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New Oscillation Theorems for Second-Order Differential Equations with Canonical and Non-Canonical Operator via Riccati Transformation

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Abstract: In this work, we prove some new oscillation theorems for second-order neutral delay differential equations of the form $(a(\xi)((v(\xi) + b(\xi)v(\vartheta(\xi)))')')' + c(\xi)G_1(v(\kappa(\xi))) + d(\xi)G_2(v(\varsigma(\xi))) = 0$ under canonical and non-canonical operators, that is, $\int_{\xi_0}^{\infty} \frac{d\xi}{a(\xi)} = \infty$ and $\int_{\xi_0}^{\infty} \frac{d\xi}{a(\xi)} < \infty$. We use the Riccati transformation to prove our main results. Furthermore, some examples are provided to show the effectiveness and feasibility of the main results.

Keywords: differential equations; second-order; neutral; delay; oscillation criteria

1. Introduction

It is well known that differential equations have many applications in research, for example, population growth, decay, Newton's law of cooling, glucose absorption by the body, spread of epidemics, Newton's second law of motion, and interacting species competition. They appear in the study of many real-world problems (see [1–3]).

Here, we mention some recent developments of oscillation theory to neutral differential equations.

In [4], Santra et al. have studied explicit criteria for the oscillation of second-order differential equations with several sub-linear positive neutral coefficients of the form

$$\left(a(\xi)\left((v(\xi) + \sum_{i=1}^{m} b_i(\xi)v^{\alpha_i}(\vartheta_i(\xi)))\right)'\right)' + c(\xi)v^\beta(\kappa(\xi)) = 0 \qquad (1)$$

and obtained some new sufficient conditions for the oscillation of (1). Santra et al. [5] have studied asymptotic behavior of a class of second-order nonlinear neutral differential equations with multiple delays of the form

$$\left(a(\xi)\left(v(\xi) + b(\xi)v(\xi - \vartheta)\right)'\right)' + \sum_{i=1}^{m} c_i(\xi)G_i(v(\xi - \kappa_i)) = 0 \qquad (2)$$

and obtained some new sufficient conditions for the oscillation of solution of (2) under a non-canonical operator with various ranges of the neutral coefficient b. In another paper [6],

Santra et al. have established some new oscillation theorems to neutral differential equations with mixed delays under a canonical operator with $0 \leq b < 1$. In [7], Bazighifan et al. have studied oscillatory properties of even-order ordinary differential equations with variable coefficients.

For more details on the oscillation theory of neutral delay differential equations, we refer the reader to the papers [8–22]. In particular, the study of oscillation of half-linear/Emden–Fowler (neutral) differential equations with deviating arguments (delayed or advanced arguments or mixed arguments) has numerous applications in physics and engineering (e.g., half-linear/Emden–Fowler differential equations arise in a variety of real-world problems, such as in the study of p-Laplace equations and chemotaxis models); see, e.g., the papers [23–34] for more details. In particular, by using different methods, the following papers were concerned with the oscillation of various classes of half-linear/Emden–Fowler differential equations and half-linear/Emden–Fowler differential equations with different neutral coefficients: the paper [24] was concerned with neutral differential equations assuming that $0 \leq b(\xi) < 1$ and $b(\xi) > 1$, where b is the neutral coefficient; the paper [25] was concerned with neutral differential equations assuming that $0 \leq b(\xi) < 1$; the paper [27] was concerned with neutral differential equations assuming that $b(\xi)$ is nonpositive; the papers [28,32] were concerned with neutral differential equations in the case where $b(\xi) > 1$; the paper [31] was concerned with neutral differential equations assuming that $0 \leq b(\xi) \leq b_0 < \infty$ and $b(\xi) > 1$; the paper [33] was concerned with neutral differential equations in the case where $0 \leq b(\xi) \leq b_0 < \infty$; the paper [34] was concerned with neutral differential equations in the case when $0 \leq b(\xi) = b_0 \neq 1$; whereas the paper [30] was concerned with differential equations with a nonlinear neutral term assuming that $0 \leq b(\xi) \leq b < 1$. These examples have the same research topic as that of this paper.

Motivated by the above studies, in this article, we obtain sufficient conditions for the oscillation of the following second-order nonlinear differential equations

$$(a(\xi)((v(\xi) + b(\xi)v(\vartheta(\xi)))'))' + c(\xi)G_1(v(\kappa(\xi))) + d(\xi)G_2(v(\varsigma(\xi))) = 0, \qquad (3)$$

where $a \in C^1([\xi_0, \infty), \mathbb{R})$, $b, c, d \in C([\xi_0, \infty), \mathbb{R})$, $G_1, G_2 \in C(\mathbb{R}, \mathbb{R})$ are continuous functions such that

(i) $a(\xi) > 0$, $0 \leq b(\xi) \leq p_0 < \infty$, $c(\xi)$, $d(\xi) \geq 0$, $c(\xi)$, $d(\xi)$ is not identically zero on any interval of the form $[\xi_*, \infty)$ for any $\xi_* \geq \xi_0$.

