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Arithmetics of Vectors of Fuzzy Sets

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Abstract: The arithmetic operations of fuzzy sets are completely different from the arithmetic operations of vectors of fuzzy sets. In this paper, the arithmetic operations of vectors of fuzzy intervals are studied by using the extension principle and a form of decomposition theorem. These two different methodologies lead to the different types of membership functions. We establish their equivalences under some mild conditions. On the other hand, the α-level sets of addition, difference and scalar products of vectors of fuzzy intervals are also studied, which will be useful for the different usage in applications.

Keywords: decomposition theorem; extension principle; fuzzy numbers; fuzzy intervals; non-normal fuzzy sets

1. Introduction

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $\mathbb{R}$ with the membership functions $\xi_{\tilde{A}}$ and $\xi_{\tilde{B}}$, respectively. The arithmetic operations $\tilde{A} \oplus \tilde{B}$, $\tilde{A} \ominus \tilde{B}$, $\tilde{A} \otimes \tilde{B}$ and $\tilde{A} \oslash \tilde{B}$ are based on the extension principle. More precisely, the membership functions are given by

$$\begin{aligned}
\xi_{\tilde{A}\oplus\tilde{B}}(z) &= \sup_{\{(x,y):z=x+y\}} \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\} \\
\xi_{\tilde{A}\ominus\tilde{B}}(z) &= \sup_{\{(x,y):z=x-y\}} \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\} \\
\xi_{\tilde{A}\otimes\tilde{B}}(z) &= \sup_{\{(x,y):z=x*y\}} \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\} \\
\xi_{\tilde{A}\oslash\tilde{B}}(z) &= \sup_{\{(x,y):z=x/y,y\neq 0\}} \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\}
\end{aligned}$$

for all $z \in \mathbb{R}$. In this paper, we consider the vectors of fuzzy sets in $\mathbb{R}$. The purpose is to study the addition, difference and scalar products of vectors of fuzzy sets.

Suppose that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ consist of fuzzy sets in $\mathbb{R}$ given by

$$\tilde{\mathbf{A}} = \left(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}\right) \text{ and } \tilde{\mathbf{B}} = \left(\tilde{B}^{(1)}, \cdots, \tilde{B}^{(n)}\right),$$

where $\tilde{A}^{(i)}$ and $\tilde{B}^{(i)}$ are fuzzy sets in $\mathbb{R}$ for $i = 1, \cdots, n$. Then, we study the addition $\tilde{\mathbf{A}} \oplus \tilde{\mathbf{B}}$, the difference $\tilde{\mathbf{A}} \ominus \tilde{\mathbf{B}}$ and the scalar product $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$.

The addition $\tilde{A}^{(i)} \oplus \tilde{B}^{(i)}$, the difference $\tilde{A}^{(i)} \ominus \tilde{B}^{(i)}$ and multiplication $\tilde{A}^{(i)} \otimes \tilde{B}^{(i)}$ regarding the components can be realized as shown above. Let $\xi_{\tilde{A}^{(i)}}$ and $\xi_{\tilde{B}^{(i)}}$ be the membership functions of $\tilde{A}^{(i)}$ and $\tilde{B}^{(i)}$, respectively, and let $\odot$ denote any one of the arithmetic operations $\oplus, \ominus, \otimes$ between $\tilde{A}^{(i)}$ and $\tilde{B}^{(i)}$. According to the extension principle, the membership function of $\tilde{A}^{(i)} \odot \tilde{B}^{(i)}$ is defined by

$$\xi_{\tilde{A}^{(i)}\odot\tilde{B}^{(i)}}(z) = \sup_{\{(x,y):z=x\circ y\}} \min\{\xi_{\tilde{A}^{(i)}}(x), \xi_{\tilde{B}^{(i)}}(y)\}$$

for all $z \in \mathbb{R}$, where the arithmetic operations $\odot \in \{\oplus, \ominus, \otimes\}$ correspond to the arithmetic operations $\circ \in \{+, -, *\}$. More detailed properties can refer to the monographs of Dubois and Prade [1] and Klir and Yuan [2]. In general, we can consider the t-norms instead of minimum functions by referring to Bede and Stefanini [3], Dubois and Prade [4], Gebhardt [5], Gomes and Barros [6], Fullér and Keresztfalvi [7], Mesiar [8], Ralescu [9], Weber [10], Wu [11–13] and Yager [14]. More precisely, the membership function of $\tilde{A}^{(i)} \odot \tilde{B}^{(i)}$ is given by

$$\xi_{\tilde{A}^{(i)} \odot \tilde{B}^{(i)}}(z) = \sup_{\{(x,y): z = x \circ y\}} t\left(\xi_{\tilde{A}^{(i)}}(x), \xi_{\tilde{B}^{(i)}}(y)\right)$$

for all $z \in \mathbb{R}$, where t is a t-norm that is a function from $[0,1] \times [0,1]$ into $[0,1]$ satisfying four axioms. It is well-known that the minimum function min is a t-norm. In this paper, we consider the general aggregation function rather than using t-norms. In this case, the membership function of $\tilde{A}^{(i)} \odot \tilde{B}^{(i)}$ is given by

$$\xi_{\tilde{A}^{(i)} \odot \tilde{B}^{(i)}}(z) = \sup_{\{(x,y): z = x \circ y\}} \mathfrak{A}\left(\xi_{\tilde{A}^{(i)}}(x), \xi_{\tilde{B}^{(i)}}(y)\right) \tag{1}$$

for all $z \in \mathbb{R}$, where $\mathfrak{A}$ is an aggregation function from $[0,1] \times [0,1]$ into $[0,1]$ without needing to satisfy some required conditions.

According to the arithmetic operations (1), the addition $\tilde{\mathbf{A}} \oplus \tilde{\mathbf{B}}$, the difference $\tilde{\mathbf{A}} \ominus \tilde{\mathbf{B}}$ and the scalar product $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$ can be naturally defined as follows

$$\begin{aligned}
\tilde{\mathbf{A}} \oplus \tilde{\mathbf{B}} &= \left(\tilde{A}^{(1)} \oplus \tilde{B}^{(1)}, \cdots, \tilde{A}^{(n)} \oplus \tilde{B}^{(n)}\right) \\
\tilde{\mathbf{A}} \ominus \tilde{\mathbf{B}} &= \left(\tilde{A}^{(1)} \ominus \tilde{B}^{(1)}, \cdots, \tilde{A}^{(n)} \ominus \tilde{B}^{(n)}\right) \\
\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}} &= \left(\tilde{A}^{(1)} \otimes \tilde{B}^{(1)}\right) \oplus \cdots \oplus \left(\tilde{A}^{(n)} \otimes \tilde{B}^{(n)}\right).
\end{aligned}$$

We can see that the scalar product $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$ is a fuzzy set in $\mathbb{R}$. The membership function of $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$ can be realized below. Let $\tilde{C}^{(i)} = \tilde{A}^{(i)} \otimes \tilde{B}^{(i)}$ for $i = 1, \cdots, n$. The membership function of $\tilde{C}^{(i)}$ can be obtained from (1). Therefore, the membership function of $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$ is given by

$$\xi_{\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}}(z) = \sup_{\{(x_1, \cdots, x_n): z = x_1 + \cdots + x_n\}} \mathfrak{A}\left(\xi_{\tilde{C}^{(1)}}(x_1), \cdots, \xi_{\tilde{C}^{(n)}}(x_n)\right),$$

where $\mathfrak{A}$ is an aggregation function from $[0,1]^n$ into $[0,1]$. In particular, the extension principle says that the aggregation function $\mathfrak{A}$ is given by the minimum function. Therefore, the membership function of $\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}$ is given by

$$\xi_{\tilde{\mathbf{A}} \bullet \tilde{\mathbf{B}}}(z) = \sup_{\{(x_1, \cdots, x_n): z = x_1 + \cdots + x_n\}} \min\{\xi_{\tilde{C}^{(1)}}(x_1), \cdots, \xi_{\tilde{C}^{(n)}}(x_n)\}.$$

We can see that $\tilde{\mathbf{A}} \oplus \tilde{\mathbf{B}}$ and $\tilde{\mathbf{A}} \ominus \tilde{\mathbf{B}}$ are still vectors of fuzzy sets. However, their membership functions cannot be obtained directly from (1). The main purpose of this paper is to propose two methodologies to define the membership functions of $\tilde{A}^{(i)} \oplus \tilde{B}^{(i)}$ and $\tilde{A}^{(i)} \ominus \tilde{B}^{(i)}$. Those methodologies can also be used to define the membership function of the scalar product $\tilde{A}^{(i)} \otimes \tilde{B}^{(i)}$.

Following the conventional way, we can use the extension principle to define the arithmetic operations of vectors of fuzzy sets. In this paper, we consider the general aggregation functions rather than using t-norms. We should mention that the decomposition theorem is a well-known result in fuzzy sets theory. Alternative, we also use the form of decomposition theorem to define the arithmetic operations of vectors of fuzzy intervals. These two methodologies can lead to the different types of membership functions. In this paper, we establish the equivalences between using the extension principle and the form of decomposition theorem under some mild conditions.

In Section 2, the concept and basic properties of non-normal fuzzy sets are presented. In Section 3, the arithmetic operations of vectors of fuzzy sets are presented using the extension principle based on the general aggregation functions. In Section 4, the arithmetic operations of vectors of fuzzy sets are presented using the form of decomposition theorem. In Section 5, many types of difference of vectors of fuzzy sets are proposed using the extension principle and the form of decomposition theorem, and their α-level sets are studied. Their equivalences are also established under some mild conditions. In Section 6, we study the addition of vectors of fuzzy sets following the same theme from Section 5. In Section 7, the scalar product of vectors of fuzzy sets are proposed, and their α-level sets are also studied.

2. Non-Normal Fuzzy Sets

Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\xi_{\tilde{A}}$. For $\alpha \in (0,1]$, the α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_\alpha = \{x \in \mathbb{R} : \xi_{\tilde{A}}(x) \geq \alpha\}. \tag{2}$$

We remark that the α-level set $\tilde{A}_\alpha$ can be an empty set when α is larger than the supremum of the membership function $\xi_{\tilde{A}}$. This ambiguity will be clarified in this section. On the other hand, the support of a fuzzy set $\tilde{A}$ is the crisp set defined by

$$\tilde{A}_{0+} = \{x \in \mathbb{R} : \xi_{\tilde{A}}(x) > 0\}.$$

The 0-level set $\tilde{A}_0$ is defined to be the closure of the support of $\tilde{A}$, i.e., $\tilde{A}_0 = \mathrm{cl}(\tilde{A}_{0+})$.

The range of membership function $\xi_{\tilde{A}}$ is denoted by $\mathcal{R}(\xi_{\tilde{A}})$ that is a subset of $[0,1]$. We see that the range $\mathcal{R}(\xi_{\tilde{A}})$ can be a proper subset of $[0,1]$ with $\mathcal{R}(\xi_{\tilde{A}}) \neq [0,1]$. For example, the range $\mathcal{R}(\xi_{\tilde{A}})$ can be some disjoint union of subintervals of $[0,1]$.

Example 1. *The membership function of a trapezoidal-like fuzzy number is given by*

$$\xi_{\tilde{A}}(x) = \begin{cases} 0.1 + 0.7 \cdot (x-1) & \text{if } 1 \leq x \leq 1.5 \\ 0.2 + 0.7 \cdot (x-1) & \text{if } 1.5 < x < 2 \\ 0.9 & \text{if } 2 \leq x \leq 3 \\ 0.2 + 0.7 \cdot (4-x) & \text{if } 3 < x < 3.5 \\ 0.1 + 0.7 \cdot (4-x) & \text{if } 3.5 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear to see that

$$\mathcal{R}(\xi_{\tilde{A}}) = [0.1, 0.45] \cup (0.55, 0.9].$$

Notice that if $\alpha \notin \mathcal{R}(\xi_{\tilde{A}})$, we still can consider the α-level set $\tilde{A}_\alpha$. Since $\mathcal{R}(\xi_{\tilde{A}}) \neq [0,1]$, it is possible that the α-level set $\tilde{A}_\alpha$ can be an empty set for some $\alpha \in [0,1]$. Therefore, when we study the properties that deal with more than two fuzzy sets, we cannot simply present the properties by saying that they hold true for each $\alpha \in [0,1]$, since some of the α-level sets can be empty. In this case, we need to carefully treat the ranges of membership functions.

Example 2. *Continuing from Example 1, we see that* $0.5 \notin \mathcal{R}(\xi_{\tilde{A}})$. *However, we still have the* 0.5*-level set* $\tilde{A}_{0.5}$. *It is clear to see that* $\tilde{A}_{0.5} = \tilde{A}_{0.45}$, *where* $0.45 \in \mathcal{R}(\xi_{\tilde{A}})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on $\mathbb{R}$, and let S be a subset of $\mathbb{R}$. Recall that the supremum $\sup_{x \in S} f(x)$ is attained if and only if there exists $x^* \in S$ such that $f(x) \leq f(x^*)$ for all $x \in S$ with $x \neq x^*$. Equivalently, the supremum $\sup_{x \in S} f(x)$ is attained if and only if

$$\sup_{x \in S} f(x) = \max_{x \in S} f(x).$$

Define $\alpha^* = \sup \mathcal{R}(\xi_{\tilde{A}})$. If $\sup \mathcal{R}(\xi_{\tilde{A}}) = \max \mathcal{R}(\xi_{\tilde{A}})$, then $\tilde{A}_{\alpha^*} \neq \emptyset$. If the supremum $\sup \mathcal{R}(\xi_{\tilde{A}})$ is not attained, then $\tilde{A}_{\alpha^*} = \emptyset$. For example, assume that

$$\xi_{\tilde{A}}(x) = \begin{cases} 1 - \frac{1}{x}, & \text{if } x \geq 1 \\ 0, & \text{if } x < 1. \end{cases}$$

It is clear to see that $\mathcal{R}(\xi_{\tilde{A}}) = [0,1)$. In this case, the supremum $\sup \mathcal{R}(\xi_{\tilde{A}})$ is not attained. However, we have $\sup \mathcal{R}(\xi_{\tilde{A}}) = 1 = \alpha^*$. In this case, the 1-level set $\tilde{A}_{\alpha^*} = \tilde{A}_1 = \emptyset$, since $\alpha^* = 1 \notin \mathcal{R}(\xi_{\tilde{A}})$.

Proposition 1. *Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\xi_{\tilde{A}}$. Define $\alpha^* = \sup \mathcal{R}(\xi_{\tilde{A}})$ and*

$$I_{\tilde{A}} = \begin{cases} [0, \alpha^*), & \textit{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \textit{ does not exist;} \\ [0, \alpha^*], & \textit{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \textit{ exists.} \end{cases} \tag{3}$$

Then $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$. Moreover, we have $\mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$ and

$$\tilde{A}_{0+} = \bigcup_{\{\alpha \in I_{\tilde{A}} : \alpha > 0\}} \tilde{A}_\alpha = \bigcup_{\{\alpha \in \mathcal{R}(\xi_{\tilde{A}}) : \alpha > 0\}} \tilde{A}_\alpha. \tag{4}$$

The interval $I_{\tilde{A}}$ presented in Proposition 1 is also called an interval range of $\tilde{A}$. We see that the interval range $I_{\tilde{A}}$ contains the actual range $\mathcal{R}(\xi_{\tilde{A}})$. The role of interval range $I_{\tilde{A}}$ can be used to say $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$. We also remark that $\mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$ and $\mathcal{R}(\xi_{\tilde{A}}) \neq I_{\tilde{A}}$ in general, since the range $\mathcal{R}(\xi_{\tilde{A}})$ can be some disjoint union of subintervals of $[0,1]$.

Example 3. *Continuing from Example 1, recall that $\mathcal{R}(\xi_{\tilde{A}}) = [0.1, 0.45] \cup (0.55, 0.9]$. We also see that $\sup \mathcal{R}(\xi_{\tilde{A}}) = \alpha^* = 0.9$. Proposition 1 says that $I_{\tilde{A}} = [0, 0.9] \neq \mathcal{R}(\xi_{\tilde{A}})$. It is clear to see that $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}} = [0, 0.9]$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}} = [0, 0.9]$.*

Therefore, the interval $I_{\tilde{A}}$ plays an important role for considering the α-level sets. In other words, the range $\mathcal{R}(\xi_{\tilde{A}})$ is not helpful for identifying the α-level sets.

Recall that $\tilde{A}$ is called a normal fuzzy set in $\mathbb{R}$ if and only if there exists $x \in \mathbb{R}$ such that $\xi_{\tilde{A}}(x) = 1$. In this case, we have $I_{\tilde{A}} = [0,1]$. However, the range $\mathcal{R}(\xi_{\tilde{A}})$ is not necessarily equal to $[0,1]$ even though $\tilde{A}$ is normal.

Let $\tilde{A}$ be a normal fuzzy set in $\mathbb{R}$. The well-known decomposition theorem says that the membership function $\xi_{\tilde{A}}$ can be expressed as

$$\xi_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x),$$

where $\chi_{\tilde{A}_\alpha}$ is the characteristic function of the α-level set $\tilde{A}_\alpha$. If $\tilde{A}$ is not normal, then we can similarly obtain the following form.

Theorem 1. (**Decomposition Theorem**) *Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$. Then the membership function $\xi_{\tilde{A}}$ can be expressed as*

$$\begin{aligned} \xi_{\tilde{A}}(x) &= \sup_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) \\ &= \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x), \end{aligned}$$

where $I_{\tilde{A}}$ is given in (3).

3. Arithmetics Using the Extension Principle

The generalized extension principle for non-normal fuzzy sets has been extensively studied in Wu [15]. In this paper, we use the extension principle to study the arithmetics of a vector of fuzzy intervals.

We denote by $\mathcal{F}_{cc}(\mathbb{R})$ the family of all fuzzy sets in $\mathbb{R}$ such that each $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfies the following conditions.

- The membership function $\xi_{\tilde{a}}$ is upper semi-continuous and quasi-concave on $\mathbb{R}$.
- The 0-level set $\tilde{a}_0$ is a compact subset of $\mathbb{R}$; that is, a closed and bounded subset of $\mathbb{R}$.

Each $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is also called a fuzzy interval. If the fuzzy interval $\tilde{a}$ is normal and the 1-level set $\tilde{a}_1$ is a singleton set $\{a\}$, where $a \in \mathbb{R}$, then $\tilde{a}$ is also called a fuzzy number with core value a. It is well-known that the α-level sets of fuzzy interval $\tilde{a}$ are all closed intervals denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ for $\alpha \in [0,1]$, which can be regarded as a closed interval with degree α. This is the reason why we call $\tilde{a}$ as a fuzzy interval.

Example 4. *The membership function of a trapezoidal fuzzy interval is given by*

$$\xi_{\tilde{a}}(r) = \begin{cases} (r - a^L)/(a_1 - a^L) & \text{if } a^L \leq r \leq a_1 \\ d^* & \text{if } a_1 < r \leq a_2 \\ (a^U - r)/(a^U - a_2) & \text{if } a_2 < r \leq a^U \\ 0 & \text{otherwise,} \end{cases}$$

which is denoted by $\tilde{a} = (d^*; a^L, a_1, a_2, a^U)$. It is clear to see that

$$\mathcal{R}(\xi_{\tilde{a}}) = [0, d^*] \text{ and } \alpha^* \equiv \sup \mathcal{R}(\xi_{\tilde{A}}) = d^*.$$

Proposition 1 says that the interval range is given by

$$I_{\tilde{a}} = [0, \alpha^*] = [0, d^*] = \mathcal{R}(\xi_{\tilde{a}}).$$

If $\alpha \notin I_{\tilde{a}} = [0, d^*]$, then the α-level set $\tilde{a}_\alpha = \emptyset$. For $\alpha \in I_{\tilde{a}}$, the α-level set $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ is given by

$$\tilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a_1 \text{ and } \tilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a_2. \tag{5}$$

Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ be two vectors in $\mathbb{R}^n$. Then, the arithmetics of vectors $\mathbf{x}$ and $\mathbf{y}$ are given by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, \cdots, x_n + y_n) \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, \cdots, x_n - y_n) \\ \mathbf{x} \bullet \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n. \end{aligned}$$

Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be two vectors of fuzzy intervals given by

$$\tilde{\mathbf{a}} = \left(\tilde{a}^{(1)}, \tilde{a}^{(2)}, \cdots, \tilde{a}^{(n)}\right) \text{ and } \tilde{\mathbf{b}} = \left(\tilde{b}^{(1)}, \tilde{b}^{(2)}, \cdots, \tilde{b}^{(n)}\right).$$

Based on the extension principle (abbreviated as EP), we study the arithmetics of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ by considering the scalar product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, the addition $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$ and the difference $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$. Given the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$, the membership functions are defined below.

