2, \dots, n$ and $m \geq 2$ is given, then the following inequality holds for all $(x_1, x_2, \dots, x_n) \in D^*$,

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x_1, x_2, \dots, x_n) + \frac{1}{mM} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \right. \\ & \quad \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{ \phi_k \} (j) \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \cdots \diamond_{\alpha_n} y_n \right| \\ &\leq \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} \|\phi_j\|_\infty \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2} \cdots \diamond_{\omega_n}} \right\|_\infty \right. \\ & \quad \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \prod_{j=1}^n |x_j - y_j| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \cdots \diamond_{\alpha_n} y_n, \right. \end{aligned} \quad (16)$$

where

$$M = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \cdots \diamond_{\alpha_n} y_n,$$

and $l_{i,j}, I_a^b \{f\}(k)$ are defined by (6) and (7), respectively.

Proof. Same as (5), we have

$$\begin{aligned} & \int_{y_n}^{x_n} \int_{y_{n-1}}^{x_{n-1}} \cdots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, v_2, \dots, v_n)}{\Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n} \Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n \\ &= \sum_{j=0}^{2^n-1} l_{n,j} I_y^x \{\phi_k\}(j), \end{aligned}$$

for all $k = 1, 2, 3, \dots, m$. Moreover, it also holds for multiple mixed integral, namely,

$$\begin{aligned} & \int_{y_n}^{x_n} \int_{y_{n-1}}^{x_{n-1}} \cdots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, v_2, \dots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\ &= \sum_{j=0}^{2^n-1} l_{n,j} I_y^x \{\phi_k\}(j), \end{aligned}$$

where $\omega_i = 1$ or 0 . Thus we can obtain

$$\begin{aligned} & \sum_{j=0}^{2^n-1} l_{n,j} I_y^x \{\phi_k\}(j) \\ &= \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \\ & \int_{y_n}^{x_n} \int_{y_{n-1}}^{x_{n-1}} \cdots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, v_2, \dots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n, \end{aligned} \quad (17)$$

where ψ_k are defined by (9). Adding up from $k = 1$ to m and multiplying

$$\lambda(y_1, y_2, \dots, y_m) \prod_{j=1, j \neq k}^m \phi_j(x_1, x_2, \dots, x_n),$$

Inequality (17) leads to

$$\begin{aligned} & \lambda(y_1, y_2, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-1} l_{n,j} I_y^x \{\phi_k\}(j) \right) \\ &= \lambda(y_1, y_2, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \left(\phi_k(x_1, x_2, \dots, x_n) \right. \right. \\ & \quad \left. \left. + \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{\phi_k\}(j) \right) \right) \\ &= m \lambda(y_1, y_2, \dots, y_m) \prod_{k=1}^m \phi_k(x_1, x_2, \dots, x_n) \\ & \quad + \lambda(y_1, y_2, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{\phi_k\}(j) \right) \\ &= \lambda(y_1, y_2, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \right. \\ & \quad \left. \int_{y_n}^{x_n} \int_{y_{n-1}}^{x_{n-1}} \cdots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, v_2, \dots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right). \end{aligned}$$

Then we can get the desired inequality by taking multiple Diamond- α_i integral with respect to x_i ,

