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# Modified Tseng's Method with Inertial Viscosity Type for Solving Inclusion Problems and Its Application to Image Restoration Problems

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**Abstract:** In this paper, we study a monotone inclusion problem in the framework of Hilbert spaces. (1) We introduce a new modified Tseng's method that combines inertial and viscosity techniques. Our aim is to obtain an algorithm with better performance that can be applied to a broader class of mappings. (2) We prove a strong convergence theorem to approximate a solution to the monotone inclusion problem under some mild conditions. (3) We present a modified version of the proposed iterative scheme for solving convex minimization problems. (4) We present numerical examples that satisfy the image restoration problem and illustrate our proposed algorithm's computational performance.

**Keywords:** inertial algorithm; Tseng's method; forward-backward algorithm; monotone inclusion problem



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## 1. Introduction

Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. A zero-point problem for monotone operators is defined as follows: find $x^* \in H$ such that

$$0 \in Tx^*, \tag{1}$$

where $T$ is a monotone operator. Barely a decade ago, many authors intensively studied the convergence of iterative methods to find a zero-point for monotone operators in the framework of Hilbert spaces. Additionally, many iterative methods have been constructed and studied to solve a zero-point problem (1), since it is connected to various optimization and nonlinear analysis issues, such as variational inequality problems, convex minimization problems, and so on. The proximal point algorithm (PPA) [1], which was constructed by Martinet in 1970, is well known as being the first algorithm to solve the problem (1). This algorithm is shown below:

$$x_{n+1} = (I + \lambda_n T)^{-1} x_n, \ \ \forall n \geq 1, \tag{2}$$

where $I$ is the identity mapping, and $\{\lambda_n\}$ is a sequence of positive real numbers. After Martinet [1] proposed the proximal point algorithm (PPA), many algorithms were developed by many authors to solve the zero-point problem. The reader can see [2–4] and the references therein for more details.

In this paper, we focus on the following monotone inclusion problem:

$$\text{find } x^* \in H \text{ such that } 0 \in (A + B)x^*, \tag{3}$$

where $A : H \to H$ and $B : H \to 2^H$ are single and multi-valued mappings, respectively. The monotone inclusion problem (3) can be written as the zero-point problem (1) by setting $T = A + B$. However, the full resolvent operator $(I + \lambda T)^{-1}$ is much harder to compute than the resolvent operators $(I + \lambda A)^{-1}$ and $(I + \lambda B)^{-1}$.

Because the monotone inclusion problem (3) is the core of image processing and many mathematical problems [5–22], many researchers have proposed and developed iterative methods for solving inclusion problems (3). The forward-backward splitting method, constructed and studied by Lions and Mercier [23] in 1979, is the most popular algorithm for solving the (3) problem. It is defined by the following iterative:

$$\begin{cases} x_{n+1} = (I + \lambda B)^{-1}(I - \lambda A)x_n, \ \ \forall n \geq 1, \end{cases} \tag{4}$$

where $x_1 \in H$ is arbitrarily chosen and $\lambda > 0$. In the algorithm (4), operators A and B are usually called the forward operator and the backward operator, respectively. For more details about forward-backward methods that have been constructed and considered to solve the inclusion problem (3), the reader is directed to [2,9,11,24–32].

To speed up the convergence rate of iteration methods, Polyak [33] introduced inertial extrapolation as an acceleration process in 1964. This method is well known as the heavy ball method. Polyak [33] used his algorithm to solve the smooth convex minimization problem. In recent years, many researchers have intensively used this useful concept for combining their algorithms with an inertial term to accelerate the speed of convergence.

In 2001, Alvarez and Attouch [34] constructed an algorithm to solve a problem of monotone operators. It combines the heavy ball method with the proximal point algorithm. The algorithm is defined as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + \lambda_n B)^{-1}w_n, \ \ \forall n \geq 1, \end{cases} \tag{5}$$

where $x_0, x_1 \in H$ are arbitrarily chosen, $\{\theta_n\} \subset [0, 1)$, and $\{\lambda_n\}$ is nondecreasing with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty. \tag{6}$$

They proved that the sequence $\{x_n\}$ generated by the algorithm (5) converges weakly to a zero-point of the monotone operator $B$.

Moudafi and Oliny [35] studied the monotone inclusion problem (3). They constructed the inertial proximal point algorithm, which combines the heavy ball method with the proximal point algorithm. The inertial proximal point algorithm is defined as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + \lambda_n B)^{-1}(x_n - \lambda_n A w_n), \ \ \forall n \geq 1, \end{cases} \tag{7}$$

where $x_0, x_1 \in H$ are arbitrarily chosen, and $A : H \to H$ and $B : H \to 2^H$ are single and multi-valued mappings, respectively. It was proven that if $\lambda_n < 2/L$ with the Lipschitz constant $L$ of the monotone operator $A$ and the condition (6) holds, then the sequence $\{x_n\}$ generated by the algorithm (7) converges weakly to a solution of the inclusion problem (3). Moreover, it has been observed that for $\theta_n > 0$, the proximal point algorithm (7) cannot be written as the forward-backward splitting method (4), since there is no estimation of operator $A$ at point $x_n$.