- For each $z \in \mathbb{R}$, the membership function of the scalar product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}(z) = \sup_{\{(\mathbf{x},\mathbf{y}): z = \mathbf{x} \bullet \mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right). \tag{6}$$

- For each $\mathbf{z} \in \mathbb{R}^n$ and for the operation $\odot \in \{\oplus, \ominus\}$ corresponding to the operation $\circ \in \{+, -\}$, the membership function of $\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}\circ\mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right). \tag{7}$$

If the aggregation function $\mathfrak{A} \equiv \min$ is taken to be the minimum function, then the above arithmetics coincide with the extension principle.

Given any fuzzy intervals $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ in $\mathbb{R}$, let

$$\alpha_i^* = \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) = \sup_{x \in \mathbb{R}} \xi_{\tilde{a}^{(i)}}(x) \text{ for } i = 1, \cdots, n$$

and

$$\beta_i^* = \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) = \sup_{x \in \mathbb{R}} \xi_{\tilde{b}^{(i)}}(x) \text{ for } i = 1, \cdots, n.$$

From Proposition 1, the interval ranges $I_{\tilde{a}^{(i)}}$ of $\tilde{a}^{(i)}$ and $I_{\tilde{b}^{(i)}}$ of $\tilde{b}^{(i)}$ are given by

$$I_{\tilde{a}^{(i)}} = \begin{cases} [0, \alpha_i^*), & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ is not attained} \\ [0, \alpha_i^*], & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ is attained} \end{cases} \tag{8}$$

and

$$I_{\tilde{b}^{(i)}} = \begin{cases} [0, \beta_i^*), & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \text{ is not attained} \\ [0, \beta_i^*], & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \text{ is attained} \end{cases} \tag{9}$$

We also write $\mathcal{R}_i \equiv \mathcal{R}(\xi_{\tilde{a}^{(i)}})$ to denote the ranges of membership functions $\xi_{\tilde{a}^{(i)}}$ for $i = 1, \cdots, n$, and write $\mathcal{R}_{n+i} \equiv \mathcal{R}(\xi_{\tilde{b}^{(i)}})$ to denote the ranges of membership functions $\xi_{\tilde{b}^{(i)}}$ for $i = 1, \cdots, n$. Let

$$\alpha^* = \sup_{(\alpha_1, \cdots, \alpha_{2n}) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_{2n}} \mathfrak{A}(\alpha_1, \cdots, \alpha_{2n}). \tag{10}$$

Example 5. *Continuing from Example 4, we consider the following trapezoidal fuzzy intervals*

$$\tilde{a}^{(1)} = (0.8; 1, 2, 3, 4) \textit{ and } \tilde{a}^{(2)} = (0.9; 2, 3, 4, 5)$$

and

$$\tilde{b}^{(1)} = (0.9; 4, 5, 6, 7) \textit{ and } \tilde{b}^{(2)} = (0.8; 3, 4, 5, 6)$$

Then, we have

$$\alpha_1^* = 0.8 = \beta_2^* \textit{ and } \alpha_2^* = 0.9 = \beta_1^*,$$

and the interval ranges are given by

$$I_{\tilde{a}^{(1)}} = [0, 0.8] = I_{\tilde{b}^{(2)}} = \mathcal{R}_1 \equiv \mathcal{R}(\xi_{\tilde{a}^{(1)}}) = \mathcal{R}_4 \equiv \mathcal{R}(\xi_{\tilde{b}^{(2)}})$$

and

$$I_{\tilde{a}^{(2)}} = [0, 0.9] = I_{\tilde{b}^{(1)}} = \mathcal{R}_2 \equiv \mathcal{R}(\xi_{\tilde{a}^{(2)}}) = \mathcal{R}_3 \equiv \mathcal{R}(\xi_{\tilde{b}^{(1)}}).$$

From (10), by taking the aggregation function $\mathfrak{A}$ as the minimum function, we have

$$\begin{aligned} \alpha^* &= \sup_{(\alpha_1, \cdots, \alpha_4) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_4} \min\{\alpha_1, \cdots, \alpha_4\} \\ &= \sup_{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in [0,0.8] \times [0,0.9] \times [0,0.9] \times [0,0.8]} \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = 0.8 \end{aligned}$$

We denote by $I_{\circledast}^{(EP)}$ and $I_{\odot}^{(EP)}$ the interval ranges of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}$, respectively, where $I_{\circledast}^{(EP)}$ and $I_{\odot}^{(EP)}$ depend on $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$. The supremum of range of membership function is given by

$$\begin{aligned} \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\odot_{EP}\tilde{\mathbf{b}}}\right) &= \sup_{\mathbf{z}\in\mathbb{R}^m} \xi_{\tilde{\mathbf{a}}\odot_{EP}\tilde{\mathbf{b}}}(\mathbf{z}) \\ &= \sup_{\mathbf{z}\in\mathbb{R}^m} \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}\circ\mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1),\cdots,\xi_{\tilde{a}^{(n)}}(x_n),\xi_{\tilde{b}^{(1)}}(y_1),\cdots,\xi_{\tilde{b}^{(n)}}(y_n)\right) \\ &= \sup_{(\alpha_1,\cdots,\alpha_{2n})\in\mathcal{R}_1\times\cdots\times\mathcal{R}_{2n}} \mathfrak{A}\left(\alpha_1,\cdots,\alpha_{2n}\right) = \alpha^*. \end{aligned} \tag{11}$$

We can similarly obtain

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\circledast_{EP}\tilde{\mathbf{b}}}\right) = \alpha^*.$$

Therefore, the definition of interval range says that

$$I_{\circledast}^{(EP)} = \begin{cases} [0,\alpha^*] & \text{if the supremum } \alpha^* = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\circledast_{EP}\tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0,\alpha^*) & \text{otherwise} \end{cases}$$

and

$$I_{\odot}^{(EP)} = \begin{cases} [0,\alpha^*] & \text{if the supremum } \alpha^* = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\odot_{EP}\tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0,\alpha^*) & \text{otherwise.} \end{cases}$$

Proposition 1 says that

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I_{\circledast}^{(EP)} \text{ and } \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I_{\circledast}^{(EP)}$$

and

$$\left(\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I_{\odot}^{(EP)} \text{ and } \left(\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I_{\odot}^{(EP)}.$$

Example 6. *Continuing from Example 5, we take the aggregation function $\mathfrak{A}$ as the minimum function. The membership function of scalar product $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \circledast_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ is given by*

$$\begin{aligned} &\xi_{(\tilde{a}^{(1)},\tilde{a}^{(2)})\circledast_{EP}(\tilde{b}^{(1)},\tilde{b}^{(2)})}(z) \\ &= \sup_{\{(x_1,x_2,y_1,y_2):z=(x_1,x_2)\bullet(y_1,y_2)\}} \min\left\{\xi_{\tilde{a}^{(1)}}(x_1),\xi_{\tilde{a}^{(2)}}(x_2),\xi_{\tilde{b}^{(1)}}(y_1),\xi_{\tilde{b}^{(2)}}(y_2)\right\}, \end{aligned}$$

and it is a continuous function. Therefore, the supremum

$$0.8 = \alpha^* = \sup \mathcal{R}\left(\xi_{(\tilde{a}^{(1)},\tilde{a}^{(2)})\circledast_{EP}(\tilde{b}^{(1)},\tilde{b}^{(2)})}\right)$$

is attained. This says that the interval range $I_{\circledast}^{(EP)}$ of scalar product $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \circledast_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ is given

$$I_{\circledast}^{(EP)} = [0,\alpha^*] = [0,0.8].$$

By considering the α-level sets, we also see that

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \circledast_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha \neq \emptyset \textit{ for } \alpha \in I_{\circledast}^{(EP)} = [0,0.8]$$

and

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \circledast_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha = \emptyset \textit{ for } \alpha \notin I_{\circledast}^{(EP)} = [0,0.8].$$

The membership function of addition $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \oplus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ *is given by*

$$\begin{aligned}&\xi_{(\tilde{a}^{(1)},\tilde{a}^{(2)})\oplus_{EP}(\tilde{b}^{(1)},\tilde{b}^{(2)})}(z_1,z_2)\\&=\sup_{\{(x_1,x_2,y_1,y_2):(z_1,z_2)=(x_1+x_2,y_1+y_2)\}}\min\left\{\xi_{\tilde{a}^{(1)}}(x_1),\xi_{\tilde{a}^{(2)}}(x_2),\xi_{\tilde{b}^{(1)}}(y_1),\xi_{\tilde{b}^{(2)}}(y_2)\right\}.\end{aligned}$$

The interval range $I_{\oplus}^{(EP)}$ *of addition* $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \oplus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ *is given by*

$$I_{\oplus}^{(EP)}=[0,\alpha^*]=[0,0.8].$$

By considering the α*-level sets, we also see that*

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \oplus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha \neq \emptyset \text{ for } \alpha\in I_{\oplus}^{(EP)}=[0,0.8]$$

and

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \oplus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha = \emptyset \text{ for } \alpha\notin I_{\oplus}^{(EP)}=[0,0.8].$$

The membership function of difference $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \ominus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ *is given by*

$$\begin{aligned}&\xi_{(\tilde{a}^{(1)},\tilde{a}^{(2)})\ominus_{EP}(\tilde{b}^{(1)},\tilde{b}^{(2)})}(z_1,z_2)\\&=\sup_{\{(x_1,x_2,y_1,y_2):(z_1,z_2)=(x_1-x_2,y_1-y_2)\}}\min\left\{\xi_{\tilde{a}^{(1)}}(x_1),\xi_{\tilde{a}^{(2)}}(x_2),\xi_{\tilde{b}^{(1)}}(y_1),\xi_{\tilde{b}^{(2)}}(y_2)\right\}.\end{aligned}$$

The interval range $I_{\ominus}^{(EP)}$ *of addition* $(\tilde{a}^{(1)},\tilde{a}^{(2)}) \ominus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})$ *is given*

$$I_{\ominus}^{(EP)}=[0,\alpha^*]=[0,0.8].$$

By considering the α*-level sets, we also see that*

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \ominus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha \neq \emptyset \text{ for } \alpha\in I_{\ominus}^{(EP)}=[0,0.8]$$

and

$$\left((\tilde{a}^{(1)},\tilde{a}^{(2)}) \ominus_{EP} (\tilde{b}^{(1)},\tilde{b}^{(2)})\right)_\alpha = \emptyset \text{ for } \alpha\notin I_{\ominus}^{(EP)}=[0,0.8].$$

For further discussion, we provide a useful lemma.

Lemma 1. (Royden [16], p. 161) *Let* X *be a topological space, and let* K *be a compact subset of* X*. Let* f *be a real-valued function defined on* X*. Then the following statements hold true.*

(i) *If* f *is upper semi-continuous, then* f *assumes its maximum on a compact subset of* X*; that is, the supremum is attained in the following sense:*

$$\sup_{x\in K} f(x)=\max_{x\in K} f(x).$$

(ii) *If* f *is lower semi-continuous, then* f *assumes its minimum on a compact subset of* X*; that is, the infimum is attained in the following sense:*

$$\inf_{x\in K} f(x)=\min_{x\in K} f(x).$$

Proposition 2. *Suppose that the aggregation function* $\mathfrak{A}:[0,1]^{2n}\to[0,1]$ *is given by*

$$\mathfrak{A}(\alpha_1,\cdots,\alpha_{2n})=\begin{cases}\min\{\alpha_1,\cdots,\alpha_{2n}\}, & \text{if } \alpha_i\in\mathcal{R}_i \text{ for } i=1,\cdots,2n\\ \text{any expression}, & \text{otherwise},\end{cases}$$

Let

$$I^* \equiv I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

Then, the following statements hold true.

(i) *We have*

$$\alpha^* = \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\}.$$

(ii) *The supremum* $\sup \mathcal{R}(\xi_{\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}})$ *is attained if and only if the supremum* $\sup I^*$ *is attained, and the supremum* $\sup \mathcal{R}(\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}})$ *is attained if and only if the supremum* $\sup I^*$ *is attained.*

(iii) *We have*

$$I_{\odot}^{(EP)} = I^* = I_{\circledast}^{(EP)}.$$

Proof. It suffices to prove the case of $\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}$, since the case of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ can be similarly obtained. From (11), we have

$$\alpha^* = \sup_{(\alpha_1, \cdots, \alpha_{2n}) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_{2n}} \min\left\{\alpha_1, \cdots, \alpha_{2n}\right\} \geq \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\}.$$

On the other hand, from (11) again, we also have

$$\begin{aligned} \alpha^* &= \sup_{\mathbf{z} \in \mathbb{R}^m} \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}\circ\mathbf{y}\}} \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} \quad (12) \\ &\leq \sup_{\mathbf{z} \in \mathbb{R}^m} \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}\circ\mathbf{y}\}} \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\} = \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\}, \end{aligned}$$

which proves part (i).

Suppose that the supremum $\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}}\right)$ is attained. From (11), there exists $\mathbf{z}^* \in \mathbb{R}^n$ such that

$$\sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}^*=\mathbf{x}\circ\mathbf{y}\}} \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} = \alpha^*. \quad (13)$$

Since the set $\{(\mathbf{x}, \mathbf{y}) : \mathbf{z}^* = \mathbf{x} \circ \mathbf{y}\}$ is closed and bounded, i.e., a compact set, and the functions $\xi_{\tilde{a}^{(i)}}$ and $\xi_{\tilde{b}^{(i)}}$ are upper semi-continuous, Lemma 1 says that the supremum in (13) is attained. In other words, there exists $(\mathbf{x}^*, \mathbf{y}^*)$ such that

$$\xi_{\tilde{a}^{(n_0)}}(x_{n_0}^*) = \alpha^* \text{ or } \xi_{\tilde{b}^{(n_0)}}(y_{n_0}^*) = \alpha^* \text{ for some } n_0 \in \{1, \cdots, n\}. \quad (14)$$

For convenience, we write $\alpha_{n+i}^* \equiv \beta_i^*$, $x_{n+i}^* \equiv y_i^*$ and $\tilde{a}^{(n+i)} \equiv \tilde{b}^{(i)}$ for $i = 1, \cdots, n$. Then, from (13) and (14), we have

$$\min\left\{\xi_{\tilde{a}^{(1)}}(x_1^*), \cdots, \xi_{\tilde{a}^{(n)}}(x_n^*), \xi_{\tilde{a}^{(n+1)}}(x_{n+1}^*), \cdots, \xi_{\tilde{a}^{(2n)}}(x_{2n}^*)\right\} = \alpha^*, \quad (15)$$

and we can say that $\xi_{\tilde{a}^{(n_1)}}(x_{n_1}^*) = \alpha^*$ for some $n_1 \in \{1, \cdots, 2n\}$. Part (i) also says that $\alpha^* = \alpha_{n_2}^*$ for some $n_2 \in \{1, \cdots, 2n\}$. Then, using (15), we have

$$\alpha_{n_2}^* = \alpha^* = \xi_{\tilde{a}^{(n_1)}}(x_{n_1}^*) \leq \xi_{\tilde{a}^{(n_2)}}(x_{n_2}^*) \leq \alpha_{n_2}^*,$$

which says that the supremum $\alpha_{n_2}^* = \sup \mathcal{R}(\xi_{\tilde{a}^{(n_2)}})$ is attained. Using (8) and (9), we obtain $I_{\tilde{a}^{(n_2)}} = [0, \alpha_{n_2}^*]$ is a closed interval, which also says that $I^* = [0, \alpha_{n_2}^*]$. Therefore, we conclude that the supremum $\sup I^*$ is also attained.

On the other hand, suppose that the supremum $\sup I^*$ is attained. Then, we have

$$I^* = [0, \alpha_{n_3}^*] = I_{\tilde{a}^{(n_3)}} \text{ for some } n_3 \in \{1, \cdots, 2n\}$$

and

$$\alpha_{n_3}^* = \min\left\{\alpha_1^*, \cdots, \alpha_{2n}^*\right\} = \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\} = \alpha^*,$$

which also says that the supremum $\alpha_{n_3}^* = \sup \mathcal{R}(\xi_{\tilde{a}^{(n_3)}})$ is attained; i.e., there exists $x_{n_3}^\circ \in \mathbb{R}$ such that $\xi_{\tilde{a}^{(n_3)}}(x_{n_3}^\circ) = \alpha_{n_3}^* = \alpha^*$. By referring to (12), there exists $\mathbf{x}^\circ \in \mathbb{R}^{2n}$ such that its n_3-component is $x_{n_3}^\circ$ and

$$\begin{aligned}\alpha^* &= \min\left\{\xi_{\tilde{a}^{(1)}}(x_1^\circ), \cdots, \xi_{\tilde{a}^{(n_3)}}(x_{n_3}^\circ) = \alpha^*, \cdots, \xi_{\tilde{a}^{(2n)}}(x_{2n}^\circ)\right\} \\ &= \min\left\{\xi_{\tilde{a}^{(1)}}(x_1^\circ), \cdots, \xi_{\tilde{a}^{(n)}}(x_n^\circ), \xi_{\tilde{b}^{(1)}}(y_1^\circ), \cdots, \xi_{\tilde{b}^{(n)}}(y_n^\circ)\right\},\end{aligned}$$

where $y_i^\circ = x_{n+i}^\circ$ for $i = 1, \cdots, n$. In this case, we have $\mathbf{z}^\circ = \mathbf{x}^\circ \circ \mathbf{y}^\circ$, which says that the supremum $\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \odot_{EP} \tilde{\mathbf{b}}}\right)$ is attained, which proves part (ii). Finally, part (iii) follows immediately from parts (i) and (ii). This completes the proof. □

4. Arithmetics Using the Form of Decomposition Theorem

The differentiation and integrals of fuzzy-number-valued functions using the form of decomposition theorem have been studied in Wu [17]. In this paper, we use the form of decomposition theorem to study the arithmetics of vector of fuzzy intervals.

Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be two vectors of fuzzy intervals with components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$, respectively, for $i = 1, \cdots, n$. Let

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}. \tag{16}$$

Then I^* is not empty, since $I_{\tilde{a}^{(i)}}$ and $I_{\tilde{b}^{(i)}}$ are intervals with left end-point 0 for $i = 1, \cdots, n$. For each $\alpha \in I^*$, the α-level sets of $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are nonempty and denoted by

$$\tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \text{ and } \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right].$$

We write

$$\left(\tilde{a}_{1\alpha}^L, \tilde{a}_{2\alpha}^L, \cdots, \tilde{a}_{n\alpha}^L\right) = \tilde{\mathbf{a}}_\alpha^L \in \mathbb{R}^n \text{ and } \left(\tilde{a}_{1\alpha}^U, \tilde{a}_{2\alpha}^U, \cdots, \tilde{a}_{n\alpha}^U\right) = \tilde{\mathbf{a}}_\alpha^U \in \mathbb{R}^n. \tag{17}$$

We also write

$$\tilde{\mathbf{a}}_\alpha = \tilde{a}_\alpha^{(1)} \times \cdots \times \tilde{a}_\alpha^{(n)} = \left[\tilde{a}_{1\alpha}^L, \tilde{a}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L, \tilde{a}_{n\alpha}^U\right] \tag{18}$$

and

$$\tilde{\mathbf{b}}_\alpha = \tilde{b}_\alpha^{(1)} \times \cdots \times \tilde{b}_\alpha^{(n)} = \left[\tilde{b}_{1\alpha}^L, \tilde{b}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{b}_{n\alpha}^L, \tilde{b}_{n\alpha}^U\right]. \tag{19}$$

In order to define the difference $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$, we consider the family $\{M_\alpha^- : \alpha \in I^* \text{ with } \alpha > 0\}$ that is formed by applying the operation $\mathbf{x} - \mathbf{y}$ to the α-level sets $\tilde{a}_\alpha^{(i)}$ and $\tilde{b}_\alpha^{(i)}$ for $i = 1, \cdots, n$, where each M_α^- is a subset of $\mathbb{R}^m$. In this paper, we study three different families described below.

