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Eigenvalues of Elliptic Functional Differential Systems via a Birkhoff–Kellogg Type Theorem †

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† Dedicated to Professor Espedito De Pascale on the occasion of his 75th birthday.

Abstract: Motivated by recent interest on Kirchhoff-type equations, in this short note we utilize a classical, yet very powerful, tool of nonlinear functional analysis in order to investigate the existence of positive eigenvalues of systems of elliptic functional differential equations subject to functional boundary conditions. We obtain a localization of the corresponding non-negative eigenfunctions in terms of their norm. Under additional growth conditions, we also prove the existence of an unbounded set of eigenfunctions for these systems. The class of equations that we study is fairly general and our approach covers some systems of nonlocal elliptic differential equations subject to nonlocal boundary conditions. An example is presented to illustrate the theory.

Keywords: positive solution; nonlocal elliptic system; functional boundary condition; cone; Birkhoff-Kellogg type theorem

MSC: primary 35J47; secondary 35B09; 35J57; 35J60; 47H10

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1. Introduction

A well known result in nonlinear analysis is the Birkhoff-Kellogg invariant-direction Theorem [1]. In the case of an infinite-dimensional normed linear space V this theorem reads as follows.

Theorem 1. *([2], Theorem 6.1). Let U be a bounded open neighborhood of 0 in an infinite-dimensional normed linear space $(V, \| \, \|)$, and let $T : \partial U \to V$ be a compact map satisfying $\|T(x)\| \geq \alpha$ for some $\alpha > 0$ for every x in ∂U. Then there exist $x_0 \in \partial U$ and $\lambda_0 \in (0, +\infty)$ such that $x_0 = \lambda_0 F(x_0)$.*

The invariant direction Theorem has been object of deep studies in the past, with applications and extensions in several directions, we refer the reader to [3–10] and references therein. In particular, we highlight that [6,8,10] provide interesting applications to the existence of eigenvalues and eigenfunctions of elliptic boundary value problems.

Here we make use of a Birkhoff-Kellogg type theorem, which is set in cones, due to Krasnosel'skiĭ and Ladyženskiĭ [11]. Before stating this result, we recall that a cone $\mathfrak{C}$ of a real Banach space $(X, \| \, \|)$ is a closed set with $\mathfrak{C} + \mathfrak{C} \subset \mathfrak{C}$, $\mu\mathfrak{C} \subset \mathfrak{C}$ for all $\mu \geq 0$ and $\mathfrak{C} \cap (-\mathfrak{C}) = \{0\}$.

Theorem 2. *([12], Theorem 2.3.6). Let $(X, \| \, \|)$ be a real Banach space, $U \subset X$ be an open bounded set with $0 \in U$, $\mathfrak{C} \subset X$ be a cone, $T : \mathfrak{C} \cap \overline{U} \to \mathfrak{C}$ be compact and suppose that*

$$\inf_{x \in \mathfrak{C} \cap \partial U} \|Tx\| > 0.$$

Then there exist $\lambda_0 \in (0, +\infty)$ and $x_0 \in \mathfrak{C} \cap \partial U$ such that $x_0 = \lambda_0 T x_0$.

By means of Theorem 2 we discuss the solvability, with respect to the parameter λ, of the following system of second order elliptic functional differential equations subject to functional boundary conditions (BCs)

$$\begin{cases} L_i u_i = \lambda f_i(x, u, Du, w_i[u]), & \text{in } \Omega, \quad i = 1, 2, \ldots, n, \\ B_i u_i = \lambda \zeta_i(x) h_i[u], & \text{on } \partial\Omega, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, L_i is a strongly uniformly elliptic operator, B_i is a first order boundary operator, $u = (u_1, \ldots, u_n)$, $Du = (\nabla u_1, \ldots, \nabla u_n)$, f_i are continuous functions, ζ_i are sufficiently regular functions, w_i and h_i are suitable compact functionals.