Lorenz and Pock [36] studied the monotone inclusion problem (3). They proposed the inertial forward-backward algorithm for monotone operators, which is defined as

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n, \ \ \forall n \geq 1, \end{cases} \tag{8}$$

where $x_0, x_1 \in H$ are arbitrarily chosen, $A : H \to H$ and $B : H \to 2^H$ are single and multi-valued mappings, respectively. They proved that the sequence $\{x_n\}$ generated by the algorithm (8) converges weakly to a solution of the monotone inclusion problem (3) with some conditions.

Kitkuan and Kumam [26] combined the forward-backward splitting method (4) with the viscosity approximation method [37] for solving the monotone inclusion problem (3). It is called the inertial viscosity forward-backward splitting algorithm, which is defined as:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n \nabla h(x_n) + (1-\alpha_n)(I+\lambda B)^{-1}(I-\lambda_n A)w_n, \ \ \forall n \geq 1, \end{cases} \tag{9}$$

where $x_0, x_1 \in H$ are arbitrarily chosen, $h : H \to \mathbb{R}$ is a differentiable function such that its gradient $\nabla h$ is a contraction with the constant $k \in (0,1)$, and $A : H \to H$ and $B : H \to 2^H$ are an inverse strongly monotone and a maximal monotone operator, respectively. They proved that the sequence $\{x_n\}$ generated by the algorithm (9) converges strongly to a solution of the monotone inclusion problem (3) under suitable conditions.

Besides solving the monotone inclusion problem using an algorithm combined with the heavy ball idea, there are many ways to solve the monotone inclusion problem. Tseng [24] introduced a powerful iterative method to solve the monotone inclusion problem (3), which is called the modified forward-backward splitting method. In short, it is known as Tseng's splitting algorithm. Let $C$ be a closed and convex subset of a real Hilbert space $H$. Tseng's splitting algorithm is defined as

$$\begin{cases} y_n = (I+\lambda_n B)^{-1}(I-\lambda_n A)x_n \\ x_{n+1} = P_C(y_n - \lambda_n(Ay_n - Ax_n)), \ \ \forall n \geq 1, \end{cases} \tag{10}$$

where $x_1 \in H$ is arbitrarily chosen, $\lambda_n$ is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, ...\}$ satisfying $\lambda\|Ay_n - Ax_n\| \leq \mu\|x_n - y_n\|$ where $\delta > 0, l \in (0,1)$, $\mu \in (0,1)$, and $P_C$ is the projection onto a closed convex subset $C$ of $H$. However, Tseng's splitting algorithm only obtains weak convergence in real Hilbert spaces.

Recently, Dilshad, Aljohani, and Akram [38] introduced and studied an iterative scheme to approximate the common solution to a split variational inclusion and a fixed-point problem of a finite collection of nonexpansive mappings. Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator $A^* : H_2 \to H_1$. Their algorithm is

$$\begin{cases} v_n = (I+\lambda G_1)^{-1}(I+\mu A^*((I+\lambda G_2)^{-1} - I)A)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{[n+1]}v_n, \ \ \forall n \geq 1, \end{cases} \tag{11}$$

where $x_1 \in H$ is arbitrarily chosen, $\lambda > 0$, $\{\alpha_n\} \in (0,1)$, $f$ is a contraction with a constant $k \in (0,1)$, and $T_{[n+1]}$ is a finite collection of nonexpansive mappings. For more detail about the split variational inclusion in other class of mappings and methods for solving them, the reader is directed to [39–42].

Based on the above idea, we introduce a new modified Tseng's method, which combines inertial and viscosity techniques to solve inclusion problems in the framework of real Hilbert spaces. The project aims to obtain algorithms with better performance and can be applied for a broader class of mappings. Furthermore, we present a modified version of the proposed iterative scheme to solve convex minimization problems. Moreover, we illustrate the computational performance of our proposed algorithms by conducting experiments that satisfy the image restoration problem.

The outline of this paper is as follows: Definitions and lemmas used to analyze our algorithm are shown in Section 2. In Section 3, we present the convergence analysis of our main theorem. Finally, we conduct experiments in which we use our proposed algorithm to solve the image restoration problem.