- We take
$$M_\alpha^- = \tilde{\mathbf{a}}_\alpha - \tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x} - \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\right\}$$
to define $\tilde{\mathbf{a}} \ominus_{DT}^\diamond \tilde{\mathbf{b}}$.
- We take
$$M_\alpha^- = \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta^{(1-)}\right) \times \cdots \times \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta^{(n-)}\right),$$
where $M_\beta^{(i-)}$ are bounded closed intervals given by
$$M_\beta^{(i-)} = \left[\min\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}, \max\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}\right].$$

for $i = 1, \cdots, n$ to define $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$.

- We take
$$M_{\alpha}^{-} = M_{\alpha}^{(1-)} \times \cdots \times M_{\alpha}^{(n-)},$$
where $M_{\alpha}^{(i-)}$ are bounded closed intervals given by
$$M_{\alpha}^{(i-)} = \left[\min\left\{\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{U}\right\}, \max\left\{\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{U}\right\}\right]$$
for $i = 1, \cdots, n$ to define $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$.

For $\ominus_{DT} \in \{\ominus_{DT}^{\diamond}, \ominus_{DT}^{\star}, \ominus_{DT}^{\dagger}\}$, based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha \in I^{*} : \alpha > 0\}} \alpha \cdot \chi_{M_{\alpha}^{-}}(\mathbf{z}).$$

Example 7. *Continuing from Example 5, we have*

$$I^{*} = I_{\tilde{a}^{(1)}} \cap I_{\tilde{a}^{(2)}} \cap I_{\tilde{b}^{(1)}} \cap I_{\tilde{b}^{(2)}} = [0, 0.8] \cap [0, 0.9] \cap [0, 0.9] \cap [0, 0.8] = [0, 0.8].$$

From (5), we have

$$\begin{aligned}
\tilde{a}_{\alpha}^{(1)} &= \left[\tilde{a}_{1\alpha}^{L}, \tilde{a}_{1\alpha}^{U}\right] = [(1-\alpha) + 2\alpha, 4(1-\alpha) + 3\alpha] = [1+\alpha, 4-\alpha] \\
\tilde{a}_{\alpha}^{(2)} &= \left[\tilde{a}_{2\alpha}^{L}, \tilde{a}_{2\alpha}^{U}\right] = [2(1-\alpha) + 3\alpha, 5(1-\alpha) + 4\alpha] = [2+\alpha, 5-\alpha] \\
\tilde{b}_{\alpha}^{(1)} &= \left[\tilde{b}_{1\alpha}^{L}, \tilde{b}_{1\alpha}^{U}\right] = [4(1-\alpha) + 5\alpha, 7(1-\alpha) + 6\alpha] = [4+\alpha, 7-\alpha] \\
\tilde{b}_{\alpha}^{(2)} &= \left[\tilde{b}_{2\alpha}^{L}, \tilde{b}_{2\alpha}^{U}\right] = [3(1-\alpha) + 4\alpha, 6(1-\alpha) + 5\alpha] = [3+\alpha, 6-\alpha].
\end{aligned}$$

Then we have

$$\tilde{\mathbf{a}}_{\alpha} = \tilde{a}_{\alpha}^{(1)} \times \tilde{a}_{\alpha}^{(2)} = \left[\tilde{a}_{1\alpha}^{L}, \tilde{a}_{1\alpha}^{U}\right] \times \left[\tilde{a}_{n\alpha}^{L}, \tilde{a}_{n\alpha}^{U}\right] = [1+\alpha, 4-\alpha] \times [2+\alpha, 5-\alpha]$$

and

$$\tilde{\mathbf{b}}_{\alpha} = \tilde{b}_{\alpha}^{(1)} \times \tilde{b}_{\alpha}^{(2)} = \left[\tilde{b}_{1\alpha}^{L}, \tilde{b}_{1\alpha}^{U}\right] \times \left[\tilde{b}_{n\alpha}^{L}, \tilde{b}_{n\alpha}^{U}\right] = [4+\alpha, 7-\alpha] \times [3+\alpha, 6-\alpha].$$

We consider three families

$$\{M_{\alpha}^{-} : \alpha \in I^{*} \text{ with } \alpha > 0\} = \{M_{\alpha}^{-} : \alpha \in (0, 0.8]\}$$

given below.

- *We take*
$$\begin{aligned}
M_{\alpha}^{-} = \tilde{\mathbf{a}}_{\alpha} - \tilde{\mathbf{b}}_{\alpha} &= \{(x_1, x_2) - (y_1, y_2) : (x_1, x_2) \in [1+\alpha, 4-\alpha] \times [2+\alpha, 5-\alpha] \\
&\qquad \text{and } (y_1, y_2) \in [4+\alpha, 7-\alpha] \times [3+\alpha, 6-\alpha]\} \\
&= [-6+2\alpha, -2\alpha] \times [-4+2\alpha, 2-2\alpha]
\end{aligned}$$
for all $\alpha \in (0, 0.8]$. The membership function of $\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}$ is given by
$$\xi_{\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}}(z_1, z_2) = \sup_{\{\alpha \in I^{*} : \alpha > 0\}} \alpha \cdot \chi_{M_{\alpha}^{-}}(z_1, z_2) = \sup_{\alpha \in (0, 0.8]} \alpha \cdot \chi_{M_{\alpha}^{-}}(z_1, z_2).$$

- *We take*
$$M_\alpha^- = \left(\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} M_\beta^{(1-)} \right) \times \left(\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} M_\beta^{(2-)} \right),$$
where $M_\beta^{(1-)}$ *and* $M_\beta^{(2-)}$ *are bounded closed intervals given by*
$$\begin{aligned} M_\beta^{(1-)} &= \left[\min\left\{\tilde{a}_{1\beta}^L - \tilde{b}_{1\beta}^L, \tilde{a}_{1\beta}^U - \tilde{b}_{1\beta}^U\right\}, \max\left\{\tilde{a}_{1\beta}^L - \tilde{b}_{1\beta}^L, \tilde{a}_{1\beta}^U - \tilde{b}_{1\beta}^U\right\}\right] \\ &= \left[\min\left\{(1+\beta)-(4+\beta), (4-\beta)-(7-\beta)\right\}, \max\left\{(1+\beta)-(4+\beta), (4-\beta)-(7-\beta)\right\}\right] \\ &= [-3,-3] = \{-3\}, \end{aligned}$$
which says that $M_\beta^{(1-)}$ *is a singleton set* $\{-3\}$ *for all* $\beta \in [0, 0.8]$. *Similarly, we can obtain*
$$M_\beta^{(2-)} = \left[\min\left\{\tilde{a}_{2\beta}^L - \tilde{b}_{2\beta}^L, \tilde{a}_{2\beta}^U - \tilde{b}_{2\beta}^U\right\}, \max\left\{\tilde{a}_{1\beta}^L - \tilde{b}_{1\beta}^L, \tilde{a}_{2\beta}^U - \tilde{b}_{2\beta}^U\right\}\right] = \{-1\}$$
for all $\beta \in [0, 0.8]$. *Therefore, we obtain*
$$M_\alpha^- = \{-3\} \times \{-1\} = \{(-3,-1)\} \tag{20}$$
for all $\alpha \in (0, 0.8]$, *which is a singleton set in* $\mathbb{R}^2$. *The membership function of* $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$ *is given by*
$$\begin{aligned} \xi_{\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}}(z_1,z_2) &= \sup_{\{\alpha\in I^*:\alpha>0\}} \alpha\cdot\chi_{M_\alpha^-}(z_1,z_2) = \sup_{\alpha\in(0,0.8]} \alpha\cdot\chi_{M_\alpha^-}(z_1,z_2) \\ &= \begin{cases} 0.8 & \textit{if } (z_1,z_2) = (-3,-1) \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

- *We take*
$$M_\alpha^- = M_\alpha^{(1-)} \times M_\alpha^{(2-)},$$
where $M_\alpha^{(1-)}$ *and* $M_\alpha^{(2-)}$ *are bounded closed intervals given by*
$$M_\alpha^{(1-)} = \left[\min\left\{\tilde{a}_{1\alpha}^L - \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U - \tilde{b}_{1\alpha}^U\right\}, \max\left\{\tilde{a}_{1\alpha}^L - \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U - \tilde{b}_{1\alpha}^U\right\}\right] = \{-3\}$$
and
$$M_\alpha^{(2-)} = \left[\min\left\{\tilde{a}_{2\alpha}^L - \tilde{b}_{2\alpha}^L, \tilde{a}_{2\alpha}^U - \tilde{b}_{2\alpha}^U\right\}, \max\left\{\tilde{a}_{1\alpha}^L - \tilde{b}_{2\alpha}^L, \tilde{a}_{2\alpha}^U - \tilde{b}_{2\alpha}^U\right\}\right] = \{-1\}.$$
Therefore, we obtain
$$M_\alpha^- = \{-3\} \times \{-1\} = \{(-3,-1)\} \tag{21}$$
for all $\alpha \in (0, 0.8]$, *which is a singleton set in* $\mathbb{R}^2$. *The membership function of* $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$ *is equal to membership function of* $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$.

In order to define the addition $\tilde{\mathbf{a}} \oplus_{DT} \tilde{\mathbf{b}}$, we consider the family $\{M_\alpha^+ : \alpha \in I^* \text{ with } \alpha > 0\}$ that is formed by applying the operation $\mathbf{x} + \mathbf{y}$ to the α-level sets $\tilde{a}_\alpha^{(i)}$ and $\tilde{b}_\alpha^{(i)}$ for $i = 1, \cdots, n$, where each M_α^+ is a subset of $\mathbb{R}^m$. In this paper, we study three different families described below.

- We take
$$M_\alpha^+ = \tilde{\mathbf{a}}_\alpha + \tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\right\}$$
to define $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$.

- We take

$$M_{\alpha}^{+}=\left(\bigcup_{\{\beta\in I^{*}:\beta\geq\alpha\}}M_{\beta}^{(1+)}\right)\times\cdots\times\left(\bigcup_{\{\beta\in I^{*}:\beta\geq\alpha\}}M_{\beta}^{(n+)}\right),$$

where $M_{\beta}^{(i+)}$ are bounded closed intervals given by

$$M_{\beta}^{(i+)}=\left[a_{i\beta}^{L}+b_{i\beta}^{L},a_{i\beta}^{U}+b_{i\beta}^{U}\right]$$

for $i=1,\cdots,n$ to define $\tilde{\mathbf{a}}\oplus_{DT}^{\star}\tilde{\mathbf{b}}$.

- We take

$$M_{\alpha}^{+}=M_{\alpha}^{(1+)}\times\cdots\times M_{\alpha}^{(n+)},$$

where $M_{\alpha}^{(i+)}$ are bounded closed intervals given by

$$M_{\alpha}^{(i+)}=\left[a_{i\alpha}^{L}+b_{i\alpha}^{L},a_{i\alpha}^{U}+b_{i\alpha}^{U}\right]$$

for $i=1,\cdots,n$ to define $\tilde{\mathbf{a}}\oplus_{DT}^{\dagger}\tilde{\mathbf{b}}$.

For $\oplus_{DT}\in\{\oplus_{DT}^{\diamond},\oplus_{DT}^{\star},\oplus_{DT}^{\dagger}\}$, based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}}(\mathbf{z})=\sup_{\{\alpha\in I^{*}:\alpha>0\}}\alpha\cdot\chi_{M_{\alpha}^{+}}(\mathbf{z}).$$

Example 8. *Continuing from Examples 5 and 7, we consider three families*

$$\{M_{\alpha}^{+}:\alpha\in I^{*}\text{ with }\alpha>0\}=\{M_{\alpha}^{+}:\alpha\in(0,0.8]\}$$

given below.

- *We take*

$$\begin{aligned}M_{\alpha}^{+}=\tilde{\mathbf{a}}_{\alpha}+\tilde{\mathbf{b}}_{\alpha}&=\{(x_1,x_2)+(y_1,y_2):(x_1,x_2)\in[1+\alpha,4-\alpha]\times[2+\alpha,5-\alpha]\\&\qquad\text{and }\ (y_1,y_2)\in[4+\alpha,7-\alpha]\times[3+\alpha,6-\alpha]\}\\&=[5+2\alpha,11-2\alpha]\times[5+2\alpha,11-2\alpha]\end{aligned}$$

for all $\alpha\in(0,0.8]$. The membership function of $\tilde{\mathbf{a}}\oplus_{DT}^{\diamond}\tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}}\oplus_{DT}^{\diamond}\tilde{\mathbf{b}}}(z_1,z_2)=\sup_{\{\alpha\in I^{*}:\alpha>0\}}\alpha\cdot\chi_{M_{\alpha}^{+}}(z_1,z_2)=\sup_{\alpha\in(0,0.8]}\alpha\cdot\chi_{M_{\alpha}^{+}}(z_1,z_2).$$

- *We take*

$$M_{\alpha}^{+}=\left(\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}}M_{\beta}^{(1+)}\right)\times\left(\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}}M_{\beta}^{(2+)}\right),$$

where $M_{\beta}^{(1+)}$ and $M_{\beta}^{(2+)}$ are bounded closed intervals given by

$$M_{\beta}^{(1+)}=\left[a_{1\beta}^{L}+b_{1\beta}^{L},a_{1\beta}^{U}+b_{1\beta}^{U}\right]=[(1+\beta)+(4+\beta),(4-\beta)+(7-\beta)]=[5+2\beta,11-2\beta]$$

for all $\beta\in[0,0.8]$. We also obtain

$$M_{\beta}^{(2+)}=\left[a_{2\beta}^{L}+b_{2\beta}^{L},a_{2\beta}^{U}+b_{2\beta}^{U}\right]=[(2+\beta)+(3+\beta),(5-\beta)+(6-\beta)]=[5+2\beta,11-2\beta].$$

for all $\beta \in [0, 0.8]$. Now, we have

$$\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} M_{\beta}^{(1+)} = \bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} [5+2\beta, 11-2\beta] = [5+2\alpha, 11-2\alpha]$$

and

$$\bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} M_{\beta}^{(2+)} = \bigcup_{\{\beta\in[0,0.8]:\beta\geq\alpha\}} [5+2\beta, 11-2\beta] = [5+2\alpha, 11-2\alpha] .$$

Therefore, we obtain

$$M_{\alpha}^{+} = [5+2\alpha, 11-2\alpha] \times [5+2\alpha, 11-2\alpha]$$

for all $\alpha \in (0, 0.8]$. The membership function of $\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}$ is equal to the membership function of $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$

- *We take*

$$M_{\alpha}^{+} = M_{\alpha}^{(1+)} \times M_{\alpha}^{(2+)},$$

where $M_{\alpha}^{(1+)}$ and $M_{\alpha}^{(2+)}$ are bounded closed intervals given by

$$M_{\alpha}^{(1+)} = \left[a_{1\alpha}^{L} + b_{1\alpha}^{L}, a_{1\alpha}^{U} + b_{1\alpha}^{U}\right] = [(1+\alpha)+(4+\alpha), (4-\alpha)+(7-\alpha)] = [5+2\alpha, 11-2\alpha]$$

and

$$M_{\alpha}^{(2+)} = \left[a_{2\alpha}^{L} + b_{2\alpha}^{L}, a_{2\alpha}^{U} + b_{2\alpha}^{U}\right] = [(2+\alpha)+(3+\alpha), (5-\alpha)+(6-\alpha)] = [5+2\alpha, 11-2\alpha] .$$

Therefore, we obtain

$$M_{\alpha}^{+} = [5+2\alpha, 11-2\alpha] \times [5+2\alpha, 11-2\alpha]$$

for all $\alpha \in (0, 0.8]$. The membership function of $\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}}$ is equal to membership function of $\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}$.

In order to define the scalar product of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$, we consider the family $\{M_{\alpha}^{\bullet} : \alpha \in I^{*} \text{ with } \alpha > 0\}$ that is formed by applying the operation $\mathbf{x} \bullet \mathbf{y}$ to the α-level sets $\tilde{a}_{\alpha}^{(i)}$ and $\tilde{b}_{\alpha}^{(i)}$ for $i = 1, \cdots, n$, where each $M_{\alpha}^{\bullet}$ is a subset of $\mathbb{R}$. In this paper, we study three different families described below.

- We take

$$M_{\alpha}^{\bullet} = \tilde{\mathbf{a}}_{\alpha} \bullet \tilde{\mathbf{b}}_{\alpha} = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_{\alpha} \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_{\alpha}\right\}$$

to define the scalar product $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$.

- We take

$$M_{\alpha}^{\bullet} = \bigcup_{\{\beta\in I^{*}:\beta\geq\alpha\}} M_{\beta},$$

where M_{β} are bounded closed intervals given by

$$M_{\beta} = \left[\min\left\{\tilde{\mathbf{a}}_{\beta}^{L} \bullet \tilde{\mathbf{b}}_{\beta}^{L}, \tilde{\mathbf{a}}_{\beta}^{U} \bullet \tilde{\mathbf{b}}_{\beta}^{U}\right\}, \max\left\{\tilde{\mathbf{a}}_{\beta}^{L} \bullet \tilde{\mathbf{b}}_{\beta}^{L}, \tilde{\mathbf{a}}_{\beta}^{U} \bullet \tilde{\mathbf{b}}_{\beta}^{U}\right\}\right]$$

to define the scalar product $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$.

- We take

$$M_{\alpha}^{\bullet} = \left[\min\left\{\tilde{\mathbf{a}}_{\alpha}^{L} \bullet \tilde{\mathbf{b}}_{\alpha}^{L}, \tilde{\mathbf{a}}_{\alpha}^{U} \bullet \tilde{\mathbf{b}}_{\alpha}^{U}\right\}, \max\left\{\tilde{\mathbf{a}}_{\alpha}^{L} \bullet \tilde{\mathbf{b}}_{\alpha}^{L}, \tilde{\mathbf{a}}_{\alpha}^{U} \bullet \tilde{\mathbf{b}}_{\alpha}^{U}\right\}\right]$$

to define the scalar product $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$.

For $\circledast_{DT} \in \{\circledast_{DT}^{\diamond}, \circledast_{DT}^{\star}, \circledast_{DT}^{\dagger}\}$, based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha\in I^{*}:\alpha>0\}} \alpha \cdot \chi_{M_{\alpha}^{\bullet}}(\mathbf{z}). \tag{22}$$

Example 9. *Continuing from Examples 5 and 7, we consider three families*

$$\{M_\alpha^\bullet : \alpha \in I^* \text{ with } \alpha > 0\} = \{M_\alpha^\bullet : \alpha \in (0, 0.8]\}$$

given below.

