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Steady Fluid–Structure Coupling Interface of Circular Membrane under Liquid Weight Loading: Closed-Form Solution for Differential-Integral Equations

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Abstract: In this paper, the problem of fluid–structure interaction of a circular membrane under liquid weight loading is formulated and is solved analytically. The circular membrane is initially flat and works as the bottom of a cylindrical cup or bucket. The initially flat circular membrane will undergo axisymmetric deformation and deflection after a certain amount of liquid is poured into the cylindrical cup. The amount of the liquid poured determines the deformation and deflection of the circular membrane, while in turn, the deformation and deflection of the circular membrane changes the shape and distribution of the liquid poured on the deformed and deflected circular membrane, resulting in the so-called fluid–structure interaction between liquid and membrane. For a given amount of liquid, the fluid–structure interaction will eventually reach a static equilibrium and the fluid–structure coupling interface is steady, resulting in a static problem of axisymmetric deformation and deflection of the circular membrane under the weight of given liquid. The established governing equations for the static problem contain both differential operation and integral operation and the power series method plays an irreplaceable role in solving the differential-integral equations. Finally, the closed-form solutions for stress and deflection are presented and are confirmed to be convergent by the numerical examples conducted.

Keywords: circular membrane; fluid–structure interaction; differential-integral equations; power series method; closed-form solution

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1. Introduction

Elastic membrane structures or structural components have applications in various fields [1–5]. These applications have provided an impetus for scholars to investigate the phenomena of large deflection of membrane [6–8]. Such investigations usually give rise to nonlinear equations with differential and even integral operation. These somewhat intractable nonlinear equations may present serious analytical difficulties when applied to boundary value problems [9–13].

The usually so-called circular membrane problem refers to the problem of axisymmetric deformation and deflection of an initially flat, peripherally fixed circular membrane subjected to transverse loads. Three main loading forms of transverse loads are involved in the existing studies: ① the uniformly distributed loads applied to the entire circular membrane [14–22], ② the uniformly distributed loads applied to the central portion of the circular membrane [23], and ③ the concentrated force applied to the center of the circular membrane [24–28]. Hencky was the first scholar to deal with the circular membrane problem concerning the first loading form of transverse loads and presented a closed-form solution in the form of power series [14]. A computational error in reference [14]

was corrected by Chien [15] and Alekseev [16], respectively. The problem originally dealt with by Hencky is usually called the well-known Hencky problem; i.e., the problem of axisymmetric deformation and deflection of an initially flat, peripherally fixed circular membrane under the action of a uniformly distributed transverse loads, where the weight of the circular membrane is usually ignored because it is usually very small in comparison with the transverse loads. The solution of the well-known Hencky problem is usually called the well-known Hencky solution, which is the first solution of the circular membrane problem and is often cited in some studies of related issues [15–22]. Chien et al. [23] analytically dealt with the symmetrical deformation of circular membrane under the action of uniformly distributed loads in its central portion, i.e., the circular membrane problem concerning the second loading form of transverse load. As for the third loading form, the concentrated force applied to the center of the circular membrane, it is, in fact, the limit case of the second loading form of transverse loads.

If an initially flat circular membrane is used as the bottom of a cylindrical cup or bucket and a certain amount of liquid is poured into the cylindrical cup or bucket, then the initially flat circular membrane will undergo axisymmetric deformation and deflection. The amount of the liquid poured determines the deformation and deflection of the circular membrane, while in turn, the deformation and deflection of the circular membrane changes the shape or distribution of the liquid over the deformed and deflected circular membrane. This results in an interaction between the liquid and the membrane, which is often referred to as a fluid–structure interaction. Obviously, for a given amount of liquid, the interaction between the liquid and the membrane will eventually reach a static equilibrium and a steady fluid–solid coupling interface will appear. Our main interest here is the static problem of axisymmetric deformation and deflection of the circular membrane under the given liquid weight loading. The closed-form solution of this static problem is expected to be used in the development of a new rain gauge [29–31]. However, such a fluid–structure coupling problem will give rise to governing equations containing both differential operation and integral operation. The power series method plays a unique and key role in solving these kinds of differential-integral equations analytically, as will be seen later.

The paper is organized as follows: in Section 2, the governing equations are established and the closed-form solutions for stress and deflection are presented. In Section 3, the numerical examples are conducted to show the differences between the presented solution and the well-known Hencky solution, and the convergence of the power-series solution for deflection and stress is verified. The concluding remarks are shown in Section 4.

2. Membrane Equation and Its Solution

An initially flat circular unstretched membrane with Young's modulus of elasticity E , Poisson's ratio ν , thickness h and radius a is fixed at the lower end of a vertically placed rigid round tube of finite length to form a cylindrical cup or bucket of inner radius a having a closed soft bottom with elastic deformation capability, and then a colored liquid with density ρ is slowly poured into the cup until the height of liquid reaches H , as shown in Figure 1, where H is the distance from the liquid level to the plane in which the initially flat circular membrane is located, w_m denotes the maximum deflection of the deflected circular membrane at static equilibrium. Based on the anticipated use of this study for rain gauge, only the case of $H \geq 0$ is considered here.

Let us take out a free body of a piece of circular membrane with radius r ($0 \leq r \leq a$) from the central portion of the whole deformed circular membrane, to study the static problem of equilibrium of this free body under the joint actions of the external force $F(r)$ produced by the transverse distributed loads $q(r)$ within r and the total force $2\pi r\sigma_r h$ produced by the membrane force $\sigma_r h$ acting on the boundary r , as shown in Figure 2; where a cylindrical coordinate system (r, φ, w) is introduced, the polar coordinate plane (r, φ) is located in the plane in which the geometric middle plane of the initially flat circular membrane is located; o denotes the origin of the cylindrical coordinate system (r, φ, w) , which is placed in the centroid of the geometric intermediate plane, r denotes the radial

coordinate, w denotes the axial coordinate of the cylindrical coordinate system (r, φ, w) as well as the transverse displacement of a point on the deflected circular membrane, θ denotes the slope angle of the deflecting membrane, and σ_r denotes the radial stress, while the angle coordinate φ is not represented in Figure 2.

Figure 1. Geometry of the circular membrane under prescribed liquid along a diameter.

Figure 2. Free body diagram of the deformed circular membrane with radius $0 \leq r \leq a$.

Obviously, the external force $F(r)$ produced by $q(r)$ within radius r is equal to the weight of the liquid within radius r , and is given by

$$F(r) = \rho g \int_0^r [w(r) + H] \cdot 2\pi r dr = 2\pi \rho g \int_0^r w(r) r dr + \rho g \pi r^2 H, \quad (1)$$

where g is the acceleration of gravity and $w(r)$ is the transverse displacement at r . Equation (1) is the usually so-called fluid-structure coupling equation at static equilibrium. The direction of the external force $F(r)$ is always perpendicular to the plane in which the initially flat circular membrane is located and vertically downward. Right here, the vertical upward force is $2\pi r \sigma_r h \sin \theta$, that is, the vertical component of the total membrane force $2\pi r \sigma_r h$ at r . Therefore, after ignoring the weight of the circular membrane, the equilibrium condition in the vertical direction, i.e., the so-called out-of-plane equilibrium equation, is given by

$$2\pi r \sigma_r h \sin \theta = F(r) = 2\pi \rho g \int_0^r w(r) r dr + \rho g \pi r^2 H, \quad (2)$$

where

$$\sin \theta \cong \tan \theta = -\frac{dw}{dr}. \quad (3)$$

Substituting Equation (3) into Equation (2) yields

$$2r\sigma_r h \frac{dw}{dr} + 2\rho g \int_0^r w(r)rdr + \rho g r^2 H = 0. \quad (4)$$

In the horizontal direction, there are two horizontal forces, the circumferential membrane force $\sigma_t h$ and the horizontal component of the radial membrane force $\sigma_r h$, where σ_t denotes the circumferential stress. Then, the equilibrium condition in the horizontal direction (i.e., the so-called in-plane equilibrium equation) is [23]

$$\frac{d}{dr}(r\sigma_r h) - \sigma_t h = 0. \quad (5)$$

Suppose that the radial strain is e_r , the circumferential strain is e_t and the radial displacement at r is $u(r)$. Then, the relations of the strain and displacement, the so-called geometric equations, may be written as [23]

$$e_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (6)$$

and

$$e_t = \frac{u}{r}. \quad (7)$$

The relations of the stress and strain (i.e., the so-called physical equations) are [23]

$$\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_t) \quad (8)$$

and

$$\sigma_t = \frac{E}{1-\nu^2}(e_t + \nu e_r). \quad (9)$$

Substituting Equations (6) and (7) into Equations (8) and (9) (to eliminate e_r and e_t in Equations (8) and (9)) yields

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] \quad (10)$$

and

$$\sigma_t = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr} + \nu\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]. \quad (11)$$

By means of Equations (10), (11) and (5), one has

$$\frac{u}{r} = \frac{1}{Eh}(\sigma_t h - \nu\sigma_r h) = \frac{1}{Eh}\left[\frac{d}{dr}(r\sigma_r h) - \nu\sigma_r h\right]. \quad (12)$$

Eliminating u from Equations (10) and (12) yields

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\sigma_r h)\right] + \frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 = 0. \quad (13)$$

Equation (13) is usually called a consistency equation. Equations (4) and (13) are two equations for the solutions of σ_r and w .

The boundary conditions, under which Equations (4) and (13) may be solved, are

$$\frac{dw}{dr} = 0 \text{ at } r = 0, \quad (14)$$

$$\frac{u}{r} = \frac{1}{Eh}\left[\frac{d}{dr}(r\sigma_r h) - \nu\sigma_r h\right] = 0 \text{ at } r = a \quad (15)$$

and

$$w = 0 \text{ at } r = a. \quad (16)$$

Let us introduce the following nondimensionalization:

$$W = \frac{w}{a}, S_r = \frac{\sigma_r}{E}, S_t = \frac{\sigma_t}{E}, x = \frac{r}{a}, H_0 = \frac{H}{a}, G = \frac{\rho g a^2}{Eh}, \quad (17)$$

and transform Equations (4), (13), (5), (14), (15) and (16) into

$$2xS_r \frac{dW}{dx} + 2G \int_0^x W(x)xdx + x^2GH_0 = 0, \quad (18)$$

$$x^2 \frac{d^2 S_r}{dx^2} + 3x \frac{dS_r}{dx} + \frac{1}{2} \left(\frac{dW}{dx} \right)^2 = 0, \quad (19)$$

$$S_t = S_r + x \frac{dS_r}{dx}, \quad (20)$$

$$\frac{dW}{dx} = 0 \text{ at } x = 0, \quad (21)$$

$$\frac{u}{r} = (1 - \nu)S_r + x \frac{dS_r}{dx} = 0 \text{ at } x = 1 \quad (22)$$

and

$$W = 0 \text{ at } x = 1. \quad (23)$$

In view of the physical phenomenon that the values of stress and deflection are both finite at $x = 0$, S_r and W can be expanded into the power series of x ; i.e., let

$$S_r = \sum_{i=0}^n c_i x^i \quad (24)$$

and

$$W = \sum_{i=0}^n d_i x^i. \quad (25)$$

After substituting Equations (24) and (25) into Equations (18) and (19), it is found, by using the mathematical software Maple 2018, that, $c_i \equiv 0$ and $d_i \equiv 0$ when $i = 1, 3, 5, \dots$, and when $i = 2, 4, 6, \dots$, the coefficients c_i and d_i can be expressed into the polynomial with regard to the coefficients c_0 and d_0 (see Appendices A and B).

