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Generalized Quasi-Einstein Manifolds in Contact Geometry

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Abstract: In this study, we investigate generalized quasi-Einstein normal metric contact pair manifolds. Initially, we deal with the elementary properties and existence of generalized quasi-Einstein normal metric contact pair manifolds. Later, we explore the generalized quasi-constant curvature of normal metric contact pair manifolds. It is proved that a normal metric contact pair manifold with generalized quasi-constant curvature is a generalized quasi-Einstein manifold. Normal metric contact pair manifolds satisfying cyclic parallel Ricci tensor and the Codazzi type of Ricci tensor are considered, and further prove that a generalized quasi-Einstein normal metric contact pair manifold does not satisfy Codazzi type of Ricci tensor. Finally, we characterize normal metric contact pair manifolds satisfying certain curvature conditions related to $\mathcal{M}$-projective, conformal, and concircular curvature tensors. We show that a normal metric contact pair manifold with generalized quasi-constant curvature is locally isometric to the Hopf manifold $S^{2n+1}(1) \times S^1$.

Keywords: generalized quasi-Einstein; contact pairs; generalized quasi-constant curvature; $\mathcal{M}$-projective curvature tensor

1. Introduction

An Einstein manifold is a Riemannian manifold (M, g), which is defined by the Ricci tensor $Ric = \lambda g$ for a non-zero constant λ. Since Einstein manifolds have important differential geometric properties and have significant physical applications therefore they are studied by geometers in a broad perspective. A Riemannian manifold M is called a quasi-Einstein manifold if the Ricci curvature tensor satisfies

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\omega(X_1)\omega(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$, where λ, β are scalars and ω is a non-zero $1-$form [1]. Quasi-Einstein manifolds are generalizations of Einstein manifolds . In the contact geometry, $\eta-$Einstein manifolds can be consider as a particular case of quasi-Einstein manifolds. When quasi-umbilical hypersurfaces were considered exact solutions of the Einstein field equations, the notion of quasi-Einstein manifold aroused [2]. As an example of quasi-Einstein manifolds, we can mention the Robertson-Walker space-times [2]. For more details on such manifolds, we refer to the reader [2–4].

The generalization of quasi-Einstein manifolds has been presented in the different perspectives. Chaki gave one of them in [5], and another was presented by Catino [6]. Catino generalized a quasi-Einstein manifold as a generalization of the concepts of Ricci solitons and quasi-Einstein manifolds. The third definition of generalized quasi-Einstein manifolds was given by De and Ghosh [7]. A Riemannian manifold (M, g) is called a generalized quasi-Einstein manifold if its Ricci tensor has following form:

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\omega(X_1)\omega(X_2) + \mu\eta(X_1)\eta(X_2)$$

where ω, η are two non-zero $1-$ forms and λ, β, μ are certain non-zero scalars [7]. The unit and orthogonal vector fields ξ_1 and ξ_2 corresponding to the $1-$forms ω and η are defined by $g(X_1, \xi_1) = \omega(X_1)$, $g(X_2, \xi_2) = \eta(X_1)$, respectively [7]. The geometric properties of generalized quasi-Einstein manifolds have been studied in [7–11]. A generalized quasi-Einstein manifold, in addition to its geometrical features, has remarkable physical applications in general relativity [12–14]. Complex $\eta-$Einstein manifolds could be considered as a special case of generalized quasi-Einstein manifolds (see [15]).

In [16], Bande and Hadjar defined a new contact structure on an $(m = 2p + 2q + 2)-$dimensional differentiable manifold M with two $1-$forms α_1, α_2. This structure was initially studied by Blair, Ludden and Yano [17] as the name of bicontact manifolds. Bande and Hadjar considered a special type of $f-$structure with complementary frames related to these contact forms and they obtained associated metric. A differentiable manifold with this structure is called a metric contact pair (MCP) manifold. Riemannian geometry of MCP manifolds is given in [18,19].

This paper is on applications of generalized quasi-Einstein manifolds in contact geometry. We consider the generalized quasi-Einstein normal metric contact pair manifolds. After presenting definitions and basic properties, we examine the existence of such manifolds. Also, we present a characterization of generalized quasi-Einstein normal metric contact pair manifolds. Moreover, we consider the notion of generalized quasi-constant curvature for normal metric contact pair manifolds and we obtain some results on the sectional curvature. We investigate a generalized quasi-Einstein normal metric contact pair manifold under some conditions for Ricci tensor. We prove that a generalized quasi-Einstein normal metric contact pair manifold does not satisfy Codazzi type of Ricci tensor. Finally, we characterize normal metric contact pair manifolds satisfying certain curvature conditions related to $\mathcal{M}$-projective, conformal, and concircular curvature tensors. We show that a normal metric contact pair manifold with generalized quasi-constant curvature is locally isometric to the Hopf manifold $S^{2n+1}(1) \times S^1$.

2. Preliminaries

Contact pairs were defined by Bande and Hadjar [16] in 2005, for details see [16,18,19]. In this section, we give some fundamental facts about contact metric pair manifolds. Also, we present some general facts and results on generalized quasi-Einstein manifolds.

Definition 1. *Let M be an $(m = 2p + 2q + 2)$ -dimensional differentiable manifold and α_1, α_2 be two 1-forms on M. If the following properties are satisfied then, the pair of (α_1, α_2) is called a contact pair of type (p, q) on M:*

- $\alpha_1 \wedge (d\alpha_1)^p \wedge \alpha_2 \wedge (d\alpha_2)^q \neq 0,$
- $(d\alpha_1)^{p+1} = 0$ *and* $(d\alpha_2)^{q+1} = 0,$

where p, q are positive integers. Then, (M, α_1, α_2) is known as a contact pair manifold [16].

The kernels of $1-$forms α_1 and α_2 define two subbundles of TM as $\mathcal{D}_1 = \{X : \alpha_1(X) = 0, X \in \Gamma(TM)\}$ and $\mathcal{D}_2 = \{X : \alpha_2(X) = 0, X \in \Gamma(TM)\}$. Also, we have two characteristic foliations of M, denoted by $\mathcal{F}_1 = \mathcal{D}_1 \cap kerd\alpha_1$ and $\mathcal{F}_2 = \mathcal{D}_2 \cap kerd\alpha_2$ respectively. $\mathcal{F}_1$ and $\mathcal{F}_2$ are completely integrable and their leaves are equipped with a contact form induced by α_2 (respectively, α_1). On the other hand, the tangent bundle of M can be split as $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2$ [19]. The horizontal sub-bundle $\mathcal{H}$ of TM can be defined as $\mathcal{H} = ker\alpha_1 \cap ker\alpha_2$.