- *We take*

$$\begin{aligned}
M_\alpha^\bullet &= \tilde{\mathbf{a}}_\alpha \bullet \tilde{\mathbf{b}}_\alpha \\
&= \{(x_1, x_2) \bullet (y_1, y_2) : (x_1, x_2) \in [1+\alpha, 4-\alpha] \times [2+\alpha, 5-\alpha] \\
&\quad \text{and } (y_1, y_2) \in [4+\alpha, 7-\alpha] \times [3+\alpha, 6-\alpha]\} \\
&= \{x_1 y_1 + x_2 y_2 : (x_1, x_2) \in [1+\alpha, 4-\alpha] \times [2+\alpha, 5-\alpha] \\
&\quad \text{and } (y_1, y_2) \in [4+\alpha, 7-\alpha] \times [3+\alpha, 6-\alpha]\} \\
&= [(1+\alpha)(4+\alpha), (4-\alpha)(7-\alpha)] + [(2+\alpha)(3+\alpha), (5-\alpha)(6-\alpha)] \\
&= \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + \alpha^2\right]
\end{aligned}$$

for all $\alpha \in (0, 0.8]$. The membership function of $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}}(z_1, z_2) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^\bullet}(z_1, z_2) = \sup_{\alpha \in (0, 0.8]} \alpha \cdot \chi_{M_\alpha^\bullet}(z_1, z_2).$$

- *We take*

$$M_\alpha^\bullet = \bigcup_{\{\beta \in [0,0.8] : \beta \geq \alpha\}} M_\beta,$$

where M_β is a bounded closed interval given by

$$\begin{aligned}
M_\beta &= \left[\min\left\{\tilde{a}_{1\beta}^L \tilde{b}_{1\beta}^L + \tilde{a}_{2\beta}^L \tilde{b}_{2\beta}^L, \tilde{a}_{1\beta}^U \tilde{b}_{1\beta}^U + \tilde{a}_{2\beta}^U \tilde{b}_{2\beta}^U\right\}, \max\left\{\tilde{a}_{1\beta}^L \tilde{b}_{1\beta}^L + \tilde{a}_{2\beta}^L \tilde{b}_{2\beta}^L, \tilde{a}_{1\beta}^U \tilde{b}_{1\beta}^U + \tilde{a}_{2\beta}^U \tilde{b}_{2\beta}^U\right\}\right] \\
&= [\min\{(1+\beta)(4+\beta) + (2+\beta)(3+\beta), (4-\beta)(7-\beta) + (5-\beta)(6-\beta)\}, \\
&\qquad \max\{(1+\beta)(4+\beta) + (2+\beta)(3+\beta), (4-\beta)(7-\beta) + (5-\beta)(6-\beta)\}] \\
&= \left[\min\left\{10 + 10\beta + \beta^2, 58 - 22\beta + \beta^2\right\}, \max\left\{10 + 10\beta + \beta^2, 58 - 22\beta + \beta^2\right\}\right] \\
&= \left[10 + 10\beta + \beta^2, 58 - 22\beta + \beta^2\right]
\end{aligned}$$

for all $\beta \in (0, 0.8]$. Therefore, we obtain

$$\begin{aligned}
M_\alpha^\bullet &= \bigcup_{\{\beta \in [0,0.8] : \beta \geq \alpha\}} M_\beta = \bigcup_{\{\beta \in [0,0.8] : \beta \geq \alpha\}} \left[10 + 10\beta + \beta^2, 58 - 22\beta + \beta^2\right] \\
&= \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + \alpha^2\right].
\end{aligned}$$

The membership function of $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ is equal to the membership function of $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$.

- *We take*

$$M_\alpha^\bullet = \left[\min\left\{\tilde{a}_{1\alpha}^L \tilde{b}_{1\alpha}^L + \tilde{a}_{2\alpha}^L \tilde{b}_{2\alpha}^L, \tilde{a}_{1\alpha}^U \tilde{b}_{1\alpha}^U + \tilde{a}_{2\alpha}^U \tilde{b}_{2\alpha}^U\right\}, \max\left\{\tilde{a}_{1\alpha}^L \tilde{b}_{1\alpha}^L + \tilde{a}_{2\alpha}^L \tilde{b}_{2\alpha}^L, \tilde{a}_{1\alpha}^U \tilde{b}_{1\alpha}^U + \tilde{a}_{2\alpha}^U \tilde{b}_{2\alpha}^U\right\}\right].$$

Then, we obtain

$$M_\alpha^\bullet = \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + \alpha^2\right].$$

Therefore, the membership function of $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$ is equal to the membership function of $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$.

We denote by $I_{\circledast}^{(DT)}$ and $I_{\odot}^{(DT)}$ the interval ranges of membership functions $\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}$ and $\xi_{\tilde{\mathbf{a}}\odot_{DT}\tilde{\mathbf{b}}}$ for $\odot_{DT} \in \{\oplus_{DT}, \ominus_{DT}\}$, respectively, where $I_{\circledast}^{(DT)}$ and $I_{\odot}^{(DT)}$ depends on $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$. We consider the supremum of range of membership function $\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}$ as follows:

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}\right) = \sup_{z\in\mathbb{R}} \xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}(z) = \sup_{z\in\mathbb{R}} \sup_{\{\alpha\in I^*:\alpha>0\}} \alpha\cdot\chi_{M_{\alpha}^{\bullet}}(z). = \sup I^* \equiv \alpha^{\diamond}. \tag{23}$$

We can similarly obtain

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\odot_{DT}\tilde{\mathbf{b}}}\right) = \sup I^* \equiv \alpha^{\diamond}.$$

Therefore, the definition of interval ranges says that

$$I_{\circledast}^{(DT)} = \begin{cases} [0,\alpha^{\diamond}] & \text{if the supremum } \alpha^{\diamond} = \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0,\alpha^{\diamond}) & \text{otherwise.} \end{cases} \tag{24}$$

and

$$I_{\odot}^{(DT)} = \begin{cases} [0,\alpha^{\diamond}] & \text{if the supremum } \alpha^{\diamond} = \mathcal{R}\left(\xi_{\tilde{\mathbf{a}}\odot_{DT}\tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0,\alpha^{\diamond}) & \text{otherwise.} \end{cases} \tag{25}$$

Proposition 1 also says that

$$\left(\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}\right)_{\alpha} \neq \emptyset \text{ for } \alpha \in I_{\circledast}^{(DT)} \text{ and } \left(\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}\right)_{\alpha} = \emptyset \text{ for } \alpha \notin I_{\circledast}^{(DT)}$$

and

$$\left(\tilde{\mathbf{a}}\odot_{DT}\tilde{\mathbf{b}}\right)_{\alpha} \neq \emptyset \text{ for } \alpha \in I_{\odot}^{(DT)} \text{ and } \left(\tilde{\mathbf{a}}\odot_{DT}\tilde{\mathbf{b}}\right)_{\alpha} = \emptyset \text{ for } \alpha \notin I_{\odot}^{(DT)}.$$

Proposition 3. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the supremum $\sup I^*$ in (16) is attained. Then*

$$I_{\circledast}^{(DT)} = I_{\odot}^{(DT)} = I^* = [0,\alpha^{\diamond}].$$

Proof. Recall the definition $I_{\tilde{a}^{(i)}}$ and $I_{\tilde{b}^{(i)}}$ in (8) and (9), respectively, for $i = 1,\cdots,n$. It is clear to see that

$$\alpha^{\diamond} = \sup I^* = \min\left\{\alpha_1^*,\cdots,\alpha_n^*,\beta_1^*,\cdots,\beta_n^*\right\}. \tag{26}$$

Since $\sup I^*$ is assumed to be attained, it follows that $I^* = [0,\alpha^{\diamond}]$. By referring to (23), we can take $z \in M_{\alpha^{\diamond}}^{\bullet} \subset \mathbb{R}$, which says that the supremum $\alpha^{\diamond}$ is attained for the range $\mathcal{R}(\xi_{\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}})$. Therefore, from (24), we have $I_{\circledast}^{(DT)} = [0,\alpha^{\diamond}] = I^*$. We can similarly obtain $I_{\odot}^{(DT)} = [0,\alpha^{\diamond}] = I^*$. This completes the proof. □

More detailed properties will be studied separately in the sequel.

Example 10. *Continuing from Examples 7–9, we consider the interval ranges $I_{\circledast}^{(DT)}$, $I_{\oplus}^{(DT)}$ and $I_{\ominus}^{(DT)}$ of $\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}$, $\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}}\ominus_{DT}\tilde{\mathbf{b}}$, respectively. Recall that $I^* = [0,0.8]$. From (23), we see that $\alpha^{\diamond} = 0.8$. Proposition 3 says that*

$$I_{\circledast}^{(DT)} = I_{\oplus}^{(DT)} = I_{\ominus}^{(DT)} = [0,0.8].$$

Therefore, it follows that

$$\left(\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}\right)_{\alpha} \neq \emptyset, \quad \left(\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}\right)_{\alpha} \neq \emptyset \text{ and } \left(\tilde{\mathbf{a}}\ominus_{DT}\tilde{\mathbf{b}}\right)_{\alpha} \neq \emptyset \text{ for } \alpha \in [0,0.8]$$

and

$$\left(\tilde{\mathbf{a}}\circledast_{DT}\tilde{\mathbf{b}}\right)_{\alpha} = \emptyset, \quad \left(\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}\right)_{\alpha} = \emptyset \text{ and } \left(\tilde{\mathbf{a}}\ominus_{DT}\tilde{\mathbf{b}}\right)_{\alpha} = \emptyset \text{ for } \alpha \notin [0,0.8].$$

5. Difference of Vectors of Fuzzy Intervals

Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be two vectors of fuzzy intervals with components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$, respectively, for $i = 1, \cdots, n$. Here we study the α-level set of $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ that is obtained from the extension principle, and the α-level sets of $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$ for $\ominus_{DT} \in \{\ominus_{DT}^{\diamond}, \ominus_{DT}^{\star}, \ominus_{DT}^{\dagger}\}$ that are obtained from the form of decomposition theorem.

5.1. Using the Extension Principle to Study the α-Level Sets of $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$

Given any aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$, recall that the membership function of difference $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}-\mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right)$$

for any $\mathbf{z} \in \mathbb{R}^n$. Let $I_{\ominus}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$. The α-level set $(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ for $\alpha \in I_{\ominus}^{(EP)}$ can be obtained by applying the results obtained in Wu [11] to the difference $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$, which is shown below. For each $\alpha \in I_{\ominus}^{(EP)}$ with $\alpha > 0$, we have

$$\begin{aligned} \left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} - \mathbf{y} : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\} \\ &= \left\{(x_1 - y_1, \cdots, x_n - y_n) : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\}. \end{aligned} \tag{27}$$

The 0-level set is given by

$$\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_0 = \tilde{\mathbf{a}}_0 - \tilde{\mathbf{b}}_0 = \left\{\mathbf{x} - \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_0 \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_0\right\}.$$

Moreover, for each $\alpha \in I_{\ominus}^{(EP)}$, the α-level sets $(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}})_\alpha$ are closed and bounded subsets of $\mathbb{R}^m$.
Now, the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by

$$\mathfrak{A}(\alpha_1, \cdots, \alpha_{2n}) = \begin{cases} \min\{\alpha_1, \cdots, \alpha_{2n}\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression}, & \text{otherwise.} \end{cases}$$

Proposition 2 says that $I_{\ominus}^{(EP)} = I^*$. Therefore, for each $\alpha \in I_{\ominus}^{(EP)} = I^*$, we have $(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}})_\alpha \neq \emptyset$, $\tilde{a}_\alpha^{(i)} \neq \emptyset$ and $\tilde{b}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$. Now, for each $\alpha \in I_{\ominus}^{(EP)}$ with $\alpha > 0$, using (27), we have

$$\begin{aligned} \left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} - \mathbf{y} : \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} \geq \alpha\right\} \\ &= \left\{\mathbf{x} - \mathbf{y} : \xi_{\tilde{a}^{(i)}}(x_i) \geq \alpha \text{ and } \xi_{\tilde{b}^{(i)}}(y_i) \geq \alpha \text{ for each } i = 1, \cdots, n\right\} \\ &= \Big\{(x_1 - y_1, \cdots, x_n - y_n) : x_i \in \tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \\ &\qquad \text{and } y_i \in \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right] \text{ for each } i = 1, \cdots, n\Big\} \\ &= \left[\tilde{a}_{1\alpha}^L - \tilde{b}_{1\alpha}^U, \tilde{a}_{1\alpha}^U - \tilde{b}_{1\alpha}^L\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L - \tilde{b}_{n\alpha}^U, \tilde{a}_{n\alpha}^U - \tilde{b}_{n\alpha}^L\right]. \end{aligned} \tag{28}$$

For the 0-level set, from (28) and (4), it is not difficult to show that

$$\begin{aligned} \left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_0 &= \mathrm{cl}\left(\bigcup_{\{\alpha \in I_{\ominus}^{(EP)} : \alpha > 0\}} \left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_\alpha\right) \\ &= \left[\tilde{a}_{10}^L - \tilde{b}_{10}^U, \tilde{a}_{10}^U - \tilde{b}_{10}^L\right] \times \cdots \times \left[\tilde{a}_{n0}^L - \tilde{b}_{n0}^U, \tilde{a}_{n0}^U - \tilde{b}_{n0}^L\right]. \end{aligned}$$

Regarding the components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$, let $I_{\ominus}^{(i)(EP)}$ be the interval range of $\tilde{a}^{(i)} \ominus_{EP} \tilde{b}^{(i)}$. From Proposition 2, we can similarly obtain $I_{\ominus}^{(i)(EP)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$. For $\alpha \in I_{\ominus}^{(i)(EP)}$, we also have

$$\left(\tilde{a}^{(i)} \ominus_{EP} \tilde{b}^{(i)}\right)_{\alpha} = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right] \text{ for } i = 1, \cdots, n. \tag{29}$$

Therefore, from (28) and (29), for $\alpha \in I_{\ominus}^{(EP)}$, we obtain

$$\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{EP} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{EP} \tilde{b}^{(n)}\right)_{\alpha}.$$

The above results are summarized in the following theorem.

Theorem 2. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any fuzzy intervals. Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by*

$$\mathfrak{A}(\alpha_1, \cdots, \alpha_{2n}) = \begin{cases} \min\{\alpha_1, \cdots, \alpha_{2n}\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression}, & \text{otherwise}, \end{cases}$$

Then, we have the following results.

(i) *Let $I_{\ominus}^{(i)(EP)}$ be the interval range of $\tilde{a}^{(i)} \ominus_{EP} \tilde{b}^{(i)}$ for $i = 1, \cdots, n$. Then, for each $\alpha \in I_{\ominus}^{(i)(EP)}$, we have*

$$\left(\tilde{a}^{(i)} \ominus_{EP} \tilde{b}^{(i)}\right)_{\alpha} = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right].$$

We also have $I_{\ominus}^{(i)(EP)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$.

(ii) *Let $I_{\ominus}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$. We have*

$$I_{\ominus}^{(EP)} \subseteq I_{\ominus}^{(i)(EP)} \text{ for } i = 1, \cdots, n, \text{ and } I_{\ominus}^{(EP)} = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

For each $\alpha \in I_{\ominus}^{(EP)}$, we also have

$$\left(\tilde{a}^{(i)} \ominus_{EP} \tilde{b}^{(i)}\right)_{\alpha} = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right]$$

and

$$\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{EP} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{EP} \tilde{b}^{(n)}\right)_{\alpha}.$$

Remark 1. *When $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be fuzzy numbers instead of fuzzy intervals, it follows that*

$$I_{\tilde{a}^{(i)}} = I_{\tilde{b}^{(i)}} = I_{\ominus}^{(i)(EP)} = I_{\ominus}^{(EP)} = [0,1] \text{ for all } i = 1, \cdots, n.$$

Therefore, Theorem 2 says that

$$\begin{aligned} \left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_{\alpha} &= \left(\tilde{a}^{(1)} \ominus_{EP} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{EP} \tilde{b}^{(n)}\right)_{\alpha} \\ &= \left[\tilde{a}_{1\alpha}^{L} - \tilde{b}_{1\alpha}^{U}, \tilde{a}_{1\alpha}^{U} - \tilde{b}_{1\alpha}^{L}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} - \tilde{b}_{n\alpha}^{U}, \tilde{a}_{n\alpha}^{U} - \tilde{b}_{n\alpha}^{L}\right]. \end{aligned}$$

for all $\alpha \in [0,1]$.

Example 11. *Continuing from Examples 5 and 7, Theorem 2 says that*

$$\left(\tilde{a}^{(1)} \ominus_{EP} \tilde{b}^{(1)}\right)_\alpha = \left[\tilde{a}^L_{1\alpha} - \tilde{b}^U_{1\alpha}, \tilde{a}^U_{1\alpha} - \tilde{b}^L_{1\alpha}\right]$$
$$= [(1+\alpha) - (7-\alpha), (4-\alpha) - (4+\alpha)] = [-6+2\alpha, -2\alpha]$$

and

$$\left(\tilde{a}^{(2)} \ominus_{EP} \tilde{b}^{(2)}\right)_\alpha = \left[\tilde{a}^L_{2\alpha} - \tilde{b}^U_{2\alpha}, \tilde{a}^U_{2\alpha} - \tilde{b}^L_{2\alpha}\right]$$
$$= [(2+\alpha) - (6-\alpha), (5-\alpha) - (3+\alpha)] = [-4+2\alpha, 2-2\alpha]$$

and

$$\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{a}^{(1)} \ominus_{EP} \tilde{b}^{(1)}\right)_\alpha \times \left(\tilde{a}^{(2)} \ominus_{EP} \tilde{b}^{(2)}\right)_\alpha = [-6+2\alpha, -2\alpha] \times [-4+2\alpha, 2-2\alpha]$$

for $\alpha \in [0, 0.8]$. *Moreover, we have* $\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ *for* $\alpha \notin [0, 0.8]$.

5.2. Using the Form of Decomposition Theorem to to Study the α*-Level Sets of* $\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}$

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. The family $\{M^-_\alpha : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M^-_\alpha = \tilde{\mathbf{a}}_\alpha - \tilde{\mathbf{b}}_\alpha.$$

We see that $\tilde{a}^{(i)}_\alpha \neq \emptyset$ and $\tilde{b}^{(i)}_\alpha \neq \emptyset$ for each $\alpha \in I^*$ and for $i = 1, \cdots, n$. Now, for $\alpha \in I^*$ with $\alpha > 0$, we have

$$\begin{aligned} M^-_\alpha &= \tilde{\mathbf{a}}_\alpha - \tilde{\mathbf{b}}_\alpha = \{\mathbf{x} - \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\} \\ &= \left\{(x_1 - y_1, \cdots, x_1 - y_1) : x_i \in \tilde{a}^{(i)}_\alpha = \left[\tilde{a}^L_{i\alpha}, \tilde{a}^U_{i\alpha}\right] \text{ and } y_i \in \tilde{b}^{(i)}_\alpha = \left[\tilde{b}^L_{i\alpha}, \tilde{b}^U_{i\alpha}\right] \text{ for } i = 1, \cdots, n\right\} \\ &= \left[\tilde{a}^L_{1\alpha} - \tilde{b}^U_{1\alpha}, \tilde{a}^U_{1\alpha} - \tilde{b}^L_{1\alpha}\right] \times \cdots \times \left[\tilde{a}^L_{n\alpha} - \tilde{b}^U_{n\alpha}, \tilde{a}^U_{n\alpha} - \tilde{b}^L_{n\alpha}\right]. \end{aligned} \tag{30}$$

Based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M^-_\alpha}(\mathbf{z}). \tag{31}$$

Let $I^{(\diamond DT)}_{\ominus}$ be the interval range of $\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}$. The α-level sets $(\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}$ for $\alpha \in I^{(\diamond DT)}_{\ominus}$ are presented below.

Proposition 4. *Suppose that the supremum* $\sup I^*$ *is attained. Then* $I^{(\diamond DT)}_{\ominus} = I^*$ *and*

$$\left(\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha = M^-_\alpha = \left[\tilde{a}^L_{1\alpha} - \tilde{b}^U_{1\alpha}, \tilde{a}^U_{1\alpha} - \tilde{b}^L_{1\alpha}\right] \times \cdots \times \left[\tilde{a}^L_{n\alpha} - \tilde{b}^U_{n\alpha}, \tilde{a}^U_{n\alpha} - \tilde{b}^L_{n\alpha}\right] \tag{32}$$

for each $\alpha \in I^*$.