The class of systems occurring in (1) is fairly general and allows us to deal with nonlocal problems of Kirchhoff-type. This is a very active area of research, a typical example of a Kirchhoff-type problem is

$$-M\Big(\int_\Omega |\nabla u|^2 \, dx\Big) \Delta u = f(x, u), \ x \in \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2}$$

which has been investigated by Ma in his survey [13]. An extension to systems of the BVP (2) has been considered by Figueiredo and Suárez [14], namely

$$\begin{cases} -M_1\big(\int_\Omega |\nabla u_1|^2 \, dx\big) \Delta u_1 = f_1(x, u_1, u_2), & x \text{ in } \Omega, \\ -M_2\big(\int_\Omega |\nabla u_2|^2 \, dx\big) \Delta u_2 = f_2(x, u_1, u_2), & x \text{ in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The approach employed in [14] is the sub-supersolution method. A similar approach has also been used in the recent papers [15,16], while variational methods have been utilized in [17–19].

Note that there has been also interest in Kirchhoff-type systems with gradient terms appearing within the nonlinearities, we mention the recent paper [20] and references therein.

The framework of (1) allows us to deal with non-homogenous BCs of functional type. In the case of nonlocal elliptic equations, non-homogeneous BCs have been investigated by Wang and An [21], Morbach and Corrêa [22] and by the author [23]. The formulation of the functionals occurring in (1) allows us to consider *multi-point* or *integral* BCs. There exists a wide literature on this topic, for brevity we refer the reader to the recent paper [23] and references therein. For further reading on the topics of non-standard elliptic systems and gradient terms appearing within the nonlinearities, we refer the reader to the recent papers [24,25].

Here we discuss, under fairly general conditions, the existence of positive eigenvalues with corresponding non-negative eigenfunctions for the system (1) and illustrate how these results can be applied in the case of nonlocal elliptic systems, see Remark 2. Our results are new and complement previous results of the author [23], by allowing the presence of gradient terms within the nonlinearities and the functionals. The results also complement the ones in [26], by considering more general nonlocal elliptic systems.

2. Eigenvalues and Eigenfunctions

In what follows, for every $\hat{\mu} \in (0,1)$ we denote by $C^{\hat{\mu}}(\overline{\Omega})$ the space of all $\hat{\mu}$-Hölder continuous functions $g : \overline{\Omega} \to \mathbb{R}$ and, for every $k \in \mathbb{N}$, we denote by $C^{k+\hat{\mu}}(\overline{\Omega})$ the space of all functions $g \in C^k(\overline{\Omega})$ such that all the partial derivatives of g of order k are $\hat{\mu}$-Hölder continuous in $\overline{\Omega}$ (for more details see ([27], Examples 1.13 and 1.14)).

We make the following assumptions on the domain Ω and the operators L_i and B_i and the functions ζ_i that occur in (1) (see ([27], Section 4 of Chapter 1)):

(1) $\Omega \subset \mathbb{R}^m$, $m \geq 2$, is a bounded domain such that its boundary $\partial\Omega$ is an $(m-1)$-dimensional $C^{2+\hat{\mu}}-$manifold for some $\hat{\mu} \in (0,1)$, such that Ω lies locally on one side of $\partial\Omega$ (see ([28], Section 6.2) for more details).

(2) L_i is a the second order elliptic operator given by

$$L_i u(x) = -\sum_{j,l=1}^{m} a_{ijl}(x)\frac{\partial^2 u}{\partial x_j \partial x_l}(x) + \sum_{j=1}^{m} a_{ij}(x)\frac{\partial u}{\partial x_j}(x) + a_i(x)u(x), \quad \text{for } x \in \Omega,$$

where $a_{ijl}, a_{ij}, a_i \in C^{\hat{\mu}}(\overline{\Omega})$ for $j,l = 1,2,\ldots,m$, $a_i(x) \geq 0$ on $\bar{\Omega}$, $a_{ijl}(x) = a_{ilj}(x)$ on $\bar{\Omega}$ for $j,l = 1,2,\ldots,m$. Moreover L_i is strongly uniformly elliptic; that is, there exists $\bar{\mu}_{i0} > 0$ such that

$$\sum_{j,l=1}^{m} a_{ijl}(x)\xi_j\xi_l \geq \bar{\mu}_{i0}\|\xi\|_e^2 \quad \text{for } x \in \Omega \text{ and } \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m,$$

where $\|\cdot\|_e$ is the Euclidean norm.

