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General Decay Rate of Solution for Love-Equation with Past History and Absorption

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Abstract: In the present paper, we consider an important problem from the point of view of application in sciences and engineering, namely, a new class of nonlinear Love-equation with infinite memory in the presence of source term that takes general nonlinearity form. New minimal conditions on the relaxation function and the relationship between the weights of source term are used to show a very general decay rate for solution by certain properties of convex functions combined with some estimates. Investigations on the propagation of surface waves of Love-type have been made by many authors in different models and many attempts to solve Love's equation have been performed, in view of its wide applicability. To our knowledge, there are no decay results for damped equations of Love waves or Love type waves. However, the existence of solution or blow up results, with different boundary conditions, have been extensively studied by many authors. Our interest in this paper arose in the first place in consequence of a query for a new decay rate, which is related to those for infinite memory ϖ in the third section. We found that the system energy decreased according to a very general rate that includes all previous results.

Keywords: nonlinear Love-equation; infinite memory; general decay rate

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1. Introduction

The Love equation is a one-dimensional mathematical model that is used to determine a many physical phenomenon. This theory is a continuation of the Euler–Bernoulli beam theory and it was developed in 1888 by Love. This kind of systems appears in the models of nonlinear Love waves or Love type waves. It is a generalization of a model introduced by [1–3].

The original Love's equation is derived in [4,5] by the energy method. Under the assumptions that the Kinetic energy per unit of length is

$$e_1 = \frac{1}{2} F\rho[\partial_t u^2 + \mu^2 w^2 \partial_t u_x^2], \tag{1}$$

and the potential energy per unite of length is

$$e_2 = \frac{1}{2} EF(u_x^2), \tag{2}$$

where F is an area of cross-section and w is a cross-section radius of gyration about the central line.

Using, in (2), the corrected form of tension, we have

$$e_2 = \frac{1}{2} F u_x (E u_x + \rho \mu^2 w^2 \partial_{tt} u_x). \tag{3}$$

Subsequently, the variational equation of motion is given by

$$\delta \int_{t_1}^{t_2} ds \int_0^L (e_1 - e_2) dx = 0, \tag{4}$$

and we then obtain the equation of extensional vibrations of rods as

$$\partial_{tt} u - \frac{E}{\rho} u_{xx} - 2\mu^2 w^2 \partial_{tt} u_{xx} = 0. \tag{5}$$

The parameters in (5) have the following meaning: u is the displacement, μ is a coefficient, E is the Young modulus of the material, and ρ is the mass density.

This type of problem describes the vertical oscillations of a rod, and it was established from Euler's variational equation of an energy functional associated with (5). A classical solution of problem (5), with null boundary conditions and asymptotic behavior are obtained using the Fourier method and method of small parameter. In this article, we consider a nonlinear Love-equation in the form

$$\begin{aligned} &\partial_{tt} y - \big(y_x + \partial_t y_x + \partial_{tt} y_x\big)_x + \int_{-\infty}^{t} \varpi(t-s) y_{xx}(s) ds \\ &= F[y] - \Big(F[y]\Big)_x + f(x,t), \quad x \in \Omega,\ 0 < t < T, \end{aligned} \tag{6}$$

where $\Omega = [0, L], L > 0$ and

$$F[y] = F\Big(x, t, y, y_x, \partial_t y, \partial_t y_x\Big) \in C^1\Big(\Omega \times \mathbb{R}^+ \times \mathbb{R}^4\Big). \tag{7}$$

The given functions ϖ, f will be specified later. With $F = F(x, t, y_1, \dots, y_4)$, we put $D_1 F = \frac{\partial F}{\partial x}$, $D_2 F = \frac{\partial F}{\partial t}$, $D_{i+2} F = \frac{\partial F}{\partial y_i}$, with $i = 1, \dots, 4$.

Equation (6) satisfies the homogeneous Dirichlet boundary conditions

$$y(0,t) = y(L,t) = 0, \qquad t > 0, \tag{8}$$

and the following initial conditions

$$y(x,-t) = y_0(x,t), \quad \partial_t y(x,0) = y_1(x). \tag{9}$$

We call the Sobolev space of order 1 on Ω the space

$$H^1(\Omega) = \{v \in L^2(\Omega),\ \partial_x v \in L^2(\Omega)\}.$$

The space H^1 is endowed with the norm that is associated to the inner product

$$\langle u, v \rangle_{1,\Omega} = \int_\Omega \Big(uv + \partial_x u \partial_x v\Big) dx,$$

and we note the corresponding norm

$$\|v\|_{1,\Omega} = \sqrt{\langle v, v\rangle_{1,\Omega}} = \Big(\int_\Omega |v|^2 dx + \int_\Omega |\partial_x v|^2 dx\Big)^{1/2}.$$

We have the generalization of such spaces. Let $m \in \mathbb{N}$. A function $v \in L^2(\Omega)$ belongs to the Sobolev space of order m, denoted $H^m(\Omega)$, if all of the derivatives of v up to order m, in the distributional sense, belong to $L^2(\Omega)$. By convention, we note $H^0(\Omega) = L^2(\Omega)$.

We denote, by $H_0^1(\Omega)$, the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. By extension, we note $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$.

In order to deal with a wave equation with infinite memory, we assume that the kernel function ϖ satisfies the following hypothesis:

Hypothesis 1. *$\varpi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is a non-increasing function such that*

$$1 - \int_0^\infty \varpi(s)ds = l > 0, \quad \varpi(0) > 0, \tag{10}$$

and there exists an increasing strictly convex function $\mathcal{H} \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, satisfying

$$\mathcal{H}(0) = \partial_t \mathcal{H}(0) = 0, \quad \text{and} \quad \lim_{t\to\infty} \partial_t \mathcal{H}(t) = \infty, \tag{11}$$

such that

$$\int_0^\infty \frac{\varpi(s)}{\mathcal{H}^{-1}(-\partial_t\varpi(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{\varpi(s)}{\mathcal{H}^{-1}(-\partial_t\varpi(s))} < \infty, \tag{12}$$

where $\mathcal{H}^{-1}$ is the inverse of $\mathcal{H}$ function.