Proof. We first consider $\alpha \in I^*$ with $\alpha > 0$. Given any $\mathbf{z} \in M^-_\alpha$, we see that $\xi_{\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) \geq \alpha$ by (31). Therefore, we obtain $\mathbf{z} \in (\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha$, which proves the inclusion $M^-_\alpha \subseteq (\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha$.

For proving another direction of inclusion, it is clear to see that $\{M^-_\alpha : \alpha \in I^* \text{ with } \alpha > 0\}$ is a nested family. Given any $\mathbf{z} \in (\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha$, i.e., $\xi_{\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) \geq \alpha$, let $\hat{\alpha} = \xi_{\tilde{\mathbf{a}} \ominus^{\diamond}_{DT} \tilde{\mathbf{b}}}(\mathbf{z})$. Assume that $\hat{\alpha} > \alpha$. Let $\epsilon = \hat{\alpha} - \alpha > 0$. According to the concept of supremum, there exists $\alpha_0 \in I^*$ satisfying $\mathbf{z} \in M^-_{\alpha_0}$ and $\hat{\alpha} - \epsilon < \alpha_0$, which implies $\alpha < \alpha_0$. This also says that $\mathbf{z} \in M^-_\alpha$, since $M^-_{\alpha_0} \subseteq M^-_\alpha$ by the nestedness.

Now, we assume that $\hat{\alpha} = \alpha$. Since I^* is an interval with left end-point 0, given any $\alpha \in I^*$ with $\alpha > 0$, we can consider the sequence $\{\alpha_s\}_{s=1}^{\infty}$ in I^* satisfying $0 < \alpha_s \uparrow \alpha$ with $\alpha > \alpha_s \in I^*$ for all s. Since $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are fuzzy intervals for $i = 1, \cdots, n$, it is well-known that

$$\tilde{a}_{\alpha}^{(i)} = \bigcap_{s=1}^{\infty} \tilde{a}_{\alpha_s}^{(i)} \text{ and } \tilde{b}_{\alpha}^{(i)} = \bigcap_{s=1}^{\infty} \tilde{b}_{\alpha_s}^{(i)} \text{ for } i = 1, \cdots, n.$$

Since

$$\tilde{\mathbf{a}}_{\alpha} = \tilde{a}_{\alpha}^{(1)} \times \cdots \times \tilde{a}_{\alpha}^{(n)} \text{ and } \tilde{\mathbf{b}}_{\alpha} = \tilde{b}_{\alpha}^{(1)} \times \cdots \times \tilde{b}_{\alpha}^{(n)},$$

we can obtain

$$\tilde{\mathbf{a}}_{\alpha} = \bigcap_{s=1}^{\infty} \tilde{\mathbf{a}}_{\alpha_s} \text{ and } \tilde{\mathbf{b}}_{\alpha} = \bigcap_{s=1}^{\infty} \tilde{\mathbf{b}}_{\alpha_s},$$

Since $M_{\alpha}^{-} = \tilde{\mathbf{a}}_{\alpha} - \tilde{\mathbf{b}}_{\alpha}$, we conclude that

$$M_{\alpha}^{-} = \bigcap_{s=1}^{\infty} M_{\alpha_s}^{-} \text{ for } \alpha \in I^* \text{ with } \alpha > 0 \text{ and } 0 < \alpha_s \uparrow \alpha \text{ with } \alpha > \alpha_s \in I^* \text{ for all } s. \tag{33}$$

Let $\epsilon_s = \alpha - \alpha_s > 0$. According to the concept of supremum, there exists $\alpha_0 \in I^*$ satisfying $\mathbf{z} \in M_{\alpha_0}^{-}$ and $\hat{\alpha} - \epsilon_s = \alpha - \epsilon_s < \alpha_0$, which implies $\alpha_0 > \alpha_s \in I^*$. This also says that $\mathbf{z} \in M_{\alpha_s}^{-}$ by the nestedness for all s. Therefore, we conclude that $\mathbf{z} \in \bigcap_{s=1}^{\infty} M_{\alpha_s}^{-}$. From (33), it follows that $\mathbf{z} \in M_{\alpha}^{-}$. Therefore, for $\alpha \in I^*$ with $\alpha > 0$, we obtain

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{a}_{1\alpha}^{L} - \tilde{b}_{1\alpha}^{U}, \tilde{a}_{1\alpha}^{U} - \tilde{b}_{1\alpha}^{L}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} - \tilde{b}_{n\alpha}^{U}, \tilde{a}_{n\alpha}^{U} - \tilde{b}_{n\alpha}^{L}\right].$$

For the 0-level set, since $I_{\ominus}^{(\diamond DT)} = I^*$ from Proposition 3, by referring to (4), it is not difficult to show that

$$\begin{aligned}\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{0} &= \operatorname{cl}\left(\bigcup_{\{\alpha \in I_{\ominus}^{(\diamond DT)}:\alpha>0\}} \left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha}\right) = \operatorname{cl}\left(\bigcup_{\{\alpha \in I^*:\alpha>0\}} \left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha}\right) \\ &= \left[\tilde{a}_{10}^{L} - \tilde{b}_{10}^{U}, \tilde{a}_{10}^{U} - \tilde{b}_{10}^{L}\right] \times \cdots \times \left[\tilde{a}_{n0}^{L} - \tilde{b}_{n0}^{U}, \tilde{a}_{n0}^{U} - \tilde{b}_{n0}^{L}\right].\end{aligned}$$

This completes the proof. □

Now, for $i = 1, \cdots, n$ and for $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ with $\alpha > 0$, we take

$$M_{\alpha}^{(i-)} = \tilde{a}_{\alpha}^{(i)} - \tilde{b}_{\alpha}^{(i)} = \left[\tilde{a}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U}\right] - \left[\tilde{b}_{i\alpha}^{L}, \tilde{b}_{i\alpha}^{U}\right] = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right]. \tag{34}$$

From (30), we see that

$$M_{\alpha}^{-} = M_{\alpha}^{(1-)} \times \cdots \times M_{\alpha}^{(n-)} \subset \mathbb{R}^n. \tag{35}$$

Let $\tilde{a}^{(i)} \ominus_{DT}^{\diamond} \tilde{b}^{(i)}$ be obtained using the form of decomposition theorem based on the family $\{M_{\alpha}^{(i-)} : \alpha \in I^* \text{ with } \alpha > 0\}$ that is defined in (34). Let $I_{\ominus}^{(i)(\diamond DT)}$ be the interval range of $\tilde{a}^{(i)} \ominus_{DT}^{\diamond} \tilde{b}^{(i)}$. Suppose that the supremum $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ is attained for each $i = 1, \cdots, n$. Then each $I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ is a bounded and closed interval with left end-point 0 for $i = 1, \cdots, n$. In this case, it is also clear to see that I^* is a bounded and closed interval with left end-point 0; that is, the supremum $\sup I^*$ is also attained. By referring to Proposition 4, we can similarly obtain $I_{\ominus}^{(i)(\diamond DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and

$$\left(\tilde{a}^{(i)} \ominus_{DT}^{\diamond} \tilde{b}^{(i)}\right)_{\alpha} = M_{\alpha}^{(i-)} = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right] \tag{36}$$

for $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and $i = 1, \cdots, n$, which also implies

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{DT}^{\diamond} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{DT}^{\diamond} \tilde{b}^{(n)}\right)_{\alpha}$$

for each $\alpha \in I^*$. The above results are summarized below.

Theorem 3. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the family $\{M_{\alpha}^{-} : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M_{\alpha}^{-} = \tilde{\mathbf{a}}_{\alpha} - \tilde{\mathbf{b}}_{\alpha}.$$

Let $I_{\ominus}^{(\diamond DT)}$ be the interval range of $\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}$, and let $I_{\ominus}^{(i)(\diamond DT)}$ be the interval range of $\tilde{a}^{(i)} \ominus_{DT}^{\diamond} \tilde{b}^{(i)}$ for $i = 1, \cdots, n$.

(i) *Suppose that the supremum $\sup I^*$ is attained. Then $I_{\ominus}^{(\diamond DT)} = I^*$ and*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{a}_{1\alpha}^{L} - \tilde{b}_{1\alpha}^{U}, \tilde{a}_{1\alpha}^{U} - \tilde{b}_{1\alpha}^{L}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} - \tilde{b}_{n\alpha}^{U}, \tilde{a}_{n\alpha}^{U} - \tilde{b}_{n\alpha}^{L}\right]$$

for each $\alpha \in I^*$.

(ii) *Suppose that the supremum $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ is attained for each $i = 1, \cdots, n$. Then*

$$I_{\ominus}^{(i)(\diamond DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} \text{ and } \left(\tilde{a}^{(i)} \ominus_{DT}^{\diamond} \tilde{b}^{(i)}\right)_{\alpha} = \left[\tilde{a}_{i\alpha}^{L} - \tilde{b}_{i\alpha}^{U}, \tilde{a}_{i\alpha}^{U} - \tilde{b}_{i\alpha}^{L}\right]$$

for each $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and each $i = 1, \cdots, n$, and

$$I_{\ominus}^{(\diamond DT)} = I^* \text{ and } \left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{DT}^{\diamond} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{DT}^{\diamond} \tilde{b}^{(n)}\right)_{\alpha}$$

for each $\alpha \in I^*$.

Remark 2. *From (8) and (9), we see that if the supremum $\sup \mathcal{R}(\xi_{\tilde{a}^{(i)}})$ and $\sup \mathcal{R}(\xi_{\tilde{b}^{(i)}})$ are attained, then $I_{\tilde{a}^{(i)}}$ and $I_{\tilde{b}^{(i)}}$ are closed intervals for all $i = 1, \cdots, n$, which also say that the supremum $\sup I^*$ and $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ for $i = 1, \cdots, n$ are attained.*

Example 12. *Continuing from Example 7, part (i) of Theorem 3 says that*

$$\begin{aligned}\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} &= \left[\tilde{a}_{1\alpha}^{L} - \tilde{b}_{1\alpha}^{U}, \tilde{a}_{1\alpha}^{U} - \tilde{b}_{1\alpha}^{L}\right] \times \left[\tilde{a}_{2\alpha}^{L} - \tilde{b}_{2\alpha}^{U}, \tilde{a}_{2\alpha}^{U} - \tilde{b}_{2\alpha}^{L}\right] \\ &= [-6 + 2\alpha, -2\alpha] \times [-4 + 2\alpha, 2 - 2\alpha]\end{aligned}$$

for $\alpha \in [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \emptyset$ for $\alpha \notin [0, 0.8]$.

5.3. Using the Form of Decomposition Theorem to to Study the α-Level Sets of $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any fuzzy intervals. The family $\{M_{\alpha}^{-} : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M_{\alpha}^{-} = \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta}^{(1-)}\right) \times \cdots \times \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta}^{(n-)}\right), \tag{37}$$

where $M_\beta^{(i-)}$ are bounded closed intervals given by

$$M_\beta^{(i-)} = \left[\min\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}, \max\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}\right]$$

for $i = 1, \cdots, n$. Based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha \in I^*: \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^-}(\mathbf{z}).$$

Let $I_\ominus^{(\star DT)}$ be the interval range of $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$. Here we study the α-level sets $(\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$ for $\alpha \in I_\ominus^{(\star DT)}$.

For $i = 1, \cdots, n$, we write

$$N_\alpha^{(i-)} \equiv \bigcup_{\{\beta \in I^*: \beta \geq \alpha\}} M_\beta^{(i-)}.$$

It is clear to see that $\{N_\alpha^{(i-)} : \alpha \in I^* \text{ for } \alpha > 0\}$ is a nested family. Since I^* is a bounded interval with left end-point 0, using the nestedness, we can show that

$$N_\alpha^{(i-)} = \bigcap_{n=1}^{\infty} N_{\alpha_n}^{(i-)} \text{ for } \alpha \in I^* \text{ with } \alpha > 0 \text{ and } 0 < \alpha_n \uparrow \alpha \text{ with } \alpha_n < \alpha \text{ for all } n. \tag{38}$$

From (37), we also see that

$$M_\alpha^- = N_\alpha^{(1-)} \times \cdots \times N_\alpha^{(n-)}.$$

Using (38), we can also obtain

$$M_\alpha^- = \bigcap_{n=1}^{\infty} M_{\alpha_n}^- \text{ for } \alpha \in I^* \text{ with } \alpha > 0 \text{ and } 0 < \alpha_n \uparrow \alpha \text{ with } \alpha_n < \alpha \text{ for all } n. \tag{39}$$

Suppose that the supremum $\sup I^*$ is attained. By applying (39) to the argument in the proof of Proposition 4, we can show that $I_\ominus^{(\star DT)} = I^*$ and

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = M_\alpha^- = N_\alpha^{(1-)} \times \cdots \times N_\alpha^{(n-)}$$

for any $\alpha \in I_\ominus^{(\star DT)} = I^*$.

Now, we consider the difference $\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}$ of components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ for $i = 1, \cdots, n$. Using the form of decomposition theorem, the membership function of $\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}$ is defined by

$$\xi_{\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}}(\mathbf{z}) = \sup_{\{\alpha \in I^*: \alpha > 0\}} \alpha \cdot \chi_{N_\alpha^{(i-)}}(\mathbf{z}).$$

Let $I_\ominus^{(i)(\star DT)}$ be the interval range of $\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}$. We also study the α-level sets $(\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)})_\alpha$ of $\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}$ for $\alpha \in I_\ominus^{(i)(\star DT)}$. Suppose that the supremum $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ is attained. Using the argument in the proof of Proposition 4 again, we can obtain $I_\ominus^{(i)(\star DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and

$$\left(\tilde{a}^{(i)} \ominus_{DT}^{\star} \tilde{b}^{(i)}\right)_\alpha = N_\alpha^{(i-)}$$

for any $\alpha \in I_\ominus^{(i)(\star DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$.

In order to obtain the compact form of the α-level sets, we propose a concept below.

Definition 1. *We say that $\tilde{a}$ is a canonical fuzzy interval if and only if $\tilde{a}$ is a fuzzy interval such that $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are continuous with respect to α on $I_{\tilde{a}}$.*

Now, we assume that $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ are any canonical fuzzy intervals. Let

$$\zeta_i^L(\beta)=\min\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\}\text{ and }\zeta_i^U(\beta)=\max\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\}.$$

Then $M_\beta^{(i-)}=[\zeta_i^L(\beta),\zeta_i^U(\beta)]$.
We also see that ζ_i^L and ζ_i^U are continuous functions on I^*. Then, for $\alpha\in I^*$ with $\alpha>0$, we can obtain

$$\begin{aligned}
N_\alpha^{(i-)}&=\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(i-)}=\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}\left[\zeta_i^L(\beta),\zeta_i^U(\beta)\right]\\
&=\left[\min_{\{\beta\in I^*:\beta\geq\alpha\}}\zeta_i^L(\beta),\max_{\{\beta\in I^*:\beta\geq\alpha\}}\zeta_i^U(\beta)\right]\\
&=\left[\min_{\{\beta\in I^*:\beta\geq\alpha\}}\min\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\},\max_{\{\beta\in I^*:\beta\geq\alpha\}}\max\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\}\right].
\end{aligned}$$

The above results are summarized below.

Theorem 4. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be any fuzzy intervals. Suppose that the family $\{M_\alpha^-:\alpha\in I^*\text{ for }\alpha>0\}$ is given by*

$$I^*=I_{\tilde{a}^{(1)}}\cap\cdots\cap I_{\tilde{a}^{(n)}}\cap I_{\tilde{b}^{(1)}}\cap\cdots\cap I_{\tilde{b}^{(n)}}$$

and

$$M_\alpha^-=\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(1-)}\right)\times\cdots\times\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(n-)}\right),$$

where $M_\beta^{(i-)}$ are bounded closed intervals given by

$$M_\beta^{(i-)}=\left[\min\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\},\max\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\}\right]$$

for $i=1,\cdots,n$. Let $I_\ominus^{(\star DT)}$ be the interval range of $\tilde{\mathbf{a}}\ominus_{DT}^\star\tilde{\mathbf{b}}$, and let $I_\ominus^{(i)(\star DT)}$ be the interval range of $\tilde{a}^{(i)}\ominus_{DT}^\star\tilde{b}^{(i)}$ for $i=1,\cdots,n$.

(i) *Suppose that the supremum* $\sup I^*$ *is attained. Then $I_\ominus^{(\star DT)}=I^*$ and*

$$\left(\tilde{\mathbf{a}}\ominus_{DT}^\star\tilde{\mathbf{b}}\right)_\alpha=M_\alpha^-=\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(1-)}\right)\times\cdots\times\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(n-)}\right)$$

for each $\alpha\in I^*$.

(ii) *Suppose that the supremum* $\sup(I_{\tilde{a}^{(i)}}\cap I_{\tilde{b}^{(i)}})$ *is attained for each $i=1,\cdots,n$. Then*

$$I_\ominus^{(i)(\star DT)}=I_{\tilde{a}^{(i)}}\cap I_{\tilde{b}^{(i)}}\text{ and }\left(\tilde{a}^{(i)}\ominus_{DT}^\star\tilde{b}^{(i)}\right)_\alpha=\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(i-)}\right)$$

for each $\alpha\in I_{\tilde{a}^{(i)}}\cap I_{\tilde{b}^{(i)}}$ and each $i=1,\cdots,n$, and

$$I_\ominus^{(\star DT)}=I^*\text{ and }\left(\tilde{\mathbf{a}}\ominus_{DT}^\star\tilde{\mathbf{b}}\right)_\alpha=\left(\tilde{a}^{(1)}\ominus_{DT}^\star\tilde{b}^{(1)}\right)_\alpha\times\cdots\times\left(\tilde{a}^{(n)}\ominus_{DT}^\star\tilde{b}^{(n)}\right)_\alpha$$

for each $\alpha\in I^*$.

Assume that $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are canonical fuzzy intervals. Then, for $i = 1, \cdots, n$, we have

$$\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}} M_\beta^{(i-)} = \left[\min_{\{\beta\in I^*:\beta\geq\alpha\}} \min\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}, \max_{\{\beta\in I^*:\beta\geq\alpha\}} \max\left\{\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L, \tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right\}\right]$$
$$= \left[\min\left\{\min_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L\right), \min_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right)\right\},\right.$$
$$\left.\max\left\{\max_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^L - \tilde{b}_{i\beta}^L\right), \max_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^U - \tilde{b}_{i\beta}^U\right)\right\}\right]$$

that are bounded and closed intervals.

Example 13. *Continuing from Example 7 by referring to (20), part (i) of Theorem 4 says that*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = M_\alpha^- = \{(-3,-1)\}$$

for $\alpha \in [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ for $\alpha \notin [0, 0.8]$.

5.4. Using the Form of Decomposition Theorem to Study the α-Level Sets of $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any fuzzy intervals. The family $\{M_\alpha^- : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M_\alpha^- = M_\alpha^{(1-)} \times \cdots \times M_\alpha^{(n-)},$$

where $M_\alpha^{(i-)}$ are bounded closed intervals given by

$$M_\alpha^{(i-)} = \left[\min\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\}, \max\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\}\right]$$

for $i = 1, \cdots, n$. Based on the form of decomposition theorem, the membership functions of $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$ and $\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}$ for $i = 1, \cdots, n$ are given by

$$\xi_{\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha\in I^*:\alpha>0\}} \alpha \cdot \chi_{M_\alpha^-}(\mathbf{z})$$

for any $\mathbf{z} \in \mathbb{R}^n$ and

$$\xi_{\tilde{a}^{(i)}\ominus_{DT}^{\dagger}\tilde{b}^{(i)}}(z) = \sup_{\{\alpha\in I^*:\alpha>0\}} \alpha \cdot \chi_{M_\alpha^{(i-)}}(z)$$

for any $z \in \mathbb{R}$, respectively. Let $I_\ominus^{(\dagger DT)}$ and $I_\ominus^{(i)(\dagger DT)}$ be the interval ranges of $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$ and $\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}$, respectively, for $i = 1, \cdots, n$. Herein we study the α-level sets $(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$ for $\alpha \in I_\ominus^{(\dagger DT)}$, and the α-level sets $(\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)})_\alpha$ of $\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}$ for $\alpha \in I_\ominus^{(i)(\dagger DT)}$. We first provide some useful lemmas.

