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Unique Fixed-Point Results for β-Admissible Mapping under (β-$\check{\psi}$)-Contraction in Complete Dislocated G_d-Metric Space

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Abstract: This paper is designed to display some results which generalize the recent results that cannot be established from the corresponding results in other spaces and do not satisfy the remarks of Jleli et al. (Fixed Point Theor Appl. 210, 2012) and Samet et al. (Int. J. Anal. Article ID 917158, 2013). We obtain unique fixed-point for mapping satisfying β-$\check{\psi}$ contraction only on a closed G_d ball in complete dislocated G_d-metric space. An example is also discussed to shed light on the main result.

Keywords: closed G_d ball; dislocated G_d-metric space; β-admissible mapping; β-$\check{\psi}$ contraction; unique fixed point

MSC: 46S40; 47H10; 54H25

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1. Introduction and Preliminaries

Fixed-point theory has several applications in different fields such as engineering, computer sciences, and social sciences and plays a vital role in the study of different aspects of mathematics. By using fixed-point theory results, a lot of methods have been constructed for the solutions of problems in sciences. Let $\mathcal{S}$ be a mapping from Y to Y. If $\mathcal{S}\mathfrak{b} = \mathfrak{b}$ for any $\mathfrak{b} \in Y$ then $\mathfrak{b}$ is known as a fixed point of $\mathcal{S}$.

One of the generalizations of a metric is G metric, which was developed by Sims and Mustafa [1]. Karapınar et al. [2] and Singh et al. [3] discussed fixed-point results in G metric spaces, which distinguish G metric spaces from other spaces. Many results in G metric spaces can be seen in [1,2,4–15].

α-admissible mapping and corresponding α-ψ contractive condition was introduced by Samet et al. [16]. They generalized the fixed-point results endowed with a partial order (see [4,17,18]). Several researchers studied and extended the results in [16] in different ways (see [8,19–23]). Recently, Shoaib et al. [24] obtained fixed-point theorems for α-ψ-locally contractive type mappings in right complete dislocated quasi G-metric spaces.

Arshad et al. [25] observed that there were mappings which had fixed points but there were no results to ensure the existence of fixed points of such mappings. They introduced a condition on closed ball to obtain common fixed points for such mappings. For further theorems on closed ball, see [14,26–28].

This paper extends the results of Karapinar et al. [2] in four different ways by using

(i) β-admissible mapping;

(ii) closed G_d ball instead of whole space;
(iii) β-$\check{\psi}$ contraction;
(iv) dislocated G_d-metric space instead of metric space.

Moreover, our contraction cannot be expressed in two variables, so there is no corresponding result in metric space for our results. This paper also generalizes the recent results given in [13–15,24]. The following definitions and results will be useful to understand the paper.

Definition 1. *[15] Let $\check{Z}$ be non-empty and $G_d : \check{Z} \times \check{Z} \times \check{Z} \to \mathbb{R}^+$. Let G_d satisfying the constraints given below:*

(i) If $G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) = 0$, then $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{l}_3$.
(ii) $G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) = G_d(\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_1) = G_d(\mathfrak{l}_3, \mathfrak{l}_1, \mathfrak{l}_2) = G_d(\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_2) = G_d(\mathfrak{l}_2, \mathfrak{l}_1, \mathfrak{l}_3) = G_d(\mathfrak{l}_3, \mathfrak{l}_2, \mathfrak{l}_1)$.
(iii) $G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) \leq G_d(\mathfrak{l}_1, \mathfrak{l}_4, \mathfrak{l}_4) + G_d(\mathfrak{l}_4, \mathfrak{l}_2, \mathfrak{l}_3)$

for all $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 \in \check{Z}$. Then $(\check{Z}, G_d)$ is said to be dislocated G_d metric space. It is noted that if in dislocated G_d-metric space $G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) = 0$ whenever $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{l}_3$, then $(\check{Z}, G_d)$ becomes a G metric space.

Example 1. *[15] Let $\check{Z} = [0, 4]$. G_d defined as $G_d = \mathfrak{l}_1 + \mathfrak{l}_2 + \mathfrak{l}_3 \ \forall \ \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \in \check{Z}$. then it can be easily check that G_d is dislocated G_d-metric space.*

Definition 2. *[15] Let $\{\mathfrak{l}_p\}$ be a sequence in dislocated G_d metric space. $\mathfrak{l} \in \check{Z}$ is the limit of $\{\mathfrak{l}_\mathfrak{p}\}$ if $\lim_{\mathfrak{p},\mathfrak{q} \to \infty} G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}, \mathfrak{l}_\mathfrak{q}) = 0$, and one says $\{\mathfrak{l}_\mathfrak{p}\}$ is G_d-convergent to $\mathfrak{l}$.*

Definition 3. *[15] Let $(\check{Z}, G_d)$ be a dislocated G_d-metric space, then*

(i) $\{\mathfrak{l}_\mathfrak{p}\}$ is $C - G_d$- sequence or Cauchy G_d sequence if for all $\varepsilon > 0$, there exists $\mathfrak{p}^\star \in \mathfrak{p} : G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{q}, \mathfrak{l}_r) < \varepsilon$ for all $\mathfrak{p}, \mathfrak{q}, r \geq \mathfrak{p}^\star$.
(ii) $(\check{Z}, G_d)$ is called complete if every C-G_d- sequence in $(\check{Z}, G_d)$ is G_d-convergent.