In the contact geometry, we have the characteristic vector field associated with the contact form. Similarly, for a contact pair (α_1, α_2) of type (p, q) we have two vector fields Z_1 and Z_2, which are uniquely determined by the following equations:

$$\alpha_1(Z_1) = \alpha_2(Z_2) = 1, \; \alpha_1(Z_2) = \alpha_2(Z_1) = 0,$$
$$i_{Z_1} d\alpha_1 = i_{Z_1} d\alpha_2 = i_{Z_2} d\alpha_2 = 0,$$

where i_X is the contraction with the vector field X [16].

Two sub-bundles of TM are defined as follows

$$T\mathcal{G}_i = kerd\alpha_i \cap ker\alpha_1 \cap ker\alpha_2, \ i = 1, 2.$$

Then, we have

$$T\mathcal{F}_1 = T\mathcal{G}_1 \oplus \mathbb{R}Z_2, \text{ and } T\mathcal{F}_2 = T\mathcal{G}_2 \oplus \mathbb{R}Z_1.$$

Therefore, we get $TM = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$. The horizontal sub-bundle can be written as $\mathcal{H} = T\mathcal{G}_1 \oplus T\mathcal{G}_2$. Also, we write $\mathcal{V} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, and $\mathcal{V}$ is called the vertical sub-bundle of TM. Consequently, the tangent bundle of M is given by $TM = \mathcal{H} \oplus \mathcal{V}$ [18].

Let X be an arbitrary vector field on M. We can write $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, where $X^{\mathcal{H}}, X^{\mathcal{V}}$ horizontal and vertical component of X, respectively. We call a vector field X as a horizontal vector field if $X \in \Gamma(\mathcal{H})$, and a vertical vector field if $X \in \Gamma(\mathcal{V})$.

Similar to almost complex structures, in the 1960s, almost contact structures were defined with tensorial viewpoint [20]. On a contact pair manifold M, an almost contact pair structure has been defined as follow by Bande and Hadjar [18].

Definition 2. *An almost contact pair structure on an $(m = 2p + 2q + 2)-$dimensional differentiable manifold M is a triple α_1, α_2, ϕ, where (α_1, α_2) is a contact pair and ϕ is a $(1,1)$ tensor field such that:*

$$\phi^2 = -I + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \ \ \phi Z_1 = \phi Z_2 = 0. \tag{1}$$

The rank of ϕ is $(2p + 2q)$ and $\alpha_1(\phi) = \alpha_2(\phi) = 0$ [18].

Definition 3. *ϕ is known as decomposable i.e., $\phi = \phi_1 + \phi_2$, if $T\mathcal{F}_i$ is invariant under ϕ [18].*

If ϕ is decomposable, then (α_i, Z_i, ϕ) induces an almost contact structure on $\mathcal{F}_j$ for $i \neq j, \ \ i, j = 1, 2$ [16]. The decomposability of ϕ does not satisfy for every almost contact pair structure. An example was given in [16], which has an almost contact pair structure, but ϕ is not decomposable. In this study, we assume that ϕ is decomposable.

Definition 4. *Let $(\alpha_1, \alpha_2, Z_1, Z_2, \phi)$ be an almost contact pair structure on a Riemannian manifold (M, g). The Riemannian metric g is called [18]*

- *compatible if $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in TM$,*
- *associated if $g(X_1, \phi X_2) = (d\alpha_1 + d\alpha_2)(X_1, X_2)$ and $g(X_1, Z_i) = \alpha_i(X_1)$, for $i = 1, 2$ and for all $X_1, X_2 \in \Gamma(TM)$.*

$4-$tuple $(\alpha_1, \alpha_2, \phi, g)$ is called a metric almost contact pair structure on a manifold M and g is an associated metric with respect to contact pair structure $(\alpha_1, \alpha_2, \phi)$. We recall $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ is a metric almost contact pair manifold.

We have the following properties for a metric almost contact pair manifold M [16]:

$$g(Z_i, Z_j) = \delta_{ij}, \ \nabla_{Z_i} Z_j = 0, \ \nabla_{Z_i}\phi = 0, \ \ \nabla_X Z_1 = -\phi_1 X, \ \nabla_X Z_2 = -\phi_2 X$$

and for every X tangent to $\mathcal{F}_i$ $i = 1, 2$.

Another major notion for an almost contact manifold is normality. Bande and Hadjar [19] studied on this notion for a metric almost contact pair manifold. They define two almost complex structures on M as [19]:

$$\mathcal{J} = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \ \ \mathcal{T} = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2.$$

Definition 5. *A metric almost contact pair manifold is said to be normal if $\mathcal{J}$ and $\mathcal{T}$ are integrable [19].*

Theorem 1. *$\mathcal{J}$ and $\mathcal{T}$ are integrable if the following equation is satisfied;*

$$[\phi, \phi](X_1, X_2) + 2d\alpha_1(X_1, X_2)Z_1 + 2d\alpha_2(X_1, X_2)Z_2 = 0,$$

for all $X_1, X_2 \in \Gamma(TM)$ [19].

From the following theorem, we have the covariant derivation of ϕ for a normal metric contact pair manifold

Theorem 2. *Let $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ be a normal metric contact pair manifold. Then we have*

$$g((\nabla_{X_1}\phi)X_2, X_3) = \sum_{i=1}^{2}(d\alpha_i(\phi X_2, X_1)\alpha_i(X_3) - d\alpha_i(\phi X_3, X_1)\alpha_i(X_2)) \tag{2}$$

for all X_1, X_2, X_3 arbitrary vector fields on M [18].

We use the following statements for the Riemann curvature;

$$R(X_1, X_2)X_3 = (\nabla^2_{X_1,X_2} - \nabla^2_{X_2,X_1})X_3 \quad ,\text{where } \nabla^2_{X_1,X_2}X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{\nabla_{X_1}X_2}X_3$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. Also, it is well known that $R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$ [20].