(ii) $\frac{G_1(u)}{u} \geq k > 0$, $\frac{G_2(v)}{v} \geq k > 0$ for $u, v \neq 0$, k is a constant.

(iii) $\vartheta \in C^1([\xi_0, \infty), \mathbb{R})$, κ, $\varsigma \in ([\xi_0, \infty), \mathbb{R})$, $\vartheta' \geq \vartheta_0 > 0$, $\vartheta(\xi) \leq \xi$, $\kappa(\xi) \leq \xi$, $\varsigma(\xi) \leq \xi$ with $\lim_{\xi \to \infty} \vartheta(\xi) = \infty = \lim_{\xi \to \infty} \kappa(\xi) = \infty = \lim_{\xi \to \infty} \varsigma(\xi)$.

The objective of our work is to establish the oscillation character of all solutions of (3) under the following canonical and non-canonical conditions:

(C1) $\int_{\xi_0}^{\infty} \frac{d\xi}{a(\xi)} = \infty$

and

(C2) $\int_{\xi_0}^{\infty} \frac{d\xi}{a(\xi)} < \infty$.

By a solution of (3), we mean a continuously differentiable function $v(\xi)$, which is defined for $\xi \geq Y^* = \min\{\vartheta(\xi_0), \kappa(\xi_0), \varsigma(\xi_0)\}$ such that $v(\xi)$ satisfies (3) for all $\xi \geq \xi_0$. In the sequel, it will always be assumed that the solutions of (3) exist on some half line $[\xi_1, \infty)$, $\xi_1 \geq \xi_0$. A solution of (3) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called non-oscillatory. Equation (3) is called oscillatory when all its solutions are oscillatory.

2. Oscillation Results under (C1)

In this section, we are proving some new oscillation results for (3) under the assumption (C1). In the whole paper, we assume that

$$z(\xi) = v(\xi) + b(\xi)v(\vartheta(\xi)). \tag{4}$$

Theorem 1. *Assume that (C1) holds, and $\kappa'(\xi) > 0$ for $\xi \geq \xi_0$, $\varsigma(\xi) \geq \kappa(\xi)$, $\vartheta(\kappa(\xi)) = \kappa(\vartheta(\xi))$ and $\vartheta(\varsigma(\xi)) = \varsigma(\vartheta(\xi))$ hold for $\xi \in [\xi_0, \infty)$. If*

(A1) $\int_{\xi_0}^{\infty} \delta(\zeta)\left[\{C(\zeta) + D(\zeta)\} - \left(1 + \frac{p_0}{\vartheta_0}\right)\frac{a(\kappa(\zeta))((\delta'(\zeta))_+)^2}{4\delta(\zeta)\kappa'(\zeta)}\right]d\zeta = \infty,$

where $C(\xi) = \min\{c(\xi), c(\vartheta(\xi))\}$, $D(\xi) = \min\{d(\xi), d(\vartheta(\xi))\}$ and $(\delta'(\xi))_+ = \max\{\delta'(\xi), 0\}$, and $\delta(\xi)$ is a positive differentiable function, then every solution of (3) is oscillatory.

Proof. Let $v(\xi)$ be a non-oscillatory solution of (3) and which is $v(\xi) > 0$ for $\xi \geq \xi_0$. Hence, there exists $\xi_1 > \xi_0$ such that $v(\xi) > 0$, $v(\vartheta(\xi)) > 0$, $v(\kappa(\xi)) > 0$ and $v(\varsigma(\xi)) > 0$ for $\xi \geq \xi_1$. Using (4), (3) becomes

$$(a(\xi)(z'(\xi)))' = -c(\xi)v(\kappa(\xi)) - d(\xi)v(\varsigma(\xi)) \leq 0, \not\equiv 0 \text{ for } \xi \geq \xi_1. \tag{5}$$

Therefore, $a(\xi)(z'(\xi))$ is non-increasing on $[\xi_1, \infty)$, that is, either $z'(\xi) > 0$ or $z'(\xi) < 0$ for $\xi \geq \xi_2 > \xi_1$. We claim that $z'(\xi) > 0$ for $\xi \geq \xi_2$. If not, there exists $\xi_3 > \xi_2$ such that $z'(\xi_2) < 0$. Then from (5), we obtain that

$$a(\xi)z'(\xi) \leq a(\xi_3)z'(\xi_3), \ \xi \geq \xi_3.$$

Hence

$$z(\xi) \leq z(\xi_3) + a(\xi_3)z'(\xi_3)\int_{\xi_3}^{\xi} \frac{d\zeta}{a(\zeta)}.$$