Lemma 2. *Let I be a closed subinterval of $[0,1]$ given by $I = [0, \gamma]$ for some $0 < \gamma \leq 1$. Let $\zeta^L : I \to \mathbb{R}$ and $\zeta^U : I \to \mathbb{R}$ be two bounded real-valued functions defined on I with $\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in I$. Suppose that the following conditions are satisfied:*

- *ζ^L is an increasing function and ζ^U is a decreasing function on I;*
- *ζ^L and ζ^U are left-continuous on $I \setminus \{0\} = (0, \gamma]$.*

Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ *for* $\alpha \in I$. *For any fixed* $x \in \mathbb{R}$, *the function*

$$\zeta(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha}(x), & \text{if } \alpha \in I \text{ with } \alpha > 0 \end{cases}$$

is upper semi-continuous on I.

Lemma 3. *Let* I *be a closed subinterval of* $[0,1]$ *given by* $I = [0,\gamma]$ *for some* $0 < \gamma \leq 1$. *For each* $i = 1, \cdots, n$, *let* $\zeta_i^L : I \to \mathbb{R}$ *and* $\zeta_i^U : I \to \mathbb{R}$ *be bounded real-valued functions defined on* I *with* $\zeta_i^L(\alpha) \leq \zeta_i^U(\alpha)$ *for each* $\alpha \in I$. *Suppose that the following conditions are satisfied:*

- ζ_i^L *are increasing function and* ζ_i^U *are decreasing function on* I *for* $i = 1, \cdots, n$;
- ζ_i^L *and* ζ_i^U *are left-continuous on* $I \setminus \{0\} = (0,\gamma]$ *for* $i = 1, \cdots, n$.

Let $M_\alpha^{(i-)} = [\zeta_i^L(\alpha), \zeta_i^U(\alpha)]$ *for* $\alpha \in I$ *and for* $i = 1, \cdots, n$, *and let* $M_\alpha = M_\alpha^{(1-)} \times \cdots \times M_\alpha^{(n-)}$. *For any fixed* $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, *the following function*

$$\zeta(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha}(\mathbf{x}), & \text{if } \alpha \in I \text{ with } \alpha > 0 \end{cases}$$

is upper semi-continuous on I.

Proof. Lemma 2 says that the functions

$$\zeta_i(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha^{(i-)}}(x_i), & \text{if } \alpha \in I \text{ with } \alpha > 0 \end{cases}$$

are upper semi-continuous on I for $i = 1, \cdots, n$. For $r \in I$, we define the sets

$$F_r = \{\alpha \in I : \zeta(\alpha) \geq r\} \text{ and } F_r^{(i-)} = \{\alpha \in I : \zeta_i(\alpha) \geq r\} \text{ for } i = 1, \cdots, n.$$

The upper semi-continuity of ζ_i says that $F_r^{(i-)}$ is a closed set for $i = 1, \cdots, n$. For $r > 0$, we want to claim $F_r = \bigcap_{i=1}^n F_r^{(i-)}$. Given any $\alpha \in F_r$, it follows that $\mathbf{x} \in M_\alpha$ and $\alpha \geq r$, i.e., $x_i \in M_\alpha^{(i-)}$ and $\alpha \geq r$ for $i = 1, \cdots, n$, which also implies $\zeta_i(\alpha) \geq r$ for $i = 1, \cdots, n$. Therefore, we obtain the inclusion $F_r \subseteq \bigcap_{i=1}^n F_r^{(i-)}$. On the other hand, suppose that $\alpha \in F_r^{(i-)}$ for $i = 1, \cdots, n$. It follows that $x_i \in M_\alpha^{(i-)}$ and $\alpha \geq r$ for $i = 1, \cdots, n$; i.e., $\mathbf{x} \in M_\alpha$ and $\alpha \geq r$. Therefore, we obtain the equality $F_r = \bigcap_{i=1}^n F_r^{(i-)}$, which also says that F_r is a closed set, since each $F_r^{(i-)}$ is a closed set for $i = 1, \cdots, n$. For $r = 0$, it is clear to see that $F_0 = I$ is a closed subinterval of $[0,1]$. Therefore, we conclude that ζ is indeed upper semi-continuous on I. This completes the proof. □

Now, we assume that the supremum $\sup I^*$ is attained. Then I^* is a bounded closed interval with $I_\ominus^{(\dagger DT)} = I^*$ by referring to Proposition 3. We also assume that $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are canonical fuzzy intervals. Under these assumptions, we claim

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta^- \text{ for } \alpha \in I_\ominus^{(\dagger DT)} \text{ with } \alpha > 0. \tag{40}$$

Let

$$\zeta_i^L(\alpha) = \min\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\} \text{ and } \zeta_i^U(\alpha) = \max\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\}.$$

Then $M_{\alpha}^{(i-)} = [\zeta_i^L(\alpha), \zeta_i^U(\alpha)]$. We also see that ζ_i^L and ζ_i^U are continuous functions on I^*. Using Lemmas 2 and 3, given any fixed $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the functions

$$\zeta(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha}(\mathbf{x}), & \text{if } \alpha \in I^* \text{ with } \alpha > 0 \end{cases}$$

and

$$\zeta_i(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha^{(i-)}}(x_i), & \text{if } \alpha \in I^* \text{ with } \alpha > 0 \end{cases}$$

are upper semi-continuous on I^* for $i = 1, \cdots, n$.

Given any $\alpha \in I_{\ominus}^{(\dagger DT)} = I^*$ with $\alpha > 0$, suppose that $\mathbf{z} \in (\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}})_\alpha$ and $\mathbf{z} \notin M_\beta^-$ for all $\beta \in I^*$ with $\beta \geq \alpha$. Then $\beta \cdot \chi_{M_\beta^-}(\mathbf{z}) < \alpha$ for all $\beta \in I^*$. Since I^* is a bounded closed interval, i.e., a compact set, and $\zeta(\beta) = \beta \cdot \chi_{M_\beta^-}(\mathbf{z})$ is upper semi-continuous on I^* as described above, the supremum of the function ζ is attained by Lemma 1. This says that

$$\xi_{\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\beta \in I^*} \zeta(\beta) = \sup_{\beta \in I^*} \beta \cdot \chi_{M_\beta^-}(\mathbf{z}) = \max_{\beta \in I^*} \beta \cdot \chi_{M_\beta^-}(\mathbf{z}) = \beta^* \cdot \chi_{M_{\beta^*}}(\mathbf{z}) < \alpha$$

for some $\beta^* \in I^*$, which violates $\mathbf{z} \in (\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}})_\alpha$. Therefore, there exists $\beta_0 \in I^*$ with $\beta_0 \geq \alpha$ satisfying $\mathbf{z} \in M_{\beta_0}^-$, which shows the following inclusion:

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha \subseteq \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta^-.$$

On the other hand, the inclusion

$$\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta^- \subseteq \left\{\mathbf{z} \in \mathbb{R}^n : \sup_{\beta \in I^*} \beta \cdot \chi_{M_\beta^-}(\mathbf{z}) \geq \alpha\right\} = \{\mathbf{z} \in \mathbb{R}^n : \xi_{\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}}(\mathbf{z}) \geq \alpha\} = (\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}})_\alpha$$

is obvious. This shows (40).

Suppose that the supremum $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ is attained. Then $I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ is also a bounded closed interval. Therefore, we can similarly obtain $I_{\ominus}^{(i)(\dagger DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and

$$\left(\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}\right)_\alpha = \bigcup_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} M_\beta^{(i-)} \text{ for } \alpha \in I_{\ominus}^{(i)(\dagger DT)} \text{ with } \alpha > 0.$$

The above results are summarized below.

Theorem 5. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that the family $\{M_\alpha : \alpha \in I^*$ for $\alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M_\alpha^- = M_\alpha^{(1-)} \times \cdots \times M_\alpha^{(n-)},$$

where $M_\alpha^{(i-)}$ are bounded closed intervals given by

$$M_\alpha^{(i-)} = \left[\min\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\}, \max\left\{\tilde{a}_{i\alpha}^L - \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U - \tilde{b}_{i\alpha}^U\right\}\right]$$

for $i = 1, \cdots, n$. *Let* $I_{\ominus}^{(\dagger DT)}$ *be the interval range of* $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$, *and let* $I_{\ominus}^{(i)(\dagger DT)}$ *be the interval range of* $\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}$ *for* $i = 1, \cdots, n$.

(i) *Suppose that the supremum* $\sup I^*$ *is attained. Then* $I_{\ominus}^{(\dagger DT)} = I^*$ *and*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta}^{-} = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} \left(M_{\beta}^{(1-)} \times \cdots \times M_{\beta}^{(n-)}\right)$$

for each $\alpha \in I^*$ *with* $\alpha > 0$, *and the* 0*-level set*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{0} = \mathrm{cl}\left(\bigcup_{\{\alpha \in I^* : \alpha > 0\}} \left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha}\right).$$

(ii) *Suppose that the supremum* $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ *is attained for each* $i = 1, \cdots, n$. *Then*

$$I_{\ominus}^{(i)(\dagger DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} \text{ and } \left(\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}\right)_{\alpha} = \bigcup_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} M_{\beta}^{(i-)}$$

for each $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ *with* $\alpha >$ *and each* $i = 1, \cdots, n$. *The* 0*-level set is*

$$\left(\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}\right)_{0} = \mathrm{cl}\left(\bigcup_{\{\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \alpha > 0\}} \left(\tilde{a}^{(i)} \ominus_{DT}^{\dagger} \tilde{b}^{(i)}\right)_{\alpha}\right).$$

Moreover, for $i = 1, \cdots, n$, *we have*

$$\begin{aligned}
&\bigcup_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} M_{\beta}^{(i-)} \\
&= \left[\min_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \min\left\{\tilde{a}_{i\beta}^{L} - \tilde{b}_{i\beta}^{L}, \tilde{a}_{i\beta}^{U} - \tilde{b}_{i\beta}^{U}\right\}, \max_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \max\left\{\tilde{a}_{i\beta}^{L} - \tilde{b}_{i\beta}^{L}, \tilde{a}_{i\beta}^{U} - \tilde{b}_{i\beta}^{U}\right\}\right] \\
&= \left[\min\left\{\min_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \left(\tilde{a}_{i\beta}^{L} - \tilde{b}_{i\beta}^{L}\right), \min_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \left(\tilde{a}_{i\beta}^{U} - \tilde{b}_{i\beta}^{U}\right)\right\},\right. \\
&\qquad \left.\max\left\{\max_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \left(\tilde{a}_{i\beta}^{L} - \tilde{b}_{i\beta}^{L}\right), \max_{\{\beta \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} : \beta \geq \alpha\}} \left(\tilde{a}_{i\beta}^{U} - \tilde{b}_{i\beta}^{U}\right)\right\}\right]
\end{aligned}$$

which are bounded and closed intervals.

Remark 3. *We remark that, in general, we cannot have the following equality:*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{DT}^{\dagger} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{DT}^{\dagger} \tilde{b}^{(n)}\right)_{\alpha} \text{ for each } \alpha \in I^*.$$

When $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ *and* $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ *are taken to be canonical fuzzy numbers instead of canonical fuzzy intervals, it follows that*

$$I_{\tilde{a}^{(i)}} = I_{\tilde{b}^{(i)}} = I_{\ominus}^{(i)(\dagger DT)} = I_{\ominus}^{(\dagger DT)} = [0,1] \text{ for all } i = 1, \cdots, n.$$

Then, we can have the following equality:

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \ominus_{DT}^{\dagger} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \ominus_{DT}^{\dagger} \tilde{b}^{(n)}\right)_{\alpha} \text{ for each } \alpha \in [0,1].$$

Example 14. *Continuing from Example 7 by referring to (21), we have* $M_{\alpha}^{-} = \{(-3,-1)\}$*. Part (i) of Theorem 5 says that*

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \bigcup_{\{\beta \in I^{*}: \beta \geq \alpha\}} M_{\beta}^{-} = \bigcup_{\{\beta \in [0,0.8]: \beta \geq \alpha\}} \{(-3,-1)\} = \{(-3,-1)\}$$

for $\alpha \in [0, 0.8]$*. Moreover, we have* $\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \emptyset$ *for* $\alpha \notin [0, 0.8]$.

5.5. The Equivalences and Fuzziness

Next, we present the equivalences between $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}$ in Theorems 2 and 3, respectively.

Theorem 6. *Let* $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ *and* $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ *be any fuzzy intervals. Suppose that* $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ *and* $\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}$ *are obtained from Theorems 2 and 3, respectively. We also assume that the supremum* $\sup I^{*}$ *is attained. Then*

$$I_{\ominus}^{(EP)} = I_{\ominus}^{(\diamond DT)} = I^{*} \textit{ and } \tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^{*}$*, we have*

$$\left(\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{a}_{1\alpha}^{L} - \tilde{b}_{1\alpha}^{U}, \tilde{a}_{1\alpha}^{U} - \tilde{b}_{1\alpha}^{L}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} - \tilde{b}_{n\alpha}^{U}, \tilde{a}_{n\alpha}^{U} - \tilde{b}_{n\alpha}^{L}\right]. \tag{41}$$

Proof. From Propositions 3 and 2, we have $I_{\ominus}^{(EP)} = I_{\ominus}^{(\diamond DT)} = I^{*}$. The equality (41) follows immediately Theorems 2 and 3, which also says that $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \ominus_{DT}^{\diamond} \tilde{\mathbf{b}}$. This completes the proof. □

We are not able to study the equivalences among $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$, $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$. However, we can study their fuzziness by considering the α-level sets. The formal definition regarding the fuzziness is given below.

Definition 2. *Let* $\tilde{a}$ *and* $\tilde{b}$ *be two fuzzy intervals with interval ranges* $I_{\tilde{a}}$ *and* $I_{\tilde{b}}$*, respectively. We say that* $\tilde{a}$ *is fuzzier than* $\tilde{b}$ *if and only if* $I_{\tilde{a}} = I_{\tilde{b}}$ *and* $\tilde{b}_{\alpha} \subseteq \tilde{a}_{\alpha}$ *for all* $\alpha \in I_{\tilde{a}}$ *with* $\alpha > 0$.

Suppose now that we plan to collect $2n$ real number data $a_1, \cdots, a_n, b_1, \cdots, b_n$ in $\mathbb{R}$. Owing to the unexpected situation, we cannot exactly obtain the desired data. Instead, we can just obtain the fuzzy data $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}, \tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ that can be described by some suitable membership functions. Now, we have two ways to calculate the difference between $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$. One is based on the extension principle to obtain $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$, and another one is based on the form of decomposition theorem to obtain $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$ for $\ominus_{DT} \in \{\ominus_{DT}^{\diamond}, \ominus_{DT}^{\star}, \ominus_{DT}^{\dagger}\}$. We claim that $\tilde{\mathbf{a}} \ominus_{EP} \tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$. In other words, we prefer to take $\tilde{\mathbf{a}} \ominus_{DT} \tilde{\mathbf{b}}$, which has less fuzziness.

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals, and let $\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}$ be obtained from Theorems 4 and 5, respectively. Suppose that the supremum $\sup I^{*}$ is attained. Then we have

$$I_{\ominus}^{(\star DT)} = I^{*} = I_{\ominus}^{(\dagger DT)}.$$

For each $\alpha \in I^{*}$ with $\alpha > 0$, we also have

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \bigcup_{\{\beta \in I^{*}: \beta \geq \alpha\}} \left(M_{\beta}^{(1-)} \times \cdots \times M_{\beta}^{(n-)}\right)$$

and

$$\left(\tilde{\mathbf{a}} \ominus_{DT}^{\star} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\bigcup_{\{\beta \in I^{*}: \beta \geq \alpha\}} M_{\beta}^{(1-)}\right) \times \cdots \times \left(\bigcup_{\{\beta \in I^{*}: \beta \geq \alpha\}} M_{\beta}^{(n-)}\right).$$

Since the inclusion

$$\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}\left(M_\beta^{(1-)}\times\cdots\times M_\beta^{(n-)}\right)\subseteq\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(1-)}\right)\times\cdots\times\left(\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(n-)}\right)$$

is obvious, it follows that

$$\left(\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}\right)_\alpha\subseteq\left(\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}\right)_\alpha \text{ for each } \alpha\in I^*=I_\ominus^{(\star DT)}=I_\ominus^{(\dagger DT)} \text{ with } \alpha>0,$$

which says that $\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}$.

On the other hand, from Theorems 6 and 4, we have

$$I_\ominus^{(\star DT)}=I^*=I_\ominus^{(\diamond DT)}.$$

For each $\alpha\in I^*$ with $\alpha>0$, we also have

$$\begin{aligned}\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}}M_\beta^{(i-)}&=\left[\min_{\{\beta\in I^*:\beta\geq\alpha\}}\min\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\},\max_{\{\beta\in I^*:\beta\geq\alpha\}}\max\left\{\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^L,\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^U\right\}\right]\\&\subseteq\left[\min_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^L-\tilde{b}_{i\beta}^U\right),\max_{\{\beta\in I^*:\beta\geq\alpha\}}\left(\tilde{a}_{i\beta}^U-\tilde{b}_{i\beta}^L\right)\right]\\&\subseteq\left[\tilde{a}_{i\alpha}^L-\tilde{b}_{i\alpha}^U,\tilde{a}_{i\alpha}^U-\tilde{b}_{i\alpha}^L\right].\end{aligned}$$

It follows that

$$\left(\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}\right)_\alpha\subseteq\left(\tilde{\mathbf{a}}\ominus_{DT}^{\diamond}\tilde{\mathbf{b}}\right)_\alpha \text{ for each } \alpha\in I^*=I_\ominus^{(\star DT)}=I_\ominus^{(\diamond DT)} \text{ with } \alpha>0,$$

which says that $\tilde{\mathbf{a}}\ominus_{DT}^{\diamond}\tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}$. The above results are summarized below.

Theorem 7. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that $\tilde{\mathbf{a}}\ominus_{EP}\tilde{\mathbf{b}}$, $\tilde{\mathbf{a}}\ominus_{DT}^{\diamond}\tilde{\mathbf{b}}$, $\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}$ are obtained from Theorem 2, Theorem 3, Theorem 4 and Theorem 5, respectively. We also assume that the supremum* $\sup I^*$ *is attained. Then*

$$I_\ominus^{(EP)}=I_\ominus^{(\diamond DT)}=I_\ominus^{(\star DT)}=I_\ominus^{(\dagger DT)}=I^*$$

and

$$\left(\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}\right)_\alpha\subseteq\left(\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}\right)_\alpha\subseteq\left(\tilde{\mathbf{a}}\ominus_{DT}^{\diamond}\tilde{\mathbf{b}}\right)_\alpha=\left(\tilde{\mathbf{a}}\ominus_{EP}\tilde{\mathbf{b}}\right)_\alpha$$

for each $\alpha\in I^*$. In other words, $\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}$, and $\tilde{\mathbf{a}}\ominus_{DT}^{\diamond}\tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}}\ominus_{DT}^{\star}\tilde{\mathbf{b}}$.

Remark 4. *Theorem 7 says that, when $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ are taken to be canonical fuzzy intervals, we may prefer to pick $\tilde{\mathbf{a}}\ominus_{DT}^{\dagger}\tilde{\mathbf{b}}$ that has less fuzziness in applications.*

6. Addition of Vectors of Fuzzy Intervals

Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be two vectors of fuzzy intervals with components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$, respectively, for $i=1,\cdots,n$. Next we study the α-level set of $\tilde{\mathbf{a}}\oplus_{EP}\tilde{\mathbf{b}}$ that is obtained from the extension principle, and the α-level sets of $\tilde{\mathbf{a}}\oplus_{DT}\tilde{\mathbf{b}}$ for $\oplus_{DT}\in\{\oplus_{DT}^{\diamond},\oplus_{DT}^{\star},\oplus_{DT}^{\dagger}\}$ that are obtained from the form of decomposition theorem.

