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A Duality Relationship Between Fuzzy Partial Metrics and Fuzzy Quasi-Metrics

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Abstract: In 1994, Matthews introduced the notion of partial metric and established a duality relationship between partial metrics and quasi-metrics defined on a set $\mathcal{X}$. In this paper, we adapt such a relationship to the fuzzy context, in the sense of George and Veeramani, by establishing a duality relationship between fuzzy quasi-metrics and fuzzy partial metrics on a set $\mathcal{X}$, defined using the residuum operator of a continuous t-norm $*$. Concretely, we provide a method to construct a fuzzy quasi-metric from a fuzzy partial one. Subsequently, we introduce the notion of fuzzy weighted quasi-metric and obtain a way to construct a fuzzy partial metric from a fuzzy weighted quasi-metric. Such constructions are restricted to the case in which the continuous t-norm $*$ is Archimedean and we show that such a restriction cannot be deleted. Moreover, in both cases, the topology is preserved, i.e., the topology of the fuzzy quasi-metric obtained coincides with the topology of the fuzzy partial metric from which it is constructed and vice versa. Besides, different examples to illustrate the exposed theory are provided, which, in addition, show the consistence of our constructions comparing it with the classical duality relationship.

Keywords: fuzzy quasi-metric; fuzzy partial metric; additive generator; residuum operator; Archimedean t-norm.

MSC: 54A40; 54D35; 54E50

1. Introduction

The concept of metric space has been extensively studied in the literature, among other reasons, due to its usefulness in many fields of Science as Physics, Biology, Computer Science, ... Indeed, it is an essential tool to quantify the proximity between objects in a real problem. Nevertheless, sometimes the nature of the problem under consideration requires a way of quantify such a proximity for which the concept of metric is too restrictive. This fact has motivated the introduction of different generalizations of the concept of metric by means of deleting or relaxing some of axioms which define it. Among others, we can find the quasi-metrics, in which the symmetry is not demanded, or the partial metrics, for those the self-distance is not necessary zero. These last ones were introduced by Matthews in [1] where, in addition, he showed a duality relationship between them and a subclass of quasi-metrics, the so-called weighted quasi-metrics.

Coming back to the restrictiveness of the notion of metric space to be used in many real problems, sometimes the considered problem involves some uncertainty, which makes it more appropriate to provide a way of measuring the proximity between objects framed in the fuzzy setting. In this direction, George and Veeramani introduced, in [2], a notion of fuzzy metric by slightly modifying a previous one given by Kramosil and Michalek in [3]. This concept has been extensively studied by different authors, both from the theoretical point of view (see, for instance, Ref. [4–12] and references therein) and by its applicability to engineering problems (see, for instance, Ref. [13–17]). Different fuzzy concepts, based on the notion of fuzzy metric due to George and Veeramani, have appeared in the literature (see, for instance, Ref. [9,18–20]). In this direction, here we adopt the concept of fuzzy quasi-metric (see Definition 5) appeared in [18], according to a modern concept of quasi-metric (see [21]). Additionally, we adopt the concept of fuzzy partial metric (see Definition 6), defined by means of the residuum operator of a continuous t-norm, appeared in [9], which, also, is according to the notion of partial metric.

The aim of this paper is to retrieve to the fuzzy setting the duality relationship between quasi-metrics and partial metrics defined on a non-empty set $\mathcal{X}$ that was established by Matthews in the classical case. To this end, we introduce a subclass of fuzzy quasi-metrics, the so-called fuzzy weighted quasi-metrics (see Definition 7). Subsequently, we provide a way to construct a fuzzy quasi-metric Q_P on $\mathcal{X}$, from a given fuzzy partial metric space $(\mathcal{X}, P, *)$ (see Theorem 2). Furthermore, as in the classical case, we show that $\mathcal{T}_{Q_P} = \mathcal{T}_P$ (see Proposition 1), and also that Q_P is weightable (see Theorem 4). On the other hand, to obtain the converse, we construct a fuzzy partial metric P_Q on $\mathcal{X}$, from a given fuzzy weighted quasi-metric space $(\mathcal{X}, Q, *, W)$ (see Theorem 3). Besides, we show that $\mathcal{T}_{P_Q} = \mathcal{T}_Q$ (see Proposition 4). In both cases, we demand on the t-norm $*$ to be Archimedean. The consistency of our constructions is detailed in Remarks 2 and 3. Several examples are provided for illustrating the theory. It is worth to mentioning that a part of the content of the paper is included in the PhD dissertation of the third author [22].