Letting $\xi \to \infty$, we obtain $z(\xi) \to -\infty$ as $\xi \to \infty$, which is a contradiction to the fact that $z'(\xi) > 0$ for $\xi \geq \xi_3$. From (3), it is easy to see that

$$(a(\xi)(z'(\xi)))' + c(\xi)v(\kappa(\xi)) + d(\xi)v(\varsigma(\xi)) + $$
$$\frac{p_0}{\vartheta'(\xi)}(a(\vartheta(\xi))(z'(\vartheta(\xi))))' + p_0c(\vartheta(\xi))v(\kappa(\vartheta(\xi)) + p_0d(\vartheta(\xi))v^\gamma(\varsigma(\vartheta(\xi))) = 0$$

for $\xi \geq \xi_3$, the above relation yields that

$$(a(\xi)(z'(\xi)))' + \frac{p_0}{\vartheta'(\xi)}(a(\vartheta(\xi))(z'(\vartheta(\xi))))' + C(\xi)z(\kappa(\xi)) + D(\xi)z(\varsigma(\xi)) \leq 0.$$

We have $\vartheta'(\xi) \geq \vartheta_0 > 0$ and $\vartheta o \kappa = \kappa o \vartheta$ and $\vartheta o \varsigma = \varsigma o \vartheta$. Then from (5), we obtain

$$(a(\xi)(z'(\xi)))' + \frac{p_0}{\vartheta_0}(a(\vartheta(\xi))(z'(\vartheta(\xi))))' + C(\xi)z(\kappa(\xi)) + D(\xi)z(\varsigma(\xi)) \leq 0. \tag{6}$$

Now, we use the general Riccati substitution

$$w(\xi) = \delta(\xi)\frac{a(\xi)z'(\xi)}{z(\kappa(\xi))} \tag{7}$$

and

$$u(\xi) = \delta(\xi)\frac{a(\vartheta(\xi))z'(\vartheta(\xi))}{z(\kappa(\xi))}, \tag{8}$$

where $\delta(\xi)$ is a positive differentiable function. Clearly, $w(\xi) > 0$ and $v(\xi) > 0$ on $[\xi_2, \infty)$. Now, differentiating (7) and from (5) we have $z'(\kappa(\xi)) \geq \frac{a(\xi)z'(\xi)}{a(\kappa(\xi))}$, and

$$
\begin{aligned}
w'(\xi) &\leq \delta(\xi) \frac{(a(\xi)z'(\xi))'}{z(\kappa(\xi))} + \frac{\delta'(\xi)}{\delta(\xi)} w(\xi) - \frac{\kappa'(\xi)w^2(\xi)}{\delta(\xi)a(\kappa(\xi))} \\
&\leq \delta(\xi) \frac{(a(\xi)z'(\xi))'}{z(\kappa(\xi))} + \frac{(\delta'(\xi))_+}{\delta(\xi)} w(\xi) - \frac{\kappa'(\xi)w^2(\xi)}{\delta(\xi)a(\kappa(\xi))}.
\end{aligned} \tag{9}
$$

Similarly, differentiating (8), by (5) and $\kappa(\xi) \leq \vartheta(\xi)$, we find $z'(\kappa(\xi)) \geq \frac{a(\vartheta(\xi))z'(\vartheta(\xi))}{a(\kappa(\xi))}$ and

$$
\begin{aligned}
u'(\xi) &\leq \delta(\xi) \frac{(a(\vartheta(\xi))z'(\vartheta(\xi)))'}{z(\kappa(\xi))} + \frac{\delta'(\xi)}{\delta(\xi)} u(\xi) - \frac{\kappa'(\xi)u^2(\xi)}{\delta(\xi)a(\kappa(\xi))} \\
&\leq \delta(\xi) \frac{(a(\vartheta(\xi))z'(\vartheta(\xi)))'}{z(\kappa(\xi))} + \frac{(\delta'(\xi))_+}{\delta(\xi)} u(\xi) - \frac{\kappa'(\xi)u^2(\xi)}{\delta(\xi)a(\kappa(\xi))}.
\end{aligned} \tag{10}
$$

From (9) and (10),

$$
\begin{aligned}
w'(\xi) + \frac{p_0}{\vartheta_0} u'(\xi) &\leq \delta(\xi) \frac{(a(\xi)z'(\xi))'}{z(\kappa(\xi))} + \frac{p_0}{\vartheta_0} \delta(\xi) \frac{(a(\vartheta(\xi))z'(\vartheta(\xi)))'}{z(\kappa(\xi))} + \frac{(\delta'(\xi))_+}{\delta(\xi)} w(\xi) \\
&\quad - \frac{\kappa'(\xi)w^2(\xi)}{\delta(\xi)a(\kappa(\xi))} + \frac{p_0}{\vartheta_0} \frac{(\delta'(\xi))_+}{\delta(\xi)} u(\xi) - \frac{p_0}{\vartheta_0} \frac{\kappa'(\xi)u^2(\xi)}{\delta(\xi)a(\kappa(\xi))}.
\end{aligned}
$$