6.1. Using the Extension Principle to Study the α-Level Sets of $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$

Given any aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$, the membership function of addition $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{(\mathbf{x},\mathbf{y}):\mathbf{z}=\mathbf{x}+\mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right)$$

for any $\mathbf{z} \in \mathbb{R}^n$. Let $I_{\oplus}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$. The α-level set $(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$ for $\alpha \in I_{\oplus}^{(EP)}$ can be obtained by applying the results obtained in Wu [11] to the addition $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$, which is shown below. For each $\alpha \in I_{\oplus}^{(EP)}$ with $\alpha > 0$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x}+\mathbf{y} : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\} \\ &= \left\{(x_1+y_1, \cdots, x_n+y_n) : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\}.\end{aligned} \tag{42}$$

The 0-level set is given by

$$\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_0 = \tilde{\mathbf{a}}_0 + \tilde{\mathbf{b}}_0 = \left\{\mathbf{x}+\mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_0 \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_0\right\}.$$

Moreover, for each $\alpha \in I_{\oplus}^{(EP)}$, the α-level sets $(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}})_\alpha$ are closed and bounded subsets of $\mathbb{R}^m$. When the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by

$$\mathfrak{A}(\alpha_1, \cdots, \alpha_{2n}) = \begin{cases} \min\{\alpha_1, \cdots, \alpha_{2n}\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression,} & \text{otherwise.} \end{cases}$$

Proposition 2 says that $I_{\oplus}^{(EP)} = I^*$. Therefore, for each $\alpha \in I_{\oplus}^{(EP)}$, we have $(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}})_\alpha \neq \emptyset$, $\tilde{a}_\alpha^{(i)} \neq \emptyset$ and $\tilde{b}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$. Now, for each $\alpha \in I_{\oplus}^{(EP)}$ with $\alpha > 0$, using (42), we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x}+\mathbf{y} : \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} \geq \alpha\right\} \\ &= \left\{\mathbf{x}+\mathbf{y} : \xi_{\tilde{a}^{(i)}}(x_i) \geq \alpha \text{ and } \xi_{\tilde{b}^{(i)}}(y_i) \geq \alpha \text{ for each } i = 1, \cdots, n\right\} \\ &= \Big\{(x_1+y_1, \cdots, x_n+y_n) : x_i \in \tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \\ &\qquad \text{and } y_i \in \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right] \text{ for each } i = 1, \cdots, n\Big\} \\ &= \left[\tilde{a}_{1\alpha}^L + \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U + \tilde{b}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L + \tilde{b}_{n\alpha}^L, \tilde{a}_{n\alpha}^U + \tilde{b}_{n\alpha}^U\right].\end{aligned} \tag{43}$$

For the 0-level set, from (43) and (4), it is not difficult to show that

$$\begin{aligned}\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_0 &= \mathrm{cl}\left(\bigcup_{\{\alpha \in I_{\oplus}^{(EP)} : \alpha > 0\}} \left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha\right) \\ &= \left[\tilde{a}_{10}^L + \tilde{b}_{10}^L, \tilde{a}_{10}^U + \tilde{b}_{10}^U\right] \times \cdots \times \left[\tilde{a}_{n0}^L + \tilde{b}_{n0}^L, \tilde{a}_{n0}^U + \tilde{b}_{n0}^U\right].\end{aligned}$$

Regarding the components $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$, let $I_{\oplus}^{(i)(EP)}$ be the interval range of $\tilde{a}^{(i)} \oplus_{EP} \tilde{b}^{(i)}$. From Proposition 2, we can similarly obtain $I_{\oplus}^{(i)(EP)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$, and, for each $\alpha \in_{\oplus}^{(i)(EP)}$, we also have

$$\left(\tilde{a}^{(i)} \oplus_{EP} \tilde{b}^{(i)}\right)_\alpha = \left[\tilde{a}_{i\alpha}^L + \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U + \tilde{b}_{i\alpha}^U\right] \text{ for } i = 1, \cdots, n. \tag{44}$$

Therefore, from (43) and (44), for $\alpha \in I_{\oplus}^{(EP)} = I^*$, we obtain

$$\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{a}^{(1)} \oplus_{EP} \tilde{b}^{(1)}\right)_\alpha \times \cdots \times \left(\tilde{a}^{(n)} \oplus_{EP} \tilde{b}^{(n)}\right)_\alpha.$$

The above results are summarized in the following theorem.

Theorem 8. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by*

$$\mathfrak{A}\left(\alpha_1, \cdots, \alpha_{2n}\right) = \begin{cases} \min\left\{\alpha_1, \cdots, \alpha_{2n}\right\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression}, & \text{otherwise}, \end{cases}$$

Then, we have the following results.

(i) *Let $I_{\oplus}^{(i)(EP)}$ be the interval range of $\tilde{a}^{(i)} \oplus_{EP} \tilde{b}^{(i)}$ for $i = 1, \cdots, n$. For each $\alpha \in I_{\oplus}^{(i)(EP)}$, we have*

$$\left(\tilde{a}^{(i)} \oplus_{EP} \tilde{b}^{(i)}\right)_\alpha = \left[\tilde{a}_{i\alpha}^L + \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U + \tilde{b}_{i\alpha}^U\right].$$

We also have $I_{\oplus}^{(i)(EP)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$.

(ii) *Let $I_{\oplus}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$. We have*

$$I_{\oplus}^{(EP)} \subseteq I_{\oplus}^{(i)(EP)} \text{ for } i = 1, \cdots, n, \text{ and } I_{\oplus}^{(EP)} = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

For each $\alpha \in I_{\oplus}^{(EP)}$, we also have

$$\left(\tilde{a}^{(i)} \oplus_{EP} \tilde{b}^{(i)}\right)_\alpha = \left[\tilde{a}_{i\alpha}^L + \tilde{b}_{i\alpha}^L, \tilde{a}_{i\alpha}^U + \tilde{b}_{i\alpha}^U\right]$$

and

$$\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{a}^{(1)} \oplus_{EP} \tilde{b}^{(1)}\right)_\alpha \times \cdots \times \left(\tilde{a}^{(n)} \oplus_{EP} \tilde{b}^{(n)}\right)_\alpha.$$

Example 15. *Continuing from Examples 5 and 7, part (ii) of Theorem 8 says that*

$$\begin{aligned} \left(\tilde{a}^{(1)} \oplus_{EP} \tilde{b}^{(1)}\right)_\alpha &= \left[\tilde{a}_{1\alpha}^L + \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U + \tilde{b}_{1\alpha}^U\right] \\ &= [(1+\alpha) + (4+\alpha), (7-\alpha) + (4-\alpha)] = [5+2\alpha, 11-2\alpha] \end{aligned}$$

and

$$\begin{aligned} \left(\tilde{a}^{(2)} \oplus_{EP} \tilde{b}^{(2)}\right)_\alpha &= \left[\tilde{a}_{2\alpha}^L - \tilde{b}_{2\alpha}^U, \tilde{a}_{2\alpha}^U - \tilde{b}_{2\alpha}^L\right] \\ &= [(2+\alpha) + (3+\alpha), (6-\alpha) + (5-\alpha)] = [5+2\alpha, 11-2\alpha] \end{aligned}$$

and

$$\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{a}^{(1)} \oplus_{EP} \tilde{b}^{(1)}\right)_\alpha \times \left(\tilde{a}^{(2)} \oplus_{EP} \tilde{b}^{(2)}\right)_\alpha = [5+2\alpha, 11-2\alpha] \times [5+2\alpha, 11-2\alpha]$$

for $\alpha \in I_{\oplus}^{(EP)} = [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ for $\alpha \notin [0, 0.8]$.

6.2. Using the Form of Decomposition Theorem to Study the α-Level Sets

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. The family $\{M_\alpha^+ : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M_\alpha^+ = \tilde{\mathbf{a}}_\alpha + \tilde{\mathbf{b}}_\alpha.$$

Since $\tilde{a}_{\alpha}^{(i)} \neq \emptyset$ and $\tilde{b}_{\alpha}^{(i)} \neq \emptyset$ for $\alpha \in I^*$ and $i = 1, \cdots, n$, given any $\alpha \in I^*$ with $\alpha > 0$, we have

$$\begin{aligned} M_{\alpha}^{+} &= \tilde{\mathbf{a}}_{\alpha} + \tilde{\mathbf{b}}_{\alpha} = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_{\alpha} \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_{\alpha}\} \\ &= \left\{(x_1 + y_1, \cdots, x_1 + y_1) : x_i \in \tilde{a}_{\alpha}^{(i)} = \left[\tilde{a}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U}\right] \text{ and } y_i \in \tilde{b}_{\alpha}^{(i)} = \left[\tilde{b}_{i\alpha}^{L}, \tilde{b}_{i\alpha}^{U}\right] \text{ for } i = 1, \cdots, n\right\} \\ &= \left[\tilde{a}_{1\alpha}^{L} + \tilde{b}_{1\alpha}^{L}, \tilde{a}_{1\alpha}^{U} + \tilde{b}_{1\alpha}^{U}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} + \tilde{b}_{n\alpha}^{L}, \tilde{a}_{n\alpha}^{U} + \tilde{b}_{n\alpha}^{U}\right]. \end{aligned} \tag{45}$$

Based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M_{\alpha}^{+}}(\mathbf{z}). \tag{46}$$

Let $I_{\oplus}^{(\diamond DT)}$ be the interval range of $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$. Suppose that the supremum $\sup I^*$ is attained. Using the similar argument in the proof of Proposition 4, we can obtain $I_{\oplus}^{(\diamond DT)} = I^*$ and the α-level sets $(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}})_{\alpha}$ of $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$ are given by

$$\left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = M_{\alpha}^{+} = \left[\tilde{a}_{1\alpha}^{L} + \tilde{b}_{1\alpha}^{L}, \tilde{a}_{1\alpha}^{U} + \tilde{b}_{1\alpha}^{U}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} + \tilde{b}_{n\alpha}^{L}, \tilde{a}_{n\alpha}^{U} + \tilde{b}_{n\alpha}^{U}\right]$$

for $\alpha \in I_{\oplus}^{(\diamond DT)}$.

Now, for $i = 1, \cdots, n$ and for $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ with $\alpha > 0$, we take

$$M_{\alpha}^{(i+)} = \tilde{a}_{\alpha}^{(i)} + \tilde{b}_{\alpha}^{(i)} = \left[\tilde{a}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U}\right] + \left[\tilde{b}_{i\alpha}^{L}, \tilde{b}_{i\alpha}^{U}\right] = \left[\tilde{a}_{i\alpha}^{L} + \tilde{b}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U} + \tilde{b}_{i\alpha}^{U}\right]. \tag{47}$$

Then, for $\alpha \in I^*$, from (45), we see that

$$M_{\alpha}^{+} = M_{\alpha}^{(1+)} \times \cdots \times M_{\alpha}^{(n+)} \subset \mathbb{R}^n.$$

Let $\tilde{a}^{(i)} \oplus_{DT}^{\diamond} \tilde{b}^{(i)}$ be obtained using the form of decomposition theorem based on the family

$$\{M_{\alpha}^{(i+)} : \alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} \text{ with } \alpha > 0\}$$

that is defined in (47). Let $I_{\oplus}^{(i)(\diamond DT)}$ be the interval range of $\tilde{a}^{(i)} \oplus_{DT}^{\diamond} \tilde{b}^{(i)}$. Suppose that the supremum $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ is attained for each $i = 1, \cdots, n$. Then the supremum $\sup I^*$ is also attained. For $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$, we can similarly obtain $I_{\oplus}^{(i)(\diamond DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ and

$$\left(\tilde{a}^{(i)} \oplus_{DT}^{\diamond} \tilde{b}^{(i)}\right)_{\alpha} = M_{\alpha}^{(i+)} = \left[\tilde{a}_{i\alpha}^{L} + \tilde{b}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U} + \tilde{b}_{i\alpha}^{U}\right] \text{ for } i = 1, \cdots, n,$$

which also implies

$$\left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \oplus_{DT}^{\diamond} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \oplus_{DT}^{\diamond} \tilde{b}^{(n)}\right)_{\alpha}.$$

The above results are summarized below.

Theorem 9. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the family $\{M_{\alpha}^{+} : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M_{\alpha}^{+} = \tilde{\mathbf{a}}_{\alpha} + \tilde{\mathbf{b}}_{\alpha}.$$

Let $I_{\oplus}^{(\diamond DT)}$ be the interval range of $\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$, and let $I_{\oplus}^{(i)(\diamond DT)}$ be the interval range of $\tilde{a}^{(i)} \oplus_{DT}^{\diamond} \tilde{b}^{(i)}$ for $i = 1, \cdots, n$.

(i) *Suppose that the supremum* $\sup I^*$ *is attained. Then* $I_{\oplus}^{(\diamond DT)} = I^*$ *and*

$$\left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{a}_{1\alpha}^{L} + \tilde{b}_{1\alpha}^{L}, \tilde{a}_{1\alpha}^{U} + \tilde{b}_{1\alpha}^{U}\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^{L} + \tilde{b}_{n\alpha}^{L}, \tilde{a}_{n\alpha}^{U} + \tilde{b}_{n\alpha}^{U}\right]$$

for each $\alpha \in I^*$. *Moreover, we have*

$$I_{\oplus}^{(EP)} = I_{\oplus}^{(\diamond DT)} = I^* \text{ and } \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}},$$

where $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}$ *is obtained from Theorem 8.*

(ii) *Suppose that the supremum* $\sup(I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}})$ *is attained for each* $i = 1, \cdots, n$. *Then*

$$I_{\oplus}^{(i)(\diamond DT)} = I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}} \text{ and } \left(\tilde{a}^{(i)} \oplus_{DT}^{\diamond} \tilde{b}^{(i)}\right)_{\alpha} = \left[\tilde{a}_{i\alpha}^{L} + \tilde{b}_{i\alpha}^{L}, \tilde{a}_{i\alpha}^{U} + \tilde{b}_{i\alpha}^{U}\right]$$

for each $\alpha \in I_{\tilde{a}^{(i)}} \cap I_{\tilde{b}^{(i)}}$ *and each* $i = 1, \cdots, n$, *and*

$$I_{\oplus}^{(\diamond DT)} = I^* \text{ and } \left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{a}^{(1)} \oplus_{DT}^{\diamond} \tilde{b}^{(1)}\right)_{\alpha} \times \cdots \times \left(\tilde{a}^{(n)} \oplus_{DT}^{\diamond} \tilde{b}^{(n)}\right)_{\alpha}$$

for each $\alpha \in I^*$.

Next, we study the addition $\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}$ by considering a family that has the same form of Theorem 4. We first need a useful property given below.

Lemma 4. *Let* $\tilde{a}$ *be a fuzzy interval with interval range* $I_{\tilde{a}}$. *Then the function* $\zeta^L(\alpha) = \tilde{a}_{\alpha}^{L}$ *is lower semi-continuous on* $I_{\tilde{a}}$, *and the function* $\zeta^U(\alpha) = \tilde{a}_{\alpha}^{U}$ *is upper semi-continuous on* $I_{\tilde{a}}$.

Theorem 10. *Let* $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ *and* $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ *be fuzzy intervals. Suppose that the family* $\{M_{\alpha} : \alpha \in I^* \text{ for } \alpha > 0\}$ *is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M_{\alpha}^{+} = \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta}^{(1+)}\right) \times \cdots \times \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta}^{(n+)}\right), \tag{48}$$

where $M_{\beta}^{(i+)}$ *are bounded closed intervals given by*

$$M_{\beta}^{(i+)} = \left[a_{i\beta}^{L} + b_{i\beta}^{L}, a_{i\beta}^{U} + b_{i\beta}^{U}\right]$$

for $i = 1, \cdots, n$. *Then*

$$\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}.$$

If we further assume that the supremum I^* *is attained, then*

$$\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}.$$

Proof. Let $\zeta_i^L(\beta) = a_{i\beta}^L + b_{i\beta}^L$ and $\zeta_i^U(\beta) = a_{i\beta}^U + b_{i\beta}^U$. Then $M_\beta^{(i)} = [\zeta_i^L(\beta), \zeta_i^U(\beta)]$. Lemma 4 say that ζ_i^L is lower semi-continuous on I^* and ζ_i^U is upper semi-continuous on I^*. Then, for $\alpha \in I^*$ with $\alpha > 0$, we can obtain

$$\begin{aligned}\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}} M_\beta^{(i+)} &= \left[\min_{\{\beta\in I^*:\beta\geq\alpha\}} \zeta_i^L(\beta), \max_{\{\beta\in I^*:\beta\geq\alpha\}} \zeta_i^U(\beta)\right] \\ &= \left[\min_{\{\beta\in I^*:\beta\geq\alpha\}} \left(a_{i\beta}^L + b_{i\beta}^L\right), \max_{\{\beta\in I^*:\beta\geq\alpha\}} \left(a_{i\beta}^U + b_{i\beta}^U\right)\right] \\ &= \left[a_{i\alpha}^L + b_{i\alpha}^L, a_{i\alpha}^U + b_{i\alpha}^U\right] = M_\alpha^{(i)}.\end{aligned}$$

Therefore, by referring to (48), we have

$$M_\alpha^+ = \left[\tilde{a}_{1\alpha}^L + \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U + \tilde{b}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L + \tilde{b}_{n\alpha}^L, \tilde{a}_{n\alpha}^U + \tilde{b}_{n\alpha}^U\right],$$

which is the same as (45). Therefore, we obtain $\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$. Now, we assume that the supremum I^* is attained. Theorems 8 and 9 say that

$$I_\oplus^{(EP)} = I_\oplus^{(\diamond DT)} = I^*$$

and

$$\left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{a}_{1\alpha}^L + \tilde{b}_{1\alpha}^L, \tilde{a}_{1\alpha}^U + \tilde{b}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L + \tilde{b}_{n\alpha}^L, \tilde{a}_{n\alpha}^U + \tilde{b}_{n\alpha}^U\right] = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha$$

for each $\alpha \in I^*$, which says that $\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}$. This completes the proof. □

Next, we study the addition $\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}}$ by considering a family that has the same form of Theorem 5. However, in this case, we need to consider the canonical fuzzy intervals rather than the fuzzy intervals.

Theorem 11. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that the family $\{M_\alpha : \alpha \in I^*$ for $\alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M_\alpha^+ = M_\alpha^{(1+)} \times \cdots \times M_\alpha^{(n+)},$$

where $M_\alpha^{(i+)}$ are bounded closed intervals given by

$$M_\alpha^{(i+)} = \left[a_{i\alpha}^L + b_{i\alpha}^L, a_{i\alpha}^U + b_{i\alpha}^U\right]$$

for $i = 1, \cdots, n$. Suppose that the supremum $\sup I^*$ *is attained. Then*

$$\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}.$$

Proof. For each $i = 1, \cdots, n$, it is clear to see that

$$\bigcup_{\{\beta\in I^*:\beta\geq\alpha\}} M_\beta^{(i+)} = \left[a_{i\alpha}^L + b_{i\alpha}^L, a_{i\alpha}^U + b_{i\alpha}^U\right] = M_\alpha^{(i+)}.$$

Since the supremum $\sup I^*$ is attained, for each $\alpha \in I^*$ with $\alpha > 0$, using the similar argument of Theorem 5, we can obtain

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha &= \bigcup_{\{\beta \in I^*:\beta \geq \alpha\}} M_\beta^{+} = \bigcup_{\{\beta \in I^*:\beta \geq \alpha\}} \left(M_\beta^{(1+)} \times \cdots \times M_\beta^{(n+)}\right) \\
&\bigcup_{\{\beta \in I^*:\beta \geq \alpha\}} \left(\left[a_{1\beta}^L + b_{1\beta}^L, a_{1\beta}^U + b_{1\beta}^U\right] \times \cdots \times \left[a_{n\beta}^L + b_{n\beta}^L, a_{n\beta}^U + b_{n\beta}^U\right]\right) \\
&= \left[a_{1\alpha}^L + b_{1\alpha}^L, a_{1\alpha}^U + b_{1\alpha}^U\right] \times \cdots \times \left[a_{n\alpha}^L + b_{n\alpha}^L, a_{n\alpha}^U + b_{n\alpha}^U\right] \\
&= \left(\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{EP} \tilde{\mathbf{b}}\right)_\alpha.
\end{aligned}$$

This completes the proof. □

We remark that Theorem 11 needs to consider the canonical fuzzy intervals rather than the fuzzy intervals, and assume that the supremum I^* is attained.