$$\begin{aligned}
& \left| \prod_{k=1}^m \phi_k(x_1, x_2, \dots, x_n) + \frac{1}{mM} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \right. \\
& \quad \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{ \phi_k \} (j) \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \dots \diamond_{\alpha_n} y_n \right| \\
&= \frac{1}{mM} \left| \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \lambda(y_1, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, \dots, x_n) \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \dots \psi_n(\omega_n) \right. \right. \\
& \quad \left. \left. \int_{y_n}^{x_n} \dots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, \dots, v_n)}{\diamond_{\omega_1} v_1 \dots \diamond_{\omega_n} v_n} \diamond_{\omega_1} v_1 \dots \diamond_{\omega_n} v_n \right) \diamond_{\alpha_1} y_1 \dots \diamond_{\alpha_n} y_n \right| \\
&\leq \frac{1}{mM} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \lambda(y_1, \dots, y_m) \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, \dots, x_n)| \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \dots \psi_n(\omega_n) \right. \\
& \quad \left. \left| \int_{y_n}^{x_n} \dots \int_{y_1}^{x_1} \frac{\partial^n \phi_k(v_1, \dots, v_n)}{\diamond_{\omega_1} v_1 \dots \diamond_{\omega_n} v_n} \diamond_{\omega_1} v_1 \dots \diamond_{\omega_n} v_n \right| \right) \diamond_{\alpha_1} y_1 \dots \diamond_{\alpha_n} y_n \\
&\leq \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, x_2, \dots, x_n)| \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \dots \psi_n(\omega_n) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2} \dots \diamond_{\omega_n}} \right\|_{\infty} \right. \\
& \quad \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \prod_{j=1}^n |x_i - y_i| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \dots \diamond_{\alpha_n} y_n \right). \\
&\quad \square
\end{aligned}$$

Remark 2. It is clear that Inequality (16) changes into (12) by taking $n = 1$ in Theorem 6.

If $n = 2$, then Theorem 6 gives the two dimensional Diamond-Alpha Ostrowski type inequalities.

Corollary 4. If λ is a positive and continuous function of two variables, $\phi_k (k = 1, 2, \dots, m) \in C([a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2}, \mathbb{R})$ are bounded, nonzero and satisfy

$$\begin{aligned}
\|\phi_k^{\Delta_1 \Delta_2}\|_{\infty} &:= \sup_{(x_1, x_2) \in D_2^*} \left| \frac{\partial^2 \phi_k(x_1, x_2)}{\Delta_1 x_1 \Delta_2 x_2} \right| < \infty, \quad \|\phi_k^{\Delta_1 \nabla_2}\|_{\infty} := \sup_{(x_1, x_2) \in D_2^*} \left| \frac{\partial^2 \phi_k(x_1, x_2)}{\Delta_1 x_1 \nabla_2 x_2} \right| < \infty \\
\|\phi_k^{\nabla_1 \Delta_2}\|_{\infty} &:= \sup_{(x_1, x_2) \in D_2^*} \left| \frac{\partial^2 \phi_k(x_1, x_2)}{\nabla_1 x_1 \Delta_2 x_2} \right| < \infty, \quad \|\phi_k^{\nabla_1 \nabla_2}\|_{\infty} := \sup_{(x_1, x_2) \in D_2^*} \left| \frac{\partial^2 \phi_k(x_1, x_2)}{\nabla_1 x_1 \nabla_2 x_2} \right| < \infty
\end{aligned}$$

then we have

$$\begin{aligned}
& \left| \prod_{k=1}^m \phi_k(x_1, x_2) + \frac{1}{mM} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) \right. \\
& \quad \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2) (-\phi_k(x_1, y_2) - \phi_k(y_1, x_2) + \phi_k(y_1, y_2)) \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right| \\
&\leq \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, x_2)| \sum_{\omega_1, \omega_2=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2}} \right\|_{\infty} \right. \\
& \quad \left. \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) |(x_1 - y_1)(x_2 - y_2)| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right), \tag{18}
\end{aligned}$$

for all $(x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$, where $D_2^* = [a_1, b_1] \times [a_2, b_2]$ and

$$M = \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2.$$

Since $\diamond_1 = \Delta$ and $\diamond_0 = \nabla$, then the corollary above gives the following corollaries when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ or 1.

