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On the Existence of Solutions of a Two-Layer Green Roof Mathematical Model

J. Ignacio Tello [1], Lourdes Tello [2],* and María Luisa Vilar [2]

[1]   Department of Fundamental Mathematics, School of Sciences, UNED, 28040 Madrid, Spain;
      jtello@mat.uned.es
[2]   Department of Applied Mathematics, ETS Arquitectura, Universidad Politécnica de Madrid,
      Av. Juan de Herrera 4, 28040 Madrid, Spain; marialuisa.vilar.guerrero@alumnos.upm.es
*   Correspondence: l.tello@upm.es

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Abstract: The aim of this article is to fill part of the existing gap between the mathematical modeling of a green roof and its computational treatment, focusing on the mathematical analysis. We first introduce a two-dimensional mathematical model of the thermal behavior of an extensive green roof based on previous models and secondly we analyze such a system of partial differential equations. The model is based on an energy balance for buildings with vegetation cover and it is presented for general shapes of roofs. The model considers a vegetable layer and the substratum and the energy exchange between them. The unknowns of the problem are the temperature of each layer described by a coupled system of two partial differential equations of parabolic type. The equation modeling the evolution of the temperature of the substratum also considers the change of phase of water described by a maximal monotone graph. The main result of the article is the proof of the existence of solutions of the system which is given in detail by using a regularization of the maximal monotone graph. Appropriate estimates are obtained to pass to the limit in a weak formulation of the problem. The result goes one step further from modeling to validate future numerical results.

Keywords: nonlinear mathematical models; green roof models; partial differential equations on manifolds; energy balance models; maximal monotone graphs

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1. Introduction

A green roof is a roof that contains a soil (growing media) and vegetation layer as its outermost surface. Green roofs are used in buildings since ancient times, they have multiple benefits, among them, the reduction of the effect of the urban heat island, the building energy savings, the storm water reduction, aesthetic effect and the acoustic benefits (see [1–3] and references therein for more details).

Two main types of green roof exist: extensive green roof and intensive green roof. In the extensive green roofs, the depth of the growing media is less than the depth in the intensive green roofs. The choice of the vegetation is also different. Sedum, small grasses, herbs and flowering herbaceous plants which do not need permanent irrigation system are the typical vegetation in an extensive green roof.

During the last few decades, green roofs have been studied from different points of view. Several mathematical models of heat transfer or heat and mass transfer for green roofs have been proposed (see [4] for a review of this type of models). Two of the most representative green roof mathematical models were proposed by del Barrio [5] and Sailor [6]. These models are based on an energy balance. In [5], a set of relevant parameters in the design of green roofs were described, among them, the leaf area index (LAI) or soil density and thickness. The energy balance mathematical model considered in [6] was validated with experimental data of a green roof installed in Florida.

Moreover, several energy balance models describing the thermal behaviour of green roofs have appeared (e.g., [2]). Most of these energy balance models are one-dimensional models, that is, they consider the substratum and the canopy as homogeneous. See [4] for a review of green roof mathematical models.

In [3] experimental results about the influence of the choice of the vegetation species, the weather conditions and the watering on the cooling potential in prefabricated green roofs were obtained. Experimental data were used in some other works to validate green roof mathematical models (see e.g., [1,6]).

Most studies agree that evapotranspiration is one of the main factors that affect the thermal behaviour of green roofs (see [4,7] and the references therein). Evapotranspiration is a combination of soil evaporation and plant transpiration. Comparing the energy balance of a green roof to the energy balance of a conventional roof, we notice that evapotranspiration is not in the second one. Evapotranspiration effect plays an important role in Biosphere models (see [8]).

The purpose of our work is to present a mathematical model of the energy balance of a green roof which includes the shape of the green roof and its mathematical treatment. A system of two coupled nonlinear parabolic equations describing the temperature in a vegetation layer and the temperature in the substrate has been considered. The model includes the main feedback mechanisms of the energy balance than previous models consider, like evapotranspiration. The proposed model in this work considers that the properties of the substratum depend on the temperature, in fact, the equation modeling the evolution of the temperature of the substratum also consider the change of phase of water described by a maximal monotone graph, denoted by γ. In consequence, the heat capacity of the substratum is not constant, it is a function depending on the temperature. This is the main difficulty in the mathematical treatment of the problem. In the proof of the existence of solutions we have used a regularization of the maximal monotone graph to avoid the lack of regularity produced by γ. The space domain is a surface $\mathcal{M}$ of $\mathbb{R}^3$, then the obtained model is a two dimensional model which allow us to include the shape of the green roof in the space domain; formulation of partial differential equations on Riemannian manifolds is required. Although most of the green roofs are flat we can also find green roofs with different shapes, spherical, cylindrical, or non-standard shape adapted to the terrain. We mention two examples of buildings whose green roofs are not contained in a plane: California Academy of Sciences by the architect Renzo Piano and la Maison Vague (Wave House) by the architect Patrick Nadeau. The model will allow us to analyze the response of the temperature to different choices of shape of the green roof, choice of the vegetation or the influence of the local climate conditions, for example.

The article is organized as follows. In Section 2, we introduce the mathematical model and the required assumptions and simplifications. In Section 3, we study the system of PDEs from a mathematical point of view. After introducing the notion of weak formulation of the equations we proceed to prove the existence of solutions. Finally, Sections 4 and 5 are devoted to discussion and conclusions respectively.

Notice that due to the shape of γ, we do not expect a regular solution T_s in the classical sense, i.e., $T_s \notin C_{t,x}^{1,2}((0,T) \times \mathcal{M})$ because the variation of T_s with respect to the heat presents a discontinuity at $T_s = 273\,\text{K}$ due to the change of phase. Since, the existence of a measurable and continuous temperature is expected, as real experiments show, a new notion of solution has to be introduced. In such direction, it is natural to present the notion of weak solution and weak formulation in appropriate functional spaces. Following the original concept of distributions and generalized functions studied by L. Schwartz and the extensive literature existing from the 40s to our days, we introduce the notion of weak solution in Definition 1, Section 3. The weak formulation is an alternative to validate the model, that can not be interpreted in terms of classical solutions and it should be understood in terms of distributions. The model is based on an energy balance, hence the temperature T_s is linked to energy. In particular, T_s is related to the latent heat of fusion, which is implicitly defined by γ and can only be interpreted in terms of distributions.

To obtain the existence of solutions, we first regularize the problem by using a regularization of γ to obtain uniform bounds of $\int_{\mathcal{M}} |\nabla T_{\epsilon s}|^2 dA$ independent of ϵ. Such estimates guarantee the pass to the limit in the weak formulation of the approximated problem to the limit one.

2. A Green Roof Mathematical Model

In this section we model the energy balance of a green roof in a given geometry, which is not necessarily flat. The model is based on an energy balance and it includes the interactions between these two layers: vegetation and substratum. The model is based on the proposed ones by del Barrio [5] and Sailor [6]. In our case, the space domain considered is a two-dimensional surface, $\mathcal{M}$.

The model describes the evolution of the temperature in two different layers: the vegetable layer and the substratum layer, denoted by T_v and T_s respectively (temperatures are given in Kelvin).

The substratum is a porous medium. It is a mixture of solid (minerals and organic material), liquid (water) and gas (air and water steam). See De Vries [9] for thermal properties of soils. We consider the graph $\gamma(u)$ in order to include the change of phase in the model and so, the heat capacity of the substratum depends on the temperature.