Hypothesis 2. *$y_0(0), y_1 \in H_0^1(\Omega) \cap H^2(\Omega)$. We need the following assumptions on source forces.*

Hypothesis 3. *$f \in H^1(\Omega \times (0, T))$.*

Hypothesis 4. *$F \in C^1\left(\Omega \times [0, T] \times \mathbb{R}^4\right)$, such that*

$$F(0, t, 0, y_2, 0, y_4) = F(1, t, 0, y_2, 0, y_4) = 0, \quad \forall\, t \in [0, T], \quad y_2, y_4 \in \mathbb{R}.$$

Hypothesis 5. *There exists a constant $m_0 \geq 0$, such that*

$$\int_\Omega |y_{0x}(., s)|^2\, dx \leq m_0^2, \quad \forall\, s > 0. \tag{13}$$

Our results are very interesting from an application point of view and, as for as, we know that there is no results for equations of Love waves or Love type waves with the presence of finite/infinite memory term ([1–3,6–12]).

Without infinite memory term, Triet et al. in [12] considered an initial boundary value problem for a nonlinear Love equation

$$\begin{aligned} u_{tt} - \frac{\partial}{\partial x}(u_x + \lambda_1 u_{xt} + u_{xtt}) + \lambda u_t &= F(x, t, u, u_x, u_t, u_{xt}) \\ &- \frac{\partial}{\partial x}\left[G(x, t, u, u_x, u_t, u_{xt})\right] + f(x, t), \quad x \in \Omega = (0, 1),\ 0 < t < T, \end{aligned} \tag{14}$$

$$u(0, t) = u(1, t) = 0, \tag{15}$$

$$u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \tag{16}$$

where $\lambda, \lambda_1 > 0$ are constants and $\tilde{u}_0, \tilde{u}_1 \in H_0^1(\Omega) \cap H^2(\Omega)$; f, F and G are given functions. First, under suitable conditions, the existence of a unique local weak solution has been proved and a blow-up result for solutions with negative initial energy is also established. A sufficient condition ensuring the global existence and exponential decay of weak solutions is given in the last section. These results will be improved in [11,12] to the Kirchhoff type.

The existence/nonexistence, exponential decay of solutions and blow-up results for viscoelastic wave equations with finite history have been extensively studied and many authors have obtained many results (see [13–21]).

Concerning problems with infinite history, we mention the work [15], in which the authors onsidered the following semi-linear hyperbolic equation, in a bounded domain of $\mathbb{R}^3$,

$$\partial_{tt}u - K(0)\Delta u - \int_0^\infty \partial_t K(s)\Delta u(t-s)ds + g(u) = f,$$

with $K(0), K(\infty) > 0, \partial_t K \leq 0$ and they proved the existence of global attractors for the problem. Next, in [22], the authors considered a fourth-order suspension bridge equation with nonlinear damping term $|\partial_t u|^{m-2}\partial_t u$ and source term $|u|^{p-2}u$. The authors found necessary and sufficient conditions for global existence and energy decay results without considering the relation between m and p. Moreover, when $p > m$, they gave a sufficient condition for finite time blow-up of solutions. The lower bound of the blow-up time is also established.

Recently, in [23], the authors studied a three-dimensional (3D) visco-elastic wave equation with nonlinear weak damping, supercritical sources, and prescribed past history

$$\partial_{tt}u - k(0)\Delta u - \int_0^\infty \partial_t k(s)\Delta u(t-s)ds + |\partial_t u|^{m-1}\partial_t u = |u|^{p-1}u, \quad t \geq 0,$$

where the relaxation function k is monotone decreasing with $k(+\infty) = 1, m \geq 1, 1 \leq p < 6$. When the source is stronger than the dissipation, i.e., $p > \max\{m, \sqrt{k(0)}\}$, they obtained some finite time blow-up results with positive initial energy. In particular, they obtained the existence of certain solutions, which blow-up in finite time for initial data at an arbitrary energy level (see [24]).

The outline of our work: in the next section, the existence results in Theorems 1 and 3 are obtained by using a new combined methods. A decay rate of energy, which is very general, is derived in the last section: Theorem 4 extends the results that were obtained in [20], where the authors established a general decay rate for relaxation functions satisfying

$$\partial_t \varpi(t) \leq -\mathcal{H}(\varpi(t)), t \geq 0, \quad \mathcal{H}(0) = 0, \tag{17}$$

for a positive function $\mathcal{H} \in C^1(\mathbb{R}^+)$ and $\mathcal{H}$ is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. This improves the conditions that were introduced by [13] on the relaxation functions:

$$\partial_t \varpi(t) \leq -\chi(\varpi(t)), \quad \chi(0) = \partial_t \chi(0) = 0, \tag{18}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. Furthermore, the authors required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \quad \lim_{s\to 0^+} \inf \frac{\chi(s)/s}{\partial_t \chi(s)} > \frac{1}{2}, \tag{19}$$

and proved a decay result for the energy in a bounded domain. In addition to these assumptions, if

$$\lim_{s\to 0^+} \sup \frac{\chi(s)/s}{\partial_t \chi(s)} < 1, \tag{20}$$

then, in this case, an explicit decay rate is given.

2. The Existence of Solution

We define the weak solution to of (6)–(9), as follows.

Definition 1. *A function y is said to be a weak solution of (6)–(9) on $[0,T]$ if*

$$y, \partial_t y, \partial_{tt} y \in L^\infty(0,T,H_0^1(\Omega)\cap H^2(\Omega)),$$

such that y satisfies the variational equation

$$\begin{aligned}&\int_\Omega \partial_{tt} y w\, dx + \int_\Omega (y_x + \partial_t y_x + \partial_{tt} y_x) w_x dx \\ &- \int_\Omega \int_0^\infty \varpi(s) y_x(t-s) ds w_x dx \\ &= \int_\Omega f w dx + \int_\Omega F[y] w dx + \int_\Omega F[y] w_x dx,\end{aligned} \tag{21}$$

for all test function $w \in H_0^1(\Omega)$, for almost all $t \in (0,T)$.

The following famous and widely used technical Lemma will play an important role in the sequel.

Lemma 1. *Ref. [25] For any $v \in C^1\left(0,T,H_0^1(\Omega)\right)$ we have*

$$\begin{aligned}&\int_\Omega \int_0^\infty \varpi(s) v_{xx}(t-s)\partial_t v(t) ds dx \\ = \;& \frac{1}{2}\partial_t \int_0^\infty \varpi(s) \int_\Omega |v_x(t-s) - v_x(t)|^2 dx ds - \frac{1}{2}\partial_t \int_0^\infty \varpi(s) ds \int_\Omega |v_x(t)|^2\, dx \\ & -\frac{1}{2} \int_0^\infty \partial_t \varpi(s) \int_\Omega |v_x(t-s) - v_x(t)|^2 dx ds + \frac{1}{2}\varpi(t) \int_\Omega |v_x(t)|^2\, dx.\end{aligned}$$

Now, we state the existence of a local solution for (6)–(9).