Corollary 5. *If the conditions in Theorem 6 hold, then we have*

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x_1, x_2, \dots, x_n) + \frac{1}{mM} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \right. \\ & \quad \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{ \phi_k \}(j) \right) \Delta_1 y_1 \Delta_2 y_2 \dots \Delta_n y_n \right| \\ & \leq \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, x_2, \dots, x_n)| \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \dots \psi_n(\omega_n) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2} \dots \diamond_{\omega_n}} \right\|_{\infty} \right. \\ & \quad \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \prod_{j=1}^n |x_i - y_i| \Delta_1 y_1 \Delta_2 y_2 \dots \Delta_n y_n, \right. \end{aligned}$$

for all $(x_1, x_2, \dots, x_n) \in D$, where

$$M = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \Delta_1 y_1 \Delta_2 y_2 \dots \Delta_n y_n.$$

Corollary 6. *If the conditions in Theorem 6 hold, then we have*

$$\begin{aligned} & \left| \prod_{k=1}^m \phi_k(x_1, x_2, \dots, x_n) + \frac{1}{mM} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \right. \\ & \quad \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2, \dots, x_n) \sum_{j=0}^{2^n-2} l_{n,j} I_y^x \{ \phi_k \}(j) \right) \nabla_1 y_1 \nabla_2 y_2 \dots \nabla_n y_n \right| \\ & \leq \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, x_2, \dots, x_n)| \sum_{\omega_1, \dots, \omega_n=0 \text{ or } 1} \psi_1(\omega_1) \dots \psi_n(\omega_n) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2} \dots \diamond_{\omega_n}} \right\|_{\infty} \right. \\ & \quad \left. \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \prod_{j=1}^n |x_i - y_i| \nabla_1 y_1 \nabla_2 y_2 \dots \nabla_n y_n, \right. \end{aligned}$$

for all $(x_1, x_2, \dots, x_n) \in D$, where

$$M = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_1}^{b_1} \lambda(y_1, y_2, \dots, y_m) \nabla_1 y_1 \nabla_2 y_2 \dots \nabla_n y_n.$$

5. Examples

In this section, we provide some examples to further illustrate our inequalities. Examples 2 and 3 are examples of Theorem 4 and Example 4 is an example of Theorem 6. In fact, there are many interesting inequalities that can be derived from Theorems 4–6 and the following three examples provide only a few special cases.

Example 2. Consider the inequality:

$$\left| x^{\frac{m(m+1)}{2}} - \frac{2}{m(b^2 - a^2)} \sum_{k=1}^m \left(\frac{b^{k+2} - a^{k+2}}{k+2} x^{\frac{m(m+1)}{2} - k} \right) \right| \leq \frac{2}{m(b^2 - a^2)} \sum_{k=1}^m \left(kb^{k-1} x^{\frac{m(m+1)}{2} - k} \right) \int_a^b y|x-y|dy, \quad 0 < a \leq x \leq b. \quad (19)$$

Proof. Taking $\mathbb{T} = \mathbb{R}$, $\lambda(y) = y$, $\phi_k(x) = x^k$ in Theorem 4, we have

$$\|\phi_k^\Delta\|_\infty = \|\phi_k^\nabla\|_\infty = \|\phi_k'\|_\infty = kb^{k-1},$$

and

$$\int_a^b \lambda(y) \diamond_\alpha y = \int_a^b y dy = \frac{b^2 - a^2}{2}, \quad \int_a^b \lambda(y) \phi_k(y) \diamond_\alpha y = \int_a^b y^{k+1} dy = \frac{b^{k+2} - a^{k+2}}{k+2}.$$

Hence we can get the desired Inequality (19) according to (12). $\square$

Example 3. Consider the inequality:

$$\left| \frac{b^{k+2} - a^{k+2}}{k+2} \left(\frac{4}{11} (b^{11} - a^{11}) - \sum_{k=1}^4 \frac{b^{12-k} - a^{12-k}}{12-k} \right) \right| \leq \int_a^b kb^{k-1} \sum_{k=1}^4 x^{11-k} \left(\int_a^b y|x-y|dy \right) dx, \quad 0 < a \leq x \leq b. \quad (20)$$

Proof. Taking $\mathbb{T} = \mathbb{R}$, $\lambda(y) = y$, $\phi_k(x) = x^k$, $m = 4$ in Theorem 4, we can get the desired Inequality (20) according to (13). $\square$