From (6) and the above inequality, it becomes

$$
\begin{aligned}
w'(\xi) + \frac{p_0}{\vartheta_0} u'(\xi) &\leq -\delta(\xi)[C(\xi) + D(\xi)] \\
&\quad + \frac{(\delta'(\xi))_+}{\delta(\xi)} w(\xi) - \frac{\kappa'(\xi)w^2(\xi)}{\delta(\xi)a(\kappa(\xi))} + \frac{p_0}{\vartheta_0} \frac{(\delta'(\xi))_+}{\delta(\xi)} u(\xi) - \frac{p_0}{\vartheta_0} \frac{\kappa'(\xi)v^2(\xi)}{\delta(\xi)a(\kappa(\xi))} \\
&\leq -\delta(\xi)[C(\xi) + D(\xi)] + (1 + \frac{p_0}{\vartheta_0}) \frac{a(\kappa(\xi))((\delta'(\xi))_+)^2}{4\delta(\xi)\kappa'(\xi)}.
\end{aligned}
$$

Integrating the above inequality from ξ_3 to ξ, we get

$$
\begin{aligned}
w(\xi) + \frac{p_0}{\vartheta_0} u(\xi) &\leq w(\xi_3) + \frac{p_0}{\vartheta_0} u(\xi_3) - \int_{\xi_3}^{\xi} \delta(\zeta) \Big[\{C(\zeta) + D(\zeta)\} \\
&\quad - (1 + \frac{p_0}{\vartheta_0}) \frac{a(\kappa(\zeta))((\delta'(\zeta))_+)^2}{4\delta(\zeta)\kappa'(\zeta)} \Big] d\zeta.
\end{aligned}
$$

that is,

$$
\int_{\xi_3}^{\xi} \delta(\zeta) \Big[\{C(\zeta) + D(\zeta)\} - (1 + \frac{p_0}{\vartheta_0}) \frac{a(\kappa(\zeta))((\delta'(\zeta))_+)^2}{4\delta(\zeta)\kappa'(\zeta)} \Big] d\zeta \leq w(\xi_3) + \frac{p_0}{\vartheta_0} u(\xi_3)
$$

which contradicts (A1). This completes the proof of the theorem. $\quad\square$

Choosing $\delta(\xi) = a(\kappa(\xi))$, by Theorem 1, we have the following results:

Corollary 1. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$. If (A2) $\limsup_{\xi \to \infty} \int_{\xi_0}^{\xi} \Big[\{a(\kappa(\zeta)C(\zeta) + a(\kappa(\zeta)D(\zeta))\} - (1 + \frac{p_0}{\vartheta_0}) \frac{\kappa'(\zeta)}{4a(\kappa(\zeta))a(\kappa(\zeta))} \Big] d\zeta = \infty$, then every solution of (3) oscillates.*

Corollary 2. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$. If*

(A3) $\liminf_{\xi \to \infty} \frac{1}{\ln(a\kappa(\xi))} \int_{\xi_0}^{\xi} \left[a(\kappa(\zeta)C(\zeta) + a(\kappa(\zeta)D(\zeta)) \right] d\zeta > \frac{\left(1 + \frac{p_0}{\vartheta_0}\right)}{4}$,

then every solution of (3) oscillates.

Corollary 3. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$. If*

(A4) $\liminf_{\xi \to \infty} \int_{\xi_0}^{\xi} \frac{\{R^2(\kappa(\zeta)C(\zeta)a(\kappa(\zeta)) + R^2(\kappa(\zeta)D(\zeta))a(\kappa(\zeta))\}}{\kappa'(\zeta)} > \frac{\left(1 + \frac{p_0}{\vartheta_0}\right)}{4}$,

then every solution of (3) oscillates.

Next, choosing $\delta(\xi) = \xi$, by Theorem 1, we have the following results.

Corollary 4. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi) \leq \varsigma(\xi)$ for $\xi \geq \xi_0$. If*

(A5) $\limsup_{\xi \to \infty} \left[\zeta C(\zeta) + \zeta D(\zeta) - (1 + \frac{p_0}{\vartheta_0}) \frac{a(\kappa(\zeta))}{s\kappa'(\zeta)} \right] d\zeta = \infty$,

then every solution of (3) oscillates.