The reminder of the paper is organized as follows. Section 2 compiles the basics used throughout the paper. Subsequently, Section 3 is devoted to obtain a fuzzy quasi-metric deduced from a fuzzy partial one in such a way that the topology is preserved and, in Section 4 is approached the conversely case. In Section 5 a brief discussion is provided. Finally, Section 6 exposes the conclusions of the present work and some lines of research to continue it.

2. Preliminaries

We begin recalling the notion of quasi-metric space that we manage throughout this paper (see [18,21]).

Definition 1. *A quasi-metric space is a pair $(\mathcal{X}, q)$ where $\mathcal{X}$ is a non-empty set, and $q : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is a mapping such that, for each $\xi, \eta, \theta \in \mathcal{X}$, the following conditions are satisfied:*

(Q1) $q(\xi, \eta) = q(\eta, \xi) = 0$ *if and only if* $\xi = \eta$ *for every* $\xi, \eta \in \mathcal{X}$.
(Q2) $q(\xi, \theta) \leq q(\xi, \eta) + q(\eta, \theta)$.

As usual, we also say that q is a quasi-metric on $\mathcal{X}$.

In a similar way that a metric, given a quasi-metric space $(\mathcal{X}, q)$, then q induces a T_0 topology $\mathcal{T}(q)$ on $\mathcal{X}$, which has as a base the family of open balls $\{B_q(\xi; \epsilon) : \xi \in \mathcal{X}, \epsilon > 0\}$, where $B_q(\xi; \epsilon) = \{\eta \in \mathcal{X} : q(\xi, \eta) < \epsilon\}$, for each $\xi \in \mathcal{X}, \epsilon > 0$.

We continue recalling the notion of partial metric space introduced by Matthews in [1].

Definition 2. *A partial metric space is a pair $(\mathcal{X}, p)$ where $\mathcal{X}$ is a non-empty set, and $p : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is a mapping such that, for each $\xi, \eta, \theta \in \mathcal{X}$, the following conditions are satisfied:*

(P1) $p(\xi, \xi) = p(\xi, \eta) = p(\eta, \eta)$ *if and only if* $\xi = \eta$.
(P2) $p(\xi, \xi) \leq p(\xi, \eta)$.
(P3) $p(\xi, \eta) = p(\eta, \xi)$.
(P4) $p(\xi, \theta) \leq p(\xi, \eta) + p(\eta, \theta) - p(\eta, \eta)$.

Again, we also say that p is a partial metric on $\mathcal{X}$.

Besides, Matthews showed in [1] that a partial metric p on a non-empty set $\mathcal{X}$ induces a T_0 topology $\mathcal{T}(p)$ on $\mathcal{X}$ which has as a base the family of open balls $\{B_p(\xi;\epsilon) : \xi \in \mathcal{X}, \epsilon > 0\}$, where $B_p(\xi;\epsilon) = \{\eta \in \mathcal{X} : p(\xi,\eta) - p(\xi,\xi) < \epsilon\}$, for each $\xi \in \mathcal{X}, \epsilon > 0$.

In addition, Matthews showed a duality relationship between partial metrics and quasi-metrics. Such a relationship is given by the fact that, from each partial metric p on a non-empty set $\mathcal{X}$ we can construct a quasi-metric q_p on $\mathcal{X}$ defining $q_p(\xi,\eta) = p(\xi,\eta) - p(\xi,\xi)$, for each $\xi,\eta \in \mathcal{X}$. In order to obtain a similar construction in the converse case, Matthews introduced, in [1], the following notion of weighted quasi-metric space.

Definition 3. *A weighted quasi-metric space is a tern $(\mathcal{X},q,w)$, where q is a quasi-metric on $\mathcal{X}$ and w is a function defined on $\mathcal{X}$ satisfying, for each $\xi,\eta \in \mathcal{X}$, the following conditions:*

(w1) $w(\xi) \geq 0$;
(w2) $q(\xi,\eta) + w(\xi) = q(\eta,\xi) + w(\eta)$.