In the vegetable layer, the energy balance depends on the following mechanisms

- solar radiation absorbed by the vegetation,
- long wave radiation exchange between vegetation and the exterior,
- radiative exchange between the two layers (vegetation and ground),
- the sensible flux between vegetation and the air surrounding the vegetation,
- evapotranspiration,
- conduction of the heat in every layer.

Following Sailor [6], the energy balance in the vegetable layer is given by the function

$$F_1 := \sigma_v[R_{av} + \epsilon_v R_i - \epsilon_v \sigma |T_v|^3 T_v] + \frac{\sigma_v \epsilon_v \epsilon_s \sigma}{\epsilon_l}(|T_s|^3 T_s - |T_v|^3 T_v) + H_v + L_v, \tag{1}$$

where

- σ_v is the fractional vegetation coverage.
- R_{av} is the absorbed solar radiation (short wave) by the vegetal layer given by

$$R_{av} = QS(t,x)\beta_v(t,x)$$

  where t is the time and $x \in \mathcal{M}$. Q is the solar constant, S is the isolation function (depending on the orientation of the surface). β_v is the coalbedo in the vegetal layer, i.e., $1 - \beta_v$ is the albedo function, that is, the fraction which is reflected.
- R_i is the long wave radiation which is absorbed.
- σ is the Stefan–Boltzman constant.
- ϵ_v is the emissivity of the vegetal layer.
- ϵ_s is the emissivity of the soil layer.
- $\epsilon_l := \epsilon_s + \epsilon_v - \epsilon_s \epsilon_v$.
- LAI is the leaf area index (see Frankenstein et al. [10]) defined as follows

$$LAI := A_0 - A_1[298 - T_s]^2$$

  where A_0 and A_1 are positive constants in the following ranges

  - $A_0 \in (2,6)$
  - $A_1 \in [0.0016, 0.0088]$.

If we consider small scales of time, the parameter LAI can be consider as constant.

- H_v is the sensible heat flux between the vegetal layer and the air (see Frankenstein et al. [10] and references therein). Following [10] we define H_v as follows

$$H_v := (e_0 + 1.1 LAI \rho_{af} c_{p,a} C_f W_{af})(T_{af} - T_v)$$

where

- e_0 windless exchange coefficient for sensible heat (2.0 W/m^2).
- ρ_{af} is the air density in the foliage (kg/m^3) near the atmosphere-foliage interface. It is a given function and for simplicity we assume homogeneous in space, i.e.,

$$\rho_{af}(t, x) = \rho_{af}(t).$$

- C_f bulk transfer coefficient;
- T_{af} is the air temperature in the foliage approximated by

$$T_{af} = (1 - \sigma_v)T_a + \sigma_v(0.3T_a + 0.6T_v + 0.3T_s)$$

and T_a is the air temperature measured at the shelter (see Frankenstein et al. [10] for more details).
- W_{af} wind speed at the air/foliage interface (m/s).
- $c_{p,a}$ specific heat of air at constant pressure.

- L_v is the latent heat flux expressed in terms of the unknown temperature. We notice that L_v depends on the stomatal resistance coefficient of the leaves of the vegetal layer.

$$L_v = l_v LAI \rho_{af} C_f W_{af} \hat{r}(q_{as} - q_{f,sat})$$

where

– l_v is the latent heat of vaporization, it is the amount of energy required to convert a unit mass of water to steam. It is measured in units of J/kg and is inversely proportional to the temperature. From Henderson-Sellers [11] it is estimated as follows (see also Sailor [6])

$$l_v = 1.911846 \times 10^6 \frac{T_v^2}{(T_v - 33.91)^2}.$$

– ρ_{af}, C_f, W_{af} have been defined previously.
– $\hat{r}$ is given by

$$\hat{r} = \frac{LAI}{LAI + r_0},$$

and r_0 is a given constant (see [10] for more details).
– q_{as} and $q_{f,sat}$ are the mixing ratio of the air within the canopy and the saturation mixing ratio at the foliage surface temperature. The explicit expression can be found in [6]. We assume that it is a given function.

For simplicity we assume

$$L_v = A_0 - A_1[298 - T_v]^2. \tag{2}$$

The energy balance in the substratum is given by F_2 (see Sailor [6])

$$F_2 := (1 - \sigma_v)[R_{as} + \epsilon_s R_i - \epsilon_s |T_s|^3 T_s] - \frac{\sigma_v \epsilon_v \epsilon_s \sigma}{\epsilon_l}(|T_s|^3 T_s - |T_v|^3 T_v) + H_s + L_s, \tag{3}$$

where

- R_{as} is the absorbed solar radiation by the substratum layer,

$$R_{as} = QS(t,x)\beta_s(t,x),$$

  and β_s represents the coalbedo function of the substratum layer.

- H_s is the sensible heat flux

$$H_s = (e_0 + \rho_{ag}c_{p,a}C_{hg}W_{af})(T_{af} - T_s)$$

  where

  - e_0 have been already defined.
  - The bulk transfer coefficient for sensible heat C_h^g is given as follows (see also [10])

$$C_{hg} = \begin{cases} \frac{A_6}{(1-16R_{ib})^{\frac{1}{2}}} & \text{if} \quad T_v < \frac{(1-0.3\sigma_v)T_s + (0.7\sigma_v - 1)T_a}{0.6\sigma_v}; \\ \frac{A_6}{1-5R_{ib}} & \text{if} \quad T_v \geq \frac{(1-0.3\sigma_v)T_s + (0.7\sigma_v - 1)T_a}{0.6\sigma_v}; \end{cases}$$

  and

$$R_{ib} := A_7 \frac{T_{af} - T_s}{T_{af} + T_s}.$$

  for $A_7 = \frac{2gZ_a}{W_{af}^2}$.

  In the model, C_{hg} is assumed as a given function.

  - T_{af} air temperature in the foliage approximated by (see Frankenstein et al. [10] and references therein)

$$T_{af} = (1 - \sigma_v)T_a + \sigma_v(0.3T_a + 0.6T_v + 0.3T_s)$$

  and T_a is the air temperature measured at the shelter (known data).
  - W_{af} wind speed at the air/foliage interface (m/s)
  - $c_{p,a}$ specific heat of air at constant pressure.

  Then

$$H_s = (\tilde{B}_0 + B_4 C_{hg})(\tilde{B}_1 + \tilde{B}_2 T_v + \tilde{B}_3 T_s).$$

  Notice that C_{hg} is assumed as a given function, therefore H_s becomes

$$H_s = B_1 + B_2 T_v + B_3 T_s. \tag{4}$$

- L_s is the latent heat flux. Following Sailor [6] we have

$$L_s = C_{e,s}l_s W_{af}\rho_{as}(q_{as} - q_s),$$

  where

  - $C_{e,s}$ is the bulk transfer coefficient, it is analogous to C_f, assumed constant.
  - l_s is the latent heat of vaporization at the ground surface temperature, assumed constant.
  - $W_{a,f}$ has been previously defined as the wind speed at the air/foliage interface and it is a given datum.
  - $\rho_{a,s}$ is the air density near the soil surface. It is a known datum.

  – q_{as} and q_s are the mixing ratio at the foliage–atmosphere interface and the mixing ratio at the ground surface. We assume they are known data. The explicit expression is presented in [6,10].