Theorem 2. *Assume that (C1) holds, $\kappa'(\xi) > 0$ for $\xi \geq \xi_0$, $\varsigma(\xi) \geq \kappa(\xi)$, $\vartheta(\kappa(\xi)) = \kappa(\vartheta(\xi))$ and $\vartheta(\varsigma(\xi)) = \varsigma(\vartheta(\xi))$ be held for $\xi \in [\xi_0, \infty)$. If*

(A6) $\limsup_{\xi \to \infty} \int_{\xi_0}^{\infty} \delta(\zeta) \left[\{C(\zeta) + D(\zeta)\} - (1 + \frac{p_0}{\vartheta_0}) \frac{a(\vartheta(\zeta))((\delta'(\zeta))_+)^2}{4\delta(\zeta)\vartheta_0} \right] d\zeta = \infty$,

where $C(\xi)$, $D(\xi)$, $(\delta'(\xi))_+$ and $\delta(\xi)$ defined in Theorem 1, then every solution of (3) oscillates.

Proof. Proof is similar to that in Theorem 1, only we have to choose

$$w(\xi) = \delta(\xi) \frac{a(\xi)z'(\xi)}{z(\vartheta(\xi))} \text{ for } \xi \geq \xi_2.$$

for the current theorem. □

Corollary 5. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$. If*
(A7) $\limsup_{\xi \to \infty} \int_{\xi_0}^{\xi} \left[a(\vartheta(\zeta))C(\zeta) + a(\vartheta(\zeta))D(\zeta) - (1 + \frac{p_0}{\vartheta_0}) \frac{\vartheta_0}{4\vartheta(\zeta)a(\vartheta(\zeta))} \right] d\zeta = \infty$,

then every solution of (3) oscillates.

Corollary 6. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$. If*
(A8) $\liminf_{\xi \to \infty} \frac{1}{\ln a(\vartheta(\xi))} \int_{\xi_0}^{\xi} \left[a(\vartheta(\zeta))C(\zeta) + a(\vartheta(\zeta))D(\zeta) \right] d\zeta > (1 + \frac{p_0}{\vartheta_0}) \frac{\vartheta_0}{4}$,

then every solution of (3) oscillates.

Next, choosing $\delta(\xi) = \xi$, from Theorem 1, we have the following result:

Corollary 7. *Assume that (C1) holds, $\kappa'(\xi) > 0$, $\varsigma(\xi) \geq \kappa(\xi)$ and $\kappa(\xi) \leq \vartheta(\xi) \leq \varsigma(\xi)$ for $\xi \geq \xi_0$. If*
(A9) $\lim_{\xi \to \infty} \left[\zeta C(\zeta) + \zeta D(\zeta) - (1 + \frac{p_0}{\vartheta_0}) \frac{a(\vartheta(\zeta))}{4s\vartheta_0} \right] d\zeta = \infty$,

then every solution of (3) oscillates.

3. Oscillation Results under (C2)

In this section, we will prove some new oscillation results for (3) under the assumption (C2).

Theorem 3. *Assume that (C2) and (A1) hold with $\kappa'(\xi) > 0$, $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$, $\varsigma(\xi) \geq \kappa(\xi)$, $\vartheta(\kappa(\xi)) = \kappa(\vartheta(\xi))$ and $\vartheta(\varsigma(\xi)) = \varsigma(\vartheta(\xi))$ holds for $\xi \in [\xi_0, \infty)$. If*
(A10) $\limsup_{\xi \to \infty} \int_{\xi_0}^{\xi} \delta(\zeta) \left[\{C(\zeta) + D(\zeta)\} - (1 + \frac{p_0}{\vartheta_0}) \frac{\vartheta_0}{4\vartheta(\zeta)(\delta(\zeta))} \right] d\zeta = \infty$,

where $C(\xi)$, $D(\xi)$, $(\delta'(\xi))_+$ and $\delta(\xi)$ are defined in Theorem 1, then every solution of (3) oscillates.

Proof. Suppose that $v(\xi)$ is a non-oscillatory solution of (3). Without loss of generality, we may assume that there exists $\xi_1 \geq \xi_0$ such that $v(\xi) > 0$, $v(\vartheta(\xi)) > 0$, $v(\kappa(\xi)) > 0$ and $v(\varsigma(\xi)) > 0$, for all $\xi \geq \xi_1$. Setting $z(\xi)$ as in Theorem 1. From (5), $a(\xi)z'(\xi)$ is nonincreasing. Consequently, $z'(\xi) > 0$ or $z'(\xi) < 0$ for $\xi \geq \xi_2 \geq \xi_1$. If $z'(\xi) > 0$, then as in Theorem 1, we get a contradiction to (A1). If $z'(\xi) < 0$, then we define

$$w(\xi) = \frac{a(\xi)z'(\xi)}{z(\xi)} \quad for \quad \xi \geq \xi_2. \tag{11}$$