Subsequently, Matthews established a way to construct a partial metric from a given weighted quasi-metric space $(\mathcal{X},q,w)$, defining a partial metric p_q on $\mathcal{X}$ given by $p_q(\xi,\eta) = q(\xi,\eta) + w(\xi)$, for each $\xi,\eta \in \mathcal{X}$.

Moreover, Matthews showed that both constructions preserve the topology. Indeed, given a partial metric space $(\mathcal{X},p)$ then, $\mathcal{T}(q_p) = \mathcal{T}(p)$. Conversely, given a weighted quasi-metric space $(\mathcal{X},q,w)$ then, $\mathcal{T}(p_q) = \mathcal{T}(q)$.

Now, we recall the notion of fuzzy metric space given by George and Veeramani in [2].

Definition 4. *A fuzzy metric space is an ordered triple $(\mathcal{X},M,*)$ such that $\mathcal{X}$ is a (non-empty) set, $*$ is a continuous t-norm (see [23]) and M is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0,\infty)$ satisfying, for all $\xi,\eta,\theta \in \mathcal{X}$ and $t,s > 0$, the following conditions:*

(GV1) $M(\xi,\eta,t) > 0$;
(GV2) $M(\xi,\eta,t) = 1$ *if and only if* $\xi = \eta$;
(GV3) $M(\xi,\eta,t) = M(\eta,\xi,t)$;
(GV4) $M(\xi,\eta,t) * M(\eta,\theta,s) \leq M(\xi,\theta,t+s)$;
(GV5) *The assignment $M_{\xi,\eta} : (0,+\infty) \to (0,1]$, given by $M_{\xi,\eta}(t) = M(\xi,\eta,t)$ for each $t > 0$, is a continuous function.*

As usual, we will say that $(M,*)$, or simply M, if confusion does not arise, is a fuzzy metric on $\mathcal{X}$.

George an Veeramani showed in [2] that every fuzzy metric M on $\mathcal{X}$ defines a topology $\mathcal{T}_M$ on X, which has as a base the family of open balls $\{B_M(\xi,\epsilon,t) : \xi \in \mathcal{X}, 0 < \epsilon < 1, t > 0\}$, where $B_M(\xi,\epsilon,t) = \{\eta \in \mathcal{X} : Q(\xi,\eta,t) > 1 - \epsilon\}$ for all $\xi \in \mathcal{X}, 0 < \epsilon < 1$ and $t > 0$.

In the next, we recall two significant examples of fuzzy metrics given in [2].

Example 1. *Let $(\mathcal{X},d)$ be a metric space and let M_d the function on $\mathcal{X} \times \mathcal{X} \times (0,\infty)$ defined by*

$$M_d(\xi,\eta,t) = \frac{t}{t + d(\xi,\eta)} \tag{1}$$

Then, $(\mathcal{X},M_d,*_M)$ is a fuzzy metric space, where $*_M$ denotes the minimum t-norm (i.e., $a *_M b = \min\{a,b\}$, for each $a,b \in [0,1]$). M_d is called the standard fuzzy metric *induced by d. The topology $\mathcal{T}_{M_d}$ coincides with the topology $\mathcal{T}(d)$ on $\mathcal{X}$ deduced from d.*

Example 2. *Let $(\mathcal{X},d)$ be a metric space and let M_d the function on $\mathcal{X} \times \mathcal{X} \times (0,\infty)$ defined by*

$$M_e(\xi,\eta,t) = e^{-\frac{d(\xi,\eta)}{t}} \tag{2}$$

Afterwards, $(\mathcal{X}, M_e, *_M)$ is a fuzzy metric space and M_e will be called the **exponential fuzzy metric** *induced by d. Again, the topology $\mathcal{T}_{M_e}$ coincides with the topology $\mathcal{T}(d)$ on $\mathcal{X}$ deduced from d.*

Gregori and Romaguera in [18] introduced two concepts of fuzzy quasi-metric. Here, we deal with the following concept which is according to Definition 1.