Lemma 1. *Let $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ be a normal metric contact pair manifold. Then we have*

$$R(X_1, Z)X_2 = -g(\phi X_1, \phi X_2)Z, \tag{3}$$

$$\begin{aligned} R(X_1, X_2, Z, X_3) = {} & d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3, X_1)\alpha_2(X_2) \\ & - d\alpha_1(\phi X_3, X_2)\alpha_1(X_1) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1), \end{aligned} \tag{4}$$

$$R(X_1, Z)Z = -\phi^2 X_1. \tag{5}$$

where $X_1, X_2, X_3 \in \Gamma(TM)$ and $Z = Z_1 + Z_2$ for Reeb vector fields Z_1, Z_2 [21].

Consider an orthonormal basis of M by

$$S = \{e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p, e_{p+1}, e_{p+2}, ..., e_{p+q}, \phi e_{p+1}, \phi e_{p+2}, ..., \phi e_{p+q}, Z_1, Z_2\}.$$

Then for all $X_1 \in \Gamma(TM)$, we get the Ricci curvature of M as

$$Ric(X_1, Z) = \sum_{i=1}^{2p+2q} d\alpha_1(\phi E_i, E_i)\alpha_1(X) + d\alpha_2(\phi E_i, E_i)\alpha_2(X)$$

where $E_i \in S$.

Lemma 2. *Let $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2, g)$ be a normal metric contact pair manifold. Then Ricci curvature of M satisfies [21]*

$$Ric(X_1, Z) = 0, \ \ for \ X_1 \in \Gamma(\mathcal{H}), \tag{6}$$

$$Ric(Z, Z) = 2p + 2q. \tag{7}$$

$$Ric(Z_1, Z_1) = 2p, \ Ric(Z_2, Z_2) = 2q, \ Ric(Z_1, Z_2) = 0. \tag{8}$$

In [7], De and Ghosh presented a theorem for the existence of a generalized quasi-Einstein Riemannian manifold.

Theorem 3. *A Riemannian manifold is a generalized quasi-Einstein manifold, if the Ricci tensor Ric satisfies the relation*

$$\begin{aligned} Ric(X_2, X_3)Ric(X_1, X_4) - Ric(X_1, X_3)Ric(X_2, X_4) &= \gamma[g(X_2, X_3)g(X_1, X_4) \\ &\quad -g(X_1, X_3)g(X_2, X_4)] \end{aligned} \tag{9}$$

where γ is a non-zero scalar [7].

The notion of quasi-constant curvature was defined by Chen and Yano [22]. De and Ghosh generalized this notion for a Riemannian manifold.

Definition 6. *Let M be a normal metric contact pair manifold. Then, M is called a normal metric contact pair manifold of generalized quasi-constant curvature if the Riemannian curvature tensor of M satisfying;*

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= A[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)] \\ &\quad B[g(X_1, X_4)\alpha_1(X_2)\alpha_1(X_3) - g(X_1, X_3)\alpha_1(X_2)\alpha_1(X_4) \\ &\quad +g(X_2, X_3)\alpha_1(X_1)\alpha_1(X_4) - g(X_2, X_4)\alpha_1(X_1)\alpha_1(X_3)] \\ &\quad C[g(X_1, X_4)\alpha_2(X_2)\alpha_2(X_3) - g(X_1, X_3)\alpha_2(X_2)\alpha_2(X_4) \\ &\quad +g(X_2, X_3)\alpha_2(X_1)\alpha_2(X_4) - g(X_2, X_4)\alpha_2(X_1)\alpha_2(X_3)] \end{aligned} \tag{10}$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$, where A, B and C are scalar functions [7].

Definition 7. *Let M be a normal metric contact pair manifold. Then, is called a manifold satisfies cyclic parallel Ricci tensor if we have*

$$(\nabla_{X_1} Ric)(X_2, X_3) + (\nabla_{X_2} Ric)(X_3, X_1) + (\nabla_{X_3} Ric)(X_1, X_2) = 0$$

for all $X_1, X_2, X_3\Gamma(TM)$ [20].

Definition 8. *Let M be a normal metric contact pair manifold. Then, M is called a manifold satisfies Codazzi type of Ricci tensor if*

$$(\nabla_{X_1} Ric)(X_2, X_3) - (\nabla_{X_2} Ric)(X_1, X_3) = 0$$

for all X_1 X_2 vector fields on M [20].

Conformal and concircular curvature tensors on contact manifolds have been studied in [23–25]. $\mathcal{M}$-projective curvature tensor on manifolds with different structures studied by many authors [26–28]. These curvature tensors on a normal metric contact pair manifold are defined as below:

Definition 9. *Let M be an $(m = 2p + 2q + 2)-$dimensional normal metric contact pair manifold. Then,*

- *$\mathcal{M}$-projective curvature tensor of M is given by [29] ,*

$$\begin{aligned} \mathcal{W}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 - \frac{1}{2(m-1)}[Ric(X_2, X_3)X_1 \\ &\quad -Ric(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2] \end{aligned} \tag{11}$$

- *conformal curvature tensor $\mathcal{C}$ of M is given by [20],*

$$\begin{aligned}\mathcal{C}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 \\ &+ \frac{scal}{(m-1)(m-2)}\left(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\right) \\ &+ \frac{1}{m-2}(g(X_1, X_3)QX_2 - g(X_2, X_3)QX_1 \\ &+ Ric(X_1, X_3)X_2 - Ric(X_2, X_3)X_1)\end{aligned}$$

- *concircular curvature tensor $\mathcal{Z}$ of M is given by [20],*

$$\mathcal{Z}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{scal}{m(m-1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]$$

for $X_1, X_2, X_3 \in \Gamma(TM)$, where Q is Ricci operator is given by $Ric(X_1, X_2) = g(QX_1, X_2)$ and scal is the scalar curvature of M.

3. Generalized Quasi-Einstein Normal Contact Pair Manifolds

In this section, we present the definition of generalized quasi-Einstein normal metric contact pair manifold. We also present some theorems on the existence and characterizations of generalized quasi-Einstein normal metric contact pair manifold.