## 2. Preliminaries

In this section, we present some notations that are used throughout this work. Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$. $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong convergence and weak convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ to $x \in H$. For any point $x$, there exists a unique nearest point for $C$, which is denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The operator $P_C$ denotes the metric projection from $H$ onto $C$. It is well known that the metric projection $P_C$ is nonlinear, and it satisfies the following:

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0, \tag{12}$$

for all $y \in C$. Next, we present several properties of operators and set-valued mappings, which are helpful later. Let $T : H \to H$ be a mapping, $Fix(T)$ denotes the set of fixed points of $T$, i.e.,

$$Fix(T) = \{x \in H \mid Tx = x\}.$$

**Proposition 1.** *Let $H$ be a real Hilbert space and $T : H \to H$ be a mapping.*

1. *$T$ is called **nonexpansive mapping** if*

$$\|Tx - Ty\| \leq \|x - y\|,$$

*for all $x, y \in H$.*

2. *$T$ is called **firmly nonexpansive mapping** if*

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2,$$

*for all $x, y \in H$.*

**Proposition 2** ([43])**.** *Let $T : H \to H$ be a mapping. Then, the following items are equivalent:*

*(i)* *$T$ is firmly nonexpansive;*
*(ii)* *$(I - T)$ is firmly nonexpansive;*
*(iii)* *$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$, for all $x, y \in H$.*

It is well known that the metric projection $P_C$ is a firmly nonexpansive mapping, i.e.,

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle, \tag{13}$$

for all $x, y \in H$.

For convenience, we let $B : H \to 2^H$ be a set-valued mapping and

$$J_\lambda^B = (I + \lambda B)^{-1},$$

be the resolvent of mapping $B$ where $\lambda > 0$. It is well known that $J_\lambda^B$ is single-valued, $D(J_\lambda^B) = H$, where $D(J_\lambda^B)$ is a domain of the operator $J_\lambda^B$, and $J_\lambda^B$ is a firmly nonexpansive mapping for all $\lambda > 0$.

**Definition 1.** *Let $B : H \to 2^H$ be a set-valued mapping with the graph $G(B)$. $B$ is called **monotone** if,*

$$\langle x - y, u - v \rangle \geq 0,$$

*for all $x, y \in H, u \in Bx$ and $v \in By$. A monotone mapping $A : H \to 2^H$ is maximal if the graph of $G(B)$ for $B$ is not properly contained in the graph of any other monotone mapping.*

**Definition 2** ([44])**.** *Let $T : H \to H$ be a mapping.*

1. *T is called L-Lipchitz continuous if a non-negative real number $L \geq 0$ exists such that*
$$\|Tx - Ty\| \leq L\|x - y\|,$$
*for all $x, y \in H$.*
2. *T is called α-inverse strongly monotone if a positive real number α exists such that*
$$\langle x - y, Tx - Ty\rangle \geq \alpha\|Tx - Ty\|^2,$$
*for all $x, y \in H$. Moreover, if T is α-inverse strongly monotone, then T is $1/\alpha$-Lipschitz continuous.*

**Lemma 1** ([43])**.** *Let H be a real Hilbert space. Then, the following equations hold:*

*(i)* $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all $x, y \in H$;*
*(ii)* $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, *for all $x, y \in H$;*
*(iii)* $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$, *for all $x, y \in H$ and $\alpha \in [0, 1]$.*

**Lemma 2** ([45])**.** *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \to 2^H$ be a maximal monotone operator. Then, for $\lambda > 0$, the following relation holds:*
$$Fix(J_\lambda^B(I - \lambda A)) = (A + B)^{-1}(0).$$

**Lemma 3** ([46])**.** *Let $a_n$ be a sequence of non-negative real numbers that satisfy the following relation:*
$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 0,$$
*where*

*(i)* $\{a_n\} \subset [0, 1], \sum \alpha_n = \infty$,
*(ii)* $\limsup \sigma_n \leq 0$,
*(iii)* $\gamma_n \geq 0\ (n \geq 1), \sum \gamma_n < \infty$.

*Then, $a_n \to 0$ as $n \to \infty$.*

**Lemma 4** ([47])**.** *Let $\{\Gamma_n\}$ be sequence of real numbers that does not decrease at infinity in the sense that the following subsequence exists: $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_i}\} < \{\Gamma_{n_i+1}\}$ for all $i \geq 0$. Additionally, consider the sequence of integers $\{\eta(n)\}_{n \geq n_0}$ defined by*
$$\eta(n) = \max\{k \leq n \mid \Gamma_k \leq \Gamma_{k+1}\}.$$

*Then $\{\eta(n)\}_{n \geq n_0}$ is a nondecreasing sequence that verifies $\lim_{n\to\infty} \eta(n) = \infty$ and for all $n \geq n_0$,*
$$\max\left\{\Gamma_{\eta(n)}, \Gamma_n\right\} \leq \Gamma_{\eta(n)+1}.$$

## 3. Results

In this section, we present a convergence analysis of the proposed algorithm, which generates sequences that converge strongly to a solution to the monotone inclusion problem (3). Throughout this section, $\Omega = (A + B)^{-1}(0)$ is used to denote the set of all the solutions to the monotone inclusion problem (3). We use the following conditions for the analysis of our method.