Definition 5. *A fuzzy quasi-metric space is a tern $(\mathcal{X}, Q, *)$, such that $\mathcal{X}$ is a non-empty set, $*$ is a continuous t-norm and Q is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0, +\infty)$ satisfying, for all $\xi, \eta, \theta \in \mathcal{X}$ and $t, s > 0$, the following conditions:*

(QGV1) $Q(\xi, \eta, t) > 0$;
(QGV2) $Q(\xi, \eta, t) = Q(\eta, \xi, t) = 1$ *if and only if* $\xi = \eta$;
(QGV3) $Q(\xi, \theta, t + s) \geq Q(\xi, \eta, t) * Q(\eta, \theta, s)$;
(QGV4) *The assignment $Q_{\xi,\eta} : (0, +\infty) \to (0, 1]$, given by $Q_{\xi,\eta}(t) = Q(\xi, \eta, t)$ for each $t > 0$, is a continuous function.*

In such a case, $(Q, *)$, or simply Q, is called a fuzzy quasi-metric on $\mathcal{X}$.

Gregori and Romaguera proved in [18] that every fuzzy quasi-metric Q on $\mathcal{X}$ generates a T_0 topology $\mathcal{T}_Q$ on $\mathcal{X}$ that has as a base the family of open sets of the form $\{B_Q(\xi, \epsilon, t) : \xi \in \mathcal{X}, 0 < \epsilon < 1, t > 0\}$, where $B_Q(\xi, \epsilon, t) = \{\eta \in \mathcal{X} : Q(\xi, \eta, t) > 1 - \epsilon\}$ for all $\xi \in \mathcal{X}$, $0 < \epsilon < 1$ and $t > 0$.

Now, we recall the concept of fuzzy partial metric space introduced by Gregori et al. in [9].

Definition 6. *A* **fuzzy partial metric space** *is an ordered triple $(\mathcal{X}, P, *)$, such that $\mathcal{X}$ is a (non-empty) set, $*$ is a continuous t-norm and P is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ satisfying, for all $\xi, \eta, \theta \in \mathcal{X}$ and $t, s > 0$, the following conditions:*

(FPGV1) $0 < P(\xi, \eta, t) \leq P(\xi, \xi, t)$;
(FPGV2) $P(\xi, \xi, t) = P(\eta, \eta, t) = P(\xi, \eta, t)$ *if and only if* $\xi = \eta$;
(FPGV3) $P(\xi, \eta, t) = P(\eta, \xi, t)$;
(FPGV4) $P(\xi, \xi, t + s) \to_* P(\xi, \theta, t + s) \geq (P(\xi, \xi, t) \to_* P(\xi, \eta, t)) * (P(\eta, \eta, s) \to_* P(\eta, \theta, s))$;
(FPGV5) *The assignment $P_{\xi,\eta} : (0, \infty) \to (0, 1]$, given by $P_{\xi,\eta}(t) = P(\xi, \eta, t)$ for each $t > 0$, is a continuous function.*

Similarly to the previous cases, Gregori et al. proved in [9] that that every fuzzy partial metric P on $\mathcal{X}$ generates a T_0 topology $\mathcal{T}_P$ on $\mathcal{X}$ which has as a base the family of open sets of the form $\{B_Q(\xi, \epsilon, t) : \xi \in \mathcal{X}, 0 < \epsilon < 1, t > 0\}$, where $B_Q(\xi, \epsilon, t) = \{\eta \in \mathcal{X} : Q(\xi, \eta, t) > 1 - \epsilon\}$ for all $\xi \in \mathcal{X}$, $0 < \epsilon < 1$ and $t > 0$.

In the previous definition, $\to_*$ denotes the residuum operator of the continuous t-norm $*$ (see, for instance, [24] in order to find a deeper treatment on it), which can be obtained by next formula:

$$a \to_* b = \begin{cases} 1, & \text{if } a \leq b; \\ \sup\{c \in [0, 1] : a * c = b\}, & \text{if } a > b. \end{cases} \tag{3}$$

To finish this section, we recall some aspects on continuous t-norms and their residuum operator, which will be useful later.