Definition 4. *[15] Open G_d ball and closed G_d ball with center $\mathfrak{l}_0 \in \check{Z}$ and radius $\check{r} > 0$ in dislocated G_d-metric space are defined as: $B_{G_d}(\mathfrak{l}_0, \check{r}) = \{\mathfrak{l} \in \check{Z} : G_d(\mathfrak{l}_0, \mathfrak{l}, \mathfrak{l}) < \check{r}\}$, $\overline{B_{G_d}(\mathfrak{l}_0, \check{r})} = \{\mathfrak{l} \in \check{Z} : G_d(\mathfrak{l}_0, \mathfrak{l}, \mathfrak{l}) \leq \check{r}\}$ respectively.*

Proposition 1. *[15] Let $(\check{Z}, G_d)$ be a dislocated G_d-metric space, then conditions given below are equivalent:*

(i) $G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}, \mathfrak{l}_\mathfrak{p}) \to 0$ as $\mathfrak{p} \to \infty$.
(ii) $G_d(\mathfrak{l}, \mathfrak{l}_\mathfrak{p}, \mathfrak{l}) \to 0$ as $\mathfrak{p} \to \infty$.
(iii) $G_d(\mathfrak{l}, \mathfrak{l}_\mathfrak{q}, \mathfrak{l}_\mathfrak{p}) \to 0$ as $\mathfrak{p}, \mathfrak{q} \to \infty$.

Definition 5. *[16] Let $\check{\psi} : [0, \infty) \to [0, \infty)$ holds the axioms:*
$(\check{\Psi}1)$ $\check{\psi}$ is non-decreasing.
$(\check{\Psi}2)$ for all $\tilde{\mathfrak{t}} > 0$, we have

$$\check{\mu}_0\left(\tilde{\mathfrak{t}}\right) = \sum_{a=0}^{\infty} \check{\psi}^a\left(\tilde{\mathfrak{t}}\right) < \infty.$$

The power a denotes the a^{th} iteration of $\check{\psi}$. All such functions form a family which is denoted by $\check{\Psi}$. $\check{\psi} \in \check{\Psi}$ is called c-comparison function.

Definition 6. *Let $\check{Z}$ be a non-empty set and $\beta : \check{Z} \times \check{Z} \times \check{Z} \to [0, \infty)$. We say that $\mathfrak{R} : \check{Z} \to \check{Z}$ is β-admissible mapping, if*

$$\beta(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3) \geq 1 \Longrightarrow \beta(\mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_2, \mathfrak{R}\mathfrak{l}_3) \geq 1, \text{ for } \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \in \check{Z}.$$

2. Main Result

Theorem 1. *Let $(\check{Z}, G_d)$ be complete dislocated G_d metric space, $\mathfrak{R} : \check{Z} \longrightarrow \check{Z}$ be a β-admissible mapping, $\check{\psi} \in \check{\Psi}$, $\check{r} > 0$ and $\mathfrak{l}_\circ \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Assume that the following assertions hold:*

$$\beta(\mathfrak{l}, k, \mathfrak{f}) G_d(\mathfrak{R}\mathfrak{l}, \mathfrak{R}k, \mathfrak{R}\mathfrak{f}) \leq \check{\psi}(\check{M}(\mathfrak{l}, k, \mathfrak{f})) \text{ for all } \mathfrak{l}, k, \mathfrak{f} \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}, \tag{1}$$

where

$$\check{M}(\mathfrak{l}, k, \mathfrak{f}) = \max \left\{ \begin{array}{c} G_d(k, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}k), \frac{G_d(\mathfrak{R}\mathfrak{l}, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}k)}{2}, \\ \frac{G_d(\mathfrak{l}, \mathfrak{R}\mathfrak{l}, k)}{2}, \frac{G_d(\mathfrak{l}, \mathfrak{R}\mathfrak{l}, \mathfrak{f})}{2}, G_d(\mathfrak{f}, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}\mathfrak{f}), \\ \frac{G_d(k, \mathfrak{R}\mathfrak{l}, \mathfrak{R}k)}{2}, \frac{G_d(\mathfrak{f}, \mathfrak{R}\mathfrak{l}, \mathfrak{R}k)}{2}, \frac{G_d(\mathfrak{R}\mathfrak{l}, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}\mathfrak{f})}{2}, \\ G_d(\mathfrak{l}, k, \mathfrak{f}), G_d(\mathfrak{l}, \mathfrak{R}\mathfrak{l}, \mathfrak{R}\mathfrak{l}), G_d(k, \mathfrak{R}k, \mathfrak{R}k), \\ \frac{G_d(\mathfrak{f}, \mathfrak{R}\mathfrak{f}, \mathfrak{R}\mathfrak{f})}{2}, \frac{G_d(k, \mathfrak{R}\mathfrak{f}, \mathfrak{R}\mathfrak{f})}{2}, \frac{G_d(\mathfrak{l}, \mathfrak{R}k, \mathfrak{R}k)}{2} \end{array} \right\} \tag{2}$$

Also

$$\sum_{a=0}^{\mathfrak{p}} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ)) \leq \check{r} \text{ ,for all } b \in \mathbb{N} \cup \{0\}. \tag{3}$$

(i) $\beta(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ) \geq 1$

(ii) *If there exists $\{\mathfrak{l}_\mathfrak{p}\}$ in $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$ such that for all $\mathfrak{p} \in \mathbb{N} \cup \{0\}$, $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}) \geq 1$ and $\mathfrak{l}_\mathfrak{p} \to e \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$, then $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}, e) \geq 1$.*

Then there exists a unique $e \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$ such that $e = \mathfrak{R}e$.

Proof. As $\mathfrak{l}_\circ \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Define a sequence

$$\mathfrak{l}_{\mathfrak{p}+1} = \mathfrak{R}\mathfrak{l}_\mathfrak{p} \quad \text{for all } \mathfrak{p} \in \mathbb{N} \cup \{0\}.$$

Let $\mathfrak{l}_{\mathfrak{p}+1} \neq \mathfrak{l}_\mathfrak{p}$, for all $\mathfrak{p} \in \mathbb{N} \cup \{0\}$, otherwise if such $\mathfrak{p}$ exists then $\mathfrak{R}(\mathfrak{l}_\mathfrak{p}) = \mathfrak{l}_\mathfrak{p}$. By using (3),

$$G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ) \leq \sum_{a=0}^{\mathfrak{p}} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ)) \leq \check{r}.$$

This implies that $\mathfrak{l}_1 \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Since $\beta(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ) \geq 1 \Rightarrow \beta(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) \geq 1$. Since $\mathfrak{R}$ is β-admissible on $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$ so $\beta(\mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1) \geq 1$.