**Assumption 1.** *A1* *Ω is nonempty.*

*A2* *A is L-Lipschitz continuous and monotone, and B is maximal monotone.*
*A3* *$\nabla h : H \to H$ is σ-Lipschitz continuous, where $\sigma \in [0, 1)$.*
*A4* *Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

**Remark 1.** *Observe that from Assumption 1 (A4) and Algorithm 1, we have from $\{\theta_n\} \in [0,1)$ that*

*(1)* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$

*(2)* $\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0.$

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**Algorithm 1** An iterative algorithm for solving inclusion problems

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**Initialization:** Given $\lambda_1 > 0$, $\mu \in (0,1)$. Let $x_1, x_2 \in H$ be arbitrary. Choose $\{\alpha_n\}$, $\nabla h$ to satisfy Assumption 1 and $\{\theta_n\}$ to satisfy Remark 1.

**Iterative Step:** Given the current iterate $x_n$, calculate the next iterate as follows:

**Step 1.** Compute

$$\begin{array}{rcl} w_n &=& x_n + \theta_n(x_n - x_{n-1}) \\ y_n &=& J^B_{\lambda_n}(I - \lambda_n A)w_n \\ s_n &=& y_n - \lambda_n(Ay_n - Aw_n) \end{array}$$

and

$$x_{n+1} = \alpha_n \nabla h(x_n) + (1-\alpha_n)s_n.$$

**Step 2.** Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\} & \text{if } Aw_n - Ay_n \neq 0; \\ \lambda_n & \text{otherwise.} \end{cases} \tag{14}$$

Replace $n$ with $n+1$ and then repeat **Step 1.**

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Next, we provide a useful lemma for analyzing our main theorem.

**Lemma 5** ([11])**.** *The sequence $\{\lambda_n\}$ generated by (14) is a non-increasing sequence and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_0, \frac{\mu}{L}\right\}.$$

**Lemma 6** ([11])**.** *Assume that Assumption (1) holds and let $\{s_n\}$ be any sequence generated by Algorithm 1. Then,*

$$\|s_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2, \tag{15}$$

*for all $p \in \Omega$ and*

$$\|s_n - y_n\| \leq \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|. \tag{16}$$

**Theorem 1.** *Assume that Assumptions (1) A1–A4 hold. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then, $x_n \to p$, where $p = P_\Omega \nabla h(p)$.*

**Proof. Step 1.** We prove that $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{s_n\}$ are bounded sequences. Assume that $p = P_\Omega \nabla h(p)$. Since [11]

$$\|s_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2, \tag{17}$$

and

$$\|s_n - y_n\| \leq \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\|. \tag{18}$$

Moreover, we observe that

$$\begin{aligned} \|y_n - p\| &\leq \|J^B_{\lambda_n}(I - \lambda_n A)w_n - p\| \\ &\leq \|w_n - p\|, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \|w_n - p\| &= \|x_n - \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{20}$$

Consider

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \nabla h(x_n) + (1-\alpha_n)s_n - p\| \\ &\leq \alpha_n\|\nabla h(x_n) - p\| + (1-\alpha_n)\|s_n - p\| \\ &\leq \alpha_n\|\nabla h(x_n) - \nabla h(p)\| + \alpha_n\|\nabla h(p) - p\| + (1-\alpha_n)\|s_n - p\| \\ &\leq \alpha_n\sigma\|x_n - p\| + \alpha_n\|\nabla h(p) - p\| + (1-\alpha_n)\|w_n - p\| \\ &\leq (1 - \alpha_n(1-\sigma))\|x_n - p\| + \alpha_n(1-\sigma)\frac{\|\nabla h(p) - p\|}{(1-\sigma)} \\ &\quad +(1-\alpha_n(1-\sigma))\theta_n\|x_n - x_{n-1}\| \\ &\leq \max\left\{\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|, (1-\sigma)\frac{\|\nabla h(p) - p\|}{(1-\sigma)}\right\} \\ &\vdots \\ &\leq \max\left\{\|x_n - p\| + \theta_1\|x_1 - x_0\|, (1-\sigma)\frac{\|\nabla h(p) - p\|}{(1-\sigma)}\right\}. \end{aligned} \tag{21}$$