First, recall that an additive generator $f_* : [0, 1] \to [0, \infty]$ of a t-norm $*$ is a strictly decreasing function which is right-continuous at 0, satisfying $f_*(1) = 0$, and such that for $a, b \in [0, 1]$ we have

$$f_*(a) + f_*(b) \in Ran(f_*) \cup [f_*(0), \infty], \tag{4}$$

and also

$$a * b = f_*^{(-1)}(f_*(a) + f_*(b)), \text{ for all } a, b \in [0, 1], \tag{5}$$

where $f_*^{(-1)}$ denotes the pseudo-inverse of the function f_* (see [24]).

This concept allows for characterizing a family of continuous t-norms, the so-called Archimedeans (i.e., those continuous t-norms $*$ such that $a * a < a$ for each $a \in (0,1)$) as shows the next theorem.

Theorem 1. *A binary operator $*$ in $[0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous additive generator f_* of $*$.*

Moreover, an additive generator f_* of a continuous Archimedean t-norm $*$ allows us to obtain a simpler formula of the $*$-residuum, as follows:

$$a \rightarrow_* b = \begin{cases} 1, & \text{if } a = 0; \\ f_*^{(-1)}\left(\max\{f_*(b) - f_*(a), 0\}\right), & \text{elsewhere.} \end{cases} \tag{6}$$

Note that the pseudo-inverse of a continuous additive generator f_* is given by

$$f_*^{(-1)}(a) = f_*^{-1}(\min\{f_*(0), a\}). \tag{7}$$

By formula (6), we conclude that, for each continuous Archimedean t-norm $*$, it is held $a \rightarrow_* b > 0$ for each $a, b \in [0,1]$.

Remark 1. *Attending to formula (6), it is obvious that given a continuous Archimedean t-norm $*$, its $*$-residuum is continuous on $(0,1] \times (0,1]$. Nevertheless, the such an affirmation is not true, in general. Indeed, the residuum operator of the non-Archimedean continuous t-norm $*_M$ is given by*

$$a \rightarrow_{*_M} b = \begin{cases} 1, & \text{if } a \leq b; \\ b, & \text{if } a > b. \end{cases}, \tag{8}$$

and one can easily observe that $\rightarrow_{*_M}$ is not continuous on $(0,1] \times (0,1]$.

Corollary 1. *Let $*$ be a continuous Archimedean t-norm, and let f_* be its continuous additive generator. Then, for every $\xi > 0$, we have that $f_*^{(-1)}(f_*(\xi)) = \xi$.*

3. From Fuzzy Partial Metrics to Fuzzy Quasi-Metrics

In this section, we provide a way of constructing a fuzzy quasi-metric from a fuzzy partial metric. To obtain such an aim, we are based on the classical techniques given by Matthews in [1].

We begin this section introducing two examples of fuzzy quasi-metric spaces. They generalize, in some sense, the exponential and standard fuzzy metric spaces deduced from a classic metric (see Section 2). Both examples will be useful later.

Example 3. *Let $(\mathcal{X}, q)$ be a quasi-metric space.*

(i) We define the fuzzy set Q_e on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$, as follows

$$Q_e(\xi, \eta, t) = e^{-\frac{q(\xi,\eta)}{t}}, \text{ for each } \xi, \eta, \theta \in \mathcal{X} \text{ and } t > 0. \tag{9}$$

After a tedious computation, one can prove that $(\mathcal{X}, Q_e, *_M)$ is a fuzzy quasi-metric space. It will be called the exponential fuzzy quasi-metric space deduced from q.

(ii) We define the fuzzy set Q_d on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ as

$$Q_d(\xi, \eta, t) = \frac{t}{t + q(\xi, \eta)}, \text{ for each } \xi, \eta, \theta \in \mathcal{X} \text{ and } t > 0. \tag{10}$$

Afterwards, $(\mathcal{X}, Q_d, *_P)$ is a fuzzy quasi-metric space (see [18]), where $*_P$ denotes the usual product t-norm (i.e., $a *_P b = a \cdot b$ for each $a, b \in [0,1]$). It is left to the reader to show that $(\mathcal{X}, Q_d, *_M)$ is also a fuzzy quasi-metric space.

Observe that both $(\mathcal{X}, Q_e, *)$ and $(\mathcal{X}, Q_p, *)$ are also fuzzy quasi-metric spaces for each continuous t-norm $*$, since $*_M \geq *$ for each t-norm $*$.

Now, we show the next theorem.