Theorem 1. *Ref. [25] Let $y_0(0), y_1 \in H_0^1(\Omega)\cap H^2(\Omega)$ be given. Assume that (Hypothesis 1)–(Hypothesis 5) hold. Subsequently, problem (6)–(9) has a unique local solution y and*

$$y, \partial_t y, \partial_{tt} y \in L^\infty(0,T_*,H_0^1(\Omega)\cap H^2(\Omega)), \tag{22}$$

for some $T_* > 0$ small enough.

Let $f \equiv 0$. Here, and in the sequel, we consider problem (6) with the boundary conditions (8) and the initial conditions (9) in the following form

$$\begin{aligned}&\partial_{tt} y - \left(y_x + \partial_t y_x + \partial_{tt} y_x\right)_x + \int_{-\infty}^t \varpi(t-s) y_{xx}(s) ds \\ &= |y|^{p-2} y + (|y_x|^{p-2} y_x)_x, \quad x \in \Omega,\ 0 < t < T,\end{aligned} \tag{23}$$

with $p \geq 2$.

We introduce the energy functional $E(t)$ that is associated to our problem (23), as follows

$$E(t) = \frac{1}{2}\int_\Omega |\partial_t y|^2 dx + \frac{1}{2}\int_\Omega |\partial_t y_x|^2 dx + J(t), \tag{24}$$

where

$$\begin{aligned}J(t) \;=\;& \frac{1}{2}\int_\Omega \left(1 - \int_0^\infty \varpi(s) ds\right)|y_x|^2 dx \\ &+ \frac{1}{2}\int_\Omega \int_0^\infty \varpi(s) |y_x(t) - y_x(t-s)|^2 ds dx \\ &+ \frac{1}{p}\int_\Omega |y_x|^p dx - \frac{1}{p}\int_\Omega |y|^p dx.\end{aligned} \tag{25}$$

Now, we introduce the stable set, as follows (see [26,27])

$$W = \{y \in H_0^1(\Omega) \cap H^2(\Omega) : I(t) > 0, J(t) < d\} \cup \{0\}, \tag{26}$$

where

$$\begin{aligned} I(t) \quad &= \quad \int_\Omega \Big(1 - \int_0^\infty \varpi(s)ds\Big)|y_x|^2 dx \\ &+ \quad \int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx \\ &+ \quad \int_\Omega |y_x|^p dx - \int_\Omega |y|^p dx. \end{aligned} \tag{27}$$

We notice that the mountain passes level d, given in (26), is defined by

$$d \quad = \quad \inf\{ \sup_{y \in H_0^1(\Omega) \cap H^2(\Omega) \setminus \{0\}, \nu \geq 0} J(\nu y)\}. \tag{28}$$

Additionally, by introducing the so called "*Nehari manifold*"

$$\mathcal{N} = \Big\{ y \in H_0^1(\Omega) \cap H^2(\Omega) \setminus \{0\} : I(t) = 0 \Big\}.$$

The potential depth d is also characterized by

$$d = \inf_{y \in \mathcal{N}} J(t). \tag{29}$$

This characterization of d shows that

$$dist\,(0, \mathcal{N}) = \min_{y \in \mathcal{N}} \|y\|_{H_0^1(\Omega) \cap H^2(\Omega)}. \tag{30}$$

It is not hard to see this Lemma.

Lemma 2. *Suppose that (Hypothesis 1) holds. Let y be solution of our equation. Subsequently, the energy functional (24) is a non-increasing function, i.e., for all $t \geq 0, \nu > 0$,*

$$\partial_t E(t) = -\int_\Omega |\partial_t y_x|^2 dx + \frac{1}{2}\int_\Omega \int_0^\infty \partial_t \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx - \frac{1}{2}\varpi(t)\int_\Omega |v_x(t)|^2\, dx \leq 0. \tag{31}$$

Proof. Multiplying (23), with $p \geq 2$, by $\partial_t y$, integrating over Ω to obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \int_\Omega |\partial_t y|^2 dx + \frac{1}{2}\partial_t \int_\Omega |y_x|^2 dx + \frac{1}{2}\partial_t \int_\Omega |\partial_t y_x) dx + \int_\Omega |\partial_t y_x|^2 dx \\ &+ \int_\Omega \int_{-\infty}^t \varpi(t-s) y_{xx}(s) \partial_t y ds dx \\ &= \frac{1}{p}\partial_t \int_\Omega |y|^p dx - \frac{1}{p}\partial_t \int_\Omega |y_x|^p dx, \end{aligned} \tag{32}$$

then, using Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \Big[\int_\Omega |\partial_t y|^2 dx + \int_\Omega |y_x|^2 dx + \int_\Omega |\partial_t y_x) dx - \frac{1}{p}\partial_t \int_\Omega |y|^p dx + \frac{1}{p}\partial_t \int_\Omega |y_x|^p dx \\ &+ \int_0^\infty \varpi(s) \int_\Omega |y_x(t-s) - y_x(t)|^2 dx ds - \int_0^\infty \varpi(s) ds \int_\Omega |y_x(t)|^2\, dx \Big] \\ &= -\int_\Omega |\partial_t y_x|^2 dx - \frac{1}{2}\int_0^\infty \partial_t \varpi(s) \int_\Omega |y_x(t-s) - y_x(t)|^2 dx ds - \frac{1}{2}\varpi(t)\int_\Omega |v_x(t)|^2\, dx, \end{aligned}$$

which completes the proof. $\square$

As in [25], we will prove the invariance of the set W, that is if for some $t_0 > 0$ we have $y(t_0) \in W$, then $y(t) \in W$, $\forall t \geq t_0$. The next Lemma ensures the existence of the potential depth.

Lemma 3. *Ref. [25] d is a positive constant.*

Lemma 4. *Ref. [25] W is a bounded neighborhood of 0 in $H_0^1(\Omega) \cap H^2(\Omega)$.*

Now, we will show that our local solution y is global in time. For this purpose it suffices to prove that the norm of the solution is bounded, independently of t. This is equivalent to proving the following Theorem.

Theorem 2. *Ref. [25] Suppose that (Hypothesis 1) and*

$$C^p l^{(1-p)} \left(\frac{2p}{p-2} E(0) \right)^{(p-2)} < l, \tag{33}$$

hold, where C is the best Poincaré's constant. If $y_0(0) \in W$, $y_1 \in H_0^1(\Omega)$, then the solution $y \in W$, $\forall t \geq 0$.

The next Theorem shows that our local solution is global in time.