$$\begin{aligned} G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) &= G_d(\mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1) \\ &\leq \beta(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) G_d(\mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1) \\ &\leq \check{\psi}(\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)). \end{aligned} \tag{4}$$

$$\begin{aligned} \check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) &= \max \left\{ \begin{array}{c} G_d(\mathfrak{l}_1, \mathfrak{R}^2\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1), \\ \frac{G_d(\mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}^2\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1)}{2}, \frac{G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{l}_1)}{2}, \\ \frac{G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{l}_1)}{2}, G_d(\mathfrak{l}_1, \mathfrak{R}^2\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1), \\ \frac{G_d(\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1)}{2}, \frac{G_d(\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1)}{2}, \\ \frac{G_d(\mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}^2\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1)}{2}, G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1), \\ G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ), G_d(\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1), \\ \frac{G_d(\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1)}{2}, \frac{G_d(\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1)}{2}, \\ \frac{G_d(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_1, \mathfrak{R}\mathfrak{l}_1)}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2), \frac{G_d(\mathfrak{l}_1, \mathfrak{l}_1, \mathfrak{l}_2)}{2}, \\ G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1), \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2} \end{array} \right\} \end{aligned} \tag{5}$$

Case 1: If $\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)$. From (4)

$$G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) \leq \check{\psi}(G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)).$$

which give contradiction to fact that $\check{\psi}(\tilde{t}) \leq \tilde{t}$.

Case 2: If $\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = \frac{G_d(\mathfrak{l}_1, \mathfrak{l}_1, \mathfrak{l}_2)}{2}$ then by using (4), we have

$$\begin{aligned}
G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_1, \mathfrak{l}_1, \mathfrak{l}_2)}{2}) \\
&\leq \check{\psi}(\frac{G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) + G_d(\mathfrak{l}_2, \mathfrak{l}_1, \mathfrak{l}_2)}{2}) \\
&\leq \check{\psi}(G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)).
\end{aligned}$$

which give contradiction to $\check{\psi}(\tilde{t}) \leq \tilde{t}$. From case 1 and case 2, (5) becomes

$$\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = \max\{G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1), \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2}\}. \tag{6}$$

Case 3: If $\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2}$ then by using (6), we have

$$\begin{aligned}
G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) &\leq \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2} \\
&\leq \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)}{2} \\
G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) &\leq G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2).
\end{aligned} \tag{7}$$

If

$$\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = \frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2}. \tag{8}$$

Using (4), (7) and (8), we have

$$\begin{aligned}
G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2)}{2}) \\
&\leq \check{\psi}(\frac{G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)}{2}) \\
&\leq \check{\psi}(\frac{G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) + G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)}{2}) \\
&\leq \check{\psi}(G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2)).
\end{aligned}$$

which give again contradiction. Hence from case 1, case 2 and case 3, we get

$$\check{M}(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) = G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1). \tag{9}$$

Now, (4) becomes

$$G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) \leq \check{\psi}(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)). \tag{10}$$

Now

$$\begin{aligned}
G_d(\mathfrak{l}_\circ, \mathfrak{l}_2, \mathfrak{l}_2) &\leq G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) \\
&\leq G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + \check{\psi}(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \\
&= \sum_{a=0}^{1} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \leq \check{r}.
\end{aligned}$$

This shows that $\mathfrak{l}_2 \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Let $\mathfrak{l}_3, \mathfrak{l}_4, ...\mathfrak{l}_h \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$ for some $h \in \mathbb{N}$. Since $\mathfrak{R}$ is β-admissible on $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. So $\beta(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) \geq 1$ this implies $\beta(\mathfrak{R}\mathfrak{l}_{h-1}, \mathfrak{R}\mathfrak{l}_h, \mathfrak{R}\mathfrak{l}_h) \geq 1$. Using (1), we have

$$\begin{aligned} G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) &= G_d(\mathfrak{R}\mathfrak{l}_{h-1}, \mathfrak{R}\mathfrak{l}_h, \mathfrak{R}\mathfrak{l}_h) \\ &\leq \beta(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) G_d(\mathfrak{R}\mathfrak{l}_{h-1}, \mathfrak{R}\mathfrak{l}_h, \mathfrak{R}\mathfrak{l}_h) \\ &\leq \check{\psi}(\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h)). \end{aligned} \tag{11}$$

From (2)

$$\begin{aligned} \check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) &= \max\{G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}), \frac{G_d(\mathfrak{l}_h, \mathfrak{l}_h, \mathfrak{l}_{h+1})}{2}, \\ G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h), &\quad \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}\}. \end{aligned} \tag{12}$$

Case 1: If $\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})$. From (11), we have

$$G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) \leq \check{\psi}(G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})).$$

which give contradiction to the fact that $\check{\psi}(\tilde{t}) \leq \tilde{t}$.

Case 2: If $\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = \frac{G_d(\mathfrak{l}_h, \mathfrak{l}_h, \mathfrak{l}_{h+1})}{2}$ then by using (11), we have

$$\begin{aligned} G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_h, \mathfrak{l}_h, \mathfrak{l}_{h+1})}{2}) \\ &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) + G_d(\mathfrak{l}_{h+1}, \mathfrak{l}_h, \mathfrak{l}_{h+1})}{2}) \\ &\leq \check{\psi}(G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})). \end{aligned}$$

which give contradiction to $\check{\psi}(\tilde{t}) \leq \tilde{t}$. From case 1 and case 2, (12) becomes

$$\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = \max\{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h), \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}\}. \tag{13}$$

Case 3: If $\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}$ then by using (13), we have

$$\begin{aligned} G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) &\leq \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2} \\ &\leq \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) + G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2} \\ G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) &\leq G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}). \end{aligned} \tag{14}$$

If

$$\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = \frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}. \tag{15}$$