Then, L_s is assumed a known function, i.e.,

$$L_s = L_s(t, x).$$

Notice that F_1 and F_2 depend on the type of vegetation, in particular, they depend on two functions which characterizes the plants used in the green roof: Leaf Area Index (LAI) and Fractional vegetation coverage (σ_v). If we consider small scales of time, these functions (LAI and σ_v) can be included in the model as constant. We notice that they are two of the relevant parameters of the model.

One of the characteristics of the energy balance of the green roof is the solar shading produced by the foliage and the cooling by evapotranspiration (these two effects are not in the energy balance of the conventional roof). We consider some relevant parameters related with vegetation: leaf area index (LAI), foliage fractional coverage (σ_v), vegetation stomatal resistance (r_s) and the vegetation coalbedo β_v. These parameters could depend on the vegetation type and the seasons. The analysis of the sensitivity of the model front fluctuations of these parameters would be a useful future research.

The model also considers the heat conduction in every layer, whose coefficients are given by k_1, k_2 respectively. Considering the heat conduction and the energy balance, under suitable boundary and initial conditions, we arrive to following system of partial differential equations

$$\begin{cases} C_v \frac{\partial T_v}{\partial t} - div(k_1(x)\nabla_\mathcal{M} T_v) = F_1(T_v, T_s), & \text{in } \mathcal{M}_T, \\ \frac{\partial \gamma(T_s)}{\partial t} - div(k_2(x)\nabla_\mathcal{M} T_s) \ni F_2(T_v, T_s), & \text{in } \mathcal{M}_T, \\ T_v = g_v, & \text{in } \partial\mathcal{M}_T, \\ T_s = g_s, & \text{in } \partial\mathcal{M}_T, \\ T_v(0, x) = T_{v0}(x), & \text{in } \mathcal{M}, \\ T_s(0, x) = T_{s0}(x), & \text{in } \mathcal{M}, \end{cases} \tag{5}$$

where

$$\mathcal{M}_T := \{(t, x) \in (0, T) \times \mathcal{M}\}$$

and

$$\partial\mathcal{M}_T := \{(t, x) \in (0, T) \times \partial\mathcal{M}\}.$$

Here, the heat capacity in every layer could depend on the moisture.

γ is defined as the graph

$$\gamma(s) = \begin{cases} \gamma_1(s - 273), & s < 273; \\ [0, L], & s = 273; \\ \gamma_2(s - 273) + L & s > 273; \end{cases} \tag{6}$$

for γ_1 and γ_2 positive numbers. These constants could depend on the moisture. See the climate model studied in [12] for more details. See also [13].

Notice that F_1 and F_2 can be expressed as follows

$$F_1(T_v, T_s) = a_0(|T_s|^3 T_s - |T_v|^3 T_v) - a_1|T_v|^3 T_v$$

$$+ [a_2 - (268 - T_s)^2](a_3 - a_4 T_v + a_5 T_s) + a_6 + a_7[298 - T_s]^2 \tag{7}$$

and

$$F_2(T_v, T_s) = a_0(|T_v|^3 T_v - |T_s|^3 T_s) - b_1 |T_s|^3 T_s + b_2 T_v + b_3 T_s + b_4. \tag{8}$$

3. Well-Posed Problem

We start the mathematical treatment of the obtained model by analyzing the existence of solutions under the following assumptions:

$(\mathcal{H}_M)$ $\mathcal{M}$ is a two-dimensional connected oriented Riemannian C^∞ manifold with compact closure and regular boundary $\partial\mathcal{M}$.

$(\mathcal{H}_S)$ $S : [0, +\infty) \times \mathcal{M} \to \mathbb{R}$, $S \in L^\infty((0, +\infty) \times \mathcal{M})$ such that there exist a positive constant s_1 verifying $0 \leq S(t, x) \leq s_1$.

$(\mathcal{H}_\beta)$ $\beta_v \in L^\infty((0, +\infty) \times \mathcal{M})$, $\beta_s \in L^\infty((0, +\infty) \times \mathcal{M})$, $0 \leq \beta_v(t, x) \leq 1$, $0 \leq \beta_s(t, x) \leq 1$.

$(\mathcal{H}_v)$ $H_v = H_v(t, x, T_v, T_s)$, $H_v \in C([0, +\infty) \times \overline{\mathcal{M}} \times \mathbb{R} \times \mathbb{R})$.

$(\mathcal{H}_s)$ $H_s = H_s(t, x, T_v, T_s)$, $H_s \in C([0, +\infty) \times \overline{\mathcal{M}} \times \mathbb{R} \times \mathbb{R})$.

$(\mathcal{H}_L)$ $L_v = L_v(t, x, T_v, T_s)$, $L_s = L_s(t, x, T_v, T_s)$, $L_v, L_s \in C([0, +\infty) \times \overline{\mathcal{M}} \times \mathbb{R} \times \mathbb{R})$.

$(\mathcal{H}_\gamma)$ γ is a maximal monotone graph defined by $\gamma(s) = \gamma_1(s - 273)$ if $s < 273$, $\gamma(s) = \gamma_2(s - 273) + L$ if $s > 273$ and $\gamma(s) = [0, L]$ if $s = 273$, where γ_1, γ_2 and L are positive constants.

$(\mathcal{H}_F)$ F_1 and F_2 are defined in (7) and (8), resp. whose coefficients $a_i = a_i(t, x)$ ($i = 2, .., 7$) and $b_j = b_j(t, x)$ ($j = 2, .., 4$) are uniformly bounded and a_0, a_1 and b_1 are positive constants.

$(\mathcal{H}_k)$ The heat conduction coefficients k_1 and k_2 satisfy

$$k_i \in W^{1,\infty}(\mathcal{M}), \quad i = 1, 2$$

and there exists a constants $k_0 > 0$ such that

$$0 < k_0 \leq k_i \leq k_0^{-1}, \quad i = 1, 2.$$

$(\mathcal{H}_g)$ The boundary conditions g_v and g_s satisfy

$$g_v, g_s \in C^2([0, T] \times \partial\mathcal{M})$$

and

$$g_v \geq 0, \qquad g_s \geq 0.$$

$(\mathcal{H}_p)$ The constants σ_v, ϵ_v, ϵ_s, σ, λ, C_v, Q are positive.