The remaining two coefficients c_0 and d_0 are called undetermined constants, which can be determined by using the boundary conditions Equations (22) and (23) as follows. From Equation (24), Equation (22) gives

$$(1 - \nu) \sum_{i=0}^n c_i + \sum_{i=1}^n i c_i = 0, \quad (26)$$

and from Equation (25), Equation (23) gives

$$\sum_{i=0}^n d_i = 0. \quad (27)$$

For a concrete problem, the values of a , h , E , ν , ρ and H are known in advance. Therefore, after substituting all expressions of c_i and d_i (which are expressed by c_0 and d_0 , see Appendices A and B) into Equations (26) and (27), we can obtain a system of equations containing only c_0 and d_0 . The undetermined constants c_0 and d_0 can be determined by solving this system of equations. Furthermore, with the known c_0 and d_0 , the other coefficients c_i and d_i ($i = 2, 4, 6, \dots$) can easily be determined and the expressions of S_r and W can also be determined. The problems dealt with here are thus solved.

3. Results and Discussions

It is obvious that the boundary condition Equation (14), i.e., $dw/dr = 0$ at $r = 0$, has not been used yet. Now, let us see whether the closed-form solution obtained in Section 2 meets this boundary condition. From Equations (17) and (25) the dimensional form of the deflection w can be written as

$$w = \sum_{i=0}^{\infty} \frac{d_i}{a^{i-1}} r^i, \quad (28)$$

and the first derivative of Equation (28) is

$$\frac{dw}{dr} = \sum_{i=1}^{\infty} \frac{id_i}{a^{i-1}} r^{i-1}. \quad (29)$$

Equation (29) gives $dw/dr = d_1$ at $r = 0$. Therefore $dw/dr \equiv 0$ at $r = 0$, due to $d_1 \equiv 0$ (see the description after Equation (25)). It indicates that Equation (14) can be automatically satisfied, which, to some extent, proves the validity of the closed-form solution obtained in Section 2.

3.1. Comparison with the Well-Known Hencky Solution

It is well known that the well-known Hencky solution applies only to the case where the transverse loads applied to the whole deflected circular membrane must, regardless of the deflection of the membrane, be uniformly distributed [14]. Obviously, the more uneven the distribution of the transverse loads is, the greater the error caused by using the well-known Hencky solution. It is not hard to imagine from Figure 1 that, for a given amount of liquid (keep the liquid level H constant), the thinner or softer the membrane is, the greater the deflection of the membrane is, while the greater the deflection of the membrane is, the more uneven the distribution of the liquid over the whole deflected circular membrane is. On the other hand, for a given circular membrane, the uniformity of liquid distribution will also change with the increase of the liquid level H . Now, let us consider a numerical example to examine the difference between using the well-known Hencky solution and the solution obtained in Section 2. When the well-known Hencky solution is used, its uniformly distributed transverse loads q are given here by $q = \rho g H$.

Suppose that a circular rubber membrane with radius $a = 20$ mm, thickness $h = 0.1$ mm, Young's modulus of elasticity and Poisson's ratio $\nu = 0.47$ is subjected to a liquid with a density of $\rho = 1 \times 10^{-6}$ kg/mm³. After the fluid–structure interaction reaches a static equilibrium, the liquid level H is assumed to be equal to 0.5 mm, 50 mm and 200 mm, respectively. The acceleration of gravity is assumed to be $g = 10$ m/s². The deflection and radial stress curves along radius are shown in Figures 3 and 4, respectively, where the solid lines represent the results calculated by the solution obtained in Section 2 and the dotted lines by the well-known Hencky solution. The concrete values of the maximum deflection and radial stress are listed in Table 1, where the “errors” are given by the absolute value of the results by the well-known Hencky solution minus the results by the solution presented in this paper and then divided by the results by the solution presented in this paper.

Figure 3. Deflection w along radius r when H takes 0.5 mm, 50 mm and 200 mm, respectively, where the solid lines by the solution presented in this paper and the dotted lines by the well-known Hencky solution.

Figure 4. Radial stress σ_r along radius r when H takes 0.5 mm, 50 mm and 200 mm, respectively, where the solid lines represent the solution presented in this paper and the dotted lines represent the well-known Hencky solution.

Table 1. Maximum deflection and radial stress values at different H calculated by the solution presented in this paper and the well-known Hencky solution.

H	Maximum Deflection [mm]			Maximum Radial Stress [MPa]		
	Presented Solution	Hencky Solution	Errors	Presented Solution	Hencky Solution	Errors
0.5	0.8169	0.6089	25.5%	0.0184	0.0091	50.8%
50	2.8733	2.7977	2.6%	0.2005	0.1911	4.7%
200	4.5161	4.5013	0.3%	0.4969	0.4947	0.4%

From Figures 3 and 4, it can be easily seen that the distance between the dotted line and the solid line decreases as the liquid level H increases. When $H = 0.5$ mm, the distance between the dotted line and the solid line is the largest and the error between the results calculated by the solution presented in this paper and the well-known Hencky solution are also the largest (see Table 1), while $H = 200$ mm, both the distance and the error are

very small. This means that when $H = 0.5$ mm, the distribution of the liquid over the whole deflected circular membrane is the most uneven, and consequently the difference between using the well-known Hencky solution and the solution presented in this paper is the most obvious (the maximum value of relative error is about 25.5% for deflection and 50.8% for radial stress; see the first row in Table 1). The main reason behind this is that the uniformly distributed transverse loads q used for the well-known Hencky solution are given by $q = \rho g H$ (where H takes 0.5 mm); while $H = 0.5$ mm, the actual height of the liquid over the whole deflected circular membrane is 0.5 mm at the edge of the circular membrane and is about 1.3169 (0.5 + 0.8169) mm (see the first column in Table 1) at the center of the circular membrane (the relative error is about $(1.3169 - 0.5)/0.5 = 163.38\%$). Therefore, the distribution of the liquid over the whole deflected circular membrane is actually very uneven. Just as stated above, the more uneven the distribution of the transverse loads is, the greater the error caused by using the well-known Hencky solution. On the other hand, when $H = 200$ mm, the actual height of the liquid over the whole deflected circular membrane is 200 mm at the edge of the circular membrane and is about 204.5161 (200 + 4.5161) mm at the center of the circular membrane (the relative error is about $(204.5161 - 200)/200 = 2.26\%$). Therefore, in this case, the distribution of the liquid over the whole deflected circular membrane is actually very uniform. In other words, in this case, the external force $F(a)$ produced by $q(r)$ within radius a , which is applied to the whole deflected circular membrane, is largely determined by $\rho g \pi a^2 H$, and the contribution of the fluid–structure interaction $2\pi \rho g \int_0^a w(r) r dr$ can be ignored, see Equation (1). In addition, the phenomenon that the results calculated by the solution presented in this paper gradually converge to the results by the well-known Hencky solution as the liquid level H increases also proves to some extent that the closed-form solution obtained in Section 2 are basically reliable, as far as the well-known Hencky solution is considered to be a reliable solution.

3.2. Verification of Convergence of the Power Series Solution

In this section, the convergence of the power series solution obtained in Section 2 will be discussed. In general, it is better to discuss the convergence of the general solution rather than that of the special solution, because the special solution will converge if the general solution converges. However, we here have to discuss the convergence of the special solution, because the discussion on the convergence of the general solution cannot be conducted due to the complexity of the coefficients c_i and d_i ($i = 2, 4, 6, \dots$) expressed by the undetermined constants c_0 and d_0 (see Appendices A and B). From the derivation in Section 2, we know that the undetermined constants c_0 and d_0 can be determined by simultaneous solutions of Equations (26) and (27); the special solutions for $S_r(x)$ and $W(x)$ can be easily obtained as long as the undetermined constants c_0 and d_0 can be determined. When calculating the undetermined constants c_0 and d_0 , we have to substitute the partial sum of former n terms of Equations (24) and (25), rather than the infinite series of Equations (24) and (25), into Equations (26) and (27), otherwise the resulting Equations (26) and (27) will contain two infinite series and are thus difficult to be solved. Therefore, it seems that the terms n will determine the values of the undetermined constants c_0 and d_0 , and different n will determine the different values of c_0 and d_0 . Hence, the discussion on the convergence of the special solution should focus on giving the variations of c_0 and d_0 with terms n . If the undetermined constants c_0 and d_0 converge as the terms n increase, then the special solution can be concluded to converge as well.

We will continue with the numerical example given in Section 3.1. A circular rubber membrane with radius $a = 20$ mm, thickness $h = 0.1$ mm, Young's modulus of elasticity $E = 7.84$ MPa and Poisson's ratio $\nu = 0.47$ is subjected to the liquid weight loading with the liquid level $H = 50$ mm. We start the numerical calculations of c_0 and d_0 from $n = 4$; that is, start from the partial sum of the former four terms of Equations (24) and (25). The variations of c_0 and d_0 with terms n are shown in Figures 5 and 6, respectively. From Figures 5 and 6, we may see that with the increase of the terms n , the values of c_0 and d_0 are gradually close to some certain values (i.e., their exact values), and are almost no

longer changed when the terms n reach around 14, which indicates that the undetermined constants c_0 and d_0 converge reasonably well. Therefore, we here show only the results of the coefficients c_i and d_i when $n \leq 24$, which are listed in Tables 2 and 3, and when $n = 24$, the variations of the coefficients c_i and d_i with i ($i = 0, 2, 4, \dots, 24$) are shown in Figures 7 and 8. From Figures 7 and 8, it can be seen that c_{24} and d_{24} are already very close to 0, which means that the values of S_r and W are already very close to their exact values when $n = 24$.

Figure 5. Variation of c_0 with n .

Figure 6. Variation of d_0 with n .

Table 2. (a) The values of c_i at different n ; (b) The values of c_i at different n ; (c) The values of c_i at different n ; (d) The value of c_{24} at $n = 24$.

(a) The values of c_i at different n				
n	c_0	c_2	c_4	c_6
4	2.5216650×10^{-2}	$-4.4711868 \times 10^{-3}$	$-4.5313948 \times 10^{-4}$	-
6	2.5484214×10^{-2}	$-4.3775807 \times 10^{-3}$	$-4.2827528 \times 10^{-4}$	$-7.5232361 \times 10^{-5}$
8	2.5550954×10^{-2}	$-4.3544054 \times 10^{-3}$	$-4.2226218 \times 10^{-4}$	$-7.3567411 \times 10^{-5}$
10	2.5569445×10^{-2}	$-4.3479638 \times 10^{-3}$	$-4.2060234 \times 10^{-4}$	$-7.3110446 \times 10^{-5}$
12	2.5574842×10^{-2}	$-4.3460754 \times 10^{-3}$	$-4.2011702 \times 10^{-4}$	$-7.2977069 \times 10^{-5}$
14	2.5576464×10^{-2}	$-4.3455055 \times 10^{-3}$	$-4.1997075 \times 10^{-4}$	$-7.2936900 \times 10^{-5}$
16	2.5576960×10^{-2}	$-4.3453305 \times 10^{-3}$	$-4.1992585 \times 10^{-4}$	$-7.2924572 \times 10^{-5}$
18	2.5577114×10^{-2}	$-4.3452760 \times 10^{-3}$	$-4.1991189 \times 10^{-4}$	$-7.2920739 \times 10^{-5}$
20	2.5577163×10^{-2}	$-4.3452589 \times 10^{-3}$	$-4.1990751 \times 10^{-4}$	$-7.2919536 \times 10^{-5}$
22	2.5577178×10^{-2}	$-4.3452535 \times 10^{-3}$	$-4.1990612 \times 10^{-4}$	$-7.2919156 \times 10^{-5}$
24	2.5577183×10^{-2}	$-4.3452518 \times 10^{-3}$	$-4.1990568 \times 10^{-4}$	$-7.2919035 \times 10^{-5}$

Table 2. Cont.