Thus, $\{x_n\}$ is bounded and $\{s_n\}$, $\{y_n\}$, and $\{w_n\}$ are bounded.
Next, we observe that

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n+1}) - p\|^2 \\ &= \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, x_n - p\rangle + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{22}$$

It follows that

$$\|(x_n - x_{n-1}) - (x_n - p)\|^2 = \|x_n - x_{n-1}\|^2 - 2\langle x_n - x_{n-1}, x_n - p\rangle + \|x_n - p\|^2,$$

and

$$2\theta_n\langle x_n - x_{n-1}, x_n - p\rangle = \theta_n\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2). \tag{23}$$

Then, by combining (22) with (23), we find that

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\ &\quad +\theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\ &\leq \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2). \end{aligned}$$

Next, by combining (17) with the above inequality, we find that

$$\begin{aligned}\|s_n - p\|^2 &\le \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\ &\quad - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2. \end{aligned} \tag{24}$$

Consider

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n\nabla h(x_n) + (1-\alpha_n)s_n - p\|^2 \\ &= \langle \alpha_n\nabla h(x_n) + (1-\alpha_n)s_n - p, x_{n+1} - p\rangle \\ &= \alpha_n\langle \nabla h(x_n) - p, x_{n+1} - p\rangle + (1-\alpha_n)\langle \nabla s_n - p, x_{n+1} - p\rangle \\ &\le \frac{\alpha_n}{2}\Big(\|\nabla h(x_n) - h(p)\|^2 + \|x_{n+1} - p\|^2\Big) + \alpha_n\langle \nabla h(p) - p, x_{n+1} - p\rangle \\ &\quad \frac{(1-\alpha_n)}{2}\Big(\|s_n - p\|^2 + \|x_{n+1} - p\|^2\Big) \\ &\le \frac{\alpha_n}{2}\Big(\|\nabla h(x_n) - h(p)\|^2 + \|x_{n+1} - p\|^2\Big) + \alpha_n\langle \nabla h(p) - p, x_{n+1} - p\rangle \\ &\quad \frac{(1-\alpha_n)}{2}\Big(\|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\ &\quad - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 + \|x_{n+1} - p\|^2\Big) \\ &\le \frac{(1-\alpha_n(1-\sigma^2))}{2}\|x_n - p\|^2 + \frac{1}{2}\|x_{n+1} - p\|^2 + \alpha_n\langle \nabla h(p) - p, x_{n+1} - p\rangle \\ &\quad +\theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 + \frac{(1-\alpha_n)}{2}\theta_n\Big(\|x_n - p\|^2 - \|x_{n-1} - p\|^2\Big) \\ &\quad - \frac{(1-\alpha_n)}{2}\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2. \end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= (1-\alpha_n(1-\sigma^2))\|x_n - p\|^2 + \alpha_n(1-\sigma^2)\left(\frac{2}{(1-\sigma^2)}\langle \nabla h(p) - p, x_{n+1} - p\rangle\right) \\ &\quad +2\theta_n(1-\alpha_n)\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\ &\quad -(1-\alpha_n)\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2. \end{aligned} \tag{26}$$

Therefore, we obtain

$$\begin{aligned}(1-\alpha_n)\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2(1-\alpha_n)\theta_n\|x_n - x_{n+1}\|^2 \\ &\quad +\alpha_n(1-\sigma^2)\left(\frac{2}{(1-\sigma^2)}\langle \nabla h(p) - p, x_{n+1} - p\rangle\right) \\ &\quad +(1-\alpha_n)\theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2). \end{aligned} \tag{27}$$

Moreover, by using (26), we obtain

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ \le\ & (1-\alpha_n(1-\sigma^2))\|x_n - p\|^2 + \alpha_n(1-\sigma^2)\Big\{\frac{2}{(1-\sigma^2)}\langle \nabla h(p) - p, x_{n+1} - p\rangle \\ & + \left(\frac{2(1-\alpha_n)}{(1-\sigma^2)}\right)\frac{\theta_n}{\alpha_n}\|x_n - x_{n+1}\|^2 \\ & + \left(\frac{(1-\alpha_n)}{(1-\sigma^2)}\right)\frac{\theta_n}{\alpha_n}\|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n-1} - p\|)\Big\}. \end{aligned} \tag{28}$$

Next, we consider two possible cases to show that $\|x_n - p\| \to 0$.

**Case 1.** Suppose that the sequence $\Gamma_n = \{\|x_n - p\|^2\}$ is non-increasing, $N \in \mathbb{N}$ exists such that $\Gamma_{n+1} \le \Gamma_n$ for each $n \ge N$. Therefore, $\Gamma_n$ converges.