Using (11), (14) and (15), we have

$$\begin{aligned} G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}) \\ &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) + G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}) \\ &\leq \check{\psi}(\frac{G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) + G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})}{2}) \\ &\leq \check{\psi}(G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1})). \end{aligned}$$

which give again contradiction. Hence from all cases, we have

$$\check{M}(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h) = G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h). \tag{16}$$

(11) become

$$\begin{aligned} & G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) \\ \leq\ & \check{\psi}(G_d(\mathfrak{l}_{h-1}, \mathfrak{l}_h, \mathfrak{l}_h)) \\ \leq\ & \cdots \leq \check{\psi}^h(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)). \end{aligned} \tag{17}$$

Now

$$\begin{aligned} G_d(\mathfrak{l}_\circ, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) &\leq G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + G_d(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_2) \\ &\quad +\ldots\ldots + G_d(\mathfrak{l}_h, \mathfrak{l}_{h+1}, \mathfrak{l}_{h+1}) \\ &\leq G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) + \check{\psi}(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \\ &\quad +\ldots\ldots + \check{\psi}^h G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1) \\ &= \sum_{a=0}^{h} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \leq \check{r}. \end{aligned}$$

This shows that $\mathfrak{l}_{h+1} \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Hence $\mathfrak{l}_\mathfrak{p} \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$, for all $\mathfrak{p} \in \mathbb{N}$ by mathematical induction. Now (17) become

$$G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}) \leq \check{\psi}^\mathfrak{p}(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \quad \text{for all } \mathfrak{p} \in \mathbb{N}.$$

As $\mathfrak{R}$ is β-admissible on $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. So $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}) \geq 1$. Now we will prove Cauchy sequence. Let $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}$ for $\varepsilon > 0$, there exists $\mathfrak{p}_\circ \in \mathbb{N}$ such that $\sum_{a \geq \mathfrak{p}_\circ} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \leq \varepsilon$ for all $\mathfrak{q} > \mathfrak{p} \geq \mathfrak{p}_\circ$.

$$\begin{aligned} G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{q}, \mathfrak{l}_\mathfrak{q}) &\leq G_d(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}) + G_d(\mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+2}, \mathfrak{l}_{\mathfrak{p}+2}) \\ +\ldots &\quad +G_d(\mathfrak{l}_{\mathfrak{q}-1}, \mathfrak{l}_\mathfrak{q}, \mathfrak{l}_\mathfrak{q}) \\ &= \sum_{a=\mathfrak{p}}^{\mathfrak{q}} \check{\psi}(G_d(\mathfrak{l}_l, \mathfrak{l}_{l+1}, \mathfrak{l}_{l+1})) \\ &\leq \sum_{a \geq \mathfrak{p}_\circ} \check{\psi}(G_d(\mathfrak{l}_l, \mathfrak{l}_{l+1}, \mathfrak{l}_{l+1})) \\ &\leq \sum_{a \geq \mathfrak{p}_\circ} \check{\psi}^a(G_d(\mathfrak{l}_\circ, \mathfrak{l}_1, \mathfrak{l}_1)) \leq \varepsilon. \end{aligned}$$

Thus, $\{\mathfrak{l}_\mathfrak{p}\}$ is a C-G_d-sequence in $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. As every closed G_d ball is closed subset. So $\{\mathfrak{l}_\mathfrak{p}\}$ is convergent in $\overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$ and the point of convergence $e \in \overline{B_{G_d}(\mathfrak{l}_\circ, \check{r})}$. Hence $\mathfrak{l}_\mathfrak{p} \to e$ as $\mathfrak{p} \to \infty$. So

$$\lim_{\mathfrak{p} \to \infty} G_d(e, \mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}) = 0.$$

By assumption, $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}, e) \geq 1$ for all $\mathfrak{p} \in \mathbb{N} \cup \{0\}$ so $\beta(\mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}e) \geq 1$. Now we must prove that $e = \mathfrak{R}(e)$.

$$\begin{aligned} G_d(\mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{R}e) &= G_d(\mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}e) \\ &\leq \beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}, e) G_d(\mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}\mathfrak{l}_\mathfrak{p}, \mathfrak{R}e) \\ &\leq \check{\psi}(\check{M}(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}, e)). \end{aligned} \tag{18}$$

$$\check{M}(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}}, e) = \max\left\{\begin{array}{c} G_d(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}+2}, \mathfrak{l}_{\mathfrak{p}+1}), \frac{G_d(\mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+2}, \mathfrak{l}_{\mathfrak{p}+1})}{2}, \\ \frac{G_d(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}})}{2}, \frac{G_d(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}+1}, e)}{2}, \\ G_d(e, \mathfrak{l}_{\mathfrak{p}+2}, \Re e), G_d(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}), \\ \frac{G_d(e, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1})}{2}, \frac{G_d(\mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+2}, \Re e)}{2}, \\ G_d(\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}}, e), \frac{G_d(e, \Re e, \Re e)}{2}, \frac{G_d(\mathfrak{l}_{\mathfrak{p}}, \Re e, \Re e)}{2} \end{array}\right\}.$$

Replace in (18) and on applying limit $\mathfrak{p} \to \infty$. We get

$$\begin{aligned} G_d(e, e, \Re e) &= \check{\psi}\left(\max\left\{G_d(e, e, \Re e), \frac{G_d(\Re e, e, \Re e)}{2}\right\}\right) \\ &\leq \check{\psi}\left(\max\left\{\begin{array}{c} G_d(e, e, \Re e), \\ \frac{G_d(\Re e, e, e) + G_d(e, e, \Re e)}{2} \end{array}\right\}\right) \\ &\leq \check{\psi}(G_d(e, e, \Re e)). \end{aligned} \tag{19}$$