Clearly, $w(\xi) < 0$. Noting that $a(\xi)z'(\xi)$ is nonincreasing, we get

$$a(\zeta)z'(\zeta) \leq a(\xi)z'(\xi) \quad for \quad s \geq \xi \geq \xi_2.$$

Dividing the above by $a(\zeta)$ and integrating it from ξ to l, we obtain

$$z(l) \leq z(\xi) + a(\xi)z'(\xi) \int_\xi^l \frac{d\zeta}{a(\zeta)} \quad for \quad \xi \geq \xi \geq \xi_2.$$

Letting $l \to \infty$ in the above inequality, we have

$$0 \leq z(\xi) + a(\xi)z'(\xi)\delta(\xi) \quad for \quad \xi \geq \xi_2.$$

Therefore,

$$\frac{a(\xi)z'(\xi)}{z(\xi)}\delta(\xi) \geq -1 \quad for \quad \xi \geq \xi_2.$$

From (6), we obtain

$$-1 \leq w(\xi)\delta(\xi) \leq 0 \quad for \quad \xi \geq \xi_2. \tag{12}$$

Similarly, we define another function

$$u(\xi) = \frac{a(\vartheta(\xi))z'(\vartheta(\xi))}{z(\xi)} \quad for \quad \xi \geq \xi_2, \tag{13}$$

obviously, $v(\xi) < 0$. Noting that $a(\xi)z'(\xi)$ is nonincreasing, we have

$$a(\vartheta(\xi))z'(\vartheta(\xi)) \geq a(\xi)z'(\xi).$$

Then $v(\xi) \geq w(\xi)$. From (12), we obtain

$$-1 \leq v(\xi)\delta(\xi) \leq 0 \quad for \quad \xi \geq \xi_2. \tag{14}$$

Differentiating (11) and (13), we get

$$w'(\xi) = \frac{(a(\xi)z'(\xi))'}{z(\xi)} - \frac{w^2(\xi)}{a(\xi)} \tag{15}$$

and

$$u'(\xi) \leq \frac{a(\vartheta(\xi))z'(\vartheta(\xi))'}{z(\xi)} - \frac{u^2(\xi)}{a(\xi)}. \tag{16}$$

From (15) and (16), we obtain

$$w'(\xi) + \frac{p_0}{\vartheta_0}u'(\xi) \leq \frac{(a(\xi)z'(\xi))'}{z(\xi)}$$
$$+ \frac{p_0}{\vartheta_0}\frac{(a(\vartheta(\xi))z'(\vartheta(\xi)))'}{z(\xi)} - \frac{w^2(\xi)}{a(\xi)} - \frac{p_0}{\vartheta_0}\frac{u^2(\xi)}{a(\xi)}. \tag{17}$$

Proceeding as in Theorem 1, (6) holds. From (6), (17) can be written as

$$w'(\xi) + \frac{p_0}{\vartheta_0}u'(\xi) \leq -\{C(\xi) + D(\xi)\} - \frac{w^2(\xi)}{a(\xi)} - \frac{p_0}{\vartheta_0}\frac{u^2(\xi)}{a(\xi)}. \tag{18}$$

Multiplying (18) by $\delta(\xi)$, and integrating on $[\xi_2, \xi]$ implies that

$$\delta(\xi)w(\xi) - \delta(\xi_2)w(\xi_2) + \int_{\xi_2}^{\xi}\frac{w(\zeta)}{a(\zeta)}d\zeta + \int_{\xi_2}^{\xi}\frac{w^2(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\delta(\xi)u(\xi)$$
$$- \frac{p_0}{\vartheta_0}\delta(\xi_2)u(\xi_2) + \frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u^2(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta$$
$$+ \int_{\xi_2}^{\xi}\delta(\zeta)\{C(\zeta) + D(\zeta)\}d\zeta \leq 0.$$

From the above inequality, we obtain

$$\delta(\xi)w(\xi) - \delta(\xi_2)w(\xi_2) + \frac{p_0}{\vartheta_0}\delta(\xi)u(\xi) - \frac{p_0}{\vartheta_0}\delta(\xi_2)u(\xi_2) + \int_{\xi_2}^{\xi}\delta(\zeta)\{C(\zeta) + D(\zeta)\}d\zeta \leq 0.$$

Thus, it follows from the above inequality that

$$\delta(\xi)w(\xi) + \frac{p_0}{\vartheta_0}\delta(\xi)u(\xi) + \int_{\xi_2}^{\xi}\delta(\zeta)\left[\{C(\zeta) + D(\zeta)\} - (1 + \frac{p_0}{\vartheta_0})\frac{\vartheta_0}{4a(\zeta)\delta(\zeta)}\right]d\zeta$$
$$\leq \delta(\xi_2)w(\xi_2) + \frac{p_0}{\vartheta_0}\delta(\xi_2)u(\xi_2).$$