Theorem 2. *Let $(\mathcal{X}, P, *)$ be a fuzzy partial metric space, where $*$ is a continuous Archimedean t-norm. Then, $(\mathcal{X}, Q_P, *)$ is a fuzzy quasi-metric space, where Q_P is the fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ given by:*

$$Q_P(\xi, \eta, t) = P(\xi, \xi, t) \to_* P(\xi, \eta, t), \tag{11}$$

for each $\xi, \eta \in \mathcal{X}, t > 0$.

Proof. We will see that Q_P fulfills Definition 5.

- **(QGV1)** As $P(\xi, \eta, t) > 0$, then $P(\xi, \xi, t) \to_* P(\xi, \eta, t) = \sup\{\theta \in [0,1] : P(\xi, \xi, t) * \theta = P(\xi, \eta, t)\} > 0$. Hence, $Q_P(\xi, \eta, 0) > 0$.
- **(QGV2)** $\xi = \eta$ implies that $P(\xi, \xi, t) = P(\xi, \eta, t)$ and $P(\eta, \eta, t) = P(\eta, \xi, t)$ for each $t > 0$. Hence, $Q_P(\xi, \eta, t) = P(\xi, \xi, t) \to_* P(\xi, \eta, t) = 1$ and $Q_P(\eta, \xi, t) = P(\eta, \eta, t) \to_* P(\eta, \xi, t) = 1$. Therefore, $Q_P(\xi, \eta, t) = 1$ and $Q_P(\eta, \xi, t) = 1$. On the other hand, if $Q_P(\xi, \eta, t) = Q_P(\eta, \xi, t) = 1$ for some $t > 0$, we have that $P(\xi, \xi, t) \to_* P(\xi, \eta, t) = P(\eta, \eta, t) \to_* P(\eta, \xi, t) = 1$. Hence, as $P(\xi, \xi, t) \geq P(\xi, \eta, t)$ and $P(\eta, \eta, t) \geq P(\eta, \xi, t)$, we have that $P(\xi, \xi, t) = P(\xi, \eta, t) = P(\eta, \xi, t) = P(\eta, \eta, t)$ for some $t > 0$, and so $\xi = \eta$.
- **(QGV3)** It is straightforward due to axiom **(PGV4)**.
- **(QGV4)** By axiom **(FPGV5)** we have that $P_{\eta,\eta}$ and $P_{\eta,\xi}$ are continuous functions on $(0, \infty)$. Furthermore, since $P(\xi, \xi, t), P(\xi, \eta, t \in (0,1]$, on account of Remark 1 we conclude that $(Q_P)_{\xi,\eta}(t) = Q(\xi, \eta, t) = P(\xi, \xi, t) \to_* P(\xi, \eta, t)$ is a continuous function due to it is a composition of continuous functions.

Hence, $(\mathcal{X}, Q_P, *)$ is a fuzzy quasi-metric space. □

We cannot remove the condition of being $*$ Archimedean in the previous theorem, as the next example shows.

Example 4. *Let $\mathcal{X} = (0,1]$. We define the fuzzy set P on $X \times X \times (0, \infty)$ as*

$$P(\xi, \eta, t) = \begin{cases} \min\{\xi, \eta\} \cdot \frac{t^2}{t+1}, & \text{if } \xi \neq \eta, t \in (0,1]; \\ \min\{\xi, \eta\} \cdot \frac{t}{t+1}, & \text{elsewhere.} \end{cases} \tag{12}$$

In [9], the authors proved that $(\mathcal{X}, P, *_M)$ is a fuzzy partial metric space. Nevertheless, if we define the fuzzy set Q_P on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ by $Q_P(\xi, \eta, t) = P(\xi, \xi, t) \to_{*_M} P(\xi, \eta, t)$, for each $\xi, \eta \in \mathcal{X}$ and $t > 0$, then Q_P does not satisfy axiom (QGV4). Indeed, on account of Example 4.2 of [9], we have that

$$Q_P\left(\frac{1}{4}, \frac{1}{2}, t\right) = P\left(\frac{1}{4}, \frac{1}{4}, t\right) \to_{*_M} P\left(\frac{1}{4}, \frac{1}{2}, t\right) = \begin{cases} \frac{t^2}{4(t+1)}, & \text{if } t \in (0,1]; \\ 1, & \text{if } t \in [1, \infty). \end{cases} \tag{13}$$

Obviously, $(Q_P)_{\frac{1}{4},\frac{1}{2}}$ is not a continuous function.