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty}\left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) > 0$, by using Remark 1, we obtain from (27) that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{29}$$

Thus, from (18), we immediately obtain

$$\lim_{n\to\infty} \|s_n - y_n\| = 0. \tag{30}$$

If we consider

$$\|s_n - w_n\| \quad \le \quad \|s_n - y_n\| + \|w_n - y_n\|, \tag{31}$$

then

$$\lim_{n\to\infty} \|s_n - w_n\| = 0. \tag{32}$$

Since $\{x_n\}$ is bounded, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p^* \in H$. By setting $T_n = J_{\lambda_n}^B (I - \lambda_n A)$, we have

$$\begin{aligned} \|(I - T_n)p^*\|^2 &= \langle (I - T_n)p^*, (I - T_n)p^* \rangle \\ &= \langle (I - T_n)p^*, p^* - w_{n_k} \rangle + \langle (I - T_n)p^*, w_{n_k} - T_n w_{n_k} \rangle \\ &\quad + \langle (I - T_n)p^*, T_n w_{n_k} - T_n p^* \rangle. \end{aligned} \tag{33}$$

Using the fact that $\|x_n - w_n\| \to 0$ and (29), we obtain

$$\lim_{n\to\infty} \|(I - T_n)p^*\| = 0. \tag{34}$$

Therefore, $p^* \in \Omega$. We can obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{2}{1-\sigma^2}\langle \nabla h(p) - p, x_{n+1} - p \rangle &= \limsup_{k\to\infty} \frac{2}{1-\sigma^2}\langle \nabla h(p) - p, x_{n_k} - p \rangle \\ &= \frac{2}{1-\sigma^2}\langle \nabla h(p) - p, p^* - p \rangle \\ &\le 0. \end{aligned} \tag{35}$$

By applying Lamma 3 and using (28) and (35) and using conditions of all parameters, we can claim that $x_n \to p = P_\Omega \nabla h(p)$.

**Case 2.** Suppose that the sequence $\Gamma_n = \{\|x_n - p\|^2\}$ is increasing. Let $\eta : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge N$ values (where $N$ is large enough). This is defined by

$$\eta(n) := \max\{k \in \mathbb{N} : \Gamma_k \le \Gamma_{k+1}\}. \tag{36}$$

Then, $\eta(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\eta(n)} \le \Gamma_{\eta(n)+1}$ for all $n \ge N$. By using (27), and the conditions of the parameters for each $n \ge N$, we have

$$
\begin{aligned}
\|w_{\eta(n)} - y_{\eta(n)}\|^2 \leq\ & \Gamma_{\eta(n)} - \Gamma_{\eta(n)+1} \\
& +2(1-\alpha_{\eta(n)})\theta_{\eta(n)}\|x_{\eta(n)} - x_{\eta(n)+1}\|^2 \\
& +\alpha_{\eta(n)}(1-\rho^2)\left(\frac{2}{(1-\rho^2)}\langle \nabla h(p) - p, x_{\eta(n)+1} - p\rangle\right) \\
& +(1-\alpha_{\eta(n)})\theta_{\eta(n)}(\Gamma_{\eta(n)} - \Gamma_{\eta(n)-1}).
\end{aligned} \tag{37}
$$

Since $\alpha_n \to 0$, we can conclude that

$$\lim_{n\to\infty} \|w_{\eta(n)} - y_{\eta(n)}\| = 0.$$

Moreover, by following the proof in Case 1, we obtain

$$\limsup_{n\to\infty} \langle \nabla h(p) - p, x_{\eta(n)+1} - p\rangle \leq 0. \tag{38}$$

Using (28), we have

$$
\begin{aligned}
\Gamma_{\eta(n)+1} \leq\ & (1-\alpha_{\eta(n)}(1-\rho^2))\Gamma_{\eta(n)} + \alpha_n(1-\rho^2)\left(\frac{2}{(1-\rho^2)}\langle \nabla h(p) - p, x_{\eta(n)+1} - p\rangle\right. \\
& +\left(\frac{2(1-\alpha_{\eta(n)})}{(1-\rho^2)}\right)\frac{\theta_{\eta(n)}}{\alpha_{\eta(n)}}\|x_{\eta(n)} - x_{\eta(n)+1}\|^2 \\
& +\left(\frac{(1-\alpha_{\eta(n)})}{(1-\rho^2)}\right)\frac{\theta_{\eta(n)}}{\alpha_{\eta(n)}}\|x_{\eta(n)} - x_{\eta(n)+1}\|\left(\sqrt{\Gamma_{\eta(n)}} + \sqrt{\Gamma_{\eta(n)-1}}\right)\Bigg).
\end{aligned} \tag{39}
$$