(3) B_i is a boundary operator given by

$$B_i u(x) = b_i(x)u(x) + \delta_i \frac{\partial u}{\partial \nu}(x) \quad \text{for } x \in \partial\Omega,$$

where ν is an outward pointing and nowhere tangent vector field on $\partial\Omega$ of class $C^{1+\hat{\mu}}$ (not necessarily a unit vector field), $\frac{\partial u}{\partial \nu}$ is the directional derivative of u with respect to ν, $b_i : \partial\Omega \to \mathbb{R}$ is of class $C^{1+\hat{\mu}}$ and moreover one of the following conditions holds:

(a) $\delta_i = 0$ and $b_i(x) \equiv 1$ (Dirichlet boundary operator).
(b) $\delta_i = 1$, $b_i(x) \equiv 0$ and $a_i(x) \not\equiv 0$ (Neumann boundary operator).
(c) $\delta_i = 1$, $b_i(x) \geq 0$ and $b_i(x) \not\equiv 0$ (Regular oblique derivative boundary operator).

(4) $\zeta_i \in C^{2-\delta_i+\hat{\mu}}(\partial\Omega)$.

It is known that, under the previous conditions (see [27], Section 4 of Chapter 1), a strong maximum principle holds, given $g \in C^{\hat{\mu}}(\bar{\Omega})$, the BVP

$$\begin{cases} L_i u(x) = g(x), & x \in \Omega, \\ B_i u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

admits a unique classical solution $u \in C^{2+\hat{\mu}}(\bar{\Omega})$ and, moreover, given $\zeta_i \in C^{2-\delta_i+\hat{\mu}}(\partial\Omega)$ the BVP

$$\begin{cases} L_i u(x) = 0, & x \in \Omega, \\ B_i u(x) = \zeta_i(x), & x \in \partial\Omega, \end{cases} \tag{5}$$

also admits a unique solution $\gamma_i \in C^{2+\hat{\mu}}(\bar{\Omega})$.

In order to investigate the solvability of the system (1), we make use of the cone of non-negative functions $\hat{P} = C(\bar{\Omega}, \mathbb{R}_+)$. The solution operator associated to the BVP (4), $K_i : C^{\hat{\mu}}(\bar{\Omega}) \to C^{2+\hat{\mu}}(\bar{\Omega})$, defined as $K_i g = u$ is linear and continuous. It is also known (see [27], Section 4 of Chapter 1) that K_i can be extended uniquely to a continuous, linear and compact operator (that we denote again by the same name) $K_i : C(\bar{\Omega}) \to C^1(\bar{\Omega})$ that leaves the cone $\hat{P}$ invariant, that is $K_i(\hat{P}) \subset \hat{P}$.

We utilize the space $C^1(\bar{\Omega}, \mathbb{R}^n)$, endowed with the norm

$$\|u\|_1 := \max\{\|u_i\|_\infty, \|\partial_{x_j} u_i\|_\infty : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\},$$

where $\|z\|_\infty = \max_{x \in \bar{\Omega}} |z(x)|$, consider the cone $P = C^1(\bar{\Omega}, \mathbb{R}^n_+)$ and define the sets

$$P_\rho := \{x \in P : \|x\|_1 < \rho\},\ \overline{P}_\rho := \{x \in P : \|x\|_1 \leq \rho\} \text{ and } \partial P_\rho := \{x \in P : \|x\|_1 = \rho\},$$

where $\rho \in (0, +\infty)$.