Theorem 3. *Suppose that (Hypothesis 1), (Hypothesis 5), $p \geq 2$ and (33) hold. If $y_0(0) \in W$, $y_1 \in H_0^1(\Omega)$, then the local solution y is global in time and such that $y \in G_T$, where*

$$G_T = \left\{ \begin{array}{ll} y: & y \in L^\infty\left(\mathbb{R}^+; H_0^1(\Omega) \cap H^2(\Omega)\right), \\ & \partial_t y \in L^\infty\left(\mathbb{R}^+; H_0^1(\Omega)\right) \end{array} \right\}. \tag{34}$$

Proof. It suffices to show that the following norm

$$\int_\Omega |\partial_t y|^2 dx + \int_\Omega |y_x|^2 dx, \tag{35}$$

is bounded independently of t.

To achieve this, we use (24), (26), and (31) to obtain

$$\begin{aligned} E(0) &\geq E(t) = J(t) + \frac{1}{2}\int_\Omega |\partial_t y|^2 dx + \frac{1}{2}\int_\Omega |\partial_t y_x|^2 dx \\ &\geq \left(\frac{p-2}{2p}\right)\left[\int_\Omega \left(1 - \int_0^\infty \varpi(s) ds\right)|y_x|^2 dx + \int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx\right] \\ &+ \frac{1}{2}\int_\Omega |\partial_t y|^2 dx + \frac{1}{p} I(t) \\ &= \left(\frac{p-2}{2p}\right)\left[l \int_\Omega |y_x|^2 dx + \int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx\right] + \frac{1}{2}\int_\Omega |\partial_t y|^2 dx + \frac{1}{p} I(t) \\ &\geq \left(\frac{l(p-2)}{2p}\right)\int_\Omega |y_x|^2 dx + \frac{1}{2}\int_\Omega |\partial_t y|^2 dx, \end{aligned}$$

since $I(t)$ and $\int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx$ are positive. Then there exists a constant $C > 0$ depending only on p and l such that

$$\int_\Omega |y_x|^2 dx + \int_\Omega |\partial_t y|^2 dx \leq C E(0).$$

This completes the proof. □

3. General Decay Rate

Theorem 4. *Suppose that (Hypothesis 1), $p \geq 2$ and (33) hold. If $y_0(0) \in H_0^1(\Omega) \cap H^2(\Omega)$, $y_1 \in H_0^1(\Omega)$. Subsequently, the energy function (24) satisfies*

$$E(t) \leq \kappa_1 \mathcal{H}_1^{-1}(\kappa t + \kappa_0), \qquad \forall t \geq 0, \tag{36}$$

where

$$\mathcal{H}_1(\tau) = \int_\tau^1 (s\partial_\tau \mathcal{H}(\kappa s))^{-1} ds, \qquad \kappa_0, \kappa_1, \kappa > 0. \tag{37}$$

We need to introduce a several Lemmas in order to prove the main Theorem 4. To this end, let us introduce the functionals

$$\varphi(t) = \int_\Omega y \partial_t y dx + \frac{1}{2} \int_\Omega |y_x|^2 dx + \int_\Omega y_x \partial_t y_x dx, \tag{38}$$

and

$$\begin{aligned} \xi(t) &= -\int_\Omega \partial_t y \int_0^\infty \varpi(s)[y(t) - y(t-s)] ds dx \\ &- \int_0^t \int_\Omega \partial_t y_x \int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)] ds dx d\tau \\ &- \int_0^t \int_\Omega \partial_{tt} y_x \int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)] ds dx d\tau. \end{aligned} \tag{39}$$

Lemma 5. *Assume that (Hypothesis 1), $p \geq 2$ and (33) hold. Subsequently, the functional $\varphi(t)$ introduced in (38) satisfies, along the solution, the estimate*

$$\begin{aligned} &\partial_t \varphi(t) \leq \int_\Omega \partial_t y^2 dx + \int_\Omega \partial_t y_x^2 dx \\ &-\Big[\frac{l}{2} - C^p \Big(\frac{2p}{(p-2)l} E(0)\Big)^{(p-2)/2}\Big] \int_\Omega |y_x|^2 dx + \frac{(1-l)}{2l} \int_\Omega \int_0^\infty \varpi(s) |y_x(t-s) - y_x|^2 ds dx, \end{aligned}$$

where C is the same constant defined in (33).

Proof.

$$\begin{aligned} \partial_t \varphi(t) = \int_\Omega \partial_t y^2 dx + \int_\Omega \partial_t y_x^2 dx - \int_\Omega y_x^2 dx + \int_\Omega |y|^p dx - \int_\Omega |y_x|^p dx \\ + \int_\Omega y_x \int_0^\infty \varpi(s) y_x(t-s) ds dx. \end{aligned}$$

The last term can be treated, as follows

$$\begin{aligned} \int_\Omega y_x \int_0^\infty \varpi(s) y_x(t-s) ds dx &\leq \frac{1}{2} \int_\Omega y_x^2 dx + \frac{1}{2} \int_\Omega \Big(\int_0^\infty \varpi(s) y_x(t-s) ds \Big)^2 dx \\ &\leq \frac{1}{2} \int_\Omega y_x^2 dx + \frac{1}{2} \int_\Omega \Big(\int_0^\infty \varpi(s) \left(|y_x(t-s) - y_x| + |y_x| \right) ds \Big)^2 dx. \end{aligned}$$

By using Cauchy–Schwarz and Young's inequalities, we obtain, for any $\nu > 0$,

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^\infty \varpi(s) \left(|y_x(t-s) - y_x| + |y_x|\right) ds \Big)^2 dx \\
&= \int_\Omega \Big(\int_0^\infty \varpi(s)|y_x(t-s) - y_x| ds \Big)^2 dx + \int_\Omega \Big(\int_0^\infty \varpi(s)|y_x| ds \Big)^2 dx \\
&\quad +2 \int_\Omega \Big(\int_0^\infty \varpi(s)|y_x(t-s) - y_x| ds \Big) \Big(\int_0^\infty \varpi(s)|y_x| ds \Big) dx \\
&\leq \Big(1 + \frac{1}{\nu}\Big) \int_\Omega \Big(\int_0^\infty \varpi(s)|y_x(t-s) - y_x| ds \Big)^2 dx \\
&\quad +(1+\nu) \int_\Omega \Big(\int_0^\infty \varpi(s)|y_x| ds \Big)^2 dx \\
&\leq \Big(1 + \frac{1}{\nu}\Big)(1-l) \int_\Omega \int_0^\infty \varpi(s)|y_x(t-s) - y_x|^2 ds dx + (1+\nu)(1-l)^2 \int_\Omega |y_x|^2 dx.
\end{aligned}
$$