By applying Lamma 3 to (39), using (38) and the conditions of all parameters, we can claim that

$$\lim_{n\to\infty} \|x_{\eta(n)+1} - p\| = 0.$$

Using Lemma 4, we obtain

$$0 \leq \|x_n - p\| \leq \Big\{\|x_n - p\|, \|x_{\eta(n)} - p\|\Big\} \leq \|x_{\eta(n)+1} - p\| \to 0 \quad \text{as} \quad n \to \infty. \tag{40}$$

Therefore, $x_n \to p = P_\Omega \nabla h(p)$, which completes the proof. □

## 4. Applications and Numerical Results

Let $F : H \to \mathbb{R}$ and $G : H \to \mathbb{R}$ be a convex function and a convex, lower-semicontinuous, and nonsmooth function, respectively. Consider the convex minimization problem that finds $\bar{x} \in H$ such that

$$F(\bar{x}) + G(\bar{x}) = \min_{x\in H}\{F(x) + G(x)\}. \tag{41}$$

Using Fermat's rule, an equivalent of the problem (41) is obtained in the form

$$0 \in \nabla F(\bar{x}) + \partial G(\bar{x}), \tag{42}$$

where $\partial G$ is a subdifferential of $G$, which is a maximal monotone. For more detail, we direct the reader to [48]. $\nabla F$ is a gradient of $F$, which is $1/L$-Lipschitz continuous [49]. By setting $A = \nabla F$ and $B = \partial G$, we obtain the following theorem:

**Algorithm 2** An iterative algorithm for solving the convex minimization problems

**Initialization:** Given $\lambda_1 > 0$, $\mu \in (0,1)$, let $x_1, x_2 \in H$ be arbitrary. Choose $\{\alpha_n\}$, $\nabla h$ to satisfy Assumption 1 and $\{\theta_n\}$ to satisfy Remark 1.

**Iterative Step:** Given the current iterate $x_n$, calculate the next iterate as follows:

**Step 1.** Compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}) \\ y_n &= J_{\lambda_n}^{\partial G}(I - \lambda_n \nabla F)w_n \\ s_n &= y_n - \lambda_n(\nabla F y_n - \nabla F w_n) \end{aligned}$$

and

$$x_{n+1} = \alpha_n \nabla h(x_n) + (1-\alpha_n)s_n.$$

**Step 2.** Update

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu\|w_n - y_n\|}{\|\nabla F w_n - \nabla F y_n\|}, \lambda_n \right\} & \text{if } \nabla F w_n - \nabla F y_n \neq 0; \\ \lambda_n & \text{otherwise.} \end{cases}$$

Replace $n$ with $n+1$ and then repeat **Step 1.**

**Theorem 2.** *Assume that Assumptions (1) A1–A4 are held. Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then, $x_n \to p$, where $p = P_\Omega \nabla h(p)$.*

In this paper, we focus on the topic of image restoration. The inversion of the following model can be used to formulate the image restoration problem:

$$y = Ax + b. \tag{43}$$

where $x \in \mathbb{R}^{n\times 1}$ is an original image, $y \in \mathbb{R}^{m\times 1}$ is the observed image, $b$ is additive noise, and $A \in \mathbb{R}^{m\times n}$. To solve the problem (43), we can transform it into the least squares minimization problem

$$\min_x \left\{ \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \right\}, \tag{44}$$

where $\lambda > 0$ is a regularization parameter. We set $G(x) = \|x\|_1$, $F(x) = \dfrac{1}{2}\|Ax - b\|_2^2$ and $\lambda_1 = 0.001$. The Lipschitz gradient of $F$ is in the form

$$\nabla F(x) = A^T(Ax - b),$$

where $A^T$ is a transpose of operator $A$. Now, an iteration is used to find the solution to the following convex minimization problem: Find $x \in \mathbb{R}^n$ such that

$$x \in \arg\min \left\{ \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \right\}, \tag{45}$$

where $A$ is a bounded linear operator and $b$ is the degraded image. Therefore, we use Theorem 2 to solve (45) by setting $h(x) = \dfrac{z^2}{12}$, $\theta_n = \dfrac{70n - 9}{100n}$ and $\alpha_n = \dfrac{1}{1000n+1}$. Next, since $G(x) = \|x\|_1$, we immediately know from [50] that

$$\begin{aligned} (I + \lambda\partial G)^{-1}(x) &= \Big( \max\{|x^1| - \lambda, 0\}\text{sign}(x^1), |x^2| - \lambda, 0\}\text{sign}(x^2), \ldots \\ &\quad \{|x^n| - \lambda, 0\}\text{sign}(x^n) \Big). \end{aligned} \tag{46}$$

In this part, we present the restoration of an image that has been corrupted by a motion blur specified by a motion length of 22 pixels and a motion orientation of 45° (blur matrix $A_1$), a Gaussian blur with a filter size of $9 \times 9$ and a standard deviation of $\sigma = 2$ (blur matrix $A_2$), an out of focus blur or a circular average filtered blurred image with a radius of $r = 5$ (blurred matrix $A_3$), and an average blur with a filter size of $9 \times 9$ (blurred matrix $A_4$), respectively. We use Algorithm 2 to restore the original grey (cameraman) and RGB (baboon) images, which are shown in Figure 1. Blurred grey images and blurred RGB images with a blurred matrix $A_1$–$A_4$ are shown in Figures 2 and 3, respectively.