We rewrite the elliptic system (1) as a fixed point problem, by considering the operators $T, \Gamma : C^1(\bar{\Omega}, \mathbb{R}^n) \to C^1(\bar{\Omega}, \mathbb{R}^n)$ given by

$$T(u) := (K_i F_i(u))_{i=1..n}, \quad \Gamma(u) := (\gamma_i h_i[u])_{i=1..n},$$

where K_i is the above mentioned extension of the solution operator associated to (4), $\gamma_i \in C^{2+\hat{\mu}}(\overline{\Omega})$ is the unique solution of the BVP (5) and

$$F_i(u)(x) := f_i(x, u(x), Du(x), w_i[u]), \text{ for } u \in C^1(\bar{\Omega}, I) \text{ and } x \in \bar{\Omega}.$$

Definition 1. *We say that λ is an* eigenvalue *of the system* (1) *if there exists $u \in C^1(\bar{\Omega})$ with $\|u\|_1 > 0$ such that the pair (u, λ) satisfies the operator equation*

$$u = \lambda(Tu + \Gamma u) = \lambda(K_i F_i(u) + \gamma_i h_i[u])_{i=1..n}. \tag{6}$$

If the pair (u, λ) satisfies (6) *we say that u is an* eigenfunction *of the system* (1) *corresponding to the eigenvalue λ. If, furthermore, the components of u are non-negative, we say that u is a* non-negative eigenfunction *of the system* (1).

We now prove our existence result, the proof is relatively straightforward and illustrates the powerfulness of Theorem 2. Note that Theorem 3 provides a precise localization of the eigenfunction in terms of its norm.

Theorem 3. *Let $\rho \in (0, +\infty)$ and assume the following conditions hold.*

(a) For every $i = 1, 2, \ldots, n$, $w_i : \overline{P}_\rho \to \mathbb{R}$ is continuous and there exist $\underline{w}_{i,\rho}, \overline{w}_{i,\rho} \in \mathbb{R}$ such that

$$\underline{w}_{i,\rho} \leq w_i[u] \leq \overline{w}_{i,\rho}, \text{ for every } u \in \overline{P}_\rho. \tag{7}$$

(b) For every $i = 1, 2, \ldots, n$, $f_i \in C(\Pi_\rho, \mathbb{R})$ and there exist $\delta_i \in C(\bar{\Omega}, \mathbb{R}_+)$ such that

$$f_i(x, u, v, w) \geq \delta_{i,\rho}(x), \text{ for every } (x, u, v, w) \in \Pi_\rho,$$

where

$$\Pi_\rho := \bar{\Omega} \times [0,\rho]^n \times [-\rho,\rho]^{m\times n} \times [\underline{w}_{i,\rho}, \overline{w}_{i,\rho}].$$

(c) *For every* $i = 1,2,\dots,n$, $\zeta_i \in C^{2-\delta_i+\hat{\mu}}(\partial\Omega)$, $\zeta_i \geq 0$, $h_i : \overline{P}_\rho \to \mathbb{R}$ *is continuous and bounded. Let* $\eta_{i,\rho} \in [0,+\infty)$ *be such that*

$$h_i[u] \geq \eta_{i,\rho}, \text{ for every } u \in \overline{P}_\rho.$$

(d) *There exist* $i_0 \in \{1,\dots,n\}$ *and* $\phi_{i_0,\rho} \in (0,+\infty)$ *such that*

$$\|K_{i_0}(\delta_{i_0,\rho}) + \eta_{i_0,\rho}\gamma_{i_0}\|_\infty \geq \phi_{i_0,\rho}. \tag{8}$$

Then the system (1) *has a positive eigenvalue with an associated eigenfunction* $u \in \partial P_\rho$.