By (12) and (14), we obtain a contradiction to (A10). This completes the proof of the theorem. □

Theorem 4. *Assume that (C2) holds, $\kappa'(\xi) > 0$, $\kappa(\xi) \leq \vartheta(\xi)$ for $\xi \geq \xi_0$, $\varsigma(\xi) \geq \kappa(\xi)$, $\vartheta(\kappa(\xi)) = \kappa(\vartheta(\xi))$ and $\vartheta(\varsigma(\xi)) = \varsigma(\vartheta(\xi))$ holds for $\xi \in [\xi_0, \infty)$. If (A1) holds and*

(A11)$\limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \delta^2(\zeta)\left[C(\zeta) + D(\zeta)\right]d\zeta = \infty$,

where $C(\xi)$, $D(\xi)$, $(\delta'(\xi))_+$ and $\delta(\xi)$ are defined in Theorem 1, then every solution of (3) oscillates.

Proof. Suppose that $v(\xi)$ is a non-oscillatory solution of (3). Without loss of generality, we may assume that there exists $\xi_1 \geq \xi_0$ such that $v(\xi) > 0$, $v(\vartheta(\xi)) > 0$, $v(\kappa(\xi)) > 0$ and $v(\varsigma(\xi)) > 0$, for all $\xi \geq \xi_1$. Setting $z(\xi)$ as in Theorem 1. From (5), $a(\xi)z'(\xi)$ is nonincreasing. Consequently $z'(\xi) > 0$ or $z'(\xi) < 0$ for $\xi \geq \xi_2 \geq \xi_1$. If $z'(\xi) > 0$, then as in Theorem 1, we get a contradiction to (A1). If $z'(\xi) < 0$, then we define $w(\xi)$ and $v(\xi)$ as

in Theorem 3, we obtain (12), (14) and (18). Multiplying (18) by $\delta^2(\xi)$, and integrating on $[\xi_2, \xi]$

$$\delta^2(\xi)w(\xi) - \delta^2(\xi_2)w(\xi_2) + \int_{\xi_2}^{\xi} 2\frac{w(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta + \int_{\xi_2}^{\xi}\frac{w^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\delta^2(\xi)u(\xi)$$
$$- \frac{p_0}{\vartheta_0}\delta(\xi_2)u(\xi_2) + 2\frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta$$
$$+ \int_{\xi_2}^{\xi}\delta^2(\zeta)\{C(\zeta) + D(\zeta)\}d\zeta \leq 0. \tag{19}$$

It follows from (C2) and (12) that

$$\delta^2(\xi)w(\xi) - \delta^2(\xi_2)w(\xi_2) + \int_{\xi_2}^{\xi} 2\frac{w(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta + \int_{\xi_2}^{\xi}\frac{w^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\delta^2(\xi)u(\xi)$$
$$- \frac{p_0}{\vartheta_0}\delta(\xi_2)u(\xi_2) + 2\frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta + \frac{p_0}{\vartheta_0}\int_{\xi_2}^{\xi}\frac{u^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta$$
$$+ \int_{\xi_2}^{\xi}\delta^2(\zeta)\{C(\zeta) + D(\zeta)\}d\zeta \leq 0.$$

From (12), we have

$$|\int_{\xi_2}^{\infty}\frac{w(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta| \leq \int_{\xi_2}^{\infty}\frac{|w(\zeta)\delta(\zeta)|}{a(\zeta)}d\zeta$$
$$\leq \int_{\xi_2}^{\infty}\frac{d\zeta}{a(\zeta)} < \infty, \int_{\xi_2}^{\xi}\frac{w^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta \leq \int_{\xi_2}^{\infty}\frac{d\zeta}{a(\zeta)} < \infty.$$

From (14), we have

$$|\int_{\xi_2}^{\infty}\frac{u(\zeta)\delta(\zeta)}{a(\zeta)}d\zeta| \leq \int_{\xi_2}^{\infty}\frac{|u(\zeta)\delta(\zeta)|}{a(\zeta)}d\zeta$$
$$\leq \int_{\xi_2}^{\infty}\frac{d\zeta}{a(\zeta)} < \infty, \int_{\xi_2}^{\xi}\frac{u^2(\zeta)\delta^2(\zeta)}{a(\zeta)}d\zeta \leq \int_{\xi_2}^{\infty}\frac{d\zeta}{a(\zeta)} < \infty.$$

From (19), we get

$$\limsup_{\xi \to \infty} \int_{\xi_0}^{\xi}\delta^2(\zeta)\{C(\zeta) + D(\zeta)\}d\zeta < \infty,$$

which contradicts (A11), and hence, the theorem is proved. $\square$

Corollary 8. *Assume that (C2) holds, $\kappa'(\xi) > 0$, $\kappa(\xi) \leq \vartheta(\xi) \leq \varsigma(\xi) \leq \xi$, for $\xi \geq \xi_0$. If $(A_4), (A_5), (A_6), (A_7)$ and (A10) holds, then every solution of (3) oscillates.*

4. Examples

In this section, we will give two examples to illustrate our main results.