(b) The values of c_i at different n				
n	c_8	c_{10}	c_{12}	c_{14}
8	$-1.5488504 \times 10^{-5}$	-	-	-
10	$-1.5357045 \times 10^{-5}$	$-3.5823498 \times 10^{-6}$	-	-
12	$-1.5318738 \times 10^{-5}$	$-3.5710160 \times 10^{-6}$	$-8.9013785 \times 10^{-7}$	-
14	$-1.5307207 \times 10^{-5}$	$-3.5676061 \times 10^{-6}$	$-8.8910802 \times 10^{-7}$	$-2.3216086 \times 10^{-7}$
16	$-1.5303669 \times 10^{-5}$	$-3.5665600 \times 10^{-6}$	$-8.8879213 \times 10^{-7}$	$-2.3206398 \times 10^{-7}$
18	$-1.5302569 \times 10^{-5}$	$-3.5662349 \times 10^{-6}$	$-8.8869395 \times 10^{-7}$	$-2.3203386 \times 10^{-7}$
20	$-1.5302224 \times 10^{-5}$	$-3.5661328 \times 10^{-6}$	$-8.8866314 \times 10^{-7}$	$-2.3202441 \times 10^{-7}$
22	$-1.5302115 \times 10^{-5}$	$-3.5661005 \times 10^{-6}$	$-8.8865339 \times 10^{-7}$	$-2.3202143 \times 10^{-7}$
24	$-1.5302080 \times 10^{-5}$	$-3.5660903 \times 10^{-6}$	$-8.8865029 \times 10^{-7}$	$-2.3202047 \times 10^{-8}$
(c) The values of c_i at different n				
n	c_{16}	c_{18}	c_{20}	c_{22}
16	$-6.2711514 \times 10^{-8}$	-	-	-
18	$-6.2702167 \times 10^{-8}$	$-1.7397329 \times 10^{-8}$	-	-
20	$-6.2699234 \times 10^{-8}$	$-1.7396410 \times 10^{-8}$	$-4.9288360 \times 10^{-9}$	-
22	$-6.2698306 \times 10^{-8}$	$-1.7396119 \times 10^{-8}$	$-4.9287442 \times 10^{-9}$	$-1.4203606 \times 10^{-9}$
24	$-6.2698011 \times 10^{-8}$	$-1.7396027 \times 10^{-8}$	$-4.9287150 \times 10^{-9}$	$-1.4203513 \times 10^{-9}$
(d) The value of c_{24} at $n = 24$				
n	c_{24}	-	-	-
24	$-4.1511468 \times 10^{-10}$	-	-	-

Table 3. (a) The values of d_i at different n ; (b) The values of d_i at different n ; (c) The values of d_i at different n ; (d) The value of d_{24} at $n = 24$.

(a) The values of d_i at different n				
n	d_0	d_2	d_4	d_6
4	1.4389897×10^{-1}	$-1.3373387 \times 10^{-1}$	$-1.0165102 \times 10^{-2}$	-
6	1.4380527×10^{-1}	$-1.3232658 \times 10^{-1}$	$-9.7095072 \times 10^{-3}$	$-1.7991832 \times 10^{-3}$
8	1.4373309×10^{-1}	$-1.3197584 \times 10^{-1}$	$-9.5986248 \times 10^{-3}$	$-1.7643172 \times 10^{-3}$
10	1.4368872×10^{-1}	$-1.3187818 \times 10^{-1}$	$-9.5679741 \times 10^{-3}$	$-1.7547332 \times 10^{-3}$
12	1.4367240×10^{-1}	$-1.3184954 \times 10^{-1}$	$-9.5590098 \times 10^{-3}$	$-1.7519351 \times 10^{-3}$
14	1.4366673×10^{-1}	$-1.3184090 \times 10^{-1}$	$-9.5563083 \times 10^{-3}$	$-1.7510923 \times 10^{-3}$
16	1.4366481×10^{-1}	$-1.3183824 \times 10^{-1}$	$-9.5554791 \times 10^{-3}$	$-1.7508337 \times 10^{-3}$
18	1.4366417×10^{-1}	$-1.3183741 \times 10^{-1}$	$-9.5552212 \times 10^{-3}$	$-1.7507533 \times 10^{-3}$
20	1.4366395×10^{-1}	$-1.3183716 \times 10^{-1}$	$-9.5551403 \times 10^{-3}$	$-1.7507281 \times 10^{-3}$
22	1.4366388×10^{-1}	$-1.3183707 \times 10^{-1}$	$-9.5551147 \times 10^{-3}$	$-1.7507201 \times 10^{-3}$
24	1.4366386×10^{-1}	$-1.3183705 \times 10^{-1}$	$-9.5551066 \times 10^{-3}$	$-1.7507175 \times 10^{-3}$
(b) The values of d_i at different n				
n	d_8	d_{10}	d_{12}	d_{14}
8	$-3.9431453 \times 10^{-4}$	-	-	-
10	$-3.9128051 \times 10^{-4}$	$-9.6550465 \times 10^{-5}$	-	-
12	$-3.9039607 \times 10^{-4}$	$-9.6267818 \times 10^{-5}$	$-2.5251202 \times 10^{-5}$	-
14	$-3.9012984 \times 10^{-4}$	$-9.6182775 \times 10^{-5}$	$-2.5223804 \times 10^{-5}$	$-6.8962634 \times 10^{-6}$
16	$-3.9004815 \times 10^{-4}$	$-9.6156686 \times 10^{-5}$	$-2.5215401 \times 10^{-5}$	$-6.8935385 \times 10^{-6}$
18	$-3.9002276 \times 10^{-4}$	$-9.6148576 \times 10^{-5}$	$-2.5212789 \times 10^{-5}$	$-6.8926917 \times 10^{-6}$
20	$-3.9001479 \times 10^{-4}$	$-9.6146031 \times 10^{-5}$	$-2.5211969 \times 10^{-5}$	$-6.8924259 \times 10^{-6}$
22	$-3.9001227 \times 10^{-4}$	$-9.6145227 \times 10^{-5}$	$-2.5211710 \times 10^{-5}$	$-6.8923419 \times 10^{-6}$
24	$-3.9001147 \times 10^{-4}$	$-9.6144970 \times 10^{-5}$	$-2.5211627 \times 10^{-5}$	$-6.8923151 \times 10^{-6}$

Table 3. Cont.

(c) The values of d_i at different n				
n	d_{16}	d_{18}	d_{20}	d_{22}
16	$-1.9420161 \times 10^{-6}$	-	-	-
18	$-1.9417401 \times 10^{-6}$	$-5.5954589 \times 10^{-7}$	-	-
20	$-1.9416535 \times 10^{-6}$	$-5.5951754 \times 10^{-7}$	$-1.6411662 \times 10^{-7}$	-
22	$-1.9416261 \times 10^{-6}$	$-5.5950858 \times 10^{-7}$	$-1.6411368 \times 10^{-7}$	$-4.8827855 \times 10^{-8}$
24	$-1.9416174 \times 10^{-6}$	$-5.5950573 \times 10^{-7}$	$-1.6411274 \times 10^{-7}$	$-4.8827547 \times 10^{-8}$
(d) The value of d_{24} at $n = 24$				
n	d_{24}	-	-	-
24	$-1.4698279 \times 10^{-8}$	-	-	-

Figure 7. The values of c_i when $n = 24$: (a) for $i = 0, 2, 4, 6, \dots, 24$; (b) for $i = 2, 4, 6, \dots, 24$. Figure 8. The values of d_i when $n = 24$: (a) for $i = 0, 2, 4, 6, \dots, 24$; (b) for $i = 2, 4, 6, \dots, 24$.

4. Concluding Remarks

In this paper, we analytically solved the problem of axisymmetric deformation and deflection of a circular membrane under liquid weight loading and presented the closed-form solution for stress and deflection. The following conclusions can be drawn from this study:

- The power series method is effective for the analytical solution to differential-integral equations.
- When the amount of liquid applied onto the circular membrane is large enough, the difference between the solution presented in this paper and the well-known Hencky solution will become relatively small. If the requirement for calculation accuracy is not too high, the problem of axisymmetric deformation of the circular membrane

under liquid self-weight loading may be treated as the well-known Hencky problem; the fluid–structure interaction may be neglected.

- iii. When the amount of liquid applied onto the circular membrane is relatively small, the solution presented in this paper will be quite different from the well-known Hencky solution. For a higher calculation accuracy, the fluid–structure interaction should be taken into account.
- iv. The numerical example conducted shows that the closed-form solution obtained in this paper has good convergence.

The work presented here could further be combined with the research and development of new rain gauges.

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Nomenclatures

a	Radius of the circular membrane
h	Thickness of the circular membrane
E	Young's modulus of elasticity
ν	Poisson's ratio
H	Height of the liquid poured into the cup
ρ	Density of the poured liquid
g	Acceleration of gravity
r	Radial coordinate of the cylindrical coordinate system
φ	Circumferential coordinate of the cylindrical coordinate system
w	Axial coordinate of the cylindrical coordinate system as well as transverse displacement of a point on the deflected circular membrane
u	Radial displacement of a point on the deflected circular membrane
w_m	Maximum deflection of the deflected circular membrane
$q(r)$	Transverse distributed loads over the circular membrane produced by the gravity of the liquid within radius r
$F(r)$	External force produced by $q(r)$ within radius r
σ_r	Radial stress
σ_t	Circumferential stress
e_r	Radial strain
e_t	Circumferential strain
θ	Slope angle of the deflected membrane
π	Pi (ratio of circumference to diameter)
W	Dimensionless transverse displacement (w/a)
S_r	Dimensionless radial stress (σ_r/E)
S_t	Dimensionless circumferential stress (σ_t/E)
H_0	Dimensionless height of liquid (H/a)
G	Dimensionless quantity ($\rho g a^2/Eh$)
x	Dimensionless radial coordinate (r/a)
c_i	Coefficients of the power series for S_r
d_i	Coefficients of the power series for W