$(\mathcal{H}_0)$ The initial data $T_{0v}, T_{0s} \in H^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$, and satisfy the boundary condition

$$T_{0v} = g_v(0, x), \quad T_{0s} = g_s(0, x), \quad \text{in } x \in \partial\mathcal{M}.$$

Notice that divergence and gradient in the diffusion terms are understood in the sense of the Riemannian metric of $\mathcal{M}$. In particular if $k_1 \equiv 1$ then the diffusion operator $div(\nabla_\mathcal{M} T_v)$ is the Laplace–Beltrami operator of T_v (see [14]). For simplicity in the notation we denote $\nabla_\mathcal{M}$ by ∇. See also [15] where an energy balance model on a Riemannian manifold was considered. For maximal monotone operators properties see [16]. Now, following [14] we define the functional spaces on manifolds,

$$L^p(\mathcal{M}) := \{u : \mathcal{M} \to \mathbb{R} \text{ measurable}, \int_\mathcal{M} |u|^p dA < +\infty\},$$

for $1 \leq p < \infty$.

$$L^\infty(\mathcal{M}) := \{u : \mathcal{M} \to \mathbb{R} \text{ measurable}, \underset{\mathcal{M}}{\text{ess sup}} |u| < +\infty\}.$$

Denote by $T_{\mathbf{p}}\mathcal{M}$ the tangent space in every point $\mathbf{p} \in \mathcal{M}$ and define the bundle tangent space $T\mathcal{M} = \cup_{p \in \mathcal{M}} T_{\mathbf{p}}\mathcal{M}$. The space $L^p(T\mathcal{M})$, also denoted by $L^p(\mathcal{M}, T\mathcal{M})$, is defined by

$$L^p(T\mathcal{M}) := \{X : \mathcal{M} \to T\mathcal{M} : \int_{\mathcal{M}} | < X, X > |^{\frac{p}{2}} dA < +\infty\}$$

where $< \cdot, \cdot >$ is the scalar product in $T\mathcal{M}$ given by the Riemannian metrics on $\mathcal{M}$.

$$W^{1,p}(\mathcal{M}) := \{u \in L^p(\mathcal{M}) : \nabla u \in L^p(T\mathcal{M})\}.$$

In particular, $H^1(\mathcal{M}) = W^{1,2}(\mathcal{M})$.

$$W^{1,\infty}(\mathcal{M}) := \{u : \mathcal{M} \to \mathbb{R} \text{ measurable}, \text{ ess} \sup_{\mathcal{M}} |u| + \text{ess} \sup_{\mathcal{M}} |\nabla u| < +\infty\}.$$

We also define $L^2(0, T : H^1(\mathcal{M}))$ as the set of measurable functions $u : (0, T) \times \mathcal{M} \to \mathbb{R}$ verifying

$$\int_0^T \int_{\mathcal{M}} |u|^2 dA dt + \int_0^T \int_{\mathcal{M}} |\nabla u|^2 dA dt < +\infty.$$

See [14,15,17] for more details about functional spaces defined on manifolds.

We now introduce the notion of weak solution.

Definition 1. *We say that $(T_v, T_s) \in [L^2(0, T : H^1(\mathcal{M}))]^2$ is a weak solution of (5) if*

(i)　$(T_v, T_s) \in [L^\infty(\mathcal{M}_T)]^2$.

(ii)　$(T_v, T_s) = (g_v, g_s)$ *for $x \in \partial\mathcal{M}$.*

(iii)　*There exists $Z \in L^\infty(\mathcal{M}_T)$, $Z \in \gamma(T_s)$ such that*

$$\int_{\mathcal{M}} C_v T_v(T, x)\psi_1(T, x)dA - \int_{\mathcal{M}} C_v T_v(0, x)\psi_1(0, x)dA - \int_0^T \int_{\mathcal{M}} C_v T_v \psi_{1t} dA dt$$

$$+ \int_0^T \int_{\mathcal{M}} k_1(x)\nabla T_v \nabla \psi_1 dA dt = \int_0^T \int_{\mathcal{M}} F_1(T_s, T_v)\psi_1 dA dt$$

and

$$\int_{\mathcal{M}} Z(T, x)\psi_2(T, x)dA - \int_{\mathcal{M}} Z(0, x)\psi_2(0, x)dA - \int_0^T \int_{\mathcal{M}} Z\psi_{2t} dA dt$$

$$+ \int_0^T \int_{\mathcal{M}} k_2(x)\nabla T_v \nabla \psi_2 dA dt = \int_0^T \int_{\mathcal{M}} F_2(T_s, T_v)\psi_2 dA dt$$

for $(\psi_1, \psi_2) \in [L^2(0, T : H_0^1(\mathcal{M})) \cap H^1(0, T : L^2(\mathcal{M}))]^2$.

We remark that although γ is multivalued, in order to solve the problem we introduce the following approximation of γ, denoted by γ_ϵ for $\epsilon > 0$ in the following way

$$\gamma_\epsilon(s) = \begin{cases} \gamma_1(s - 273), \text{ if } s < 273; \\ -((\gamma_2 - \gamma_1)\epsilon + 2L)\dfrac{(s - 273)^3}{\epsilon^3} + (3L + 2\epsilon(\gamma_2 - \gamma_1))\dfrac{(s - 273)^2}{\epsilon^2} + \gamma_1(s - 273), \\ \quad \text{if } 273 \leq s \leq 273 + \epsilon; \\ \gamma_2(s - 273) + L, \text{ if } s > 273 + \epsilon. \end{cases} \tag{9}$$

Notice that $\gamma_\epsilon \in C^1$ and there exists a positive constant $c(\epsilon)$ such that $|\gamma_\epsilon''| < c(\epsilon)$ a.e.

We denote the inverse of γ_ϵ by φ_ϵ, i.e., φ_ϵ satisfies

$$\varphi_\epsilon(\gamma_\epsilon(s)) = s.$$

Consider the approximated problem

$$
\begin{cases}
C_v \frac{\partial T_{\epsilon v}}{\partial t} - div(k_1(x)\nabla_{\mathcal{M}} T_{\epsilon v}) = F_1(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)), & \text{in } \mathcal{M}_T, \\
\frac{\partial \Gamma_\epsilon}{\partial t} - div(k_2(x)\nabla_{\mathcal{M}} \varphi_\epsilon(\Gamma_\epsilon)) = F_2(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)), & \text{in } \mathcal{M}_T, \\
T_{\epsilon v} = g_v, & \text{in } \partial \mathcal{M}_T, \\
\Gamma_\epsilon = \gamma_\epsilon(g_s), & \text{in } \partial \mathcal{M}_T \\
T_{\epsilon v}(0, x) = T_{v0}(x), & \text{in } \mathcal{M}, \\
\Gamma_\epsilon(0, x) = \gamma_\epsilon(T_{s0}), & \text{in } \mathcal{M}.
\end{cases}
\tag{10}
$$

We now introduce the notion of weak solution of problem (10)

Definition 2. *We say that* $(T_{\epsilon v}, \Gamma_\epsilon) \in [C(0, T : L^2(\mathcal{M}))]^2$ *is a weak solution of (10) if*

(i) $(T_{\epsilon v}, \Gamma_\epsilon) \in [L^\infty(\mathcal{M}_T) \cap L^2(0, T : H^1(\mathcal{M}))]^2$,

(ii) $(T_{\epsilon v}, \Gamma_\epsilon) = (g_v, \gamma_\epsilon(g_s))$ *for* $x \in \partial \mathcal{M}$, *such that*

$$
\int_{\mathcal{M}} T_{\epsilon v}(T, x) \psi_1(T, x) dA - \int_{\mathcal{M}} T_{\epsilon v}(0, x) \psi_1(0, x) dA - \int_0^T \int_{\mathcal{M}} T_{\epsilon v} \psi_{1t} dA dt
$$

$$
+ \int_0^T \int_{\mathcal{M}} k_1(x) \nabla T_{\epsilon v} \nabla \psi_1 dA dt = \int_0^T \int_{\mathcal{M}} F_1(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)) \psi_1 dA dt
$$

and

$$
\int_{\mathcal{M}} \Gamma_\epsilon(T, x) \psi_2(T, x) dA - \int_{\mathcal{M}} \Gamma_\epsilon(0, x) \psi_2(0, x) dA - \int_0^T \int_{\mathcal{M}} \Gamma_\epsilon \psi_{2t} dA dt
$$

$$
+ \int_0^T \int_{\mathcal{M}} k_2(x) \nabla \varphi_\epsilon(\Gamma_\epsilon) \nabla \psi_2 dA dt = \int_0^T \int_{\mathcal{M}} F_2(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)) \psi_2 dA dt
$$

for $(\psi_1, \psi_2) \in [L^2(0, T : H_0^1(\mathcal{M})) \cap H^1(0, T : L^2(\mathcal{M}))]^2$.