Example 1. *Let us consider the following equation*

$$[\xi(v(\xi) + b(\xi)v(\lambda_1(\xi))')']' + c(\xi)G_1(v(\lambda_2(\xi)) + d(\xi)G_2(v(\lambda_3(\xi)) = 0, \quad \xi \geq 1 \tag{20}$$

where $a(\xi) = \xi$, $\vartheta(\xi) = \lambda_1(\xi)$, $G_1(v) = v(1 + v^2) = G_2(v)$, $0 \leq b(\xi) \leq p_0 < \infty$, $\delta(\xi) = 1$, $c(\xi) = \gamma\xi = d(\xi)$, $\kappa(\xi) = \lambda_2(\xi)$, $\varsigma(\xi) = \lambda_3(\xi)$ and $\gamma > 0$, $\vartheta_0 = \lambda_1$, $\lambda_1 \leq \lambda_2 \leq \lambda_3$,

$$\limsup_{\xi \to \infty} \int_{\xi_0}^{\xi}\left[\{\zeta C(\zeta) + \zeta D(\zeta)\} - \left(1 + \frac{p_0}{\vartheta_0}\right)\frac{a(\vartheta(\zeta))}{4s\vartheta_0}\right]d\zeta$$

$$= \limsup_{\xi \to \infty} \int_1^\xi \left[2\gamma - \frac{1 + \frac{p_0}{\lambda_1}}{4} \right] d\zeta = \infty,$$

where $2\gamma > \left(\frac{1 + \frac{p_0}{\lambda_1}}{4} \right)$. Therefore, by Corollary 7, all solutions of (20) oscillate.

Example 2. *Let us consider the following equation*

$$[\xi^2 (v(\xi) + b(\xi)v(\xi - \vartheta))')']' + \xi G_1 (v(\xi - \kappa)) + \xi G_2 (v(\xi - \varsigma)) = 0, \quad \xi \geq 1 \quad (21)$$

where $a(\xi) = \xi^2$, $G_1(v) = v(1 + v^2) = G_2(v)$, $0 \leq b(\xi) \leq p_0 < \infty$, $\delta(\xi) = \frac{1}{\xi}$, $c(\xi) = \xi = d(\xi)$, $\vartheta(\xi) = \xi - \vartheta$, $\kappa(\xi) = \xi - \kappa = \varsigma(\xi)$ *and* $\kappa'(\xi) = 1 = \vartheta'(\xi)$, $\vartheta_0 = 1$, $c(\xi) = \xi - \vartheta = d(\xi)$, $a(\xi) = \frac{1}{1 - \frac{1}{\xi}}$

$$\limsup_{\xi \to \infty} \int_{\xi_0}^\xi \left[\{R(\kappa(\zeta))C(\zeta) + R(\varsigma(\zeta))D(\zeta)\} - \left(1 + \frac{p_0}{\vartheta_0} \right) \frac{\kappa'(\zeta)}{4a(\kappa(\zeta))a(\kappa(\zeta))} \right] d\zeta$$

$$= \limsup_{\xi \to \infty} \int_1^\xi \left[2\left\{ (s - \vartheta)\left(1 - \frac{1}{s - \kappa} \right) \right\} - \frac{1 + p_0}{4(s - \kappa)^2}\left(1 - \frac{1}{s - \kappa} \right) \right] d\zeta = \infty,$$

$$= \limsup_{\xi \to \infty} \int_{\xi_0}^\xi \left[\delta(\zeta)c(\zeta) + \delta(\zeta)d(\zeta) - \frac{1 + \frac{p_0}{\vartheta_0}}{4a(\zeta)\delta(\zeta)} \right] d\zeta.$$

$$= \limsup_{\xi \to \infty} \int_1^\xi \left[2\left(1 - \frac{\vartheta}{s} \right) - \frac{1 + p_0}{4s} \right] d\zeta = \infty.$$

Therefore, by Corollary 8, all solutions of (21) oscillate.

5. Conclusions

In this paper, we defined some new general Riccati transformations to study the oscillation of second-order differential equations of neutral type with two nonlinear functions G_1 and G_2 and proved new oscillation theorems under canonical and non-canonical operators with the help of the general Riccati transformation. It would be of attentiveness to analyze the oscillation of (3) for sub-linear, super-linear and integral neutral coefficients; for more details, we refer the reader to the papers [24,27,28,30–34].

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