Again, contradiction to $\check{\psi}(\tilde{t}) < \tilde{t}$. Hence $\check{\psi}(0) = 0 \Rightarrow G_d(e, e, \Re e) = 0 \Rightarrow e = Te$. For uniqueness, consider $e = Te$ and $\ddot{d} = T\ddot{d}$

$$\begin{aligned} G_d(\ddot{d}, e, e) &= G_d(\Re \ddot{d}, \Re e, \Re e) \\ &\leq \beta(\ddot{d}, e, e) G_d(\Re \ddot{d}, \Re e, \Re e) \\ &\leq \check{\psi}(M(\ddot{d}, e, e)). \end{aligned} \tag{20}$$

$$\check{M}(\ddot{d}, e, e) = \max\left\{\begin{array}{c} G_d(e, \ddot{d}, e), \frac{G_d(\ddot{d}, \ddot{d}, e)}{2}, \\ G_d(\ddot{d}, \ddot{d}, \ddot{d}), G_d(e, e, e) \end{array}\right\}.$$

If $\check{M}(\ddot{d}, e, e) = G_d(e, e, e)$.

$$\begin{aligned} G_d(e, e, e) &= G_d(\Re e, \Re e, \Re e) \\ &\leq \beta(e, e, e) G_d(\Re e, \Re e, \Re e) \\ &\leq \check{\psi}(G_d(e, e, e)). \end{aligned}$$

which give contradiction. Similarly

$$G_d(\ddot{d}, \ddot{d}, \ddot{d}) \leq \check{\psi}(G_d(\ddot{d}, \ddot{d}, \ddot{d})).$$

and

$$G_d(\ddot{d}, e, e) \leq \check{\psi}(G_d(e, \ddot{d}, e)).$$

Give a contradiction. Hence (20) become

$$\begin{aligned} G_d(\ddot{d}, e, e) &\leq \check{\psi}\left(\frac{G_d(\ddot{d}, \ddot{d}, e)}{2}\right). \\ &\leq \check{\psi}\left(\frac{G_d(\ddot{d}, e, e) + G_d(e, \ddot{d}, e)}{2}\right). \\ &\leq \check{\psi}(G_d(\ddot{d}, e, e)). \end{aligned}$$

Again contradiction. Hence $\check{\psi}(0) = 0$ implies that $G_d(\ddot{d}, e, e) = 0$. So $e = \ddot{d}$.

□

Example 2. *Let $\check{Z} = [0, 2]$, Let $G_d : \check{Z} \times \check{Z} \times \check{Z} \to \mathbb{R}^+$ be a mapping defined by*

$$G_d = \mathfrak{l} + k + \mathfrak{f}, \textit{ for all } \mathfrak{l}, k, \mathfrak{f} \in \check{Z}.$$

It can be easily check that G_d is dislocated G_d-metric space. Let $\Re : \check{Z} \to \check{Z}$ be defined by

$$\Re(\iota) = \begin{cases} \frac{\iota}{2} & if\ \iota \in [0, \frac{3}{4}] \\ 2-\iota & if\ \iota \in (\frac{3}{4}, 2] \end{cases}.$$

Let $\iota_\circ = \frac{1}{4}$ and $\check{r} = \frac{7}{4}$ such that $\overline{B_{G_d}(\iota_\circ, \check{r})} = [0, \frac{3}{4}]$. Now we define a mapping $\beta : \check{Z} \times \check{Z} \times \check{Z} \to [0, \infty)$ by

$$\beta(\iota, k, \mathfrak{f}) = \begin{cases} \frac{8}{7} & if\ \iota, k, \mathfrak{f} \in [0, \frac{3}{4}] \\ 5 & if\ \iota, k, \mathfrak{f} \in (\frac{3}{4}, 2] \end{cases}.$$

It is clear that $\beta(\iota, k, \mathfrak{f}) > 1 \Rightarrow \beta(\Re\iota, \Re k, \Re\mathfrak{f}) > 1$. Hence $\Re$ is an β-admissible on $\check{Z}$. Let for all $\tilde{t} \geq 0$, $\check{\psi}(\tilde{t}) = \frac{5}{7}\tilde{t}$. Let $\iota, k, \mathfrak{f} \in [\frac{3}{4}, 2]$. Let $\iota = 1, k = 1.5, \mathfrak{f} = 2$

$$\begin{aligned} \beta(\iota, k, \mathfrak{f}) G_d(\Re\iota, \Re k, \Re\mathfrak{f}) &= \beta(1, 1.5, 2) \times G_d(\Re 1, \Re 1.5, \Re 2) \\ &= 5 \times G_d(1, 0.5, 0) \\ &= 5 \times (1.5) = 7.5. \end{aligned} \tag{21}$$

$$\begin{aligned} \check{M}(\iota, k, \mathfrak{f}) &= \check{M}(1, 1.5, 2) \\ &= \max \left\{ \begin{array}{c} G_d(1.5, 1, 0.5), \frac{G_d(1,1,0.5)}{2}, \\ \frac{G_d(1,1,1.5)}{2}, \frac{G_d(1,1,2)}{2}, \\ G_d(2, 1, 0), \frac{G_d(1.5,1,1.5)}{2}, \\ \frac{G_d(2,1,0.5)}{2}, \frac{G_d(1,1,0)}{2}, \\ G_d(1, 1.5, 2), G_d(1, 1, 1), \\ G_d(1.5, 0.5, 0.5), \frac{G_d(2,0,0)}{2} \\ , \frac{G_d(1.5,0,0)}{2}, \frac{G_d(1,0.5,0.5)}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} 3, 1.25, 1.75, 2, 3, 1.5, 1.75, \\ 1, 4.5, 3, 2.5, 1, 0.75, 1 \end{array} \right\} \\ &= 4.5. \end{aligned}$$

$$\check{\psi}(\check{M}(\iota, k, \mathfrak{f})) = \check{\psi}(4.5) = 3.2142. \tag{22}$$