Proof. Due to the assumptions above, the operator $T+\Gamma$ maps $\overline{P}_\rho$ into P and is compact (by construction, the map F is continuous and bounded and Γ is a finite rank operator). Take $u \in \partial P_\rho$, then for every $x \in \bar{\Omega}$ we have

$$K_{i_0}F_{i_0}u(x) + \gamma_{i_0}(x)h_{i_0}[u] \geq K_{i_0}(\delta_{i_0,\rho})(x) + \eta_{i_0,\rho}\gamma_{i_0}(x). \tag{9}$$

Taking the supremum for $x \in \bar{\Omega}$ in (9) we obtain

$$\|Tu + \Gamma u\|_1 \geq \|T_{i_0}u + \Gamma_{i_0}u\|_\infty \geq \|K_{i_0}(\delta_{i_0,\rho}) + \eta_{i_0,\rho}\gamma_{i_0}\|_\infty \geq \phi_{i_0,\rho}. \tag{10}$$

Note that the RHS of (10) does not depend on the particular u chosen. Therefore we have

$$\inf_{u\in\partial P_\rho} \|Tu + \Gamma u\|_1 \geq \phi_{i_0,\rho} > 0,$$

and the result follows by Theorem 2. □

Remark 1. *Note that we have chosen to use inequalities in* (7)–(8)*; this is due that, in applications, it is often easier and somewhat more efficient to use estimates on the nonlienarieties involved. Furthermore note that, in our reasoning, what really matters is that some positivity occurs in one component of the system, either in the nonlinearity* f_i *or in the functional* h_i.

The following Corollary provides a sufficient condition for the existence of an unbounded set of eigenfunctions for the system (1).

Corollary 1. *In addition to the hypotheses of Theorem 3, assume that* ρ *can be chosen arbitrarily in* $(0,+\infty)$*. Then for every* ρ *there exists a non-negative eigenfunction* $u_\rho \in \partial P_\rho$ *of the system* (1) *to which corresponds a* $\lambda_\rho \in (0,+\infty)$.

Remark 2. *To illustrate the applicability of the above results to the context of nonlocal elliptic equations, we focus on the case of Kirchhoff-type systems with Dirichlet BCs of the type*

$$\begin{cases} -\tilde{w}_i[u]\Delta u_i = \lambda \tilde{f}_i(x,u,Du), & \text{in } \Omega, \quad i = 1,2,\dots,n, \\ u_i = \lambda\zeta_i(x)h_i[u], & \text{on } \partial\Omega, \quad i = 1,2,\dots,n. \end{cases} \tag{11}$$

Note that system (11) can fit within the framework of (1), by considering

$$f_i(x,u,Du,w_i[u]) = \tilde{f}_i(x,u,Du)w_i[u], \text{ where } w_i[u] = (\tilde{w}_i[u])^{-1}.$$

We also observe that the setting (11) permits to address several classes of problems in a unified way (rather than a case-to-case study), this can be done by considering different functionals $\tilde{w}_i$ and h_i. We highlight the following cases (the list is not exhaustive):

(1) *The choice of $n=1$, $\tilde{w}_1[u] \equiv 1$, $\tilde{f}_1(x,u,Du) = e^{u_1}$ and $h_1[u] \equiv 0$ yields the classical Gelfand problem (see for example [29] and references therein), while fixing $\tilde{w}_1[u] = \int_\Omega e^{u_1}\,dx$, $\tilde{f}_1(x,u,Du) = e^{u_1}$ and $h_1[u] \equiv 0$ yields the celebrated mean field problem (see for example [30] and references therein).*

(2) *The choice of $n=2$, $\tilde{w}_i[u] = M_i\big(\int_\Omega |\nabla u_i|^2\,dx\big)$, $h_i[u] \equiv 0$ and $\tilde{f}_i$ not depending on Du leads to the class of systems studied in [14].*

(3) *The case of $\tilde{f}_i$ not depending on Du, with $\tilde{w}_i$ and h_i acting on the cone of non-negative functions $C(\bar{\Omega}, \mathbb{R}^n_+)$, has been studied by the author in [23].*

The following example provides a system of the type (11) that cannot be handled by the theory of [14–19], due to the presence of gradient terms in the nonlinearities, and by the results in [20], due to the presence of the nonlocal BCs. It also illustrates, in contrast to previous results on Kirchhoff-type systems known to the author, that it is possible to consider some interaction between the gradient terms of the components of the system occurring within the nonlocal part of the differential equation or within the nonlocal BCs.