Appendix A

$$\begin{aligned}
c_2 &= -\frac{G^2(H_0+d_0)^2}{64c_0^2}, \\
c_4 &= -\frac{G^3(H_0+d_0)^2}{6144c_0^5}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2), \\
c_6 &= -\frac{G^4(H_0+d_0)^2}{4718592c_0^8}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(13GH_0^2 + 26GH_0d_0 + 13Gd_0^2 - 40c_0^2), \\
c_8 &= -\frac{G^5(H_0+d_0)^2}{1509949440c_0^{11}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(85G^2H_0^4 + 340G^2H_0^3d_0 + 85G^2d_0^4 \\
&\quad + 510G^2H_0^2d_0^2 + 340G^2H_0d_0^3 - 510GH_0^2c_0^2 - 1024GH_0c_0^2d_0 - 512Gc_0^2d_0^2 + 448c_0^4), \\
c_{10} &= -\frac{G^6(H_0+d_0)^2}{724775731200c_0^{14}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(925G^3H_0^6 + 5550G^3H_0^5d_0 \\
&\quad + 13875G^3H_0^4d_0^2 + 18500G^3H_0^3d_0^3 + 13875G^3H_0^2d_0^4 + 5550G^3H_0d_0^5 + 925G^3d_0^6 \\
&\quad - 8252G^2H_0^4c_0^2 - 33008G^2H_0^3c_0^2d_0 - 49512G^2H_0^2c_0^2d_0^2 - 33008G^2H_0c_0^2d_0^3 \\
&\quad - 8252G^2c_0^2d_0^4 + 17344GH_0^4c_0^4 + 34688GH_0c_0^4d_0 + 17344Gc_0^4d_0^2 - 5376c_0^6), \\
c_{12} &= -\frac{G^7(H_0+d_0)^2}{974098582732800c_0^{17}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(30125G^4H_0^8 \\
&\quad + 241000G^4H_0^7d_0 + 843500G^4H_0^6d_0^2 + 1687000G^4H_0^5d_0^3 + 2108750G^4H_0^4d_0^4 \\
&\quad + 1687000G^4H_0^3d_0^5 + 843500G^4H_0^2d_0^6 + 241000G^4H_0d_0^7 + 30125G^4d_0^8 \\
&\quad - 355344G^3H_0^6c_0^2 - 2132064G^3H_0^5c_0^2d_0 - 5330160G^3H_0^4c_0^2d_0^2 \\
&\quad - 7106880G^3H_0^3c_0^2d_0^3 - 5330160G^3H_0^2c_0^2d_0^4 - 2132064G^3H_0c_0^2d_0^5 \\
&\quad - 355344G^3c_0^2d_0^6 + 1217664G^2H_0^4c_0^4 + 4870656G^2H_0^3c_0^4d_0 \\
&\quad + 7305984G^2H_0^2c_0^4d_0^2 + 4870656G^2H_0c_0^4d_0^3 + 1217664G^2c_0^4d_0^4 \\
&\quad - 1178624GH_0^2c_0^6 - 2357248GH_0c_0^6d_0 - 1178624Gc_0^6d_0^2 + 135168c_0^8), \\
c_{14} &= -\frac{G^8(H_0+d_0)^2}{6982338641028710400c_0^{20}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(5481025G^5H_0^{10} \\
&\quad + 54810250G^5H_0^9d_0 + 246646125G^5H_0^8d_0^2 + 657723000G^5H_0^7d_0^3 \\
&\quad + 1151015250G^5H_0^6d_0^4 + 1381218300G^5H_0^5d_0^5 + 1151015250G^5H_0^4d_0^6 \\
&\quad + 657723000G^5H_0^3d_0^7 + 246646125G^5H_0^2d_0^8 + 54810250G^5H_0d_0^9 + 5481025G^5d_0^{10} \\
&\quad - 80340520G^4H_0^8c_0^2 - 642724160G^4H_0^7c_0^2d_0 - 2249534560G^4H_0^6c_0^2d_0^2 \\
&\quad - 4499069120G^4H_0^5c_0^2d_0^3 - 5623836400G^4H_0^4c_0^2d_0^4 - 4499069120G^4H_0^3c_0^2d_0^5 \\
&\quad - 2249534560G^4H_0^2c_0^2d_0^6 - 642724160G^4H_0c_0^2d_0^7 - 80340520G^4c_0^2d_0^8 \\
&\quad + 384915840G^3H_0^6c_0^4 + 2309495040G^3H_0^5c_0^4d_0 + 5773737600G^3H_0^4c_0^4d_0^2 \\
&\quad + 7698316800G^3H_0^3c_0^4d_0^3 + 5773737600G^3H_0^2c_0^4d_0^4 + 2309495040G^3H_0c_0^4d_0^5 \\
&\quad + 384915840G^3c_0^4d_0^6 - 674397184G^2H_0^4c_0^6 - 2697588736G^2H_0^3c_0^6d_0 \\
&\quad - 4046383104G^2H_0^2c_0^6d_0^2 - 2697588736G^2H_0c_0^6d_0^3 - 674397184G^2c_0^6d_0^4 \\
&\quad + 334532608GH_0^2c_0^8 + 669065216GH_0c_0^8d_0 + 334532608Gc_0^8d_0^2 - 14057472c_0^{10}), \\
c_{16} &= -\frac{G^9(H_0+d_0)^2}{8043654114465074380800c_0^{23}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
&\quad (165851725G^6H_0^{12} + 1990220700G^6H_0^{11}d_0 + 10946213850G^6H_0^{10}d_0^2 \\
&\quad + 36487379500G^6H_0^9d_0^3 + 82096603875G^6H_0^8d_0^4 + 131354566200G^6H_0^7d_0^5 \\
&\quad + 153246993900G^6H_0^6d_0^6 + 131354566200G^6H_0^5d_0^7 + 82096603875G^6H_0^4d_0^8 \\
&\quad + 36487379500G^6H_0^3d_0^9 + 10946213850G^6H_0^2d_0^{10} + 1990220700G^6H_0d_0^{11} \\
&\quad + 165851725G^6d_0^{12} - 2904572016G^5H_0^{10}c_0^2 - 29045720160G^5H_0^9c_0^2d_0 \\
&\quad - 130705740720G^5H_0^8c_0^2d_0^2 - 348548641920G^5H_0^7c_0^2d_0^3 \\
&\quad - 609960123360G^5H_0^6c_0^2d_0^4 - 731952148032G^5H_0^5c_0^2d_0^5 \\
&\quad - 609960123360G^5H_0^4c_0^2d_0^6 - 348548641920G^5H_0^3c_0^2d_0^7 \\
&\quad - 130705740720G^5H_0^2c_0^2d_0^8 - 29045720160G^5H_0c_0^2d_0^9 \\
&\quad - 2904572016G^5c_0^2d_0^{10} + 17934897216G^4H_0^8c_0^4 \\
&\quad + 143479177728G^4H_0^7c_0^4d_0 + 502177122048G^4H_0^6c_0^4d_0^2 \\
&\quad + 1004354244096G^4H_0^5c_0^4d_0^3 + 1255442805120G^4H_0^4c_0^4d_0^4 \\
&\quad + 1004354244096G^4H_0^3c_0^4d_0^5 + 502177122048G^4H_0^2c_0^4d_0^6 \\
&\quad + 143479177728G^4H_0c_0^4d_0^7 + 17934897216G^4c_0^4d_0^8)
\end{aligned}$$

$$\begin{aligned}
& -46544832512G^3H_0^6c_0^6 - 279268995072G^3H_0^5c_0^6d_0 \\
& -698172487680G^3H_0^4c_0^6d_0^2 - 930896650240G^3H_0^3c_0^6d_0^3 \\
& -698172487680G^3H_0^2c_0^6d_0^4 - 279268995072G^3H_0c_0^6d_0^5 \\
& -46544832512G^3c_0^6d_0^6 + 46777135104G^2H_0^4c_0^8 \\
& + 187108540416G^2H_0^3c_0^8d_0 + 280662810624G^2H_0^2c_0^8d_0^2 \\
& + 187108540416G^2H_0c_0^8d_0^3 + 46777135104G^2c_0^8d_0^4 \\
& - 12516261888GH_0^2c_0^{10} - 25032523776GH_0c_0^{10}d_0 \\
& - 12516261888Gc_0^{10}d_0^2 + 187432960c_0^{12}) \\
c_{18} = & -\frac{G^{10}(H_0+d_0)^2}{11582861924829707108352000c_0^{26}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (6440470375G^7H_0^{14} + 90166585250G^7H_0^{13}d_0 + 586082804125G^7H_0^{12}d_0^2 \\
& + 2344331216500G^7H_0^{11}d_0^3 + 6446910845375G^7H_0^{10}d_0^4 \\
& + 12893821690750G^7H_0^9d_0^5 + 19340732536125G^7H_0^8d_0^6 \\
& + 22103694327000G^7H_0^7d_0^7 + 19340732536125G^7H_0^6d_0^8 \\
& + 12893821690750G^7H_0^5d_0^9 + 6446910845375G^7H_0^4d_0^{10} \\
& + 2344331216500G^7H_0^3d_0^{11} + 586082804125G^7H_0^2d_0^{12} \\
& + 90166585250G^7H_0d_0^{13} + 6440470375G^7d_0^{14} \\
& - 131152225780G^6H_0^{12}c_0^2 + 992711744640G^5c_0^4d_0^{10} \\
& - 1573826709360G^6H_0^{11}c_0^2d_0 - 8656046901480G^6H_0^{10}c_0^2d_0^2 \\
& - 28853489671600G^6H_0^9c_0^2d_0^3 - 64920351761100G^6H_0^8c_0^2d_0^4 \\
& - 103872562817760G^6H_0^7c_0^2d_0^5 - 121184656620720G^6H_0^6c_0^2d_0^6 \\
& - 103872562817760G^6H_0^5c_0^2d_0^7 - 64920351761100G^6H_0^4c_0^2d_0^8 \\
& - 28853489671600G^6H_0^3c_0^2d_0^9 - 8656046901480G^6H_0^2c_0^2d_0^{10} \\
& - 1573826709360G^6H_0c_0^2d_0^{11} - 131152225780G^6c_0^2d_0^{12} \\
& + 992711744640G^5H_0^{10}c_0^4 + 992711744640G^5H_0^9c_0^4d_0 \\
& + 44672028508800G^5H_0^8c_0^4d_0^2 + 119125409356800G^5H_0^7c_0^4d_0^3 \\
& + 208469466374400G^5H_0^6c_0^4d_0^4 + 250163359649280G^5H_0^5c_0^4d_0^5 \\
& + 208469466374400G^5H_0^4c_0^4d_0^6 + 119125409356800G^5H_0^3c_0^4d_0^7 \\
& + 44672028508800G^5H_0^2c_0^4d_0^8 + 9927117446400G^5H_0c_0^4d_0^9 \\
& - 3445151675648G^4H_0^8c_0^6 - 27561213405184G^4H_0^7c_0^6d_0 \\
& - 96464246918144G^4H_0^6c_0^6d_0^2 + 5444708495360G^3H_0^6c_0^8 \\
& - 192928493836288G^4H_0^5c_0^6d_0^3 - 241160617295360G^4H_0^4c_0^6d_0^4 \\
& - 192928493836288G^4H_0^3c_0^6d_0^5 - 96464246918144G^4H_0^2c_0^6d_0^6 \\
& - 27561213405184G^4H_0c_0^6d_0^7 - 3445151675648G^4c_0^6d_0^8 \\
& + 32668250972160G^3H_0^5c_0^8d_0 + 81670627430400G^3H_0^4c_0^8d_0^2 \\
& + 108894169907200G^3H_0^3c_0^8d_0^3 + 81670627430400G^3H_0^2c_0^8d_0^4 \\
& + 32668250972160G^3H_0c_0^8d_0^5 + 5444708495360G^3c_0^8d_0^6 \\
& - 3333820760064G^2H_0^4c_0^{10} + 493015269376GH_0^2c_0^{12} \\
& - 13335283040256G^2H_0^3c_0^{10}d_0 - 20002924560384G^2H_0^2c_0^{10}d_0^2 \\
& - 13335283040256G^2H_0c_0^{10}d_0^3 - 3333820760064G^2c_0^{10}d_0^4 \\
& + 986030538752GH_0c_0^{12}d_0 + 493015269376Gc_0^{12}d_0^2 - 2549088256c_0^{14}) \\
c_{20} = & -\frac{G^{11}(H_0+d_0)^2}{40771673975400569021399040000c_0^{29}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (624247690625G^8H_0^{16} + 9987963050000G^8H_0^{15}d_0 + 74909722875000G^8H_0^{14}d_0^2 \\
& + 349578706750000G^8H_0^{13}d_0^3 + 1136130796937500G^8H_0^{12}d_0^4 \\
& + 2726713912650000G^8H_0^{11}d_0^5 + 8034067778343750G^8H_0^8d_0^8 \\
& + 4998975506525000G^8H_0^{10}d_0^6 + 7141393580750000G^8H_0^9d_0^7 \\
& + 7141393580750000G^8H_0^7d_0^9 + 4998975506525000G^8H_0^6d_0^{10} \\
& + 2726713912650000G^8H_0^5d_0^{11} + 74909722875000G^8H_0^2d_0^{14} \\
& + 1136130796937500G^8H_0^4d_0^{12} + 349578706750000G^8H_0^3d_0^{13} \\
& + 9987963050000G^8H_0d_0^{15} + 624247690625G^8d_0^{16} - 14489792053000G^7H_0^{14}c_0^2)
\end{aligned}$$