Remark 5. *When $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be canonical fuzzy numbers instead of canonical fuzzy intervals, it follows that*

$$I_{\oplus}^{(EP)} = I_{\oplus}^{(\diamond DT)} = I_{\oplus}^{(\star DT)} = I_{\oplus}^{(\dagger DT)} = I^* = [0,1],$$

which also says that the supremum $\sup I^*$ is attained. Therefore, the above theorems are applicable.

Example 16. *Using Theorem 11 and Example 15, we see that*

$$\left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha = [5+2\alpha, 11-2\alpha] \times [5+2\alpha, 11-2\alpha] \text{ for } \alpha \in [0, 0.8]$$

and

$$\left(\tilde{\mathbf{a}} \oplus_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \oplus_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin [0, 0.8].$$

7. Scalar Product of Vectors of Fuzzy Intervals

In the sequel, we are going to use the extension principle by referring to (6) to study the scalar product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, and use the form of decomposition theorem by referring to (22) to study the scalar product $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$.

7.1. Using the Extension Principle

Given any aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$, the membership function of scalar product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}(z) = \sup_{\{(\mathbf{x},\mathbf{y}):z=\mathbf{x}\bullet\mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right)$$

for any $z \in \mathbb{R}$. Let $I_{\circledast}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$. The α-level set $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ for $\alpha \in I_{\circledast}^{(EP)}$ can be obtained by applying the results obtained in Wu [11] to the scalar product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, which is shown below. For each $\alpha \in I_{\circledast}^{(EP)}$ with $\alpha > 0$, we have

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} \bullet \mathbf{y} : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\} \\
&= \left\{x_1 y_1 + \cdots + x_n y_n : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\}.
\end{aligned} \tag{49}$$

The 0-level set is given by

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_0 = \tilde{\mathbf{a}}_0 \bullet \tilde{\mathbf{b}}_0 = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_0 \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_0\right\}.$$

Moreover, for each $\alpha \in I_{\circledast}^{(EP)}$, the α-level sets $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_\alpha$ are closed and bounded subsets of $\mathbb{R}^m$. Now, the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by

$$\mathfrak{A}\left(\alpha_1, \cdots, \alpha_{2n}\right) = \begin{cases} \min\left\{\alpha_1, \cdots, \alpha_{2n}\right\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression,} & \text{otherwise.} \end{cases}$$

Proposition 2 says that $I_{\circledast}^{(EP)} = I^*$. Therefore, for each $\alpha \in I_{\circledast}^{(EP)}$, we have $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_\alpha \neq \emptyset$, $\tilde{a}_\alpha^{(i)} \neq \emptyset$ and $\tilde{b}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$. Now, for each $\alpha \in I_{\circledast}^{(EP)}$ with $\alpha > 0$, using (49), we have

$$\begin{aligned} \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} \bullet \mathbf{y} : \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} \geq \alpha\right\} \\ &= \left\{\mathbf{x} \bullet \mathbf{y} : \xi_{\tilde{a}^{(i)}}(x_i) \geq \alpha \text{ and } \xi_{\tilde{b}^{(i)}}(y_i) \geq \alpha \text{ for each } i = 1, \cdots, n\right\} \\ &= \Big\{x_1 y_1 + \cdots + x_n y_n : x_i \in \tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \\ &\qquad \text{and } y_i \in \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right] \text{ for each } i = 1, \cdots, n\Big\} \\ &= \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n), \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n)\right], \end{aligned} \tag{50}$$

where $\tilde{\mathbf{a}}_\alpha$ and $\tilde{\mathbf{b}}_\alpha$ are given in (18) and (19). For the 0-level set, from (50) and (4), it is not difficult to show that

$$\begin{aligned} \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_0 &= \mathrm{cl}\left(\bigcup_{\{\alpha \in I_{\circledast}^{(EP)} : \alpha > 0\}} \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha\right) \\ &= \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_0, \tilde{\mathbf{b}}_0)} (x_1 y_1 + \cdots + x_n y_n), \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_0, \tilde{\mathbf{b}}_0)} (x_1 y_1 + \cdots + x_n y_n)\right]. \end{aligned}$$

Definition 3. *Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\xi_{\tilde{A}}$. We say that $\tilde{A}$ is nonnegative when $\xi_{\tilde{A}}(x) = 0$ for each $x < 0$.*

It is clear to see that a fuzzy interval $\tilde{a}$ is nonnegative if and only if $\tilde{a}_\alpha^L \geq 0$ for each $\alpha \in I_{\tilde{a}}$. Suppose that $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are nonnegative fuzzy intervals. Then

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{a}_{1\alpha}^L \tilde{b}_{1\alpha}^L + \cdots + \tilde{a}_{n\alpha}^L \tilde{b}_{n\alpha}^L, \tilde{a}_{1\alpha}^U \tilde{b}_{1\alpha}^U + \cdots + \tilde{a}_{n\alpha}^U \tilde{b}_{n\alpha}^U\right].$$

The above results are summarized in the following theorem.

Theorem 12. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any fuzzy intervals. Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is given by*

$$\mathfrak{A}\left(\alpha_1, \cdots, \alpha_{2n}\right) = \begin{cases} \min\left\{\alpha_1, \cdots, \alpha_{2n}\right\}, & \text{if } \alpha_i \in \mathcal{R}_i \text{ for } i = 1, \cdots, 2n \\ \text{any expression,} & \text{otherwise,} \end{cases}$$

Let $I_{\circledast}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$. Then $I_{\circledast}^{(EP)} = I^*$, and, for each $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}, \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}\right],$$

where $\tilde{\mathbf{a}}_\alpha$ *and* $\tilde{\mathbf{b}}_\alpha$ *are given in* (18) *and* (19)*. Suppose that* $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ *and* $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ *are nonnegative fuzzy intervals. Then*

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right],$$

where $\tilde{\mathbf{a}}^L_\alpha$, $\tilde{\mathbf{a}}^U_\alpha$, $\tilde{\mathbf{b}}^L_\alpha$ *and* $\tilde{\mathbf{b}}^U_\alpha$ *are given in* (17)*.*

Example 17. *Continuing from Examples 5 and 7, Theorem 12 says that we need to calculate*

$$\min_{((x_1,x_2),(y_1,y_2))\in([\tilde{a}^L_{1\alpha},\tilde{a}^U_{1\alpha}]\times[\tilde{a}^L_{2\alpha},\tilde{a}^U_{2\alpha}],[\tilde{b}^L_{1\alpha},\tilde{b}^U_{1\alpha}]\times[\tilde{b}^L_{2\alpha},\tilde{b}^U_{2\alpha}])} (x_1,x_2)\bullet(y_1,y_2)$$

and

$$\max_{((x_1,x_2),(y_1,y_2))\in([\tilde{a}^L_{1\alpha},\tilde{a}^U_{1\alpha}]\times[\tilde{a}^L_{2\alpha},\tilde{a}^U_{2\alpha}],[\tilde{b}^L_{1\alpha},\tilde{b}^U_{1\alpha}]\times[\tilde{b}^L_{2\alpha},\tilde{b}^U_{2\alpha}])} (x_1,x_2)\bullet(y_1,y_2).$$

In other words, given any fixed $\alpha \in [0, 0.8]$*, we want to calculate*

$$\begin{array}{ll} \min/\max & x_1y_1 + x_2y_2 \\ \textit{subject to} & 1+\alpha \le x_1 \le 4-\alpha \\ & 2+\alpha \le x_2 \le 5-\alpha \\ & 4+\alpha \le y_1 \le 7-\alpha \\ & 3+\alpha \le y_2 \le 6-\alpha. \end{array}$$

Since $\alpha \in [0, 0.8]$*, the minimum is*

$$(1+\alpha)(4+\alpha) + (2+\alpha)(3+\alpha) = 10 + 10\alpha + \alpha^2$$

and the maximum is

$$(4-\alpha)(7-\alpha) + (5-\alpha)(6-\alpha) = 58 - 22\alpha + 2\alpha^2.$$

Therefore, Theorem 12 says that

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + 2\alpha^2\right]$$

for $\alpha \in [0, 0.8]$*. Moreover, we have* $\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ *for* $\alpha \notin [0, 0.8]$*.*

7.2. Using the Form of Decomposition Theorem

Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. The family $\{M^\bullet_\alpha : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M^\bullet_\alpha = \tilde{\mathbf{a}}_\alpha \bullet \tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\right\}.$$

Since $\tilde{a}^{(i)}_\alpha \neq \emptyset$ and $\tilde{b}^{(i)}_\alpha \neq \emptyset$ for $\alpha \in I^*$ and $i = 1, \cdots, n$, given any $\alpha \in I^*$ with $\alpha > 0$, we have

$$M^\bullet_\alpha = \tilde{\mathbf{a}}_\alpha \bullet \tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\right\} = \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}, \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}\right].$$

Based on the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}}\circledast^\diamond_{DT}\tilde{\mathbf{b}}}(z) = \sup_{\{\alpha\in I^*:\alpha>0\}} \alpha \cdot \chi_{M^\bullet_\alpha}(z).$$

Let $I_{\circledast}^{(\diamond DT)}$ be the interval range of $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$. Suppose that the supremum $\sup I^*$ is attained. Using the similar argument in the proof of Proposition 4, we can obtain $I_{\circledast}^{(\diamond DT)} = I^*$ and the α-level sets $(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ is given by

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = M_\alpha^\bullet = \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}, \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}\right]$$

for $\alpha \in I_{\circledast}^{(\diamond DT)}$. The above results are summarized below.

Theorem 13. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any fuzzy intervals. Suppose that the family $\{M_\alpha : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M_\alpha^\bullet = \tilde{\mathbf{a}}_\alpha \bullet \tilde{\mathbf{b}}_\alpha.$$

We also assume that the supremum $\sup I^*$ is attained. Then $I_{\circledast}^{(\diamond DT)} = I^*$, and, for $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}, \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}\right].$$

When $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be nonnegative fuzzy intervals, we have

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}_\alpha^L \bullet \tilde{\mathbf{b}}_\alpha^L, \tilde{\mathbf{a}}_\alpha^U \bullet \tilde{\mathbf{b}}_\alpha^U\right].$$

Example 18. *By referring to Example 17, Theorems 12 and 13 say that*

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + 2\alpha^2\right]$$

for $\alpha \in [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ for $\alpha \notin [0, 0.8]$.

Next, we study the scalar product $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ by considering a different family that has the similar form of Theorem 4. Recall that $\tilde{a}$ is a canonical fuzzy interval in a universal set U if and only if $\tilde{a}$ is a fuzzy interval such that $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are continuous with respect to α on $I_{\tilde{a}}$.

Theorem 14. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that the family $\{M_\alpha^\bullet : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} \text{ and } M_\alpha^\bullet = \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta\right),$$

where M_β is a bounded closed interval given by

$$M_\beta = \left[\min\left\{\tilde{\mathbf{a}}_\beta^L \bullet \tilde{\mathbf{b}}_\beta^L, \tilde{\mathbf{a}}_\beta^U \bullet \tilde{\mathbf{b}}_\beta^U\right\}, \max\left\{\tilde{\mathbf{a}}_\beta^L \bullet \tilde{\mathbf{b}}_\beta^L, \tilde{\mathbf{a}}_\beta^U \bullet \tilde{\mathbf{b}}_\beta^U\right\}\right].$$

We also assume that the supremum $\sup I^*$ is attained. Then $I_{\circledast}^{(\star DT)} = I^*$, and, for $\alpha \in I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha &= M_\alpha^\bullet \\ &= \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}_\beta^L \bullet \tilde{\mathbf{b}}_\beta^L, \tilde{\mathbf{a}}_\beta^U \bullet \tilde{\mathbf{b}}_\beta^U\right\}, \max_{\{\beta \in I^* : \beta \geq \alpha\}} \max\left\{\tilde{\mathbf{a}}_\beta^L \bullet \tilde{\mathbf{b}}_\beta^L, \tilde{\mathbf{a}}_\beta^U \bullet \tilde{\mathbf{b}}_\beta^U\right\}\right].\end{aligned} \tag{51}$$

When $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be nonnegative canonical fuzzy intervals, we have

$$\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right].$$

Proof. We define two functions ζ^L and ζ^U on I^* as follows:

$$\zeta^L(\beta) = \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\} \text{ and } \zeta^U(\beta) = \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}.$$

Then ζ^L and ζ^U are continuous on I^*, since we consider the canonical fuzzy intervals. We also see that $M_\beta = [\zeta^L(\beta), \zeta^U(\beta)]$. Using the similar argument of Theorem 4, we can obtain $I^{(\star DT)}_{\circledast} = I^*$, and, for $\alpha \in I^*$, we also have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= M^\bullet_\alpha = \bigcup_{\{\beta \in I^*: \beta \geq \alpha\}} M_\beta = \left[\min_{\{\beta \in I^*: \beta \geq \alpha\}} \zeta^L(\beta), \max_{\{\beta \in I^*: \beta \geq \alpha\}} \zeta^U(\beta)\right] \\ &= \left[\min_{\{\beta \in I^*: \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta \in I^*: \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right].\end{aligned} \tag{52}$$

This completes the proof. □

Example 19. *Continuing from Examples 5 and 7, we can obtain*

$$\min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\} = 10 + 10\beta + \beta^2$$

and

$$\max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\} = 58 - 22\beta + 2\beta^2.$$

Using (51), we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= \left[\min_{\{\beta \in [0,0.8]: \beta \geq \alpha\}} \left(10 + 10\beta + \beta^2\right), \max_{\{\beta \in [0,0.8]: \beta \geq \alpha\}} \left(58 - 22\beta + 2\beta^2\right)\right] \\ &= \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + 2\alpha^2\right]\end{aligned}$$

for $\alpha \in [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \emptyset$ for $\alpha \notin [0, 0.8]$.

Theorem 15. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be any canonical fuzzy intervals. Suppose that the family $\{M_\alpha : \alpha \in I^* \text{ for } \alpha > 0\}$ is given by*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}$$

and

$$M^\bullet_\alpha = \left[\min\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}, \max\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}\right].$$

We also assume that the supremum $\sup I^*$ is attained. Then $I^{(\dagger DT)}_{\circledast} = I^*$, and, for $\alpha \in I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= \bigcup_{\{\beta \in I^*: \beta \geq \alpha\}} M^\bullet_\beta \\ &= \left[\min_{\{\beta \in I^*: \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta \in I^*: \beta \geq \alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right].\end{aligned}$$

When $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ are taken to be nonnegative canonical fuzzy intervals, we have

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{\mathbf{a}}_{\alpha}^{L} \bullet \tilde{\mathbf{b}}_{\alpha}^{L}, \tilde{\mathbf{a}}_{\alpha}^{U} \bullet \tilde{\mathbf{b}}_{\alpha}^{U}\right].$$

Proof. Using the similar argument of Theorem 5, we can obtain $I_{\circledast}^{(\star DT)} = I^*$. For $\alpha \in I^*$, we also have

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \bigcup_{\{\beta \in I^*: \beta \geq \alpha\}} M_{\beta}^{\bullet}.$$

By referring to (52), we complete the proof. □

Example 20. *By referring to Example 19, Theorems 14 and 15 say that*

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}\right)_{\alpha} = \left[10 + 10\alpha + \alpha^2, 58 - 22\alpha + 2\alpha^2\right]$$

for $\alpha \in [0, 0.8]$. Moreover, we have $\left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \emptyset$ for $\alpha \notin [0, 0.8]$.

7.3. The Equivalences and Fuzziness

Next, we present the equivalences among $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$ for $\circledast_{DT} \in \{\circledast_{DT}^{\diamond}, \circledast_{DT}^{\star}, \circledast_{DT}^{\dagger}\}$.

Theorem 16. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be fuzzy intervals. Suppose that $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ are obtained from Theorems 12 and 13, respectively. We also assume that the supremum $\sup I^*$ is attained. Then*

$$I_{\circledast}^{(EP)} = I_{\circledast}^{(\diamond DT)} = I^* \text{ and } \tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_{\alpha}, \tilde{\mathbf{b}}_{\alpha})} \mathbf{x} \bullet \mathbf{y}, \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_{\alpha}, \tilde{\mathbf{b}}_{\alpha})} \mathbf{x} \bullet \mathbf{y}\right].$$

Theorem 17. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$ are obtained from Theorems 14 and 15, respectively. We also assume that the supremum $\sup I^*$ is attained. Then*

$$I_{\circledast}^{(\star DT)} = I_{\circledast}^{(\dagger DT)} = I^* \text{ and } \tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}\right)_{\alpha} &= \left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} \\ &= \left[\min_{\{\beta \in I^*: \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}_{\beta}^{L} \bullet \tilde{\mathbf{b}}_{\beta}^{L}, \tilde{\mathbf{a}}_{\beta}^{U} \bullet \tilde{\mathbf{b}}_{\beta}^{U}\right\}, \max_{\{\beta \in I^*: \beta \geq \alpha\}} \max\left\{\tilde{\mathbf{a}}_{\beta}^{L} \bullet \tilde{\mathbf{b}}_{\beta}^{L}, \tilde{\mathbf{a}}_{\beta}^{U} \bullet \tilde{\mathbf{b}}_{\beta}^{U}\right\}\right].\end{aligned}$$

Theorem 18. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be nonnegative canonical fuzzy intervals. Suppose that $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$, $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$ are obtained from Theorem 12, Theorem 13, Theorem 14 and Theorem 15, respectively. We also assume that the supremum $\sup I^*$ is attained. Then*

$$I_{\circledast}^{(EP)} = I_{\circledast}^{(\diamond DT)} = I_{\circledast}^{(\star DT)} = I_{\circledast}^{(\dagger DT)} = I^* \text{ and } \tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}\right)_{\alpha} = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{\mathbf{a}}_{\alpha}^{L} \bullet \tilde{\mathbf{b}}_{\alpha}^{L}, \tilde{\mathbf{a}}_{\alpha}^{U} \bullet \tilde{\mathbf{b}}_{\alpha}^{U}\right].$$

The equivalence between $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ cannot be guaranteed. The following theorem compares their fuzziness.

Theorem 19. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ are obtained from Theorems 13 and 14, respectively. We also assume that the supremum $\sup I^*$ is attained. Then $I_{\circledast}^{(\diamond DT)} = I_{\circledast}^{(\star DT)} = I^*$ and $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$.*

Proof. For $\alpha \in I^*$ with $\alpha > 0$, it is clear to see that

$$\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y} \leq \min_{\{\beta \in I^* : \beta \geq \alpha\}} \min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\beta, \tilde{\mathbf{b}}_\beta)} \mathbf{x} \bullet \mathbf{y} \leq \min_{\{\beta \in I^* : \beta \geq \alpha\}} \min \left\{ \tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta \right\}$$

and

$$\max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y} \geq \max_{\{\beta \in I^* : \beta \geq \alpha\}} \max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\beta, \tilde{\mathbf{b}}_\beta)} \mathbf{x} \bullet \mathbf{y} \geq \max_{\{\beta \in I^* : \beta \geq \alpha\}} \max \left\{ \tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta \right\}.$$

From Theorems 16 and 17, we obtain

$$\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha \subseteq \left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha$$

for each $\alpha \in I^*$ with $\alpha > 0$, which says that $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$. This completes the proof. □

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