Definition 10. *Let M be a normal metric contact pair manifold. Then, M is called generalized quasi-Einstein normal metric contact pair manifold if the Ricci curvature of M has the following form;*

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\alpha_1(X_1)\alpha_1(X_2) + \mu\alpha_2(X_1)\alpha_2(X_2)$$

for functions λ, β, μ on M and all $X_1, X_2 \in \Gamma(TM)$.

If we set $X_1 = X_2 = Z_1$ and $X_1 = X_2 = Z_2$, respectively, we obtain $\beta = 2p - \lambda$ and $\mu = 2q - \lambda$. Thus, the Ricci curvature of generalized quasi-Einstein normal metric contact pair manifold is given by

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + (2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + (2q - \lambda)\alpha_2(X_1)\alpha_2(X_2) \tag{12}$$

for all $X_1, X_2 \in \Gamma(TM)$. Therefore, the scalar curvature is

$$scal = 2(\lambda + 1)(p + q). \tag{13}$$

Let X be an arbitrary vector field on M. We can write $X = X^{\mathcal{H}} + X^{\mathcal{V}}$. Since the Ricci curvature is a linear tensor we have

$$Ric(X_1, X_2) = Ric(X_1^{\mathcal{H}}, X_2^{\mathcal{H}}) + Ric(X_1^{\mathcal{V}}, X_2^{\mathcal{V}}).$$

Considering the decomposition of tangent bundle mentioned above (see [24] for details), we get

$$Ric(X_1^{\mathcal{V}}, X_2^{\mathcal{V}}) = 2p\alpha_1(X_1)\alpha_1(X_2) + 2q\alpha_2(X_1)\alpha_2(X_2).$$

Thus, we reach following useful result.

Proposition 1. *A normal metric contact pair manifold is a generalized quasi-Einstein manifold if and only if the horizontal bundle is Einstein, that is for a function λ on M, we have $Ric(X_1^{\mathcal{H}}, X_2^{\mathcal{H}}) = \lambda g(X_1^{\mathcal{H}}, X_2^{\mathcal{H}})$.*

Assume that (9) is satisfied on a normal metric contact pair manifold M. By setting $X_1 = X_4 = Z$ and $X_2, X_3 \in \Gamma(\mathcal{H})$, then from (6), we have

$$Ric(X_2, X_3) = \frac{2\gamma}{m-2} g(X_2, X_3).$$

Thus, from the Proposition1 M is a generalized quasi-Einstein manifold. Using (13), we get $\gamma = \frac{1}{2}(scal - m + 2)$ and hence we state;

Corollary 1. *Let M be normal metric contact pair manifold with scalar curvature $scal \neq m-2$. If we have the relation*

$$\begin{aligned} Ric(X_2, X_3)Ric(X_1, X_4) - Ric(X_1, X_3)Ric(X_2, X_4) &= \frac{1}{2}(scal - m + 2)[g(X_2, X_3)g(X_1, X_4) \\ &\quad - g(X_1, X_3)g(X_2, X_4)] \end{aligned}$$

on M for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$, then M is a generalized quasi-Einstein manifold.

Let π be a plane section in $T_Q M$ for any $Q \in M$. The sectional curvature of π is given as $Sec(\pi) = Sec(u \wedge v)$, where u, v orthonormal vector fields . For any $(p+q)$-dimensional subspace $\mathcal{L} \subset T_Q M$, $2 \leq p + q \leq m$, its scalar curvature $scal(\mathcal{L})$ is denoted by

$$scal(\mathcal{L}) = \sum_{1 \leq i,j \leq p+q} Sec(E_i \wedge E_j) \tag{14}$$

where $E_1, ..., E_n$ is any orthonormal basis of $\mathcal{L}$ [30]. When $\mathcal{L} = T_Q M$, the scalar curvature is just the scalar curvature $scal(Q)$ of M at $Q \in M$.

The characterizations of Einstein [31,32], quasi-Einstein [33] and generalized quasi-Einstein [11,34] manifolds have been obtained by using the sectional curvature of subspaces of tangent bundle. Analogous to the proof of the Theorem 2.2 of [11], we have following assertion immediately.

Theorem 4. *An $(m = 2p + 2q + 2)-$dimensional normal metric contact pair manifold is a generalized quasi-Einstein manifold if and only if there exist a function λ on M satisfying*

$$\begin{aligned} scal(P) + p + q - \lambda &= \sec(P^{\perp}), \;\; Z_1, Z_2 \in T_Q P^{\perp} \\ scal(N) + p + q &= \sec(N^{\perp}), \;\; Z_1, Z_2 \in T_Q N^{\perp} \\ scal(R) + q - p &= \sec(P^{\perp}), \;\; Z_1 \in T_Q R, Z_2 \in T_Q R^{\perp} \end{aligned}$$

where $(p + q + 1)-$plane sections P, R and $(p + q)-$plane section N; $P^{\perp}, N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in $T_Q M$, respectively.

We consider the normal metric contact pair manifold is of generalized quasi-constant curvature. In the following proposition, we derive some relations on sectional curvature of M.

Proposition 2. *Let M be a normal metric contact pair manifold of generalized quasi-constant curvature. Then, we have the following:*

- *the sectional curvature of horizontal bundle is A,*
- *the sectional curvature of plane section spanned by $X \in \Gamma(\mathcal{H})$ and Z is $2A + B + C$,*
- *the sectional curvature of plane section spanned by $X \in \Gamma(\mathcal{H})$ and Z_1, Z_2 is $A + B$ and $A + C$, respectively.*

Proof. Let take $X_1 = X_4 = X, X_2 = X_4 = X'$, where X, X' unit and mutually orthogonal horizontal vector fields. Then from (10), we obtain

$$\begin{aligned} sec(X, X') &= A[g(X', X')g(X, X) - g(X, X')g(X', X)] \\ &= A \end{aligned}$$

For $X_1 = X_4 = X, X_2 = X_4 = Z$ for unit horizontal vector field X, we get

$$\begin{aligned} sec(X, Z) &= A[g(Z, Z)g(X, X) - g(X, X')g(X', X) - g(X, Z)g(Z, X)] \\ &\quad +B(g(X, X)\alpha_1(Z)\alpha_1(Z) + Cg(X, X)\alpha_2(Z)\alpha_2(Z) \\ &= 2A + B + C \end{aligned}$$

Similarly, we can derive the other assertions. □

From above proposition , we get

Corollary 2. *In a normal metric contact pair manifold of generalized quasi-constant curvature, we have*

$$sec(X, Z) = sec(X, Z_1) + sec(X, Z_2)$$

for any horizontal and unit vector field X.