Subsequently,

$$
\begin{aligned}
\partial_t \varphi(t) \leq& \int_\Omega \partial_t y^2 dx + \int_\Omega \partial_t y_x^2 dx + \Big[\frac{1}{2} + \frac{1}{2}(1+\nu)(1-l)^2 - 1\Big] \int_\Omega |y_x|^2 dx \\
&+\frac{1}{2}\Big(1 + \frac{1}{\nu}\Big)(1-l) \int_\Omega \int_0^\infty \varpi(s)|y_x(t-s) - y_x|^2 ds dx \\
&+ \int_\Omega |y|^p dx.
\end{aligned} \tag{40}
$$

By the continuous embedding for $p \geq 2$, we have

$$
\begin{aligned}
&\int_\Omega |y|^p dx \leq C^p \Big(\int_\Omega |y_x|^2 dx \Big)^{p/2} \\
&\leq C^p \Big(\int_\Omega |y_x|^2 dx \Big)^{(p-2)/2} \int_\Omega |y_x|^2 dx \\
&\leq C^p \Big(\frac{2p}{(p-2)l} E(0) \Big)^{(p-2)/2} \int_\Omega |y_x|^2 dx.
\end{aligned} \tag{41}
$$

Using (33) and choosing $\nu = \frac{l}{1-l}$, we obtain

$$
\begin{aligned}
\partial_t \varphi(t) \quad \leq \quad & \int_\Omega \partial_t y^2 dx + \int_\Omega \partial_t y_x^2 dx \\
&-\Big[\frac{l}{2} - C^p l^{1-p/2} \Big(\frac{2p}{(p-2)} E(0) \Big)^{(p-2)/2}\Big] \int_\Omega |y_x|^2 dx \\
&+\frac{(1-l)}{2l} \int_\Omega \int_0^\infty \varpi(s)|y_x(t-s) - y_x|^2 ds dx.
\end{aligned}
$$

□

Lemma 6. *Assume that (Hypothesis 1), (Hypothesis 5), and $p \geq 2$ hold. Subsequently, for $\nu < (1-l)$, the functional introduced in (40) satisfies, along the solution, the estimate*

$$
\begin{aligned}
\partial_t \xi(t) \quad \leq \quad & -a \int_\Omega |y_x|^2 dx - ((1-l) - \nu) \int_\Omega \partial_t y^2 dx \\
&+ \quad b \int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 ds dx \\
&+ \quad \frac{c\varpi(0)}{4\nu} \int_\Omega \int_0^\infty (-\partial_t \varpi(s))|y(t) - y(t-s)|^2 ds dx,
\end{aligned}
$$

where

$$a = c(\nu)\Big(1 + 2(1-l)^2 - \Big(\frac{2p}{(p-2)l}E(0)\Big)^{(p-2)/2}\Big) > 0,$$

and for all $\nu > 0$

$$b = \frac{1-l}{4\nu} + (2\nu + \frac{1}{4\nu})(1-l) + 2\nu(1-l)^{p-1}c\Big(\frac{8}{l}E(0) + 2m_0^2\Big)^{(p-2/2)} > 0.$$

Proof. We have

$$\begin{aligned}
\partial_t \xi(t) &= \int_\Omega y_x \int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)]dsdx \\
&- \int_\Omega \left(\int_0^\infty \varpi(s) y_x(t-s)ds\right)\left(\int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)]ds\right)dx \\
&- \int_\Omega |y|^{p-2}y \int_0^\infty \varpi(s)[y(t) - y(t-s)]dsdx \\
&+ \int_\Omega |y_x|^{p-2}y_x \int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)]dsdx \\
&- \int_\Omega \partial_t y \int_0^\infty \partial_t\varpi(s)[y(t) - y(t-s)]dsdx - (1-l)\int_\Omega \partial_t y^2 dx.
\end{aligned}$$

For any $\nu > 0$, we have

$$\begin{aligned}
&\int_\Omega y_x \int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)]dsdx \\
&\le \nu \int_\Omega |y_x|^2 dx + \frac{1-l}{4\nu}\int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \left(\int_0^\infty \varpi(s) y_x(t-s)ds\right)\left(\int_0^\infty \varpi(s)[y_x(t) - y_x(t-s)]ds\right)dx \\
&\le 2\nu(1-l)^2 \int_\Omega |y_x|^2 dx + (2\nu + \frac{1}{4\nu})(1-l)\int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx.
\end{aligned}$$

Furthermore, by using (13), we have the following estimate

$$\begin{aligned}
&\int_\Omega |y_x(t) - y_x(t-s)|^2 dx \\
&\le 2\int_\Omega |y_x(t)|^2 dx + 2\int_\Omega |y_x(t-s)|^2 dx \\
&\le 4\sup_{s>0}\int_\Omega |y_x(s)|^2 dx + 2\sup_{\tau<0}\int_\Omega |y_x(\tau)|^2 dx \\
&\le 4\sup_{s>0}\int_\Omega |y_x(s)|^2 dx + 2\sup_{\tau>0}\int_\Omega |y_{0x}(\tau)|^2 dx \\
&\le 8E(0) + 2m_0^2.
\end{aligned} \tag{42}$$

Now, because $p \ge 2$, we have, by using (42) and the previous estimate,

$$\begin{aligned}
&\int_\Omega |y|^{p-2}y\int_0^\infty \varpi(s)[y(t)-y(t-s)]dsdx\\
&\le \nu\int_\Omega\Big|\int_0^\infty \varpi(s)|y(t)-y(t-s)|ds\Big|^p dx + c(\nu)\int_\Omega |y|^p dx\\
&\le \nu(1-l)^{p-1}\int_\Omega\int_0^\infty \varpi(s)|y(t)-y(t-s)|^p dsdx + c(\nu)\int_\Omega |y|^p dx\\
&\le \nu(1-l)^{p-1}c\int_0^\infty \varpi(s)\Big(\int_\Omega |y_x(t)-y_x(t-s)|^2dx\Big)^{p/2}ds + c(\nu)\int_\Omega |y|^p dx\\
&\le \nu(1-l)^{p-1}c\Big(\frac{8}{1}E(0)+2m_0^2\Big)^{(p-2)/2}\int_0^\infty \varpi(s)\int_\Omega |y_x(t)-y_x(t-s)|^2dx\\
&\quad +c(\nu)\int_\Omega |y|^p dx\\
&\le \nu(1-l)^{p-1}c\Big(\frac{8}{1}E(0)+2m_0^2\Big)^{(p-2)/2}\int_0^\infty \varpi(s)\int_\Omega |y_x(t)-y_x(t-s)|^2dx\\
&\quad +c(\nu)\Big(\frac{2p}{(p-2)l}E(0)\Big)^{(p-2)/2}\int_\Omega |y_x|^2dx.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\int_\Omega |y_x|^{p-2}y_x\int_0^\infty \varpi(s)[y_x(t)-y_x(t-s)]dsdx\\
&\le \nu\int_\Omega\Big|\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|ds\Big|^p dx + c(\nu)\int_\Omega |y_x|^p dx\\
&\le \nu(1-l)^{p-1}\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^p dsdx + c(\nu)\int_\Omega |y_x|^p dx\\
&\le \nu(1-l)^{p-1}c\int_0^\infty \varpi(s)\Big(\int_\Omega |y_x(t)-y_x(t-s)|^2dx\Big)^{p/2}ds + c(\nu)\int_\Omega |y_x|^p dx\\
&\le \nu(1-l)^{p-1}c\Big(\frac{8}{1}E(0)+2m_0^2\Big)^{(p-2)/2}\int_0^\infty \varpi(s)\int_\Omega |y_x(t)-y_x(t-s)|^2dx\\
&\quad +c(\nu)\int_\Omega |y_x|^2dx.
\end{aligned}$$