$$\begin{aligned}
& -202857088742000G^7H_0^{13}c_0^2d_0 - 1318571076823000G^7H_0^{12}c_0^2d_0^2 \\
& -5274284307292000G^7H_0^{11}c_0^2d_0^3 - 14504281845053000G^7H_0^{10}c_0^2d_0^4 \\
& -29008563690106000G^7H_0^9c_0^2d_0^5 - 43512845535159000G^7H_0^8c_0^2d_0^6 \\
& -49728966325896000G^7H_0^7c_0^2d_0^7 - 43512845535159000G^7H_0^6c_0^2d_0^8 \\
& -29008563690106000G^7H_0^5c_0^2d_0^9 - 14504281845053000G^7H_0^4c_0^2d_0^{10} \\
& -5274284307292000G^7H_0^3c_0^2d_0^{11} - 1318571076823000G^7H_0^2c_0^2d_0^{12} \\
& -202857088742000G^7H_0c_0^2d_0^{13} - 14489792053000G^7c_0^2d_0^{14} \\
& +129976428743936G^6H_0^{12}c_0^4 + 1559717144927232G^6H_0^{11}c_0^4d_0 \\
& +8578444297099776G^6H_0^{10}c_0^4d_0^2 + 28594814323665920G^6H_0^9c_0^4d_0^3 \\
& +64338332228248320G^6H_0^8c_0^4d_0^4 + 102941331565197312G^6H_0^7c_0^4d_0^5 \\
& +120098220159396864G^6H_0^6c_0^4d_0^6 + 102941331565197312G^6H_0^5c_0^4d_0^7 \\
& +64338332228248320G^6H_0^4c_0^4d_0^8 + 28594814323665920G^6H_0^3c_0^4d_0^9 \\
& +8578444297099776G^6H_0^2c_0^4d_0^{10} + 1559717144927232G^6H_0c_0^4d_0^{11} \\
& +129976428743936G^6c_0^4d_0^{12} - 567342708341248G^5H_0^{10}c_0^6 \\
& -5673427083412480G^5H_0^9c_0^6d_0 - 25530421875356160G^5H_0^8c_0^6d_0^2 \\
& -68081125000949760G^5H_0^7c_0^6d_0^3 - 119141968751662080G^5H_0^6c_0^6d_0^4 \\
& -142970362501994496G^5H_0^5c_0^6d_0^5 - 119141968751662080G^5H_0^4c_0^6d_0^6 \\
& -68081125000949760G^5H_0^3c_0^6d_0^7 - 25530421875356160G^5H_0^2c_0^6d_0^8 \\
& -5673427083412480G^5H_0c_0^6d_0^9 - 567342708341248G^5c_0^6d_0^{10} \\
& +1244939899486208G^4H_0^8c_0^8 + 9959519195889664G^4H_0^7c_0^8d_0 \\
& +34858317185613824G^4H_0^6c_0^8d_0^2 + 69716634371227648G^4H_0^5c_0^8d_0^3 \\
& +87145792964034560G^4H_0^4c_0^8d_0^4 + 69716634371227648G^4H_0^3c_0^8d_0^5 \\
& +34858317185613824G^4H_0^2c_0^8d_0^6 + 9959519195889664G^4H_0c_0^8d_0^7 \\
& +1244939899486208G^4c_0^8d_0^8 - 1276001808646144G^3H_0^6c_0^{10} \\
& -7656010851876864G^3H_0^5c_0^{10}d_0 - 19140027129692160G^3H_0^4c_0^{10}d_0^2 \\
& -25520036172922880G^3H_0^3c_0^{10}d_0^3 - 19140027129692160G^3H_0^2c_0^{10}d_0^4 \\
& -7656010851876864G^3H_0c_0^{10}d_0^5 - 1276001808646144G^3c_0^{10}d_0^6 \\
& +493067700273152G^2H_0^4c_0^{12} + 1972270801092608G^2H_0^3c_0^{12}d_0 \\
& +2958406201638912G^2H_0^2c_0^{12}d_0^2 + 1972270801092608G^2H_0c_0^{12}d_0^3 \\
& +493067700273152G^2c_0^{12}d_0^4 - 40655185248256GH_0^2c_0^{14} \\
& -81310370496512GH_0c_0^{14}d_0 - 40655185248256Gc_0^{14}d_0^2 + 70447529984c_0^{16} \\
c_{22} = & -\frac{G^{12}(H_0+d_0)^2}{172219550872092003546389544960000c_0^{32}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (73847994008125G^9H_0^{18} + 1329263892146250G^9H_0^{17}d_0 \\
& +11298743083243125G^9H_0^{16}d_0^2 - 1924297443407700G^8H_0^{16}c_0^2 \\
& +60259963110630000G^9H_0^{15}d_0^3 + 225974861664862500G^9H_0^{14}d_0^4 \\
& +632729612661615000G^9H_0^{13}d_0^5 + 1370914160766832500G^9H_0^{12}d_0^6 \\
& +2350138561314570000G^9H_0^{11}d_0^7 + 3231440521807533750G^9H_0^{10}d_0^8 \\
& +3590489468675037500G^9H_0^9d_0^9 + 3231440521807533750G^9H_0^8d_0^{10} \\
& +2350138561314570000G^9H_0^7d_0^{11} + 1370914160766832500G^9H_0^6d_0^{12} \\
& +632729612661615000G^9H_0^5d_0^{13} + 225974861664862500G^9H_0^4d_0^{14} \\
& +60259963110630000G^9H_0^3d_0^{15} + 11298743083243125G^9H_0^2d_0^{16} \\
& +1329263892146250G^9H_0d_0^{17} + 73847994008125G^9d_0^{18} \\
& -30788759094523200G^8H_0^{15}c_0^2d_0 - 230915693208924000G^8H_0^{14}c_0^2d_0^2 \\
& -1077606568308312000G^8H_0^{13}c_0^2d_0^3 - 3502221347002014000G^8H_0^{12}c_0^2d_0^4 \\
& -8405331232804833600G^8H_0^{11}c_0^2d_0^5 - 15409773926808861600G^8H_0^{10}c_0^2d_0^6 \\
& -22013962752584088000G^8H_0^9c_0^2d_0^7 - 24765708096657099000G^8H_0^8c_0^2d_0^8 \\
& -22013962752584088000G^8H_0^7c_0^2d_0^9 - 15409773926808861600G^8H_0^6c_0^2d_0^{10} \\
& -8405331232804833600G^8H_0^5c_0^2d_0^{11} - 3502221347002014000G^8H_0^4c_0^2d_0^{12} \\
& -1077606568308312000G^8H_0^3c_0^2d_0^{13} - 230915693208924000G^8H_0^2c_0^2d_0^{14} \\
& -30788759094523200G^8H_0c_0^2d_0^{15} - 1924297443407700G^8c_0^2d_0^{16}
\end{aligned}$$

$$\begin{aligned}
& +19965729788090944G^7H_0^{14}c_0^4 + 279520217033273216G^7H_0^{13}c_0^4d_0 \\
& +1816881410716275904G^7H_0^{12}c_0^4d_0^2 + 7267525642865103616G^7H_0^{11}c_0^4d_0^3 \\
& +19985695517879034944G^7H_0^{10}c_0^4d_0^4 + 39971391035758069888G^7H_0^9c_0^4d_0^5 \\
& +59957086553637104832G^7H_0^8c_0^4d_0^6 + 68522384632728119808G^7H_0^7c_0^4d_0^7 \\
& +59957086553637104832G^7H_0^6c_0^4d_0^8 + 39971391035758069888G^7H_0^5c_0^4d_0^9 \\
& +19985695517879034944G^7H_0^4c_0^4d_0^{10} + 7267525642865103616G^7H_0^3c_0^4d_0^{11} \\
& +1816881410716275904G^7H_0^2c_0^4d_0^{12} + 279520217033273216G^7H_0c_0^4d_0^{13} \\
& +19965729788090944G^7c_0^4d_0^{14} - 105258286088052736G^6H_0^{12}c_0^6 \\
& -1263099433056632832G^6H_0^{11}c_0^6d_0 - 6947046881811480576G^6H_0^{10}c_0^6d_0^2 \\
& -23156822939371601920G^6H_0^9c_0^6d_0^3 - 52102851613586104320G^6H_0^8c_0^6d_0^4 \\
& -83364562581737766912G^6H_0^7c_0^6d_0^5 - 97258656345360728064G^6H_0^6c_0^6d_0^6 \\
& -83364562581737766912G^6H_0^5c_0^6d_0^7 - 52102851613586104320G^6H_0^4c_0^6d_0^8 \\
& -23156822939371601920G^6H_0^3c_0^6d_0^9 - 6947046881811480576G^6H_0^2c_0^6d_0^{10} \\
& -1263099433056632832G^6H_0c_0^6d_0^{11} - 105258286088052736G^6c_0^6d_0^{12} \\
& +298141919987527680G^5H_0^{10}c_0^8 + 2981419199875276800G^5H_0^9c_0^8d_0 \\
& +13416386399438745600G^5H_0^8c_0^8d_0^2 + 35777030398503321600G^5H_0^7c_0^8d_0^3 \\
& +62609803197380812800G^5H_0^6c_0^8d_0^4 + 75131763836856975360G^5H_0^5c_0^8d_0^5 \\
& +62609803197380812800G^5H_0^4c_0^8d_0^6 + 35777030398503321600G^5H_0^3c_0^8d_0^7 \\
& +13416386399438745600G^5H_0^2c_0^8d_0^8 + 2981419199875276800G^5H_0c_0^8d_0^9 \\
& +298141919987527680G^5c_0^8d_0^{10} - 440975944806858752G^4H_0^8c_0^{10} \\
& -3527807558454870016G^4H_0^7c_0^{10}d_0 - 12347326454592045056G^4H_0^6c_0^{10}d_0^2 \\
& -24694652909184090112G^4H_0^5c_0^{10}d_0^3 - 30868316136480112640G^4H_0^4c_0^{10}d_0^4 \\
& -24694652909184090112G^4H_0^3c_0^{10}d_0^5 - 12347326454592045056G^4H_0^2c_0^{10}d_0^6 \\
& -3527807558454870016G^4H_0c_0^{10}d_0^7 - 440975944806858752G^4c_0^{10}d_0^8 \\
& +304616960657457152G^3H_0^6c_0^{12} + 1827701763944742912G^3H_0^5c_0^{12}d_0 \\
& +4569254409861857280G^3H_0^4c_0^{12}d_0^2 + 6092339213149143040G^3H_0^3c_0^{12}d_0^3 \\
& +4569254409861857280G^3H_0^2c_0^{12}d_0^4 + 1827701763944742912G^3H_0c_0^{12}d_0^5 \\
& +304616960657457152G^3c_0^{12}d_0^6 - 75883562624090112G^2H_0^4c_0^{14} \\
& -303534250496360448G^2H_0^3c_0^{14}d_0 - 455301375744540672G^2H_0^2c_0^{14}d_0^2 \\
& -303534250496360448G^2H_0c_0^{14}d_0^3 - 75883562624090112G^2c_0^{14}d_0^4 \\
& +3486326154330112GH_0^2c_0^{16} + 6972652308660224GH_0c_0^{16}d_0 \\
& +3486326154330112Gc_0^{16}d_0^2 - 1972530839552c_0^{18})
\end{aligned}$$