Theorem 5. *A normal metric contact pair manifold of generalized quasi-constant curvature is a generalized quasi-Einstein manifold with coefficients $\lambda = A(m-1) + B + C,\ \beta = B(m-2)$, and $\mu = C(m-2)$.*

Proof. Let M be a normal metric contact pair manifold of generalized quasi-constant curvature. Consider an orthonormal basis of M as

$$S = \{e_1, e_2, ..., e_p, \phi_1 e_1, \phi_1 e_2, ..., \phi_1 e_p, e_{p+1}, e_{p+2}, ..., e_{p+q}, \phi_2 e_{p+1}, \phi_2 e_{p+2}, ..., \phi_2 e_{p+q}, Z_1, Z_2\}.$$

By taking sum of (10) from $i = 1$ to $i = 2p + 2q + 2$ for $X_2 = X_3 = E_i \in S$, we obtain

$$\begin{aligned} \sum_{i=1}^{2p+2q+2} R(X_1, E_i, E_i, X_4) &= \sum_{i=1}^{2p+2q+2} \{A[g(E_i, E_i)g(X_1, X_4) - g(X_1, E_i)g(E_i, X_4)] \\ &\quad B[g(X_1, X_4)\alpha_1(E_i)\alpha_1(E_i) - g(X_1, E_i)\alpha_1(E_i)\alpha_1(X_4) \\ &\quad +g(E_i, E_i)\alpha_1(X_1)\alpha_1(X_4) - g(E_i, X_4)\alpha_1(X_1)\alpha_1(E_i)] \\ &\quad C[g(X_1, X_4)\alpha_2(E_i)\alpha_2(E_i) - g(X_1, E_i)\alpha_2(E_i)\alpha_2(X_4) \\ &\quad +g(E_i, E_i)\alpha_2(X_1)\alpha_2(X_4) - g(E_i, X_4)\alpha_2(X_1)\alpha_2(E_i)]\}. \end{aligned}$$

For $1 \leq i \leq 2p + 2q$ since $\alpha_j(E_i) = 0, j = 1, 2$ and $\sum_{i=1}^{2p+q+2} g(X_1, E_i)g(E_i, X_2) = g(X_1, X_2)$ we get

$$\begin{aligned} Ric(X_1, X_4) &= [A(m-1) + B + C]g(X_1, X_4) + B(m-2)\alpha_1(X_1)\alpha_1(X_4) \\ &\quad +C(m-2)\alpha_2(X_1)\alpha_2(X_4) \end{aligned}$$

which completes the proof. □

4. Normal Metric Contact Pair Manifold Satisfying Certain Conditions on Ricci Tensor

De and Mallick[9] proved that a generalized quasi-Einstein Riemann manifold satisfies cyclic parallel Ricci tensor if generators of the manifolds are Killing vector fields. As we know that the characteristic vector fields of a normal metric contact pair manifold Z_1, Z_2 are Killing vector fields [35]. Thus, by easy computations, we get

$$(\nabla_{X_1}\alpha_1)X_2 + (\nabla_{X_2}\alpha_1)X_1 = 0, \ \ (\nabla_{X_1}\alpha_2)X_2 + (\nabla_{X_2}\alpha_2)X_1 = 0 \tag{15}$$

for all $X_1, X_2 \in \Gamma(TM)$. On the other hand, we have

$$(\nabla_{X_1}Ric)(X_2, X_3) = \nabla_{X_1}Ric(X_2, X_3) - Ric(\nabla_{X_1}X_2, X_3) - Ric(X_2, \nabla_{X_1}X_3)$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$ [20]. Then, from (12) we obtain

$$\begin{aligned}(\nabla_{X_1}Ric)(X_2, X_3) \ \ = \ \ & X_1[\lambda]g(\phi X_2, \phi X_3) \\ & +(2p-\lambda)\big[\big((\nabla_{X_1}\alpha_1)X_3\big)\,\alpha_1(X_3) + \big((\nabla_{X_1}\alpha_1)X_2\big)\,\alpha_1(X_2)\big] \\ & (2q-\lambda)\big[\big((\nabla_{X_1}\alpha_1)X_3\big)\,\alpha_2(X_2) + \big((\nabla_{X_1}\alpha_1)X_2\big)\,\alpha_2(X_3)\big]\end{aligned} \tag{16}$$

where $X_1[\lambda]$ is the derivation of λ in the direction of X_1. Thus, from (15), we obtain

$$\begin{aligned}& (\nabla_{X_1}Ric)(X_2, X_3) + (\nabla_{X_2}Ric)(X_3, X_1) + (\nabla_{X_3}Ric)(X_1, X_2) \\ = \ & X_1[\lambda]g(\phi X_2, \phi X_3) + X_2[\lambda]g(\phi X_3, \phi X_1) + X_3[\lambda]g(\phi X_1, \phi X_2).\end{aligned}$$

As a consequence , we can state the following theorem.

Theorem 6. *Let M be a generalized quasi-Einstein normal metric contact pair manifold. If λ is constant then M satisfies cyclic parallel Ricci tensor.*

In [9], it has been proved that if a generalized quasi-Einstein Riemann manifold satisfies Codazzi type of Ricci tensor, then the associated 1-forms are closed.