Hence from (21) *and* (22), $\beta(\iota, k, \mathfrak{f}) G_d(\Re\iota, \Re k, \Re\mathfrak{f}) \leq \check{\psi}(\check{M}(\iota, k, \mathfrak{f}))$. *Let $\iota, k, \mathfrak{f} \in \overline{B_{G_d}(\iota_\circ, \check{r})} = [0, \frac{3}{4}]$*

$$\begin{aligned} \beta(\iota, k, \mathfrak{f}) G_d(\Re\iota, \Re k, \Re\mathfrak{f}) &= \frac{8}{7} G_d(\frac{\iota}{2}, \frac{k}{2}, \frac{\mathfrak{f}}{2}) \\ &= \frac{8}{7}(\frac{\iota}{2} + \frac{k}{2} + \frac{\mathfrak{f}}{2}) \\ \beta(\iota, k, \mathfrak{f}) G_d(\Re\iota, \Re k, \Re\mathfrak{f}) &= \frac{4}{7}(\iota + k + \mathfrak{f}). \end{aligned} \tag{23}$$

$$\check{M}(\iota, k, \mathfrak{f}) = \max \left\{ \begin{array}{c} \frac{6k+\iota}{4}, \frac{3\iota+2k}{8}, \frac{3\iota+2k}{4}, \\ \frac{3\iota+2\mathfrak{f}}{4}, \frac{\iota+6\mathfrak{f}}{4}, \frac{3k+\iota}{4}, \\ \frac{\iota+k+2\mathfrak{f}}{4}, \frac{3\iota+2\mathfrak{f}}{8}, \\ \iota + k + \mathfrak{f}, 2\iota, 2k, \\ \mathfrak{f}, \frac{\mathfrak{f}+k}{2}, \frac{\iota+k}{2} \end{array} \right\}$$

Now,

$$\begin{aligned} 0 &\leq \frac{6k+\mathfrak{l}}{4}, \frac{\mathfrak{l}+6\mathfrak{f}}{4} \leq \frac{21}{16}. \qquad 0 \leq 2\mathfrak{l}, 2k \leq \frac{3}{2}. \\ 0 &\leq \frac{3\mathfrak{l}+2k}{4}, \frac{3\mathfrak{l}+2\mathfrak{f}}{4} \leq \frac{15}{16}. \quad 0 \leq \frac{3\mathfrak{l}+2k}{8}, \frac{3\mathfrak{l}+2\mathfrak{f}}{8} \leq \frac{15}{32}. \\ 0 &\leq \frac{3k+\mathfrak{l}}{4}, \frac{\mathfrak{l}+k+2\mathfrak{f}}{4}, \frac{\mathfrak{f}+k}{2}, \frac{\mathfrak{l}+k}{2}, \mathfrak{f} \leq \frac{3}{4}. \\ 0 &\leq \mathfrak{l}+k+\mathfrak{f} \leq \frac{9}{4}. \end{aligned}$$

From above inequalities, it is clear that maximum value is $\mathfrak{l}+k+\mathfrak{f}$. *i.e.,* $M(\mathfrak{l},k,\mathfrak{f}) = \mathfrak{l}+k+\mathfrak{f}$.

$$\check{\psi}(\check{M}(\mathfrak{l},k,\mathfrak{f})) = \frac{5}{7}(\mathfrak{l}+k+\mathfrak{f}). \tag{24}$$

Hence from (23) *and* (24), $\beta(\mathfrak{l},k,\mathfrak{f})G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}k,\mathfrak{R}\mathfrak{f}) \leq \check{\psi}(\check{M}(\mathfrak{l},k,\mathfrak{f}))$. *So the contraction holds for* $\overline{B_{G_d}(\mathfrak{l}_\circ,\check{r})} = [0,\frac{3}{4}]$. *Also*

$$\begin{aligned} & \sum_{a=0}^{\mathfrak{p}} \check{\psi}^a(G_d(\mathfrak{l}_\circ,\mathfrak{R}\mathfrak{l}_\circ,\mathfrak{R}\mathfrak{l}_\circ)) \\ = & \sum_{a=0}^{\mathfrak{p}} \check{\psi}^a(G_d(\frac{1}{4},\mathfrak{R}\frac{1}{4},\mathfrak{R}\frac{1}{4})) \\ = & \frac{1}{2}\sum_{a=0}^{b}(\frac{5}{7})^a = \frac{7}{4} = \check{r}. \end{aligned}$$

Hence all the constraints of main result holds. We have $\{\mathfrak{l}_\mathfrak{p}\}$ *in* $\overline{B_{G_d}(\mathfrak{l}_\circ,\check{r})}$, $\beta(\mathfrak{l}_\mathfrak{p},\mathfrak{l}_{\mathfrak{p}+1},\mathfrak{l}_{\mathfrak{p}+1}) \geq 1$ *and* $\{\mathfrak{l}_\mathfrak{p}\} \to 0 \in \overline{B_{G_d}(\mathfrak{l}_\circ,\check{r})}$. *Also* $\beta(\mathfrak{l}_\mathfrak{p},\mathfrak{l}_\mathfrak{p},0) \geq 1$ *for all* $\mathfrak{p} \in \mathbb{N} \cup \{0\}$. *Moreover,* $\mathfrak{R}(0) = 0$.

Corollary 1. *Let* $(\check{Z},G_d)$ *be complete dislocated* G_d *metric space,* $\mathfrak{R}:\check{Z} \longrightarrow \check{Z}$ *be a* β*-admissible mapping and* $\check{\psi} \in \check{\Psi}$. *Assume that the following assertions hold:*

$$\begin{aligned} & \beta(\mathfrak{l},k,\mathfrak{f})G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}k,\mathfrak{R}\mathfrak{f}) \\ \leq & \check{\psi}(\check{M}(\mathfrak{l},k,\mathfrak{f})), \end{aligned}$$