Theorem 1. *Assume the hypotheses* $(\mathcal{H}_M)$–$(\mathcal{H}_p)$ *then, for every initial datum* (T_v^0, T_s^0) *satisfying* $(\mathcal{H}_0)$ *the problem (5) has a weak solution*

$$
(T_v, T_s) \in [C(0, T : L^2(\mathcal{M})) \cap L^\infty(\mathcal{M}_T) \cap L^2(0, T : H^1(\mathcal{M}))]^2.
$$

The proof of Theorem 1 is organized into several steps.
We consider the function $w = (w_1, w_2)$ defined as follows

$$
\begin{cases}
-div(k_1(x)\nabla w_1) = 0, & x \in \mathcal{M}, \\
w_1 = g_v, & x \in \partial \mathcal{M},
\end{cases}
\tag{11}
$$

and

$$
\begin{cases}
-div(k_2(x)\nabla w_2) = 0, & x \in \mathcal{M}, \\
w_2 = g_s, & x \in \partial \mathcal{M}.
\end{cases}
\tag{12}
$$

Notice that under assumptions $(\mathcal{H}_k)$–$(\mathcal{H}_g)$ we have that

$$
w \in [L^2(0, T : W^{2,p}(\mathcal{M})) \cap H^1(0, T : W^{1,p}(\mathcal{M}))]^2, \qquad \text{for any } p < \infty,
$$

see Gilbart and Trudinger [18], Theorem 9.15 p. 241.
We first obtain the following a-priori estimates.

Lemma 1. *Let* $(T_{\epsilon v}, \Gamma_\epsilon)$ *the solution to (10), and* $T_{\epsilon s} = \varphi_\epsilon(\Gamma_\epsilon)$ *then, under assumptions of Theorem 1, then, for any* $T < \infty$ *we have that*

$$
(T_{\epsilon v}, T_{\epsilon s}) \in [L^\infty(\mathcal{M}_T)]^2,
$$

moreover there exists a positive constant c_∞ independent of ϵ and T such that

$$\|T_{\epsilon v}\|_{L^\infty(\mathcal{M})}, \ \|T_{\epsilon s}\|_{L^\infty(\mathcal{M})} \le c_\infty.$$

Proof. We denote by P_v and P_s the following functions

$$P_v(T_{\epsilon v}, T_{\epsilon s}) = -a_1 |T_{\epsilon v}|^3 T_{\epsilon v} + [a_2 - (268 - T_{\epsilon s})^2](a_3 - a_4 T_{\epsilon v} + a_5 T_{\epsilon s}) + a_6 + a_7 [298 - T_{\epsilon s}]^2$$

and

$$P_s(T_{\epsilon v}, T_{\epsilon s}) = -b_1 |T_{\epsilon s}|^3 T_{\epsilon s} + b_2 T_{\epsilon v} + b_3 T_{\epsilon s} + b_4.$$

Let

$$k := \max\{ \sup_{(t,x)\in\mathcal{M}_T} \{|w_1|\}, \ \sup_{(t,x)\in\mathcal{M}_T} \{|w_2|\}, k_0\}.$$

where $k_0 > 0$ satisfies, for $|s| \le k_0 + 1$

$$P_v(T_{\epsilon v}, s) < 0, \text{ for any } T_{\epsilon v} > k_0, \quad P_v(T_{\epsilon v}, s) > 0 \text{ for any } T_{\epsilon v} < -k_0,$$

and

$$P_s(s, T_{\epsilon s}) < 0, \text{ for any } T_{\epsilon s} > k_0, \quad P_v(s, T_{\epsilon s}) > 0 \quad \text{ for any } T_{\epsilon s} < -k_0.$$

Since

$$|T_{\epsilon s}| \le (T_{\epsilon s} - k)_+ + (-k - T_{\epsilon s})_+ + k$$

we have

$$F_1(T_{\epsilon v}, T_{\epsilon s})(T_{\epsilon v} - k)_+ \le \alpha_1(T_{\epsilon v}, T_{\epsilon s})[(T_{\epsilon v} - k)_+^2 + (-T_{\epsilon v} - k)_+^2 + (T_{\epsilon s} - k)_+^2 + (-T_{\epsilon s} - k)_+^2].$$

In similar way we obtain

$$F_2(T_{\epsilon v}, T_{\epsilon s})(T_{\epsilon s} - k)_+ \le \alpha_2(T_{\epsilon v}, T_{\epsilon s})[(T_{\epsilon s} - k)_+^2 + (-T_{\epsilon s} - k)_+^2 + (T_{\epsilon v} - k)_+^2 + (-T_{\epsilon v} - k)_+^2],$$

$$-F_1(T_{\epsilon v}, T_{\epsilon s})(-T_{\epsilon v} - k)_+ \le \alpha_3(T_{\epsilon s}, T_{\epsilon v})[(T_{\epsilon s} - k)_+^2 + (-T_{\epsilon s} - k)_+^2 + (T_{\epsilon v} - k)_+^2 + (-T_{\epsilon v} - k)_+^2]$$

and

$$-F_2(T_{\epsilon v}, T_{\epsilon s})(-T_{\epsilon s} - k)_+ \le \alpha_4(T_{\epsilon v}, T_{\epsilon s})[(T_{\epsilon v} - k)_+^2 + (-T_{\epsilon v} - k)_+^2 + (T_{\epsilon s} - k)_+^2 + (-T_{\epsilon s} - k)_+^2].$$

We introduce the truncation function τ defined by

$$\tau(s) = \begin{cases} -k, & \text{if } s < -k - 1, \\ s, & \text{if } -k - 1 \le s \le k + 1, \\ k + 1, & \text{if } k + 1 < s, \end{cases}$$

and consider the problem (10) where F_1 and F_2 are replaced by $\tilde{F}_1$ and $\tilde{F}_2$

$$\tilde{F}_1 = a_0(|\tau(T_{\epsilon s})|^3 T_{\epsilon s} - |\tau(T_{\epsilon v})|^3 T_{\epsilon v}) - a_1 |\tau(T_{\epsilon v})|^3 T_{\epsilon v}$$

$$+ [a_2 - (268 - \tau(T_{\epsilon s}))^2](a_3 - a_4 T_{\epsilon v} + a_5 T_{\epsilon s}) + a_6 + a_7 [298 - \tau(T_{\epsilon s})]^2.$$

and

$$\tilde{F}_2(T_{\epsilon v}, T_{\epsilon s}) = a_0(|\tau(T_{\epsilon v})|^3 T_{\epsilon v} - |\tau(T_{\epsilon s})|^3 T_{\epsilon s}) - b_1 |\tau(T_{\epsilon s})|^3 T_{\epsilon s} + b_2 T_{\epsilon v} + b_3 T_{\epsilon s} + b_4.$$