Suppose that Ricci tensor Ric of a normal metric contact pair manifold M is Codazzi type. Then, from (15) and (16) we obtain

$$\begin{aligned}& (2p-\lambda)[((\nabla_{X_1}\alpha_1)X_2 - (\nabla_{X_2}\alpha_1)X_1)\alpha_1(X_3) \\ & +((\nabla_{X_1}\alpha_1)X_3\alpha_1(X_2) + (\nabla_{X_2}\alpha_1)X_3\alpha_1(X_1))] \\ & +(2q-\lambda)[((\nabla_{X_1}\alpha_1)X_2 - (\nabla_{X_2}\alpha_1)X_1)\alpha_2(X_3) \\ & +((\nabla_{X_1}\alpha_1)X_3\alpha_2(X_2) + (\nabla_{X_2}\alpha_1)X_3\alpha_2(X_1))] = 0.\end{aligned}$$

Let take $X_3 = Z_1$, then we get

$$(2p-\lambda)((\nabla_{X_1}\alpha_1)X_2 - (\nabla_{X_2}\alpha_1)X_1) = 0$$

which implies $\lambda = 2p$ or $(\nabla_{X_1}\alpha_1)X_2 - (\nabla_{X_2}\alpha_1)X_1 = 0$. If $\lambda = 2p$ then the manifold is not generalized quasi-Einstein, so this case is not possible. In the other case we obtain

$$0 = (\nabla_{X_1}\alpha_1)X_2 - (\nabla_{X_2}\alpha_1)X_1 = d\alpha_1(X_1, X_2) = 0$$

and so α_1 is closed. Similarly, by choosing $X_3 = Z_2$ we obtain α_2 is closed. As we know contact pairs (α_1, α_2) are not closed. So, our assumption is not valid. Finally, we conclude that

Theorem 7. *A generalized quasi-Einstein normal metric contact pair manifold does not satisfy Codazzi type of Ricci tensor.*

5. Normal Metric Contact Pair Manifold Satisfying Certain Curvature Conditions

Curvature tensors give us many geometric properties of contact manifolds. Some properties of normal metric contact pair manifold satisfying certain conditions of curvature tensors were given in [21,24]. In this section, we examine the $\mathcal{M}$-projective curvature tensor $\mathcal{W}$, conformal curvature tensor $\mathcal{C}$ and concircular curvature tensor $\mathcal{Z}$ on a normal metric contact pair manifold.

From (11), we have

$$\begin{aligned} \mathcal{W}(X_1, Z)Z &= \frac{m}{2m-1}X_1 - \frac{1}{m-1}QX_1 &(17)\\ \mathcal{W}(X_1, X_2)Z &= R(X_1, X_2)Z &(18)\\ \mathcal{W}(X_1, Z)X_2 &= \left[\frac{(2m-1)(m-2)}{2(m-1)}g(X_1, X_2) + \frac{1}{2(m-1)}Ric(X_1, X_2)\right]Z &(19)\end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(\mathcal{H})$. Also, since $Ric(X_1, X_2) = g(QX_1, X_2)$, where Q is the Ricci operator, we have

$$\begin{aligned} \mathcal{W}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) - \tfrac{1}{2(m-1)}[Ric(X_2, X_3)g(X_1, X_4) \\ &\quad - Ric(X_1, X_3)g(X_2, X_4) + g(X_2, X_3)Ric(X_1, X_4) \\ &\quad - g(X_1, X_3)Ric(X_2, X_4)]. \end{aligned} \tag{20}$$

for all $X_1, X_2, X_3 \in \Gamma(M)$. M is called $\mathcal{M}$-projectively flat if $\mathcal{W}$ vanishes identically on M.

Theorem 8. *A generalized quasi-Einstein normal metric contact pair manifold is $\mathcal{M}$-projectively flat if and only if it is of generalized quasi-constant curvature.*

Proof. Suppose that M is a generalized quasi-Einstein manifold. Then, from (12) and (11) we have

$$\begin{aligned} \mathcal{W}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) \\ &\quad - \tfrac{\lambda}{m-1}[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)] \\ &\quad - \tfrac{2p-\lambda}{2(m-1)}[g(X_1, X_4)\alpha_1(X_2)\alpha_1(X_3) - g(X_1, X_3)\alpha_1(X_2)\alpha_1(X_4) \\ &\quad + g(X_2, X_3)\alpha_1(X_1)\alpha_1(X_4) - g(X_2, X_4)\alpha_1(X_1)\alpha_1(X_3)] \\ &\quad - \tfrac{2q-\lambda}{2(m-1)}[g(X_1, X_4)\alpha_2(X_2)\alpha_2(X_3) - g(X_1, X_3)\alpha_2(X_2)\alpha_2(X_4) \\ &\quad + g(X_2, X_3)\alpha_2(X_1)\alpha_2(X_4) - g(X_2, X_4)\alpha_2(X_1)\alpha_2(X_3)]. \end{aligned} \tag{21}$$

Thus, it is seen that M is $\mathcal{M}$-projectively flat if and only if M is of generalized quasi-constant curvature with coefficients $A = \frac{\lambda}{m-1}$, $B = \frac{2p-\lambda}{2(m-1)}$ and $C = \frac{2q-\lambda}{2(m-1)}$. □

The Riemann manifolds satisfying $R(X_1, X_2) \cdot R = 0$ are called semi-symmetric, where $R(X_1, X_2)$ acts on R as a derivation. Semi-symmetric contact manifolds were studied by Perrone [36]. Similarly, if $\mathcal{W}(X_1, X_2) \cdot R = 0$ then M is called $\mathcal{M}$-projectively semi-symmetric. $\mathcal{W}(X_1, X_2) \cdot R$ is defined as

$$\begin{aligned} (\mathcal{W}(X_1, X_2) \cdot R)(X_3, X_4)X_5 &= \mathcal{W}(X_1, X_2) \cdot R(X_3, X_4)X_5 - R(\mathcal{W}(X_1, X_2), X_3, X_4)X_5 \\ &\quad - R(X_3, \mathcal{W}(X_1, X_2)X_4)X_5 - R(X_3, X_4)\mathcal{W}(X_1, X_2)X_5 \end{aligned} \tag{22}$$

for all $X_1, X_2, X_3, X_4, X_5 \in \Gamma(TM)$. Also, we have

$$(\mathcal{W}(X_1, X_2) \cdot Ric)(X_3, X_4) = -Ric(\mathcal{W}(X_1, X_2), X_3) - Ric(X_3, \mathcal{W}(X_1, X_2)X_4). \tag{23}$$

If $\mathcal{W}(X_1, X_2) \cdot Ric = 0$ then M is called $\mathcal{M}$-projectively Ricci semi-symmetric.

Theorem 9. *A normal metric contact pair manifold is $\mathcal{M}$-projectively semi-symmetric if and only if M is a generalized quasi-Einstein manifold.*

Proof. From (22) and using (17)–(19) we obtain

$$\begin{aligned}(\mathcal{W}(X_1,Z)\cdot R)(X_3,X_4)X_5 &= KR(X_1,X_3,X_4,X_5)Z + LR(X_3,X_4,X_5,QX_1)Z \\ &\quad -(Kg(X_1,X_3)+LRic(X_1,X_3))g(X_4,X_5)Z \\ &\quad +(Kg(X_1,X_5)+LRic(X_1,X_5))R(X_3,X_4)Z\end{aligned}$$

where $K = \frac{(2m-1)(m-2)}{2(m-1)}$ and $L = \frac{1}{2(m-1)}$.