Example 1. *Take $\Omega = \{x \in \mathbb{R}^2 : \|x\|_e < 1\}$ and consider the system*

$$\begin{cases} -\big(e^{u_2(0)} + \int_\Omega |\nabla u_1|^2\,dx\big)\Delta u_1 = \lambda e^{u_1}(1+|\nabla u_2|^2), & \text{in } \Omega, \\ -e^{(\int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2\,dx)}\Delta u_2 = \lambda u_2^2 |\nabla u_1|^2, & \text{in } \Omega, \\ u_1 = \lambda h_1[(u_1,u_2)],\ u_2 = \lambda h_2[(u_1,u_2)], & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where

$$h_1[(u_1,u_2)] = \Big(\frac{\partial u_1}{\partial x_2}(0)\Big)^2 + \Big(\frac{\partial u_2}{\partial x_1}(0)\Big)^2 \text{ and } h_2[(u_1,u_2)] = (u_1(0))^2 + \int_\Omega |\nabla u_2|^2\,dx.$$

Denote by $\hat{1}$ the function equal to 1 on $\bar{\Omega}$. Note that for $i=1,2$, $K_i(\hat{1}) = \frac{1}{4}(1 - x_1^2 - x_2^2)$, where $x = (x_1,x_2)$, and $\|K_i(\hat{1})\|_\infty = \frac{1}{4}$. Furthermore note that we may take $\gamma_1 = \gamma_2 = \hat{1}$.

We fix $\rho \in (0,+\infty)$ and consider

$$\begin{aligned} f_1(u_1,u_2,\nabla u_1,\nabla u_2, w_1[(u_1,u_2)]) &:= e^{u_1}(1+|\nabla u_2|^2) w_1[(u_1,u_2)], \\ f_2(u_1,u_2,\nabla u_1,\nabla u_2, w_2[(u_1,u_2)]) &:= u_2^2 |\nabla u_1|^2 w_2[(u_1,u_2)], \end{aligned}$$

where

$$\begin{aligned} w_1[(u_1,u_2)] &:= \Big(e^{u_2(0)} + \int_\Omega |\nabla u_1|^2\,dx\Big)^{-1}, \\ w_2[(u_1,u_2)] &:= e^{-(\int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2\,dx)}. \end{aligned}$$

In this case we may take

$$[\underline{w}_{1,\rho}, \overline{w}_{1,\rho}] = [(2\pi\rho^2 + e^\rho)^{-1}, 1],\ [\underline{w}_{2,\rho}, \overline{w}_{2,\rho}] = [e^{-4\pi\rho^2}, 1],$$

$$\delta_{1,\rho}(x) \equiv (2\pi\rho^2 + e^\rho)^{-1},\ \delta_{2,\rho}(x) \equiv 0,\ \eta_{1,\rho} = \eta_{2,\rho} = 0,$$

and therefore we get

$$\|K_1(\delta_{1,\rho}) + \eta_{1,\rho}\gamma_1\|_\infty = (8\pi\rho^2 + 4e^\rho)^{-1} = \phi_{1,\rho} > 0.$$

Thus we can apply Corollary 1, obtaining uncountably many pairs (u_ρ, λ_ρ) of non-negative eigenfunctions and positive eigenvalues for the system (12).

3. Conclusions

We have illustrated how a classical Birkhoff-Kellogg type theorem can be applied to provide new results on the existence of positive eigenvalues with corresponding non-negative eigenfunctions for systems of elliptic functional differential equations subject to functional BCs. As a special case we investigated the case of Kirchhoff-type systems, providing a concrete example in which all the constants that occur in the theory can be computed.

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