The reconstructed grey images corrupted by blurred matrixes $A_1$–$A_4$ are shown in Figure 4, and the reconstructed RGB images corrupted by blurred matrixes $A_1$–$A_4$ are shown in Figure 5.

In order to measure the quality of the restored images, we use the signal-to-noise ratio:

$$\mathrm{SNR} = 20 \log \frac{\|x\|_2}{\|x - x_{n+1}\|_2}, \tag{47}$$

where $x$ is an original image. The behavior of SNR for the Algorithm 2 of all cases for grey and RGB images are shown in Figures 6 and 7, respectively.

**Figure 1.** Original images.

**Figure 2.** Blurred grey images with blurred matrixes $A_1$–$A_4$, respectively.

**Figure 3.** Blurred RGB images with blurred matrixes $A_1$–$A_4$, respectively.

**Figure 4.** Reconstructed grey images corrupted by blur matrixes $A_1$–$A_4$, respectively.

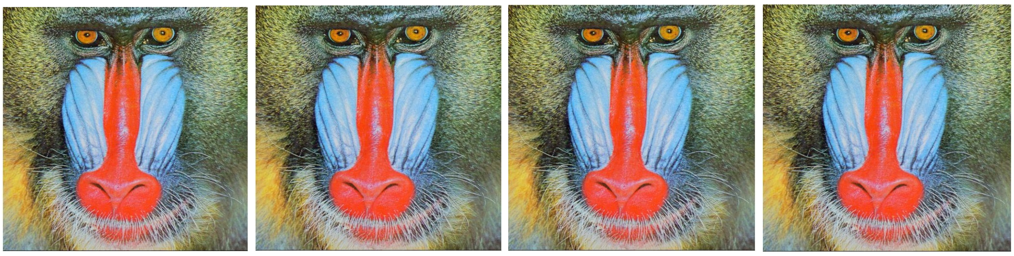

**Figure 5.** Reconstructed RGB images corrupted by blurred matrixes $A_1$–$A_4$, respectively.

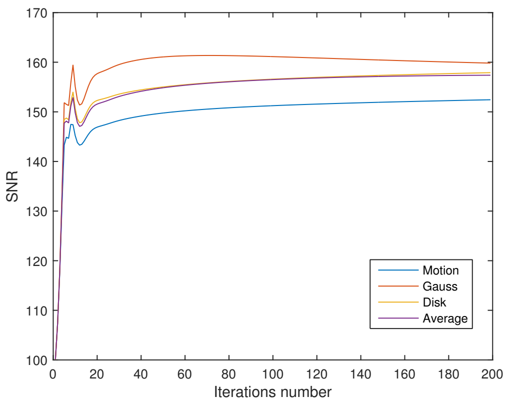


**Figure 6.** The behavior of SNR for the Algorithm 2 of all cases for grey images.

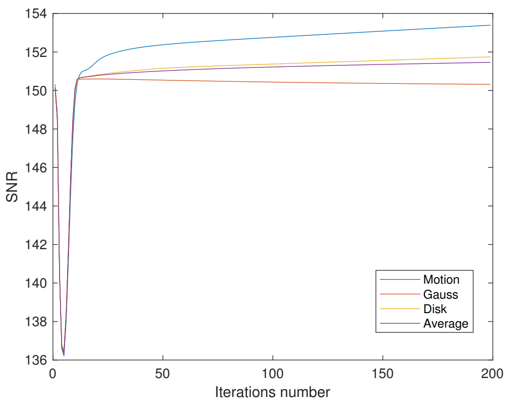


**Figure 7.** The behavior of SNR for the Algorithm 2 of all cases for RGB images.

## 5. Conclusions

In this paper, we proposed a modified Tseng's method that combines inertial and viscosity techniques to solve monotone inclusion problems in real Hilbert spaces. We also established a strong convergence theorem. Our modifications improve the practicality of the algorithm, which means that it performs better and can be applied for a more expansive mapping class. Moreover, we used our algorithm to solve some parts of image recovery problems.

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