Example 4. Consider the inequality:

$$\left| x_1^3 x_2^3 - P_1(N)(x_1^3 x_2^2 + x_1^2 x_2^3) + P_2(N)x_1^2 x_2^2 - P_3(N)(x_1^3 x_2 + x_1 x_2^3) + P_4(N)x_1 x_2 \right| \leq \frac{1}{2M} \left(x_1^2 x_2^2 + 4N^2 x_1 x_2 \right) \int_{a_2}^{b_2} \int_{a_1}^{b_1} y_1 y_2 |(x_1 - y_1)(x_2 - y_2)| \diamond_{\frac{1}{2}} y_1 \diamond_{\frac{1}{2}} y_2, \quad (x_1, x_2) \in [1, N]_{\mathbb{N}}^2$$

where $N \geq 3$, $N \in \mathbb{N}$, $M = f_0(N)$ and

$$P_0(N) = \frac{N^4 + 2N^3 - 3N^2 - 8N - 4}{16}, \quad P_1(N) = \frac{2N^5 + 5N^4 - 8N^3 - 11N^2 - 12N - 12}{96P_0(N)},$$

$$P_2(N) = \frac{4N^6 + 12N^5 + 7N^4 + 6N^3 - 11N^2 - 6}{288P_0(N)}, \quad P_3(N) = \frac{N^6 + 3N^5 - 5N^4 - 7N^3 - 8N - 8}{64P_0(N)},$$

$$P_4(N) = \frac{N^8 + 4N^7 + -10N^6 + 4N^5 + N^4 - 32N^3 - 16}{128P_0(N)}.$$

Proof. Taking $\mathbb{T} = \mathbb{N}$, $n = m = 2$, $\phi_k(x_1, x_2) = x_1^k x_2^k$, $\lambda(y_1, y_2) = y_1 y_2$, $a_1 = a_2 = 1$, $b_1 = b_2 = N$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$ in Theorem 6, the left-hand side of Inequality (18) transforms into

$$\begin{aligned}
& \left| \prod_{k=1}^m \phi_k(x_1, x_2) + \frac{1}{mM} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) \right. \\
& \left. \sum_{k=1}^m \left(\prod_{j \neq k} \phi_j(x_1, x_2) (-\phi_k(x_1, y_2) - \phi_k(y_1, x_2) + \phi_k(y_1, y_2)) \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right| \\
&= \left| x_1^3 x_2^3 + \frac{1}{2M} \int_1^N \int_1^N y_1 y_2 \sum_{k=1}^2 \left(x_1^{3-k} x_2^{3-k} (-x_1^k y_2^k - x_2^k y_1^k + y_1^k y_2^k) \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right| \\
&= \left| x_1^3 x_2^3 + \frac{1}{2M} \int_1^N \int_1^N \sum_{k=1}^2 \left(-x_1^3 x_2^{3-k} y_1 y_2^{k+1} - x_1^{3-k} x_2^3 y_1^{k+1} y_2 \right. \right. \\
& \quad \left. \left. + x_1^{3-k} x_2^{3-k} y_1^{k+1} y_2^{k+1} \right) \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right| \\
&= \left| x_1^3 x_2^3 + \frac{1}{2M} \int_1^N \int_1^N -x_1^3 x_2^2 y_1 y_2^2 - x_1^2 x_2^3 y_1^2 y_2 + x_1^2 x_2^2 y_1^2 y_2^2 \right. \\
& \quad \left. - x_1^3 x_2 y_1 y_2^3 - x_1 x_2^3 y_1^3 y_2 + x_1 x_2 y_1^3 y_2^3 \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right| \\
&= \left| x_1^3 x_2^3 + \frac{1}{2M} \left(\sum_{\omega_1, \omega_2=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \int_1^N \int_1^N -x_1^3 x_2^2 y_1 y_2^2 - x_1^2 x_2^3 y_1^2 y_2 + x_1^2 x_2^2 y_1^2 y_2^2 \right. \right. \\
& \quad \left. \left. - x_1^3 x_2 y_1 y_2^3 - x_1 x_2^3 y_1^3 y_2 + x_1 x_2 y_1^3 y_2^3 \diamond_{\omega_1} y_1 \diamond_{\omega_2} y_2 \right) \right| \\
&= \left| x_1^3 x_2^3 - \frac{1}{2M} (x_1^3 x_2^2 + x_1^2 x_2^3) \frac{N(N+1)N(N+1)(2N+1) - 12(N^3 + N^2 + N + 1)}{48} \right. \\
& \quad + \frac{1}{2M} x_1^2 x_2^2 \frac{N^2(N+1)^2(2N+1)^2 - 6(N^4 + 2N^2 + 1)}{24} \\
& \quad - \frac{1}{2M} (x_1^3 x_2 + x_1 x_2^3) \frac{N^3(N+1)^3 - 8(N^4 + N^3 + N + 1)}{32} \\
& \quad \left. + \frac{1}{2M} x_1 x_2 \frac{N^4(N+1)^4 - 16(N^6 + 2N^3 + 1)}{64} \right| \\
&= \left| x_1^3 x_2^3 - P_1(N)(x_1^3 x_2^2 + x_1^2 x_2^3) + P_2(N)x_1^2 x_2^2 - P_3(N)(x_1^3 x_2 + x_1 x_2^3) + P_4(N)x_1 x_2 \right|. \quad (21)
\end{aligned}$$