We illustrate the construction presented in Theorem 2 applying it to some particular cases of fuzzy partial metric space.

Example 5. *Let $(\mathcal{X}, p)$ be a partial metric space. First, recall that, following the Matthews' construction we have that q_p is a quasi-metric on $\mathcal{X}$, where $q_p(\xi,\eta) = p(\xi,\eta) - p(\xi,\xi)$ for each $\xi,\eta \in \mathcal{X}$.*

(i) By Proposition 3.3 in [9], $(\mathcal{X}, P_e, *_P)$ is a fuzzy partial metric space, where $P_e(\xi,\eta,t) = e^{-\frac{p(\xi,\eta)}{t}}$, for each $\xi,\eta \in \mathcal{X}, t > 0$. Since $*_P$ is a continuous Archimedean t-norm then, by Theorem 2, we have that $(\mathcal{X}, Q_{P_e}, *_P)$ is a fuzzy quasi-metric space, where Q_{P_e} is given by

$$Q_{P_e}(\xi,\eta,t) = P_e(\xi,\xi,t) \rightarrow_* P_e(\xi,\eta,t), \tag{14}$$

for each $\xi,\eta \in \mathcal{X}, t > 0$. It will be called the exponential fuzzy partial metric deduced from p.

Recall that an additive generator of $*_P$ is f_{*_P} given by $f_{*_P}(a) = -\log(a)$, for $a \in [0,1]$. So, on account of formula (6) we have, for each $a,b \in [0,1]$, that

$$a \rightarrow_{*_P} b = \begin{cases} 1, & \text{if } a \leq b; \\ \frac{b}{a}, & \text{if } b > a. \end{cases} \tag{15}$$

Then, for each $\xi,\eta \in \mathcal{X}, t > 0$, we obtain

$$P_e(\xi,\xi,t) \rightarrow_* P_e(\xi,\eta,t) = \frac{e^{-\frac{p(\xi,\eta)}{t}}}{e^{-\frac{p(\xi,\xi)}{t}}} = e^{-\frac{p(\xi,\eta)-p(\xi,\xi)}{t}} = e^{-\frac{q_P(\xi,\eta)}{t}}. \tag{16}$$

Thus, $Q_{P_e}(\xi,\eta,t) = e^{-\frac{q_P(\xi,\eta)}{t}}$, for each $\xi,\eta \in \mathcal{X}, t > 0$.

(ii) By Proposition 3.4 in [9], $(\mathcal{X}, P_d, *_H)$ is a fuzzy partial metric space, where $P_d(\xi,\eta,t) = \frac{t}{t+p(\xi,\eta)}$, for each $\xi,\eta \in \mathcal{X}, t > 0$, and $*_H$ denotes the Hamacher product t-norm, which is given by the following expression:

$$a *_H b = \begin{cases} 0, & \text{if } a = b = 0; \\ \frac{ab}{a+b-ab}, & \text{elsewhere}, \end{cases} \tag{17}$$

for each $a,b \in [0,1]$. It will be called the standard fuzzy partial metric deduced from p.

In [24], it was pointed out that the function $f_{*_H}(a) = \frac{1-a}{a}$ is an additive generator of $*_H$ and so, on account of formula (7), the function $f_H^{(-1)}(\eta) = \frac{1}{1+\eta}$ is its pseudo-inverse. Attending to these observations and taking into account formula (6), the expression of the $*_H$-residuum is given by

$$a \rightarrow_{*_H} b = \begin{cases} 1, & \text{if } a \leq b; \\ \frac{ab}{ab+a-b}, & \text{if } a > b. \end{cases} \tag{18}$$

Because $*_H$ is a continuous Archimedean t-norm then, by Theorem 2, we have that $(\mathcal{X}, Q_{P_d}, *_H)$ is a fuzzy quasi-metric space, where Q_{P_d} is given by

$$Q_{P_d}(\xi,\eta,t) = P_d(\xi,\xi,t) \rightarrow_* P_d(\xi,\eta,t), \tag{19}$$

for each $\xi,\eta \in \mathcal{X}, t > 0$.