$$\begin{aligned}
c_{24} = & -\frac{G^{13}(H_0+d_0)^2}{859719997953483281703576608440320000c_0^{35}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (10470550396840625G^{10}H_0^{20} + 209411007936812500G^{10}H_0^{19}d_0 \\
& +1989404575399718750G^{10}H_0^{18}d_0^2 + 11936427452398312500G^{10}H_0^{17}d_0^3 \\
& +50729816672692828125G^{10}H_0^{16}d_0^4 + 162335413352617050000G^{10}H_0^{15}d_0^5 \\
& +405838533381542625000G^{10}H_0^{14}d_0^6 + 811677066763085250000G^{10}H_0^{13}d_0^7 \\
& +1318975233490013531250G^{10}H_0^{12}d_0^8 + 1758633644653351375000G^{10}H_0^{11}d_0^9 \\
& +1934497009118686512500G^{10}H_0^{10}d_0^{10} + 1758633644653351375000G^{10}H_0^9d_0^{11} \\
& +1318975233490013531250G^{10}H_0^8d_0^{12} + 811677066763085250000G^{10}H_0^7d_0^{13} \\
& +405838533381542625000G^{10}H_0^6d_0^{14} + 162335413352617050000G^{10}H_0^5d_0^{15} \\
& +50729816672692828125G^{10}H_0^4d_0^{16} + 11936427452398312500G^{10}H_0^3d_0^{17} \\
& +1989404575399718750G^{10}H_0^2d_0^{18} + 209411007936812500G^{10}H_0d_0^{19} \\
& +10470550396840625G^{10}d_0^{20} - 302621805505979000G^9H_0^{18}c_0^2 \\
& -5447192499107622000G^9H_0^{17}c_0^2d_0 - 46301136242414787000G^9H_0^{16}c_0^2d_0^2 \\
& -246939393292878864000G^9H_0^{15}c_0^2d_0^3 - 926022724848295740000G^9H_0^{14}c_0^2d_0^4 \\
& -2592863629575228072000G^9H_0^{13}c_0^2d_0^5 - 5617871197412994156000G^9H_0^{12}c_0^2d_0^6 \\
& -9630636338422275696000G^9H_0^{11}c_0^2d_0^7 - 13242124965330629082000G^9H_0^{10}c_0^2d_0^8 \\
& -14713472183700698980000G^9H_0^9c_0^2d_0^9 - 13242124965330629082000G^9H_0^8c_0^2d_0^{10} \\
& -9630636338422275696000G^9H_0^7c_0^2d_0^{11} - 5617871197412994156000G^9H_0^6c_0^2d_0^{12}
\end{aligned}$$

$$\begin{aligned}
& -2592863629575228072000G^9H_0^5c_0^2d_0^{13} - 926022724848295740000G^9H_0^4c_0^2d_0^{14} \\
& -246939393292878864000G^9H_0^3c_0^2d_0^{15} - 46301136242414787000G^9H_0^2c_0^2d_0^{16} \\
& -5447192499107622000G^9H_0c_0^2d_0^{17} - 302621805505979000G^9c_0^2d_0^{18} \\
& +3566092425128384960G^8H_0^{16}c_0^4 + 57057478802054159360G^8H_0^{15}c_0^4d_0 \\
& +427931091015406195200G^8H_0^{14}c_0^4d_0^2 + 1997011758071895577600G^8H_0^{13}c_0^4d_0^3 \\
& +6490288213733660627200G^8H_0^{12}c_0^4d_0^4 + 15576691712960785505280G^8H_0^{11}c_0^4d_0^5 \\
& +28557268140428106759680G^8H_0^{10}c_0^4d_0^6 + 40796097343468723942400G^8H_0^9c_0^4d_0^7 \\
& +45895609511402314435200G^8H_0^8c_0^4d_0^8 + 40796097343468723942400G^8H_0^7c_0^4d_0^9 \\
& +28557268140428106759680G^8H_0^6c_0^4d_0^{10} + 15576691712960785505280G^8H_0^5c_0^4d_0^{11} \\
& +6490288213733660627200G^8H_0^4c_0^4d_0^{12} + 1997011758071895577600G^8H_0^3c_0^4d_0^{13} \\
& +427931091015406195200G^8H_0^2c_0^4d_0^{14} + 57057478802054159360G^8H_0c_0^4d_0^{15} \\
& +3566092425128384960G^8c_0^4d_0^{16} - 22064961114736215040G^7H_0^{14}c_0^6 \\
& -308909455606307010560G^7H_0^{13}c_0^6d_0 - 2007911461440995568640G^7H_0^{12}c_0^6d_0^2 \\
& -8031645845763982274560G^7H_0^{11}c_0^6d_0^3 - 22087026075850951255040G^7H_0^{10}c_0^6d_0^4 \\
& -44174052151701902510080G^7H_0^9c_0^6d_0^5 - 66261078227552853765120G^7H_0^8c_0^6d_0^6 \\
& -75726946545774690017280G^7H_0^7c_0^6d_0^7 - 66261078227552853765120G^7H_0^6c_0^6d_0^8 \\
& -44174052151701902510080G^7H_0^5c_0^6d_0^9 - 22087026075850951255040G^7H_0^4c_0^6d_0^{10} \\
& -8031645845763982274560G^7H_0^3c_0^6d_0^{11} - 2007911461440995568640G^7H_0^2c_0^6d_0^{12} \\
& -308909455606307010560G^7H_0c_0^6d_0^{13} - 22064961114736215040G^7c_0^6d_0^{14} \\
& +76935695715733790720G^6H_0^{12}c_0^8 + 923228348588805488640G^6H_0^{11}c_0^8d_0 \\
& +5077755917238430187520G^6H_0^{10}c_0^8d_0^2 + 16925853057461433958400G^6H_0^9c_0^8d_0^3 \\
& +38083169379288226406400G^6H_0^8c_0^8d_0^4 + 60933071006861162250240G^6H_0^7c_0^8d_0^5 \\
& +71088582841338022625280G^6H_0^6c_0^8d_0^6 + 60933071006861162250240G^6H_0^5c_0^8d_0^7 \\
& +38083169379288226406400G^6H_0^4c_0^8d_0^8 + 16925853057461433958400G^6H_0^3c_0^8d_0^9 \\
& +5077755917238430187520G^6H_0^2c_0^8d_0^{10} + 923228348588805488640G^6H_0c_0^8d_0^{11} \\
& +76935695715733790720G^6c_0^8d_0^{12} - 150811020027895349248G^5H_0^{10}c_0^{10} \\
& -1508110200278953492480G^5H_0^9c_0^{10}d_0 - 6786495901255290716160G^5H_0^8c_0^{10}d_0^2 \\
& -18097322403347441909760G^5H_0^7c_0^{10}d_0^3 - 31670314205858023342080G^5H_0^6c_0^{10}d_0^4 \\
& -38004377047029628010496G^5H_0^5c_0^{10}d_0^5 - 31670314205858023342080G^5H_0^4c_0^{10}d_0^6 \\
& -18097322403347441909760G^5H_0^3c_0^{10}d_0^7 - 6786495901255290716160G^5H_0^2c_0^{10}d_0^8 \\
& -1508110200278953492480G^5H_0c_0^{10}d_0^9 - 150811020027895349248G^5c_0^{10}d_0^{10} \\
& +156446903803372371968G^4H_0^8c_0^{12} + 1251575230426978975744G^4H_0^7c_0^{12}d_0 \\
& +4380513306494426415104G^4H_0^6c_0^{12}d_0^2 + 8761026612988852830208G^4H_0^5c_0^{12}d_0^3 \\
& +10951283266236066037760G^4H_0^4c_0^{12}d_0^4 + 8761026612988852830208G^4H_0^3c_0^{12}d_0^5 \\
& +4380513306494426415104G^4H_0^2c_0^{12}d_0^6 + 1251575230426978975744G^4H_0c_0^{12}d_0^7 \\
& +156446903803372371968G^4c_0^{12}d_0^8 - 74687799413960605696G^3H_0^6c_0^{14} \\
& -448126796483763634176G^3H_0^5c_0^{14}d_0 - 1120316991209409085440G^3H_0^4c_0^{14}d_0^2 \\
& -1493755988279212113920G^3H_0^3c_0^{14}d_0^3 - 1120316991209409085440G^3H_0^2c_0^{14}d_0^4 \\
& -448126796483763634176G^3H_0c_0^{14}d_0^5 - 74687799413960605696G^3c_0^{14}d_0^6 \\
& +12147792181792342016G^2H_0^4c_0^{16} + 48591168727169368064G^2H_0^3c_0^{16}d_0 \\
& +72886753090754052096G^2H_0^2c_0^{16}d_0^2 + 48591168727169368064G^2H_0c_0^{16}d_0^3 \\
& +12147792181792342016G^2c_0^{16}d_0^4 - 309052965610586112GH_0^2c_0^{18} \\
& -618105931221172224GH_0c_0^{18}d_0 - 309052965610586112Gc_0^{18}d_0^2 \\
& +55837796073472c_0^{20})
\end{aligned}$$

Appendix B

$$\begin{aligned}
d_2 &= -\frac{G(H_0+d_0)}{4c_0}, \\
d_4 &= -\frac{G^2(H_0+d_0)}{512c_0^4}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2), \\
d_6 &= -\frac{G^3(H_0+d_0)}{147456c_0^7}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(5GH_0^2 + 10GH_0d_0 + 5Gd_0^2 - 8c_0^2), \\
d_8 &= -\frac{G^4(H_0+d_0)}{75497472c_0^{10}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(55G^2H_0^4 + 220G^2H_0^3d_0 + 55G^2d_0^4 \\
&\quad + 330G^2H_0^2d_0^2 + 220G^2H_0d_0^3 - 224GH_0^2c_0^2 - 448GH_0c_0^2d_0 - 224Gc_0^2d_0^2 + 64c_0^4), \\
d_{10} &= -\frac{G^5(H_0+d_0)}{30198988800c_0^{13}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(525G^3H_0^6 + 3150G^3H_0^5d_0 \\
&\quad + 7875G^3H_0^4d_0^2 + 10500G^3H_0^3d_0^3 + 7875G^3H_0^2d_0^4 + 3150G^3H_0d_0^5 - 14128G^2H_0c_0^2d_0^3 \\
&\quad + 525G^3d_0^6 - 3532G^2H_0^4c_0^2 - 14128G^2H_0^3c_0^2d_0 - 21192G^2H_0^2c_0^2d_0^2 - 3532G^2c_0^2d_0^4 \\
&\quad + 4544GH_0^2c_0^4 + 9088GH_0c_0^4d_0 + 4544Gc_0^4d_0^2 - 256c_0^6), \\
d_{12} &= -\frac{G^6(H_0+d_0)}{34789235097600c_0^{16}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(15375G^4H_0^8 \\
&\quad + 123000G^4H_0^7d_0 + 430500G^4H_0^6d_0^2 + 861000G^4H_0^5d_0^3 + 1076250G^4H_0^4d_0^4 \\
&\quad + 861000G^4H_0^3d_0^5 + 430500G^4H_0^2d_0^6 + 123000G^4H_0d_0^7 + 15375G^4d_0^8 \\
&\quad - 145364G^3H_0^6c_0^2 - 872184G^3H_0^5c_0^2d_0 - 872184G^3H_0c_0^2d_0^5 \\
&\quad - 2180460G^3H_0^4c_0^2d_0^2 - 2907280G^3H_0^3c_0^2d_0^3 - 2180460G^3H_0^2c_0^2d_0^4 \\
&\quad - 145364G^3c_0^2d_0^6 + 360864G^2H_0^4c_0^4 + 1443456G^2H_0^3c_0^4d_0 \\
&\quad + 2165184G^2H_0^2c_0^4d_0^2 + 1443456G^2H_0c_0^4d_0^3 + 360864G^2c_0^4d_0^4 \\
&\quad - 194304GH_0^2c_0^6 - 388608GH_0c_0^6d_0 - 194304Gc_0^6d_0^2 + 2048c_0^8), \\
d_{14} &= -\frac{G^7(H_0+d_0)}{109099041266073600c_0^{19}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2)(1278825G^5H_0^{10} \\
&\quad + 12788250G^5H_0^9d_0 + 57547125G^5H_0^8d_0^2 + 153459000G^5H_0^7d_0^3 \\
&\quad + 268553250G^5H_0^6d_0^4 + 322263900G^5H_0^5d_0^5 + 268553250G^5H_0^4d_0^6 \\
&\quad + 153459000G^5H_0^3d_0^7 + 57547125G^5H_0^2d_0^8 + 12788250G^5H_0d_0^9 \\
&\quad + 1278825G^5d_0^{10} - 15624872G^4H_0^8c_0^2 - 124998976G^4H_0^7c_0^2d_0 \\
&\quad - 437496416G^4H_0^6c_0^2d_0^2 - 874992832G^4H_0^5c_0^2d_0^3 \\
&\quad - 1093741040G^4H_0^4c_0^2d_0^4 - 874992832G^4H_0^3c_0^2d_0^5 \\
&\quad - 437496416G^4H_0^2c_0^2d_0^6 - 124998976G^4H_0c_0^2d_0^7 \\
&\quad - 15624872G^4c_0^2d_0^8 + 58775168G^3H_0^6c_0^4 + 352651008G^3H_0^5c_0^4d_0 \\
&\quad + 881627520G^3H_0^4c_0^4d_0^2 + 1175503360G^3H_0^3c_0^4d_0^3 \\
&\quad + 881627520G^3H_0^2c_0^4d_0^4 + 352651008G^3H_0c_0^4d_0^5 \\
&\quad + 58775168G^3c_0^4d_0^6 - 71685120G^2H_0^4c_0^6 \\
&\quad - 286740480G^2H_0^3c_0^6d_0 - 430110720G^2H_0^2c_0^6d_0^2 \\
&\quad - 286740480G^2H_0c_0^6d_0^3 - 71685120G^2c_0^6d_0^4 + 17813504GH_0^2c_0^8 \\
&\quad + 35627008GH_0c_0^8d_0 + 17813504Gc_0^8d_0^2 - 32768c_0^{10}), \\
d_{16} &= -\frac{G^8(H_0+d_0)}{223434836512918732800c_0^{22}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
&\quad (71612125G^6H_0^{12} + 859345500G^6H_0^{11}d_0 + 4726400250G^6H_0^{10}d_0^2 \\
&\quad + 15754667500G^6H_0^9d_0^3 + 35448001875G^6H_0^8d_0^4 + 56716803000G^6H_0^7d_0^5 \\
&\quad + 66169603500G^6H_0^6d_0^6 + 56716803000G^6H_0^5d_0^7 + 35448001875G^6H_0^4d_0^8 \\
&\quad + 15754667500G^6H_0^3d_0^9 + 4726400250G^6H_0^2d_0^{10} + 859345500G^6H_0d_0^{11} \\
&\quad + 71612125G^6d_0^{12} - 1074406560G^5H_0^{10}c_0^2 - 10744065600G^5H_0^9c_0^2d_0 \\
&\quad - 48348295200G^5H_0^8c_0^2d_0^2 - 128928787200G^5H_0^7c_0^2d_0^3 \\
&\quad - 225625377600G^5H_0^6c_0^2d_0^4 - 270750453120G^5H_0^5c_0^2d_0^5 \\
&\quad - 225625377600G^5H_0^4c_0^2d_0^6 - 128928787200G^5H_0^3c_0^2d_0^7 \\
&\quad - 48348295200G^5H_0^2c_0^2d_0^8 + 305919828480G^4H_0^5c_0^4d_0^3 \\
&\quad - 10744065600G^4H_0c_0^2d_0^9 - 1074406560G^4c_0^2d_0^{10} + 5462854080G^4H_0^8c_0^4 \\
&\quad + 43702832640G^4H_0^7c_0^4d_0 + 152959914240G^4H_0^6c_0^4d_0^2 \\
&\quad + 382399785600G^4H_0^4c_0^4d_0^4 + 305919828480G^4H_0^3c_0^4d_0^5 \\
&\quad + 152959914240G^4H_0^2c_0^4d_0^6 - 217917767680G^3H_0^3c_0^6d_0^3 \\
&\quad + 43702832640G^4H_0c_0^4d_0^7 + 5462854080G^4c_0^4d_0^8 - 10895888384G^3H_0^6c_0^6)
\end{aligned}$$