The last term can be estimated, as follows

$$\begin{aligned}
&-\int_\Omega \partial_t y\int_0^\infty \partial_t\varpi(s)[y(t)-y(t-s)]dsdx\\
&\le \nu\int_\Omega |\partial_t y|^2dx + \frac{c\varpi(0)}{4\nu}\int_\Omega\int_0^\infty (-\partial_t\varpi(s))|y(t)-y(t-s)|^2dsdx.
\end{aligned}$$

A combination of all estimates gives

$$\begin{aligned}
\partial_t\xi(t) \;\le\;& \nu\int_\Omega |y_x|^2dx + \frac{1-l}{4\nu}\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2dsdx\\
+\;& 2\nu(1-l)^2\int_\Omega |y_x|^2dx + (2\nu+\frac{1}{4\nu})(1-l)\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2dsdx\\
+\;& \nu(1-l)^{p-1}c\Big(\frac{8}{1}E(0)+2m_0^2\Big)^{(p-2/2)}\int_0^\infty \varpi(s)\int_\Omega |y_x(t)-y_x(t-s)|^2dx\\
& +c(\nu)\Big(\frac{2p}{(p-2)l}E(0)\Big)^{(p-2)/2}\int_\Omega |y_x|^2dx\\
+\;& \nu(1-l)^{p-1}c\Big(\frac{8}{1}E(0)+2m_0^2\Big)^{(p-2/2)}\int_0^\infty \varpi(s)\int_\Omega |y_x(t)-y_x(t-s)|^2dx\\
& +c(\nu)\int_\Omega |y_x|^2dx\\
+\;& \nu\int_\Omega |\partial_t y|^2dx + \frac{c\varpi(0)}{4\nu}\int_\Omega\int_0^\infty (-\partial_t\varpi(s))|y(t)-y(t-s)|^2dsdx - (1-l)\int_\Omega \partial_t y^2dx.
\end{aligned}$$

Afterwards, for $\nu < (1-l)$

$$\begin{aligned} \partial_t \xi(t) &\leq a\int_\Omega |y_x|^2 dx - ((1-l)-\nu)\int_\Omega \partial_t y^2 dx \\ &+ b\int_\Omega \int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 ds dx \\ &+ \frac{c\varpi(0)}{4\nu}\int_\Omega \int_0^\infty (-\partial_t \varpi(s))|y(t)-y(t-s)|^2 ds dx, \end{aligned}$$

where by (33), we have

$$a = c(\nu)\Big(1 + 2(1-l)^2 - \Big(\frac{2p}{(p-2)l}E(0)\Big)^{(p-2)/2}\Big) > 0,$$

and for all $\nu > 0$, we have

$$b = \frac{1-l}{4\nu} + (2\nu + \frac{1}{4\nu})(1-l) + 2\nu(1-l)^{p-1}c\Big(\frac{8}{l}E(0) + 2m_0^2\Big)^{(p-2/2)} > 0.$$

□

Let us define the Lyapunov functional

$$L(t) = \varepsilon_1 E(t) + \varphi(t) + \varepsilon_2 \xi(t), \quad \varepsilon_1, \varepsilon_2 > 0. \tag{43}$$

We need the next Lemma, which means that there is an equivalence between the Lyapunov and energy functions

Lemma 7. *There exist positive real numbers c_1 and c_2, such that*

$$L \sim E. \tag{44}$$

Proof. By (43), we have

$$\begin{aligned} |L(t) - \varepsilon_1 E(t)| &\leq |\varphi(t)| + \varepsilon_2|\xi(t)| \\ &\leq \int_\Omega |y\partial_t y| dx + \frac{1}{2}\int_\Omega |y_x|^2 dx + \int_\Omega |y_x \partial_t y_x| dx \\ +\varepsilon_2 & \int_\Omega \Big|\partial_t y \int_0^\infty \varpi(s)[y(t)-y(t-s)]ds\Big| dx \\ +\varepsilon_2 & \int_0^t \int_\Omega \Big|\partial_t y_x \int_0^\infty \varpi(s)[y_x(t)-y_x(t-s)]ds d\tau\Big| dx \\ +\varepsilon_2 & \int_0^t \int_\Omega \Big|\partial_{tt} y_x \int_0^\infty \varpi(s)[y_x(t)-y_x(t-s)]ds d\tau\Big| dx. \end{aligned}$$

Thanks to Hölder and Young's inequalities, we have

$$\begin{aligned} \int_\Omega |y\partial_t y| dx &\leq \Big(\int_\Omega |y|^2 dx\Big)^{1/2}\Big(\int_\Omega |\partial_t y|^2 dx\Big)^{1/2} \\ &\leq \frac{1}{2}\Big(\int_\Omega |y|^2 dx\Big) + \frac{1}{2}\Big(\int_\Omega |\partial_t y|^2 dx\Big) \\ &\leq \frac{c}{2}\Big(\int_\Omega |y_x|^2 dx\Big) + \frac{1}{2}\Big(\int_\Omega |\partial_t y|^2 dx\Big). \end{aligned}$$

Similarly, we have

$$\int_\Omega |y_x \partial_t y_x| dx \leq \frac{1}{2}\left(\int_\Omega |y_x|^2 dx\right) + \frac{1}{2}\left(\int_\Omega |\partial_t y_x|^2 dx\right),$$

and

$$\begin{aligned}&\int_\Omega \left|\partial_t y \int_0^\infty \varpi(s)[y(t)-y(t-s)]ds\right| dx \\ &\leq \frac{1}{2}\left(\int_\Omega |\partial_t y|^2 dx\right) + \frac{c}{2}\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 dsdx.\end{aligned}$$