Let take X_1, X_3, X_5 horizontal vector fields and $X_4 = Z$, from (3)–(7), we get

$$(\mathcal{W}(X_1,Z)\cdot R)(X_3,Z)X_5 = -(Kg(X_1,X_5)+LRic(X_1,X_5))X_3.$$

Thus, we conclude that $(\mathcal{W}(X_1,Z)\cdot R)(X_3,X_4)X_5 = 0$ if and only if horizontal bundle of M is Einstein. From Proposition 1, we obtain

$$Ric(X_1,X_5) = -\frac{K}{L}g(X_1,X_5) + (2p+\frac{K}{L})\alpha_1(X_1)\alpha_1(X_5) + (2q+\frac{K}{L})\alpha_2(X_1)\alpha_2(X_5) \tag{24}$$

Therefore, the manifold is a generalized quasi-Einstein. □

Theorem 10. *An $(m = 2p+2q+2)$-dimensional normal metric contact pair manifold satisfies $\mathcal{W}\cdot Ric = 0$ if and only M is generalized quasi-Einstein manifold.*

Proof. For $X_1, X_3, X_4 \in \Gamma(TM)$ from (23) we get

$$\begin{aligned}(\mathcal{W}(X_1,Z)\cdot Ric)(X_3,X_4) &= -Ric([Kg(X_1,X_3)+LRic(X_1,X_3)]Z,X_4) \\ &\quad -Ric(X_3,[Kg(X_1,X_4)+LRic(X_1,X_4)]Z).\end{aligned}$$

Let take X_1, X_4 vector fields and $X_3 = Z$ from (6), (7), we obtain

$$(\mathcal{W}(X_1,Z)\cdot Ric)(X_3,X_4) = -(2p+2q)[Kg(X_1,X_4)+LRic(X_1,X_4)].$$

Therefore, $(\mathcal{W}(X_1,Z)\cdot Ric)(X_3,X_4) = 0$ if and only if horizontal bundle is Einstein. From Proposition (1) we get (24), which completes the proof. □

Blair, Bande and Hadjar [21] studied on conformal flatness of normal metric contact pair manifolds and they proved following theorem.

Theorem 11. *A conformally flat normal metric contact pair manifold is locally isometric to the Hopf manifold $S^{2q+1}(1)\times S^1$ [21].*

Thus, we get following results, for a generalized quasi-Einstein normal metric contact pair manifold.

Theorem 12. *Let M be a generalized quasi-Einstein normal metric contact pair manifold. If M is of generalized quasi-constant curvature with coefficients $A = \frac{\lambda m - m + 2}{(m-1)(m-2)}$, $B = \frac{2p-\lambda}{m-2}$ and $C = \frac{2q-\lambda}{m-2}$, then it is locally isometric to the Hopf manifold $S^{2q+1}(1)\times S^1$.*

Proof. Let M be a generalized quasi-Einstein normal metric contact pair manifold. Then, we have

$$\begin{aligned}\mathcal{C}(X_1,X_2,X_3,X_4) &= R(X_1,X_2,X_3,X_4)\\ &+\frac{scal}{(m-1)(m-2)}[g(X_2,X_3)g(X_1,X_4)-g(X_1,X_3)g(X_2,X_4)]\\ &-\frac{2\lambda}{m-2}[g(X_2,X_3)g(X_1,X_4)-g(X_1,X_3)g(X_2,X_4)]\\ &-\frac{2p-\lambda}{m-2}[g(X_1,X_4)\alpha_1(X_2)\alpha_1(X_3)-g(X_1,X_3)\alpha_1(X_2)\alpha_1(X_4)\\ &+g(X_2,X_3)\alpha_1(X_1)\alpha_1(X_4)-g(X_2,X_4)\alpha_1(X_1)\alpha_1(X_3)]\\ &-\frac{2q-\lambda}{m-2}[g(X_1,X_4)\alpha_2(X_2)\alpha_2(X_3)-g(X_1,X_3)\alpha_2(X_2)\alpha_2(X_4)\\ &-g(X_2,X_4)\alpha_2(X_1)\alpha_2(X_3)].\end{aligned}$$

Suppose that M is of generalized quasi-constant curvature with coefficients $A=\frac{\lambda m-m+2}{(m-1)(m-2)}$, $B=\frac{2p-\lambda}{m-2}$ and $C=\frac{2q-\lambda}{m-2}$. Then, we get $\mathcal{C}=0$ which means that M is conformally flat. Thus, the Theorem 6.4 M is locally isometric to the Hopf manifold $S^{2q+1}(1)\times S^1$. □

By using the definition of $\mathcal{M}$-projective curvature tensor and conformal curvature tensor, we have

$$\begin{aligned}\mathcal{C}(X_1,X_2)X_3 &= \tfrac{2(m-1)}{m-2}\mathcal{W}(X_1,X_2)X_3-\tfrac{m}{m-2}R(X_1,X_2,)X_3\\ &+\tfrac{scal}{(m-1)(m-2)}[g(X_2,X_3)X_1-g(X_1,X_3)X_2].\end{aligned}\tag{25}$$

Let M be a $\mathcal{M}$-projectively flat normal metric contact pair manifold , then, from (25), M is conformally flat if and only if

$$R(X_1,X_2,)X_3=\frac{scal}{m(m-1)}[g(X_2,X_3)X_1-g(X_1,X_3)X_2],$$

which means M is concircular flat. Finally, we conclude that

Theorem 13. *Let M be $\mathcal{M}$-projectively flat normal metric contact pair manifold. If M is also concircularly flat then it is locally isometric to Hopf manifold $S^{2q+1}(1)\times S^1$.*