where

$$\check{M}(\mathfrak{l},k,\mathfrak{f}) = \max\left\{\begin{array}{c} G_d(k,\mathfrak{R}^2\mathfrak{l},\mathfrak{R}k), \frac{G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}^2\mathfrak{l},\mathfrak{R}k)}{2}, \\ \frac{G_d(\mathfrak{l},\mathfrak{R}\mathfrak{l},k)}{2}, \frac{G_d(\mathfrak{l},\mathfrak{R}\mathfrak{l},\mathfrak{f})}{2}, \\ G_d(\mathfrak{f},\mathfrak{R}^2\mathfrak{l},\mathfrak{R}\mathfrak{f}), \frac{G_d(k,\mathfrak{R}\mathfrak{l},\mathfrak{R}k)}{2} \\ \frac{G_d(\mathfrak{f},\mathfrak{R}\mathfrak{l},\mathfrak{R}k)}{2}, \frac{G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}^2\mathfrak{l},\mathfrak{R}\mathfrak{f})}{2}, G_d(\mathfrak{l},k,\mathfrak{f}), \\ G_d(\mathfrak{l},\mathfrak{R}\mathfrak{l},\mathfrak{R}\mathfrak{l}), G_d(k,\mathfrak{R}k,\mathfrak{R}k), \frac{G_d(\mathfrak{f},\mathfrak{R}\mathfrak{f},\mathfrak{R}\mathfrak{f})}{2}, \\ \frac{G_d(k,\mathfrak{R}\mathfrak{f},\mathfrak{R}\mathfrak{f})}{2}, \frac{G_d(\mathfrak{l},\mathfrak{R}k,\mathfrak{R}k)}{2} \end{array}\right\}.$$

(i) *There exists* $\mathfrak{l}_\circ \in \check{Z}$ *such that* $\beta(\mathfrak{l}_\circ,\mathfrak{R}\mathfrak{l}_\circ,\mathfrak{R}\mathfrak{l}_\circ) \geq 1$;

(ii) *If there exists* $\{\mathfrak{l}_\mathfrak{p}\}$ *in* $\check{Z}$ *such that for all* $\mathfrak{p} \in \mathbb{N} \cup \{0\}$, $\beta(\mathfrak{l}_\mathfrak{p},\mathfrak{l}_{\mathfrak{p}+1},\mathfrak{l}_{\mathfrak{p}+1}) \geq 1$ *and* $\mathfrak{l}_\mathfrak{p} \to u \in \check{Z}$, *then* $\beta(\mathfrak{l}_\mathfrak{p},\mathfrak{l}_\mathfrak{p},u) \geq 1$.

Then there exists a unique $e \in \check{Z}$ *such that* $e = \mathfrak{R}e$.

Corollary 2. *Let $(\check{Z}, G_d)$ be complete dislocated G_d metric space, $\mathfrak{R} : \check{Z} \longrightarrow \check{Z}$ be a mapping and $\check{\psi} \in \check{\Psi}$. Assume that the following assertions hold:*

$$G_d(\mathfrak{R}\mathfrak{l}, \mathfrak{R}k, \mathfrak{R}\mathfrak{f}) \leq \check{\psi}(\check{M}(\mathfrak{l}, k, \mathfrak{f})),$$

where

$$\check{M}(\mathfrak{l}, k, \mathfrak{f}) = \max \left\{ \begin{array}{c} G_d(k, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}k), \frac{G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}^2\mathfrak{l},\mathfrak{R}k)}{2}, \frac{G_d(\mathfrak{l},\mathfrak{R}\mathfrak{l},k)}{2}, \\ \frac{G_d(\mathfrak{l},\mathfrak{R}\mathfrak{l},\mathfrak{f})}{2}, G_d(\mathfrak{f}, \mathfrak{R}^2\mathfrak{l}, \mathfrak{R}\mathfrak{f}), \frac{G_d(k,\mathfrak{R}\mathfrak{l},\mathfrak{R}k)}{2}, \\ \frac{G_d(\mathfrak{f},\mathfrak{R}\mathfrak{l},\mathfrak{R}k)}{2}, \frac{G_d(\mathfrak{R}\mathfrak{l},\mathfrak{R}^2\mathfrak{l},\mathfrak{R}\mathfrak{f})}{2}, G_d(\mathfrak{l}, k, \mathfrak{f}), \\ G_d(\mathfrak{l}, \mathfrak{R}\mathfrak{l}, \mathfrak{R}\mathfrak{l}), G_d(k, \mathfrak{R}k, \mathfrak{R}k), \frac{G_d(\mathfrak{f},\mathfrak{R}\mathfrak{f},\mathfrak{R}\mathfrak{f})}{2}, \\ \frac{G_d(k,\mathfrak{R}\mathfrak{f},\mathfrak{R}\mathfrak{f})}{2}, \frac{G_d(\mathfrak{l},\mathfrak{R}k,\mathfrak{R}k)}{2} \end{array} \right\}.$$

Then there exists a unique $e \in \check{Z}$ such that $e = \mathfrak{R}e$.

Corollary 3. *Let $(\check{Z}, G_d)$ be complete dislocated G_d metric space, $\mathfrak{R} : \check{Z} \longrightarrow \check{Z}$ be a β-admissible mapping and $\check{\psi} \in \check{\Psi}$. Assume that the following assertions hold:*

$$\beta(\mathfrak{l}, k, \mathfrak{f})G_d(\mathfrak{R}\mathfrak{l}, \mathfrak{R}k, \mathfrak{R}\mathfrak{f}) \leq \check{\psi}(G_d(\mathfrak{l}, k, \mathfrak{f})),$$

(i) there exists $\mathfrak{l}_\circ \in \check{Z}$ such that $\beta(\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ, \mathfrak{R}\mathfrak{l}_\circ) \geq 1$;

(ii) If there exists $\{\mathfrak{l}_\mathfrak{p}\}$ in $\check{Z}$ such that for all $\mathfrak{p} \in \mathbb{N} \cup \{0\}$, $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_{\mathfrak{p}+1}, \mathfrak{l}_{\mathfrak{p}+1}) \geq 1$ and $\mathfrak{l}_\mathfrak{p} \to u \in \check{Z}$, then $\beta(\mathfrak{l}_\mathfrak{p}, \mathfrak{l}_\mathfrak{p}, u) \geq 1$.

Then there exists a unique $e \in \check{Z}$ such that $e = \mathfrak{R}e$.