Notice that, thanks to Mean Value Theorem,

$$a_0(|\tau(T_{es})|^3 T_{es} - |\tau(T_{ev})|^3 T_{ev})(T_{ev} - k)_+ \leq 4a_0(k+1)^3(T_{ev} - k)_+(T_{es} - k)_+$$

and

$$a_0(|\tau(T_{ev})|^3 T_{ev} - |\tau(T_{es})|^3 T_{es})(T_{es} - k)_+ \leq 4a_0(k+1)^3(T_{es} - k)_+(T_{ev} - k)_+$$

and their symmetric once we multiply by $-(-T_{ev} - k)_+$ and $-(-T_{ev} - k)_+$. We multiply by $(T_{ev} - k)_+$ and $-(-T_{ev} - k)_+$ in the first equation of (10) and by $(T_{es} - k)_+$ and $-(-T_{es} - k)_+$ in the second equation to obtain, after integration by parts

$$\frac{d}{dt} \int_{\mathcal{M}} [(T_{ev} - k)_+^2 + (-T_{ev} - k)_+^2 + (T_{es} - k)_+^2 + (-T_{es} - k)_+^2] dA$$

$$\leq \int_{\mathcal{M}} \alpha(\tau(T_{ev}), \tau(T_{es}))[(T_{ev} - k)_+^2 + (-T_{ev} - k)_+^2 + (T_{es} - k)_+^2 + (-T_{es} - k)_+^2] dA.$$

Then, Gronwall's Lemma and the choice of the truncation function end the proof. □

Lemma 2. *Let $(T_{ev}, \Gamma_\epsilon)$ the solution to (10), and $T_{es} = \varphi_\epsilon(\Gamma_\epsilon)$ then, under assumptions of Theorem 1, there exists $c > 0$ independent of ϵ and T such that*

$$\int_0^T \int_{\mathcal{M}} (|\nabla T_{ev}|^2 + |T_{ev}|^2) dA dt + \int_0^T \int_{\mathcal{M}} (|\nabla T_{es}|^2 + |T_{es}|^2) dA dt \leq c(T+1).$$

Proof. We multiply the first equation in (10) by $T_{ev} - w_1$ and integrate by parts over $\mathcal{M}$ to get

$$\frac{C_v}{2} \frac{d}{dt} \int_{\mathcal{M}} (T_{ev} - w_1)^2 dA + \int_{\mathcal{M}} k_1(x) |\nabla(T_{ev} - w_1)|^2 dA \leq \int_{\mathcal{M}} F_1(T_{ev}, T_{es})(T_{ev} - w_1) dA$$

$$-C_v \int_{\mathcal{M}} w_{1t}(T_{ev} - w_1) dA.$$

In view of Lemma 1 we have that

$$|F_1(T_{ev}, T_{es}) T_{ev}| \leq c.$$

After integration we obtain

$$\frac{C_v}{2} \int_{\mathcal{M}} (T_{ev} - w_1)^2 dA + \int_{\mathcal{M}_T} k_1(x) |\nabla(T_{ev} - w_1)|^2 dA \leq c(T+1).$$

In the same way, we multiply by $T_{es} - w_2$ in the second equation in (10). Then, after integration over $\mathcal{M}$ we have

$$\int_{\mathcal{M}} \gamma_\epsilon'(T_{es})(T_{es})_t(T_{es} - w_2) dA + \int_{\mathcal{M}} k_2(x) |\nabla(T_{es} - w_2)|^2 dA \leq \int_{\mathcal{M}} F_2(T_{ev}, T_{es})(T_{es} - w_2) dA.$$

The subscript t represents the time derivative. We can express the first term by using

$$\gamma_\epsilon'(T_{es})(T_{es})_t(T_{es} - w_2) = \frac{d}{dt} [\Phi_\epsilon(T_{es}) - (w_2 - 273)\gamma_\epsilon(T_{es})] + \gamma_\epsilon(T_{es}) w_{2t}$$

for $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$ defined as follows

- If $u < 273$

$$\Phi_\epsilon(u) = \frac{1}{2} \gamma_1 (u - 273)^2,$$

- if $273 \leq u \leq 273 + \epsilon$,

$$\Phi_\epsilon(u) = -((\gamma_2 - \gamma_1)\epsilon + 2L)\frac{3(u - 273)^4}{4\epsilon^3} + (3L + 2(\gamma_2 - \gamma_1)\epsilon)\frac{2(u - 273)^3}{3\epsilon^2} + \frac{\gamma_1}{2}(u - 273)^2,$$

- if $u > 273 + \epsilon$,

$$\Phi_\epsilon(u) = \frac{\gamma_2}{2}(u - 273)^2 + \frac{1}{12}(\gamma_2 - \gamma_1)\epsilon^2 + \frac{1}{2}\epsilon L.$$

Φ_ϵ verifies that $\Phi'_\epsilon(u) = \gamma'_\epsilon(u)(u - 273)$ for all $u \in \mathbb{R}$. Notice that

$$|\gamma_\epsilon(T_{\epsilon s})w_{2t}| \leq c.$$

Therefore, thanks to Lemma 1 we have

$$\frac{d}{dt}\int_\mathcal{M}[\Phi_\epsilon(T_{\epsilon s}) - \hat{w}_2\gamma_\epsilon(T_{\epsilon s})]\,dA + \int_\mathcal{M}k_2(x)|\nabla(T_{\epsilon s} - w_2)|^2 dA \leq c.$$

where $\hat{w}_2 = w_2 - 273$. After integration over $(0, t)$, and thanks to Lemma 1 we obtain,

$$\int_0^T\int_\mathcal{M}(|\nabla T_{\epsilon s}|^2 + T_{\epsilon s}^2)\,dA\,dt \leq c(T + 1)$$

and the proof ends. □

Lemma 3. *Let* $(T_{\epsilon v}, \Gamma_\epsilon)$ *the solution to* (10)*, and* $T_{\epsilon s} = \varphi_\epsilon(\Gamma_\epsilon)$ *then, there exist constants* c_0 *and* c_1 *independent of* ϵ *and* T *such that*

$$\int_0^T\int_\mathcal{M}(|T_{\epsilon v t}|^2 + |T_{\epsilon s t}|^2)\,dA\,dt \leq c_0 T + c_1,$$

and

$$\int_\mathcal{M}(|\nabla T_{\epsilon v}|^2 + |\nabla T_{\epsilon s}|^2)\,dA \leq c_0 T + c_1.$$

Proof. We multiply the first two equations in (10) by $T_{\epsilon v t} - w_{1t}$ and $T_{\epsilon s t} - w_{2t}$ respectively to obtain

$$C_v\int_\mathcal{M}|T_{\epsilon v t}|^2 dA + \frac{1}{2}\frac{d}{dt}\int_\mathcal{M}k_1(x)|\nabla T_{\epsilon v}|^2 dA \leq \int_\mathcal{M}F_1 T_{\epsilon v t}\,dA + \int_\mathcal{M}k_1(x)\nabla T_{\epsilon v}\nabla w_{1t}\,dA,$$

and

$$\int_\mathcal{M}\gamma'_\epsilon|T_{\epsilon s t}|^2 dA + \frac{1}{2}\frac{d}{dt}\int_\mathcal{M}k_2(x)|\nabla T_{\epsilon s}|^2 dA \leq \int_\mathcal{M}F_2 T_{\epsilon s t}\,dA + \int_\mathcal{M}k_2(x)\nabla T_{\epsilon s}\nabla w_{2t}\,dA.$$