The two last terms can be estimated, as follows

$$\begin{aligned}&\int_0^t\int_\Omega \left|\partial_t y_x \int_0^\infty \varpi(s)[y_x(t)-y_x(t-s)]dsd\tau\right| dx \\ &\leq \frac{1}{2}\int_0^t\left(\int_\Omega |\partial_t y_x|^2 dx\right)d\tau + \frac{1}{2}\int_0^t\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 dsdxd\tau,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_\Omega \left|\partial_{tt} y_x \int_0^\infty \varpi(s)[y_x(t)-y_x(t-s)]dsd\tau\right| dx \\ &\leq \frac{1}{2}\int_0^t\left(\int_\Omega |\partial_{tt} y_x|^2 dx\right)d\tau + \frac{1}{2}\int_0^t\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 dsdxd\tau.\end{aligned}$$

Hence, there exists a constant $C > 0$, such that

$$|L(t) - \varepsilon_1 E(t)| \quad \leq \quad CE(t).$$

Therefore, we can choose ε_1, so that

$$L(t) \sim E(t). \tag{45}$$

□

Lemma 8. *Assume that (Hypothesis 1) hold. Susequently, there exist strictly positive constants λ and c, such that*

$$\partial_t L(t) \leq -\lambda E(t) + c\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 dsdx. \tag{46}$$

Proof. By Lemmas 2, 5, and 6, we have

$$\begin{aligned}\partial_t L(t) \quad &= \quad \varepsilon_1\partial_t E(t) + \partial_t\varphi(t) + \varepsilon_2\partial_t\xi(t) \\ &\leq \quad -[\varepsilon_2[(1-l)-\nu]-1]\int_\Omega |\partial_t y|^2 dx - (\varepsilon_1 - 1)\int_\Omega |\partial_t y_x|^2 dx \\ &- \quad \left[\frac{l}{2} - \left(\frac{2p}{(p-2)l}E(0)\right)^{(p-2)/2} - \varepsilon_2 a\right]\int_\Omega |y_x|^2 dx \\ &+ \quad \left[\frac{(1-l)}{2l} + b\varepsilon_2\right]\int_\Omega\int_0^\infty \varpi(s)|y_x(t-s)-y_x|^2 dsdx \\ &+ \quad \left[\frac{\varepsilon_1}{2} - \varepsilon_2\frac{c\varpi(0)}{4\nu}\right]\int_\Omega\int_0^\infty \partial_t\varpi(s)|y_x(t)-y_x(t-s)|^2 dsdx,\end{aligned}$$

where, by (33), we have

$$a = c(\nu)\Big(1 + 2(1-l)^2 - \Big(\frac{2p}{(p-2)l}E(0)\Big)^{(p-2)/2}\Big) > 0,$$

and for all $\nu > 0$

$$b = \frac{1-l}{4\nu} + (2\nu + \frac{1}{4\nu})(1-l) + 2\nu(1-l)^{p-1}c\Big(\frac{8}{l}E(0) + 2m_0^2\Big)^{(p-2/2)} > 0.$$

Now, we choose ν and, once this constant is fixed, we can select $\varepsilon_1, \varepsilon_2$ small enough that give, for $p \geq 2$, the existence of constants $c, \lambda > 0$, such that (46) holds true. □

Lemma 9. *Assume that (Hypothesis 1) hold. Subsequently, there exist $\gamma, \gamma_0 > 0$, such that for all $t > 0$*

$$\int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx \leq \frac{-\gamma \partial_t E(t)}{\partial_t \mathcal{H}(\gamma_0 E(t))} + \gamma\gamma_0 E(t). \tag{47}$$

Proof. Let $\mathcal{H}^*$ be the convex conjugate of $\mathcal{H}$ in the sense of Young (see [28] pp. 61–64), then

$$\begin{aligned} \mathcal{H}^*(s) &= s(\partial_t\mathcal{H})^{-1}(s) - \mathcal{H}[(\partial_t\mathcal{H})^{-1}(s)] \\ &\leq s(\partial_t\mathcal{H})^{-1}(s), \quad s \in (0, \partial_t\mathcal{H}(r)), \end{aligned} \tag{48}$$

and satisfies the following Young's inequality

$$AB \leq \mathcal{H}^*(A) + \mathcal{H}(B), \quad A \in (0, \partial_t\mathcal{H}(r)), B \in (0, r], \tag{49}$$

for

$$B = \mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)$$

$$A = \frac{r_1\partial_t\mathcal{H}(\gamma_0 E(t))\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)}.$$

Afterwards, for $r_1, r_2 > 0$, we have

$$\begin{aligned} &\int_\Omega \int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx \\ &= \frac{1}{r_1\partial_t\mathcal{H}(\gamma_0 E(t))}\int_0^\infty \Big\{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big) \\ &\times \quad \frac{r_1\partial_t\mathcal{H}(\gamma_0 E(t))\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)}\Big\}ds \\ &\leq -\frac{r_2}{r_1\partial_t\mathcal{H}(\gamma_0 E(t))}\int_\Omega\int_0^\infty \partial_t\varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx \\ &+ \frac{1}{r_1\partial_t\mathcal{H}(\gamma_0 E(t))}\int_0^\infty \mathcal{H}^*\Big(\frac{r_1\partial_t\mathcal{H}(\gamma_0 E(t))\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)}\Big)ds. \end{aligned}$$

By (31), (48), we have

$$\begin{aligned} &\int_\Omega\int_0^\infty \varpi(s)|y_x(t) - y_x(t-s)|^2 dsdx - \leq \frac{2r_2}{r_1\partial_t\mathcal{H}(\gamma_0 E(t))}\partial_t E(t) \\ &+ \int_0^\infty \Big\{\frac{\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)} \\ &\times \partial_t\mathcal{H}^{-1}\Big(\frac{r_1\partial_t\mathcal{H}(\gamma_0 E(t))\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(-r_2\partial_t\varpi(s)\int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)}\Big)\Big\}ds. \end{aligned}$$