Remark 1. *By taking non-empty proper subsets of $\check{M}(\mathfrak{l}, k, \mathfrak{f})$ instead of $\check{M}(\mathfrak{l}, k, \mathfrak{f})$ in Theorem 1, we can obtain different new results.*

Remark 2. *Different new results in ordered complete dislocated G-metric space can be obtained by expressing contraction endowed with an order.*

3. Application

In this section, we investigate the solution of integral equation:

$$\mathfrak{l}(t) = \int_a^b H(t, s)K(s, \mathfrak{l}(s))ds; \quad t \in [a, b]. \tag{25}$$

Let $\check{Z} = (C[a, b], R)$ represents the family of all continuous functions from $[a, b]$ to R.
Define $\mathfrak{R} : \check{Z} \to \check{Z}$ by

$$\mathfrak{R}\mathfrak{l}(t) = \int_a^b H(t, s)K(s, \mathfrak{l}(s))ds; \quad t \in [a, b]. \tag{26}$$

Theorem 2. *Consider Equation (25) and assume that:*

1. $H : [a, b] \times [a, b] \to [0, \infty)$ *is a continuous mapping,*
2. $K : [a, b] \times R \to R$ *where K is continuous function,*
3. $\max_{t\in[a,b]} \int_a^b H(t, s)ds < \lambda$, *for some $\lambda \in (0, 1)$.*
4. *For all $\mathfrak{l}(s), k(s) \in \check{Z}$; $s \in [a, b]$ we have*

$$|K(s, \mathfrak{l}(s)) - K(s, k(s))| \leq |\mathfrak{l}(s) - k(s)|.$$

Then Equation (25) has a solution.

Proof. Let $\check{Z}$ and $\Re$ be as defined above. For all $\mathfrak{l}, k, \mathfrak{f} \in \check{Z}$ define the dislocated G_d metric space on $\check{Z}$ by

$$G(\mathfrak{l}, k, \mathfrak{f}) = d(\mathfrak{l}, k) + d(k, \mathfrak{f}) + d(\mathfrak{l}, \mathfrak{f}) \tag{27}$$

where

$$d(\mathfrak{l}, k) = \sup_{t \in [a,b]} |\mathfrak{l}(t) - k(t)|. \tag{28}$$

Evidently that $(\check{Z}, G_d)$ is a complete dislocated G_d metric space, since $(\check{Z}, d)$ is complete dislocated metric space.

Now, Let $\mathfrak{l}(t), k(t) \in \check{Z}$, then by (26), (27) and (28), we have

$$\begin{aligned}
|\Re\mathfrak{l}(t) - \Re k(t)| &= \Big| \int_a^b H(t,s)\big[K(s,\mathfrak{l}(s)) - K(s,k(s))\big] ds \Big| \\
&\leq \int_a^b H(t,s)|K(s,\mathfrak{l}(s)) - K(s,k(s)|ds \\
&\leq \int_a^b H(t,s)|\mathfrak{l}(s) - k(s)|ds \\
&\leq \int_a^b H(t,s) \sup_{s \in [a,b]} |\mathfrak{l}(s) - k(s)|ds \\
&= \sup_{t \in [a,b]} |\mathfrak{l}(t) - k(t)| \int_a^b H(t,s)ds \\
&\leq \lambda \sup_{t \in [a,b]} |\mathfrak{l}(t) - k(t)|.
\end{aligned}$$

Hence,

$$\sup_{t \in [a,b]} |\Re\mathfrak{l}(t) - \Re k(t)| \leq \lambda \sup_{t \in [a,b]} |\mathfrak{l}(t) - k(t)|. \tag{29}$$

Similarly, we have

$$\sup_{t \in [a,b]} |\Re k(t) - \Re\mathfrak{f}(t)| \leq \lambda \sup_{t \in [a,b]} |k(t) - \mathfrak{f}(t)| \tag{30}$$

and

$$\sup_{t \in [a,b]} |\Re\mathfrak{l}(t) - \Re\mathfrak{f}(t)| \leq \lambda \sup_{t \in [a,b]} |\mathfrak{l}(t) - \mathfrak{f}(t)|. \tag{31}$$

Therefore, from (29), (30) and (31), we have

$$\begin{aligned}
&\sup_{t \in [a,b]} |\Re\mathfrak{l}(t) - \Re k(t)| + \\
&\sup_{t \in [a,b]} |\Re k(t) - \Re\mathfrak{f}(t)| + \\
&\sup_{t \in [a,b]} |\Re\mathfrak{l}(t) - \Re\mathfrak{f}(t)| \\
&\quad \leq \lambda \Big[\sup_{t \in [a,b]} |\mathfrak{l}(t) - k(t)| + \\
&\sup_{t \in [a,b]} |k(t) - \mathfrak{f}(t)| + \\
&\sup_{t \in [a,b]} |\mathfrak{l}(t) - \mathfrak{f}(t)| \Big]
\end{aligned}$$

which implies

$$G(\Re\mathfrak{l}, \Re k, \Re\mathfrak{f}) \leq \lambda G(\mathfrak{l}, k, \mathfrak{f}).$$

Taking $\check{\psi} : [0, \infty) \to [0, \infty)$ by $\check{\psi}(t) = \lambda t$ for all $t > 0$ and $\beta : \check{Z} \times \check{Z} \times \check{Z} \to [0, \infty)$ by

$$\beta(\mathfrak{l}, k, \mathfrak{f}) = \begin{cases} 1, & \text{if } \mathfrak{l} \neq k \neq \mathfrak{f} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have

$$\beta(\mathfrak{l}, k, \mathfrak{f}) G(\mathfrak{R}\mathfrak{l}, \mathfrak{R}k, \mathfrak{R}\mathfrak{f}) \leq \check{\psi}(G(\mathfrak{l}, k, \mathfrak{f}))$$

Thus, all the assumptions of Corollary 3 are satisfied and the $\mathfrak{R}$ has fixed point in $\check{Z}$ as a solution of (25). □

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