Since

$$\gamma'_\epsilon > \min\{\gamma_1, \gamma_2\}$$

and thanks to Young inequality we get

$$\frac{C_v}{2}\int_\mathcal{M}|T_{\epsilon v t}|^2 dA + \frac{1}{2}\frac{d}{dt}\int_\mathcal{M}k_1(x)|\nabla T_{\epsilon v}|^2 dA \leq c\int_\mathcal{M}|F_1|^2 dA$$

$$+ c\int_\mathcal{M}|\nabla T_{\epsilon v}|^2 dA + c\int_\mathcal{M}|\nabla w_{1t}|^2 dA,$$

and

$$\min\{\gamma_1, \gamma_2\}\int_\mathcal{M}|T_{\epsilon s t}|^2 dA + \frac{1}{2}\frac{d}{dt}\int_\mathcal{M}k_2(x)|\nabla T_{\epsilon s}|^2 dA \leq c\int_\mathcal{M}|F_2|^2 dA$$

$$+ c\int_\mathcal{M}|\nabla T_{\epsilon s}|^2 dA + c\int_\mathcal{M}|\nabla w_{2t}|^2 dA.$$

After integration over $(0, T)$ and thanks to previous lemmas and assumption $\mathcal{H}_g$, we get, for $T < \infty$

$$\int_0^T \int_{\mathcal{M}} |T_{\epsilon vt}|^2 dA + \int_{\mathcal{M}} |\nabla T_{\epsilon v}|^2 dA \le c_0 T + c_1,$$

and

$$\int_0^T \int_{\mathcal{M}} |T_{\epsilon st}|^2 dA + \int_{\mathcal{M}} |\nabla T_{\epsilon s}|^2 dA \le c_0 T + c_1,$$

which ends the proof. $\square$

Proof of Theorem 1. The proof is divided into two different steps. In the first step we prove the existence and uniqueness of solution to the approximated problem (10). In the second step we prove the convergence of the approximated solution to (10) to the solution of (5).

Step 1. We consider the equation satisfied by $(T_{\epsilon v}, \Gamma_\epsilon)$,

$$C_v \frac{\partial T_{\epsilon v}}{\partial t} - div(k_1(x) \nabla T_{\epsilon v}) = F_1(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)), \tag{13}$$

$$\frac{\partial \Gamma_\epsilon}{\partial t} - div(k_2(x) \nabla \varphi_\epsilon(\Gamma_\epsilon)) = F_2(T_{\epsilon v}, \varphi_\epsilon(\Gamma_\epsilon)), \tag{14}$$

with boundary conditions

$$T_{\epsilon v} = g_v, \quad \Gamma_\epsilon = \gamma_\epsilon(g_s), \quad \text{in } \partial \mathcal{M} \tag{15}$$

and initial data

$$T_{\epsilon v} = T_{v0}, \quad \Gamma_\epsilon(0, x) = \gamma_\epsilon(T_{s0}), \quad \text{in } \partial \mathcal{M}. \tag{16}$$

We construct a fixed point argument in the following way. Let

$$\mathcal{A} := \{ u \in L^2(0, T : L^2(\mathcal{M})), \quad |u| \le c_\infty \}$$

where c_∞ is defined in Lemma 1. Let

$$E_\epsilon : \mathcal{A} \to L^2(0, T : L^2(\mathcal{M}))$$

be defined by

$$E_\epsilon(\hat{T}_{\epsilon v}, \hat{\Gamma}_\epsilon) = (T_{\epsilon v}, \Gamma_\epsilon),$$

where $(T_{\epsilon v}, \Gamma_\epsilon)$ is the solution of the ϵ-problem

$$C_v \frac{\partial T_{\epsilon v}}{\partial t} - div(k_1(x) \nabla T_{\epsilon v}) = F_1(\hat{T}_{\epsilon v}, \varphi_\epsilon(\hat{\Gamma}_\epsilon)), \tag{17}$$

$$\frac{\partial \Gamma_\epsilon}{\partial t} - div(k_2(x) \varphi_\epsilon'(\hat{\Gamma}_\epsilon) \nabla \Gamma_\epsilon) = F_2(\hat{T}_{\epsilon v}, \varphi_\epsilon(\hat{\Gamma}_\epsilon)), \tag{18}$$

with boundary conditions (15) and initial data (16).

Notice that, since $\hat{F}_1$ and $\hat{F}_2$ are uniformly bounded,

$$0 < \frac{1}{c} < \varphi_\epsilon' < c < \infty,$$

T_{0v}, $T_{0s} \in H^1(\mathcal{M})$, and g_v, $g_s \in C^2((0,T) \times \partial\mathcal{M})$ we have that there exists a unique solution to (17), (18) satisfying

$$\int_0^T \int_{\mathcal{M}} \left[|T_{\epsilon vt}|^2 + |\Gamma_{\epsilon t}|^2 \right] dA dt + \int_{\mathcal{M}} \left[|\nabla T_{\epsilon v}|^2 + |\nabla \Gamma_\epsilon|^2 \right] dA \leq c(\epsilon), \tag{19}$$

and

$$\|T_{\epsilon v}\|_{L^\infty(\mathcal{M})}, \ \|\Gamma_\epsilon\|_{L^\infty(\mathcal{M})} \leq c_\infty \tag{20}$$

for time T small enough. The proof of the estimates (19) and (20) is similar to the proofs of Lemmas 2 and 3 therefore we omit the details.

Thanks to the Aubin–Lions–Simon Lemma (see Simon [19])

$$(T_{\epsilon v}, \Gamma_\epsilon) \in C(0,T : L^2(\mathcal{M}))^2$$

and (19) and (20) yields $E_\epsilon(\mathcal{A}) \subset \mathcal{A}$. Standard computations, shows the continuity of

$$E_\epsilon : \mathcal{A} \to [L^\infty(0,T : H^1(\mathcal{M})) \cap H^1(0,T : L^2(\mathcal{M})) \cap L^\infty(\mathcal{M}_T)]^2.$$

Since

$$H^1(\mathcal{M}) \hookrightarrow L^p(\mathcal{M}), \quad p < \infty$$

is a compact embedding, we have that

$$L^\infty(0,T : H^1(\mathcal{M})) \cap H^1(0,T : L^2(\mathcal{M})) \cap L^\infty(\mathcal{M}_T) \hookrightarrow L^p(0,T : L^p(\mathcal{M}))$$

for any $p < \infty$ is a compact embedding as well as to $C(0,T : L^p(\mathcal{M}))$. Therefore, for any sequence $(T_{j\epsilon v}, \Gamma_{j\epsilon})$ $(j = 1 \ldots \infty)$ defined by

$$(T_{j\epsilon v}, \Gamma_{j\epsilon}) = E_\epsilon(T_{(j-1)\epsilon v}, \Gamma_{(j-1)\epsilon})$$

for

$$(T_{1\epsilon v}, \Gamma_{1\epsilon}) \in \mathcal{A},$$

we have that there exists a sub-sequence $(T_{n\epsilon v}, \gamma_{n\epsilon})$ such that

$$(T_{n\epsilon v}, \Gamma_{n\epsilon}) \to (T_{\epsilon v}^*, \Gamma_\epsilon^*)$$