The right-hand side of Inequality (18) transforms into

$$\begin{aligned}
& \frac{1}{mM} \sum_{k=1}^m \left(\prod_{j \neq k} |\phi_j(x_1, x_2)| \sum_{\omega_1, \omega_2=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \left\| \phi_k^{\diamond_{\omega_1} \diamond_{\omega_2}} \right\|_{\infty} \right. \\
& \quad \left. \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) |(x_1 - y_1)(x_2 - y_2)| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right. \\
&= \frac{1}{2M} \sum_{k=1}^2 \left(x_1^{3-k} x_2^{3-k} \frac{1}{4} \left(\|\phi_k^{\Delta_1 \Delta_2}\|_{\infty} + \|\phi_k^{\Delta_1 \nabla_2}\|_{\infty} + \|\phi_k^{\nabla_1 \Delta_2}\|_{\infty} + \|\phi_k^{\nabla_1 \nabla_2}\|_{\infty} \right) \right. \\
& \quad \left. \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lambda(y_1, y_2) |(x_1 - y_1)(x_2 - y_2)| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right. \\
&= \frac{1}{2M} \sum_{k=1}^2 \left(\frac{x_1^{3-k} x_2^{3-k}}{4} \left(((N+1)^k - N^k)((N+1)^k - N^k) \right. \right. \\
& \quad + ((N+1)^k - N^k)(N^k - (N-1)^k) + (N^k - (N-1)^k)((N+1)^k - N^k) \\
& \quad \left. \left. + (N^k - (N-1)^k)(N^k - (N-1)^k) \right) \int_{a_2}^{b_2} \int_{a_1}^{b_1} y_1 y_2 |(x_1 - y_1)(x_2 - y_2)| \diamond_{\alpha_1} y_1 \diamond_{\alpha_2} y_2 \right. \\
&= \frac{1}{2M} \left(x_1^2 x_2^2 + 4N^2 x_1 x_2 \right) \int_{a_2}^{b_2} \int_{a_1}^{b_1} y_1 y_2 |(x_1 - y_1)(x_2 - y_2)| \diamond_{\frac{1}{2}} y_1 \diamond_{\frac{1}{2}} y_2. \quad (22)
\end{aligned}$$

We can complete the proof by combining Inequalities (21) and (22). $\square$

6. Conclusions

In this paper, we give the definition of multiple Diamond-Alpha integral with different α . Some Ostrowski type inequalities with more functions and higher dimensions are established.

The new integral has wide applications; it can represent multiple mixed integrals and generalize inequalities effectively. Moreover, it will play a key role in the process of researching Diamond-Alpha dynamic equations, which is also the aim in the following work.

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