$$\begin{aligned}
& -65375330304G^3H_0^5c_0^6d_0 - 163438325760G^3H_0^4c_0^6d_0^2 \\
& -163438325760G^3H_0^2c_0^6d_0^4 - 65375330304G^3H_0c_0^6d_0^5 \\
& -10895888384G^3c_0^6d_0^6 + 43822227456G^2H_0^2c_0^8d_0^2 \\
& +7303704576G^2H_0^4c_0^8 + 29214818304G^2H_0^3c_0^8d_0 \\
& +29214818304G^2H_0c_0^8d_0^3 + 7303704576G^2c_0^8d_0^4 - 866254848GH_0^2c_0^{10} \\
& -1732509696GH_0c_0^{10}d_0 - 866254848Gc_0^{10}d_0^2 + 262144c_0^{12}) \\
d_{18} = & -\frac{G^9(H_0+d_0)}{289571548120742677708800c_0^{25}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (2596581625G^7H_0^{14} + 36352142750G^7H_0^{13}d_0 + 236288927875G^7H_0^{12}d_0^2 \\
& +945155711500G^7H_0^{11}d_0^3 + 2599178206625G^7H_0^{10}d_0^4 \\
& +5198356413250G^7H_0^9d_0^5 + 7797534619875G^7H_0^8d_0^6 \\
& +7797534619875G^7H_0^7d_0^7 + 8911468137000G^7H_0^6d_0^8 \\
& +5198356413250G^7H_0^5d_0^9 + 2599178206625G^7H_0^4d_0^{10} \\
& +236288927875G^7H_0^3d_0^{11} + 36352142750G^7H_0^2d_0^{12} \\
& -46223906500G^6H_0^{12}c_0^2 - 554686878000G^6H_0^{11}c_0^2d_0 \\
& -3050777829000G^6H_0^{10}c_0^2d_0^2 + 2596581625G^7d_0^{14} + 945155711500G^7H_0^3d_0^{11} \\
& -10169259430000G^6H_0^9c_0^2d_0^3 - 22880833717500G^6H_0^8c_0^2d_0^4 \\
& -36609333948000G^6H_0^7c_0^2d_0^5 - 42710889606000G^6H_0^6c_0^2d_0^6 \\
& -36609333948000G^6H_0^5c_0^2d_0^7 - 22880833717500G^6H_0^4c_0^2d_0^8 \\
& -10169259430000G^6H_0^3c_0^2d_0^9 - 3050777829000G^6H_0^2c_0^2d_0^{10} \\
& -554686878000G^6H_0c_0^2d_0^{11} - 46223906500G^6c_0^2d_0^{12} \\
& +297379154880G^5H_0^{10}c_0^4 + 297379154880G^5H_0^9c_0^4d_0 \\
& +13382061969600G^5H_0^8c_0^4d_0^2 + 35685498585600G^5H_0^7c_0^4d_0^3 \\
& +62449622524800G^5H_0^6c_0^4d_0^4 + 74939547029760G^5H_0^5c_0^4d_0^5 \\
& +62449622524800G^5H_0^4c_0^4d_0^6 + 35685498585600G^5H_0^3c_0^4d_0^7 \\
& +13382061969600G^5H_0^2c_0^4d_0^8 + 2973791548800G^5H_0c_0^4d_0^9 \\
& +297379154880G^5c_0^4d_0^{10} - 838832062208G^4H_0^8c_0^6 \\
& -6710656497664G^4H_0^7c_0^6d_0 - 23487297741824G^4H_0^6c_0^6d_0^2 \\
& -46974595483648G^4H_0^5c_0^6d_0^3 - 58718244354560G^4H_0^4c_0^6d_0^4 \\
& -46974595483648G^4H_0^3c_0^6d_0^5 - 23487297741824G^4H_0^2c_0^6d_0^6 \\
& -6710656497664G^4H_0c_0^6d_0^7 - 838832062208G^4c_0^6d_0^8 \\
& +995841536000G^3H_0^6c_0^8 + 5975049216000G^3H_0^5c_0^8d_0 \\
& +14937623040000G^3H_0^4c_0^8d_0^2 + 19916830720000G^3H_0^3c_0^8d_0^3 \\
& +14937623040000G^3H_0^2c_0^8d_0^4 + 5975049216000G^3H_0c_0^8d_0^5 \\
& +995841536000G^3c_0^8d_0^6 - 388074553344G^2H_0^4c_0^{10} \\
& -1552298213376G^2H_0^3c_0^{10}d_0 - 2328447320064G^2H_0^2c_0^{10}d_0^2 \\
& -1552298213376G^2H_0c_0^{10}d_0^3 - 388074553344G^2c_0^{10}d_0^4 \\
& +22064398336GH_0^2c_0^{12} + 44128796672GH_0c_0^{12}d_0 \\
& +22064398336Gc_0^{12}d_0^2 - 1048576c_0^{14}) \\
d_{20} = & -\frac{G^{10}(H_0+d_0)}{463314476993188284334080000c_0^{28}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (118334925625G^8H_0^{16} + 1893358810000G^8H_0^{15}d_0 \\
& +14200191075000G^8H_0^{14}d_0^2 + 66267558350000G^8H_0^{13}d_0^3 \\
& +215369564637500G^8H_0^{12}d_0^4 + 516886955130000G^8H_0^{11}d_0^5 \\
& +947626084405000G^8H_0^{10}d_0^6 + 1353751549150000G^8H_0^9d_0^7 \\
& +1522970492793750G^8H_0^8d_0^8 + 1353751549150000G^8H_0^7d_0^9 \\
& +947626084405000G^8H_0^6d_0^{10} + 516886955130000G^8H_0^5d_0^{11} \\
& +215369564637500G^8H_0^4d_0^{12} + 1893358810000G^8H_0^3d_0^{13} \\
& +66267558350000G^8H_0^2d_0^{14} + 14200191075000G^8H_0d_0^{15} \\
& +118334925625G^8d_0^{16} - 2438867435750G^7H_0^{14}c_0^2 \\
& -34144144100500G^7H_0^{13}c_0^2d_0 - 221936936653250G^7H_0^{12}c_0^2d_0^2
\end{aligned}$$