By the fact that $\mathcal{H}^{-1}$ is concave and $\mathcal{H}^{-1}(0) = 0$, the function $h(s) = \frac{s}{\mathcal{H}^{-1}(s)}$, such that, for $0 \le s_1 < s_2$, we have

$$h(s_1) \le h(s_2).$$

Therefore, using (43) to obtain

$$\begin{aligned}
&\partial_t \mathcal{H}^{-1}\Big(\frac{r_1 \partial_t \mathcal{H}(\gamma_0 E(t))\varpi(s) \int_\Omega |y_x(t) - y_x(t-s)|^2 dx}{\mathcal{H}^{-1}\Big(- r_2 \partial_t \varpi(s) \int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)}\Big) \\
&= \partial_t \mathcal{H}^{-1}\Big[\frac{r_1 \partial_t \mathcal{H}(\gamma_0 E(t))\varpi(s)}{-r_2 \partial_t \varpi(s)} h\Big(- r_2 \partial_t \varpi(s) \int_\Omega |y_x(t) - y_x(t-s)|^2 dx\Big)\Big] \\
&\le \partial_t \mathcal{H}^{-1}\Big[\frac{r_1 \partial_t \mathcal{H}(\gamma_0 E(t))\varpi(s)}{-r_2 \partial_t \varpi(s)} h\Big(- r_2 \partial_t \varpi(s) \Big(\frac{8}{l} E(0) + 2m_0^2\Big)\Big)\Big] \\
&\le \partial_t \mathcal{H}^{-1}\Big[\frac{r_1 \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \partial_t \mathcal{H}(\gamma_0 E(t))\varpi(s)}{\mathcal{H}^{-1}\Big(- r_2 \partial_t \varpi(s) \Big(\frac{8}{l} E(0) + 2m_0^2\Big)\Big)}\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_\Omega \int_0^\infty \varpi(s) |y_x(t) - y_x(t-s)|^2 ds dx \\
&\le -\frac{2r_2}{r_1 \partial_t \mathcal{H}(\gamma_0 E(t))} \partial_t E(t) \\
&+ \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \int_0^\infty \Big\{ \frac{\varpi(s)}{\mathcal{H}^{-1}\Big(- r_2 \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \partial_t \varpi(s)\Big)} \\
&\times \partial_t \mathcal{H}^{-1}\Big[\frac{r_1 \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \partial_t \mathcal{H}(\gamma_0 E(t))\varpi(s)}{\mathcal{H}^{-1}\Big(- r_2 \partial_t \varpi(s) \Big(\frac{8}{1-l} E(0) + 2m_0^2\Big)\Big)}\Big]\Big\} ds.
\end{aligned}$$

By (Hypothesis 1), we have

$$\sup_{s \in \mathbb{R}^+} \frac{\varpi(s)}{\mathcal{H}^{-1}\Big(- \partial_t \varpi(s)\Big)} = \kappa_1 < \infty,$$

and

$$\int_0^\infty \frac{\varpi(s)}{\mathcal{H}^{-1}\Big(- \partial_t \varpi(s)\Big)} = \kappa_2 < \infty.$$

Because $\partial_t \mathcal{H}^{-1}$ is nondecreasing, we choose r_1, r_2, such that

$$\begin{aligned}
&\int_\Omega \int_0^\infty \varpi(s) |y_x(t) - y_x(t-s)|^2 ds dx \\
&\le -\frac{2\kappa_2}{\partial_t \mathcal{H}(\gamma_0 E(t))} \partial_t E(t) + \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \partial_t \mathcal{H}^{-1} \partial_t \mathcal{H}(\gamma_0 E(t)) \int_0^\infty \frac{\varpi(s)}{\mathcal{H}^{-1}\Big(- \partial_t \varpi(s)\Big)} \\
&\le -\frac{2\kappa_2}{\partial_t \mathcal{H}(\gamma_0 E(t))} \partial_t E(t) + \Big(\frac{8}{l} E(0) + 2m_0^2\Big) \gamma_0 E(t).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 4. Multiplying (46) by $\partial_t\mathcal{H}(\gamma_0 E(t))$ and using results in (47)

$$\begin{aligned}\partial_t\mathcal{H}(\gamma_0 E(t))\partial_t L(t) &\leq -\varepsilon_1\partial_t\mathcal{H}(\gamma_0 E(t))E(t)+c\partial_t\mathcal{H}(\gamma_0 E(t))\int_\Omega\int_0^\infty \varpi(s)|y_x(t)-y_x(t-s)|^2 ds dx\\ &\leq -[\varepsilon_1-c\gamma\gamma_0]\partial_t\mathcal{H}(\gamma_0 E(t))E(t)-c\gamma\partial_t E(t).\end{aligned}$$

We choose γ_0 small enough, so that $\varepsilon_1-c\gamma\gamma_0>0$.

Put

$$g(t)=\partial_t\mathcal{H}(\gamma_0 E(t))L(t)+c\gamma E(t)\sim E(t),$$

then,

$$\partial_t g(t)\leq -\kappa g(t)\partial_t\mathcal{H}(\gamma_0 g(t)),$$

which implies that $\partial_t(\mathcal{H}_1(g))\geq\kappa$, where

$$\mathcal{H}_1(\tau)=\int_\tau^1\frac{1}{s\partial_t\mathcal{H}(\gamma_0 s)}ds,\quad 0<\tau<1.$$

Integrating $\partial_t(\mathcal{H}_1(g))\geq\kappa$ over $[0,t]$, we get

$$g(t)\leq\mathcal{H}_1^{-1}(\kappa t+\kappa_0),$$

the equivalence between $E(t)$ and $g(t)$ gives the result. □

4. Conclusions

By imposing a new appropriate conditions (Hypothesis 1)–(Hypothesis 5), which seems not be used in the literature, with the help of some special results, we obtained an unusual a decay rate result while using properties of convex functions combined with some estimates, extending some earlier results known in the existing literature. The main results in this manuscript are the following. Theorem 3 for the global existence of solutions and Theorem 4 for the general decay rate.

A class of symmetric regularized long wave equations, which is known in abbreviation as (SRLWEs), is given by

$$\partial_{tt}y-y_{xx}-\partial_{tt}y_{xx}=-y\partial_t y_x-u_x\partial_t u. \tag{50}$$

Equation (50) was proposed as a model for propagation of weakly nonlinear ion acoustic and space charge waves; it is explicitly symmetric in the x and t derivatives and is very similar to the regularized long wave equation, which describes shallow water waves and plasma drift waves. The SRLWE and its symmetric version also arises in many other areas of mathematical physics. We remark that Equation (50) is special form of the equation that is discussed in (6), in which $F[y]=-y\partial_t y_x-u_x\partial_t u$.

Our research falls within the scope of the modern Time-partial differential equations interests; it is considered among the issues that have wide applications in science and engineering that are related to the energy systems. The importance of this research, although it is theoretical, lies in the following: we found that viscoelastic damping term causing the decrease in energy and decreasing followed the infinite memory, depending on initial data. It will be very interesting if one considers numerical studies. It will be our next research project.

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