in $L^p(0,T : L^p(\mathcal{M}))$ strong and

$$(T_{n\epsilon v}, \Gamma_{n\epsilon}) \rightharpoonup (T_{\epsilon v}^*, \Gamma_\epsilon^*)$$

weakly in $L^p(0,T : H^1(\mathcal{M}))$. Then, it results that

$$\int_0^T \int_{\mathcal{M}} k_2(x) \varphi_\epsilon'(\Gamma_{(n-1)\epsilon}) \nabla \Gamma_{n\epsilon} \nabla \psi_2 dA dt \to \int_0^T \int_{\mathcal{M}} k_2(x) \varphi_\epsilon'(\Gamma_\epsilon^*) \nabla \Gamma_\epsilon^* \nabla \psi_2 dA dt$$

and

$$\int_0^T \int_{\mathcal{M}} |\varphi_\epsilon(\Gamma_{(n-1)\epsilon})|^q \varphi_\epsilon(\Gamma_{(n-1)\epsilon}) \psi_2 dA dt \to \int_0^T \int_{\mathcal{M}} |\varphi_\epsilon(\Gamma_\epsilon^*)|^q \varphi_\epsilon(\Gamma_\epsilon^*) \psi_2 dA dt.$$

Therefore Γ_ϵ^* satisfies the weak formulation of (10) given in (2) in $(0,T)$. In view of Lemma 1 we may extend the solution up to $T < \infty$.

Uniqueness of solutions of (10) is a consequence of regularity of the functions γ_ϵ, F_1 and F_2.

Step 2. Let $(T_{\epsilon v}, \Gamma_{\epsilon})$ be the solution to (10), then, we denote by $T_{\epsilon s}$ the inverse of Γ_{ϵ}^*, i.e.,

$$T_{\epsilon s} = \varphi_{\epsilon}(\Gamma_{\epsilon}^*).$$

Thanks to Lemmas 1–3 and Aubin–Lions–Simon Lemma, there exists a subsequence, $(\tilde{T}_{\epsilon v}, \tilde{T}_{\epsilon s})$ such that

$$\tilde{T}_{\epsilon s} = \varphi_{\epsilon}(\tilde{\Gamma}_{\epsilon}^*)$$

satisfying

$$(\tilde{T}_{\epsilon v}, \tilde{T}_{\epsilon s}) \to (T_v^*, T_s^*)$$

strongly in $L^p(0, T : L^p(\mathcal{M}))$ and $C(0, T : L^p(\mathcal{M}))$ for any $p < \infty$ and

$$(\tilde{T}_{\epsilon v}, \tilde{T}_{\epsilon s}) \rightharpoonup (T_v^*, T_s^*)$$

weakly in $L^p(0, T : H^1(\mathcal{M}))$ for any $p < \infty$.

Since any $Z_1 \in \gamma(\tilde{T}_{\epsilon s})$ and $Z_2 \in \gamma(\tilde{T}_{\epsilon s} - \epsilon)$ are uniformly bounded in $L^\infty(\mathcal{M}_T)$, there exist $\overline{\gamma}_{\epsilon}$ and $\underline{\gamma}_{\epsilon}$ such that

$$\overline{\gamma}_{\epsilon} \in \gamma(\tilde{T}_{\epsilon s})$$

$$\underline{\gamma}_{\epsilon} \in \gamma(\tilde{T}_{\epsilon s} - \epsilon)$$

satisfying

$$\underline{\gamma}_{\epsilon} \leq \gamma_{\epsilon}(\tilde{T}_{\epsilon s}) \leq \overline{\gamma}_{\epsilon}. \tag{21}$$

In view of the boundedness in $L^\infty(\mathcal{M}_T)$ of $\overline{\gamma}_{\epsilon}$ and $\underline{\gamma}_{\epsilon}$ we have that there exists a subsequence of ϵ (denoted with the same sub index) such that

$$\overline{\gamma}_{\epsilon} \rightharpoonup \overline{\gamma}^* \tag{22}$$

and

$$\underline{\gamma}_{\epsilon} \rightharpoonup \underline{\gamma}^* \tag{23}$$

weakly in $L^p(\mathcal{M}_T)$ for any $p < \infty$.

Thanks to [20] Proposition 20.32, p. 300 (see also [16]), since γ is maximal monotone, we have

$$\underline{\gamma}^*, \overline{\gamma}^* \in \gamma(T_s^*). \tag{24}$$

In view of the boundedness of $\gamma_{\epsilon}(\tilde{T}_{\epsilon s})$ there exists a subsequence, denoted by $\tilde{\gamma}_{\epsilon}$ such that

$$\tilde{\gamma}_{\epsilon} \rightharpoonup \gamma^{**}$$

weakly in $L^p(\mathcal{M}_T)$ for any $p < \infty$. The inequality (21) proves

$$\underline{\gamma}^* \leq \gamma^{**} \leq \overline{\gamma}^*. \tag{25}$$

Since γ is maximal, (24) and (25) prove

$$\gamma^{**} \in \gamma(T_s^*).$$

Since $\tilde{T}_{\epsilon v}$ converges strongly in $C(0, T : L^2(\mathcal{M}))$ and γ is maximal monotone we proceed as before to obtain the convergence of a subsequence (of the previous subsequence) of $\gamma_{\epsilon}(\tilde{T}_{\epsilon s}(T, \cdot))$ to $\gamma^{***} \in \gamma(T_s^*(T, \cdot))$.

We now take limits in the weak formulation of problem (10) to conclude the result. □

4. Discussion

A two-layers model has been proposed to analyze the thermal behaviour of green roofs. The model is based on an energy balance on the vegetation layer and the soil layer. The model incorporates the evapotranspiration, that is, the combination of soil evaporation and plants transpiration, this effect is one of the differences between a green roof and a conventional roof. The modeling has considered some simplifications in the reaction terms F_1 and F_2, the case where the reaction terms are more general functions will be considered in the future including non-polynomial growth terms. The shape of the green roof is defined in a two dimensional spatial domain which leads to formulate the model as a system of partial differential equations on manifolds. One of the difficulties of the mathematical analysis of the model is the multivalued term $\gamma(T_s)$ in the balance on the substratum. The multivalued term and the coefficient C_v present a dependence of the humidity w, assumed known. Further development must consider w as unknown of a third equation, which will allow us to decrease the data collection.

We have proved the existence of solutions (temperature in every layer) in a suitable functional space and we have obtained some estimates of them. This mathematical treatment open new questions about this mathematical model. The model will allow to analyze the impact of fluctuations of the parameters, among them, the leaf area index (LAI), fractional vegetation coverage (σ_v).

5. Conclusions

In this work, we have proposed and analyzed a mathematical model of the thermal behavior of the green roof. The system consists of two coupled parabolic equations describing the temperature in a vegetation layer and the temperature in the substrate. Two features of this nonlinear model are remarkable in comparison with other previous models in the study of green roofs: the space domain which is a surface (and so, a two dimensional space variable) and the term γ to describe the effect of the change of phases in the substratum. The other terms in the energy balance model were expressed as polynomials with power less or equal to four.

The second result of the article is the mathematical proof of the existence of solutions which is presented in detail. Due to the shape of γ, classical solutions are not expected if the temperature falls bellow 273 K, therefore weak solutions have to be introduced.

Future work of this two-layer energy balance model will be the analysis of the relevant parameters (e.g., LAI), the inclusion of non-polynomial reaction terms and the numerical approximation of the solutions.

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