$$\begin{aligned}
& -887747746613000G^7H_0^{11}c_0^2d_0^3 - 2438867435750G^7c_0^2d_0^{14} \\
& -2441306303185750G^7H_0^{10}c_0^2d_0^4 - 4882612606371500G^7H_0^9c_0^2d_0^5 \\
& -7323918909557250G^7H_0^8c_0^2d_0^6 - 8370193039494000G^7H_0^7c_0^2d_0^7 \\
& -7323918909557250G^7H_0^6c_0^2d_0^8 - 4882612606371500G^7H_0^5c_0^2d_0^9 \\
& -2441306303185750G^7H_0^4c_0^2d_0^{10} - 887747746613000G^7H_0^3c_0^2d_0^{11} \\
& -221936936653250G^7H_0^2c_0^2d_0^{12} - 34144144100500G^7H_0c_0^2d_0^{13} \\
& +19021198932112G^6H_0^{12}c_0^4 + 228254387185344G^6H_0^{11}c_0^4d_0 \\
& +1255399129519392G^6H_0^{10}c_0^4d_0^2 - 69963325870208G^5H_0^{10}c_0^6 \\
& +4184663765064640G^6H_0^9c_0^4d_0^3 + 9415493471395440G^6H_0^8c_0^4d_0^4 \\
& +15064789554232704G^6H_0^7c_0^4d_0^5 + 17575587813271488G^6H_0^6c_0^4d_0^6 \\
& +15064789554232704G^6H_0^5c_0^4d_0^7 + 9415493471395440G^6H_0^4c_0^4d_0^8 \\
& +4184663765064640G^6H_0^3c_0^4d_0^9 + 1255399129519392G^6H_0^2c_0^4d_0^{10} \\
& +228254387185344G^6H_0c_0^4d_0^{11} + 19021198932112G^6c_0^4d_0^{12} \\
& -699633258702080G^5H_0^9c_0^6d_0 - 3148349664159360G^5H_0^8c_0^6d_0^2 \\
& -8395599104424960G^5H_0^7c_0^6d_0^3 - 14692298432743680G^5H_0^6c_0^6d_0^4 \\
& -17630758119292416G^5H_0^5c_0^6d_0^5 - 14692298432743680G^5H_0^4c_0^6d_0^6 \\
& -8395599104424960G^5H_0^3c_0^6d_0^7 - 3148349664159360G^5H_0^2c_0^6d_0^8 \\
& -699633258702080G^5H_0c_0^6d_0^9 - 69963325870208G^5c_0^6d_0^{10} \\
& +123048169851904G^4H_0^8c_0^8 - 92437077204992G^3H_0^6c_0^{10} \\
& +984385358815232G^4H_0^7c_0^8d_0 + 3445348755853312G^4H_0^6c_0^8d_0^2 \\
& +6890697511706624G^4H_0^5c_0^8d_0^3 + 8613371889633280G^4H_0^4c_0^8d_0^4 \\
& +6890697511706624G^4H_0^3c_0^8d_0^5 + 3445348755853312G^4H_0^2c_0^8d_0^6 \\
& +984385358815232G^4H_0c_0^8d_0^7 + 123048169851904G^4c_0^8d_0^8 \\
& -554622463229952G^3H_0^5c_0^{10}d_0 - 1386556158074880G^3H_0^4c_0^{10}d_0^2 \\
& -1848741544099840G^3H_0^3c_0^{10}d_0^3 - 1386556158074880G^3H_0^2c_0^{10}d_0^4 \\
& -554622463229952G^3H_0c_0^{10}d_0^5 - 92437077204992G^3c_0^{10}d_0^6 \\
& +21592071405568G^2H_0^4c_0^{12} - 582969982976G^2H_0^2c_0^{14} \\
& +86368285622272G^2H_0^3c_0^{12}d_0 + 129552428433408G^2H_0^2c_0^{12}d_0^2 \\
& +86368285622272G^2H_0c_0^{12}d_0^3 + 21592071405568G^2c_0^{12}d_0^4 \\
& -1165939965952GH_0c_0^{14}d_0 - 582969982976Gc_0^{14}d_0^2 + 4194304c_0^{16}) \\
d_{22} = & -\frac{G^{11}(H_0+d_0)}{3587907309835250073883115520000c_0^{31}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (26479758670625G^9H_0^{18} + 476635656071250G^9H_0^{17}d_0 \\
& +4051403076605625G^9H_0^{16}d_0^2 - 620277594363700G^8H_0^{16}c_0^2 \\
& +21607483075230000G^9H_0^{15}d_0^3 + 81028061532112500G^9H_0^{14}d_0^4 \\
& +226878572289915000G^9H_0^{13}d_0^5 + 491570239961482500G^9H_0^{12}d_0^6 \\
& +842691839933970000G^9H_0^{11}d_0^7 + 1158701279909208750G^9H_0^{10}d_0^8 \\
& +1287445866565787500G^9H_0^9d_0^9 + 1158701279909208750G^9H_0^8d_0^{10} \\
& +842691839933970000G^9H_0^7d_0^{11} + 491570239961482500G^9H_0^6d_0^{12} \\
& +226878572289915000G^9H_0^5d_0^{13} + 81028061532112500G^9H_0^4d_0^{14} \\
& +21607483075230000G^9H_0^3d_0^{15} + 4051403076605625G^9H_0^2d_0^{16} \\
& +476635656071250G^9H_0d_0^{17} + 26479758670625G^9d_0^{18} \\
& -9924441509819200G^8H_0^{15}c_0^2d_0 - 74433311323644000G^8H_0^{14}c_0^2d_0^2 \\
& -347355452843672000G^8H_0^{13}c_0^2d_0^3 - 1128905221741934000G^8H_0^{12}c_0^2d_0^4 \\
& -2709372532180641600G^8H_0^{11}c_0^2d_0^5 - 4967182975664509600G^8H_0^{10}c_0^2d_0^6 \\
& -7095975679520728000G^8H_0^9c_0^2d_0^7 - 7982972639460819000G^8H_0^8c_0^2d_0^8 \\
& -7095975679520728000G^8H_0^7c_0^2d_0^9 - 4967182975664509600G^8H_0^6c_0^2d_0^{10} \\
& -2709372532180641600G^8H_0^5c_0^2d_0^{11} - 1128905221741934000G^8H_0^4c_0^2d_0^{12} \\
& -347355452843672000G^8H_0^3c_0^2d_0^{13} - 74433311323644000G^8H_0^2c_0^2d_0^{14} \\
& -9924441509819200G^8H_0c_0^2d_0^{15} - 620277594363700G^8c_0^2d_0^{16} \\
& +5691966058468800G^7H_0^{14}c_0^4 + 79687524818563200G^7H_0^{13}c_0^4d_0 \\
& +517968911320660800G^7H_0^{12}c_0^4d_0^2 + 2071875645282643200G^7H_0^{11}c_0^4d_0^3
\end{aligned}$$

$$\begin{aligned}
& +5697658024527268800G^7H_0^{10}c_0^4d_0^4 + 11395316049054537600G^7H_0^9c_0^4d_0^5 \\
& +17092974073581806400G^7H_0^8c_0^4d_0^6 + 19534827512664921600G^7H_0^7c_0^4d_0^7 \\
& +17092974073581806400G^7H_0^6c_0^4d_0^8 + 11395316049054537600G^7H_0^5c_0^4d_0^9 \\
& +5697658024527268800G^7H_0^4c_0^4d_0^{10} + 2071875645282643200G^7H_0^3c_0^4d_0^{11} \\
& +517968911320660800G^7H_0^2c_0^4d_0^{12} + 79687524818563200G^7H_0c_0^4d_0^{13} \\
& +5691966058468800G^7c_0^4d_0^{14} - 25932933890948096G^6H_0^{12}c_0^6 \\
& -311195206691377152G^6H_0^{11}c_0^6d_0 - 1711573636802574336G^6H_0^{10}c_0^6d_0^2 \\
& -5705245456008581120G^6H_0^9c_0^6d_0^3 - 12836802276019307520G^6H_0^8c_0^6d_0^4 \\
& -20538883641630892032G^6H_0^7c_0^6d_0^5 - 23962030915236040704G^6H_0^6c_0^6d_0^6 \\
& -20538883641630892032G^6H_0^5c_0^6d_0^7 - 12836802276019307520G^6H_0^4c_0^6d_0^8 \\
& -5705245456008581120G^6H_0^3c_0^6d_0^9 - 1711573636802574336G^6H_0^2c_0^6d_0^{10} \\
& -311195206691377152G^6H_0c_0^6d_0^{11} - 25932933890948096G^6c_0^6d_0^{12} \\
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& +2759504445362810880G^5H_0^8c_0^8d_0^2 + 7358678520967495680G^5H_0^7c_0^8d_0^3 \\
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& +12877687411693117440G^5H_0^4c_0^8d_0^6 + 7358678520967495680G^5H_0^3c_0^8d_0^7 \\
& +2759504445362810880G^5H_0^2c_0^8d_0^8 + 613223210080624640G^5H_0c_0^8d_0^9 \\
& +61322321008062464G^5c_0^8d_0^{10} - 71614989018365952G^4H_0^8c_0^{10} \\
& -572919912146927616G^4H_0^7c_0^{10}d_0 - 2005219692514246656G^4H_0^6c_0^{10}d_0^2 \\
& -4010439385028493312G^4H_0^5c_0^{10}d_0^3 - 5013049231285616640G^4H_0^4c_0^{10}d_0^4 \\
& -4010439385028493312G^4H_0^3c_0^{10}d_0^5 - 2005219692514246656G^4H_0^2c_0^{10}d_0^6 \\
& -572919912146927616G^4H_0c_0^{10}d_0^7 - 71614989018365952G^4c_0^{10}d_0^8 \\
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& +529681428558643200G^3H_0^2c_0^{12}d_0^4 + 211872571423457280G^3H_0c_0^{12}d_0^5 \\
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& -20103378075910144G^2H_0c_0^{14}d_0^3 - 5025844518977536G^2c_0^{14}d_0^4 \\
& +63454920572928G^2H_0^2c_0^{16} + 126909841145856GH_0c_0^{16}d_0 \\
& +63454920572928G^2c_0^{16}d_0^2 - 67108864c_0^{18}) \\
d_{24} = & -\frac{G^{12}(H_0+d_0)}{4133269220930208085113349079040000c_0^{34}}(GH_0^2 + 2GH_0d_0 + Gd_0^2 - 8c_0^2) \\
& (891919562511250G^{10}H_0^{20} + 17838391250225000G^{10}H_0^{19}d_0 \\
& +169464716877137500G^{10}H_0^{18}d_0^2 + 1016788301262825000G^{10}H_0^{17}d_0^3 \\
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& +34570802242936050000G^{10}H_0^{14}d_0^6 + 69141604485872100000G^{10}H_0^{13}d_0^7 \\
& +112355107289542162500G^{10}H_0^{12}d_0^8 + 149806809719389550000G^{10}H_0^{11}d_0^9 \\
& +164787490691328505000G^{10}H_0^{10}d_0^{10} + 149806809719389550000G^{10}H_0^9d_0^{11} \\
& +112355107289542162500G^{10}H_0^8d_0^{12} + 69141604485872100000G^{10}H_0^7d_0^{13} \\
& +34570802242936050000G^{10}H_0^6d_0^{14} + 13828320897174420000G^{10}H_0^5d_0^{15} \\
& +4321350280367006250G^{10}H_0^4d_0^{16} + 1016788301262825000G^{10}H_0^3d_0^{17} \\
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& -421340131086057450G^9H_0^{17}c_0^2d_0 - 3581391114231488325G^9H_0^{16}c_0^2d_0^2 \\
& -19100752609234604400G^9H_0^{15}c_0^2d_0^3 - 71627822284629766500G^9H_0^{14}c_0^2d_0^4 \\
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& -744929351760149571600G^9H_0^{11}c_0^2d_0^7 - 1024277858670205660950G^9H_0^{10}c_0^2d_0^8 \\
& -1138086509633561845500G^9H_0^9c_0^2d_0^9 - 1024277858670205660950G^9H_0^8c_0^2d_0^{10} \\
& -744929351760149571600G^9H_0^7c_0^2d_0^{11} - 434542121860087250100G^9H_0^6c_0^2d_0^{12} \\
& -200557902396963346200G^9H_0^5c_0^2d_0^{13} - 71627822284629766500G^9H_0^4c_0^2d_0^{14} \\
& -19100752609234604400G^9H_0^3c_0^2d_0^{15} - 3581391114231488325G^9H_0^2c_0^2d_0^{16} \\
& -421340131086057450G^9H_0c_0^2d_0^{17} - 23407785060336525G^9c_0^2d_0^{18}
\end{aligned}$$

$$\begin{aligned}
& +247234276374827672G^8H_0^{16}c_0^4 + 3955748421997242752G^8H_0^{15}c_0^4d_0 \\
& +29668113164979320640G^8H_0^{14}c_0^4d_0^2 + 138451194769903496320G^8H_0^{13}c_0^4d_0^3 \\
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& +1979852085209619997376G^8H_0^6c_0^4d_0^{10} + 1079919319205247271296G^8H_0^5c_0^4d_0^{11} \\
& +449966383002186363040G^8H_0^4c_0^4d_0^{12} + 138451194769903496320G^8H_0^3c_0^4d_0^{13} \\
& +29668113164979320640G^8H_0^2c_0^4d_0^{14} + 3955748421997242752G^8H_0c_0^4d_0^{15} \\
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& -490292489097847141888G^7H_0^{11}c_0^6d_0^3 - 1348304345019079640192G^7H_0^{10}c_0^6d_0^4 \\
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& +152409464354373632G^2c_0^{16}d_0^4 - 884560174776320GH_0^2c_0^{18} \\
& -1769120349552640GH_0c_0^{18}d_0 - 884560174776320Gc_0^{18}d_0^2 + 134217728c_0^{20})
\end{aligned}$$

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