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References

1. Chaki, M.C. On quasi Einstein manifolds. *Publ. Math.* **2000**, *57*, 297–306.
2. De, U.C.; Ghosh, G.C. On quasi Einstein manifolds. *Periodica Mathematica Hungarica* **2004**, *48*, 223–231. [CrossRef]
3. Ghosh, G.C.; De, U.C.; Binh, T.Q. Certain curvature restrictions on a quasi Einstein manifolds. *Publ. Math. Debr.* **2006**, *69*, 209–217.
4. Tripathi, M.M.; Kim, J. On $N(k)-$quasi Einstein manifolds. *Commun. Korean Math. Soc.* **2007**, *22*, 411. [CrossRef]
5. Chaki, M.C. On Generalized quasi-Einstein manifold. *Publ. Math. Debrecen* **2001**, *58*, 638–691.
6. Catino, G. Generalized quasi-Einstein manifolds with harmonic Weyl tensor. *Mathematische Zeitschrift* **2012**, *271*, 751–756. [CrossRef]
7. De, U.C.; Ghosh, G.C. On generalized quasi Einstein manifolds. *Kyungpook Math. J.* **2004**, *44*, 607. [CrossRef]
8. De, U.C.; Mallick, S. On the existence of generalized quasi-Einstein manifolds. *Arch. Math.* **2011**, *47*, 279–291.

9. De, U.C.; Mallick, S. On generalized quasi-Einstein manifolds admitting certain vector fields. *Filomat* **2015**, *29*, 599–609. [CrossRef]
10. Prakasha, D.G.; Venkatesha, H. Some results on generalized quasi-Einstein manifolds. *Chin. J. Math.* **2014**, *18*, 563803 [CrossRef]
11. Sular, S.; Özgür, C. Characterizations of generalized quasi-Einstein manifolds. *An. Univ. Ovidius Constanta-Seria Mat.* **2012**, *20*, 407–416. [CrossRef]
12. De, U.C.; Shenawy, S. Generalized quasi-Einstein GRW space-times. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950124. [CrossRef]
13. Güler, S.; Demirbağ, S.A. A study of generalized quasi Einstein spacetimes with applications in general relativity. *Int. J. Theor. Phys.* **2016**, *55* 548–562. [CrossRef]
14. Mallick, S.; De, U.C. On a Class of Generalized quasi-Einstein Manifolds with applications to relativity. *Acta Univ. Palacki. Olomuc. Fac. Rerum Nat. Math.* **2016**, *55*, 111–127.
15. Turgut Vanlı, A.; Ünal, İ. On complex η-Einstein normal complex contact metric manifolds. *Commun. Math. Appl.* **2017**, *8*, 301–313.
16. Bande, G.; Hadjar, A. Contact pairs. *Tohoku Math. J.* **2005**, *57*, 247–260. [CrossRef]
17. Blair, D.E.; Ludden, G.D.; Yano, K. Geometry of complex manifolds similar to the Calabi-Eckmann manifolds. *J. Differ. Geom.* **1974**, *9*, 263–274. [CrossRef]
18. Bande, G.; Hadjar, A. Contact pair structures and associated metrics. *Differ. Geom.-Proc. VIII Int. Colloq.* **2009**, 266–275. [CrossRef]
19. Bande, G.; Hadjar, A. On normal contact pairs. *Int. J. Math.* **2010**, *21*, 266–275. [CrossRef]
20. Yano, K.; Kon, M. Structures on Manifolds. *World Sci.* **1985**. [CrossRef]
21. Bande, G.; Blair, D.E.; Hadja, A. Bochner and conformal flatness of normal metric contact pairs. *Ann. Glob. Anal. Geom.* **2015**, *48*, 47–56. [CrossRef]
22. Chen, B.Y.; Yano, K. Hypersurfaces of a conformally flat space. *Tensor (NS)* **1972**, *26*, 318–322.
23. Unal, İ.; Sarı, R.; Turgut Vanlı, A. Concircular curvature tensor on generalized kenmotsu manifolds. *Gümüşhane Üniversitesi Fen Bilimleri Enstitüsü Dergisi* **2018**, *CMES 2018*, 99–105.
24. Ünal, İ. Some flatness conditions on normal metric contact pairs. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 1–10
25. Turgut Vanlı, A.; Ünal, İ. Conformal, concircular, quasi-conformal and conharmonic flatness on normal complex contact metric manifolds. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750067. [CrossRef]
26. Prakasha D.G.; Fernandez, L.M.; Mirji, K. The $\mathcal{M}$-projective curvature tensor field on generalized (κ, μ)-paracontact metric manifolds. *Georgian Math. J.* **2020**, *27*, 141–147. [CrossRef]
27. Ayar, G.; Chaubey, S.K. M-projective curvature tensor over cosymplectic manifolds. *Differ. Geom. Dyn. Syst.* **2019**, *21*, 23–33.
28. Zengin, F.Ö. On m-projectively flat LP-Sasakian manifolds. *Ukrainian Math. J.* **2014**, *65*, 1725–1732. [CrossRef]
29. Pokhariyal, G.P.; Mishra, R.S. Curvature tensors and their relativistics significance. *Yokohama Math. J.* **1970**, *18*, 105–108.
30. Chen, B.Y. Some new obstructions to minimal and Lagrangian isometric immersions. *Jpn. J. Math.* **2000**, *26*, 105–127. [CrossRef]
31. Chen, B.Y.; Dillen, F.; Verstraelen, L.; Vrancken, L. Characterizations of Riemannian space forms, Einstein spaces and conformally flat spaces. *Proc. Am. Math. Soc.* **2000**, *128*, 589–598. [CrossRef]
32. Dumitru, D. On Einstein spaces of odd dimension. *Bul. Univ. Transilvania Brasov.* **2007**, *14*, 95–97.
33. Bande, G.; Blair, D.E.; Hadja, A. Characterization of quasi-Einstein manifolds. *An. Stiint. Univ. AL. I. Cuza Iasi. Mat. (NS)* **2007**, *53*, 67–72.
34. Dumitru, D. A characterization of generalized quasi-Einstein manifolds. *Novi Sad J. Math.* **2012**, *142*, 89–94.
35. Bande, G.; Hadjar, A. Symmetry in the geometry of metric contact pairs. *Mathematische Nachrichten* **2013**, *286*, 1701–1709. [CrossRef]
36. Perrone, D. Contact Riemannian manifolds satisfying $R(X, \xi)R = 0$. *Yokohama Math. J.* **1992**, *39*, 141–149.