On account of formula (18) we have, for each $\xi,\eta \in \mathcal{X}, t > 0$, that

$$\begin{aligned} P_d(\xi,\xi,t) \rightarrow_* P_d(\xi,\eta,t) &= \frac{\frac{t}{t+p(\xi,\eta)} \cdot \frac{t}{t+p(\xi,\xi)}}{\frac{t}{t+p(\xi,\eta)} \cdot \frac{t}{t+p(\xi,\xi)} + \frac{t}{t+p(\xi,\xi)} - \frac{t}{t+p(\xi,\eta)}} = \\ &= \frac{1}{1 + \frac{t+p(\xi,\eta)}{t} - \frac{t+p(\xi,\xi)}{t}} = \frac{t}{t+p(\xi,\eta)-p(\xi,\xi)} = \frac{t}{t+q_P(\xi,\eta)}. \end{aligned} \tag{20}$$

Thus, $Q_{P_d}(\xi,\eta,t) = \frac{t}{t+q_p(\xi,\eta)}$, *for each* $\xi,\eta \in \mathcal{X}, t > 0$.

Example 6. *Let* $\mathcal{X} = (0,1]$ *and consider the partial metric* p_m *defined on* $\mathcal{X}$, *where* $p_m(\xi,\eta) = \max\{\xi,\eta\}$ *for each* $\xi,\eta \in \mathcal{X}$. *We define the fuzzy set* P_m *on* $\mathcal{X} \times \mathcal{X} \times (0,\infty)$ *given by*

$$P_m(\xi,\eta,t) = 1 - p_m(\xi,\eta), \text{ for each } \xi,\eta \in \mathcal{X} \text{ and } t > 0. \tag{21}$$

It is left to the reader to show that $(\mathcal{X}, P_m, *_{\mathfrak{L}})$ *is a fuzzy partial metric space, where* $*_{\mathfrak{L}}$ *denotes the Lukasievicz t-norm, which is given by* $a *_{\mathfrak{L}} b = \max\{a+b-1, 0\}$.

Recall that an additive generator of $*_{\mathfrak{L}}$ *is* $f_{*_{\mathfrak{L}}}$ *given by* $f_{*_{\mathfrak{L}}}(a) = 1 - a$ *for each* $a \in [0,1]$. *Accordingly, on account of formula (6), the residuum operator of* $*_{\mathfrak{L}}$ *is given by*

$$a \rightarrow_{*_{\mathfrak{L}}} b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{if } a > b. \end{cases} \tag{22}$$

Taking into account that $*_{\mathfrak{L}}$ *is a continuous Archimedean t-norm then, by Theorem 2 we conclude that* $(\mathcal{X}, Q_{P_m}, *_{\mathfrak{L}})$ *is a fuzzy quasi-metric space, where* Q_{P_m} *is given by*

$$Q_{P_m}(\xi,\eta,t) = P_m(\xi,\xi,t) \rightarrow_{*_{\mathfrak{L}}} P_m(\xi,\eta,t), \tag{23}$$

for each $\xi,\eta,\in \mathcal{X}$ *and* $t > 0$.

By formula (23) we have, for each $\xi,\eta,\in \mathcal{X}$ *and* $t > 0$, *that*

$$\begin{aligned} P_m(\xi,\xi,t) \rightarrow_{*_{\mathfrak{L}}} P_m(\xi,\eta,t) = 1 - (1 - p_m(\xi,\xi)) + 1 - p_m(\xi,\eta) = \\ = 1 - (p_m(\xi,\xi) - p_m(\xi,\eta)) = 1 - q_{p_m}(\xi,\eta). \end{aligned} \tag{24}$$

Thus, $Q_{P_m}(\xi,\eta,t) = 1 - q_{p_m}(\xi,\eta)$, *for each* $\xi,\eta,\in \mathcal{X}$ *and* $t > 0$.

Remark 2. *Observe, in the previous examples, that we obtain the same fuzzy quasi-metric, both if we construct the exponential (or standard) fuzzy quasi-metric deduced from* q_p *and if we construct the fuzzy quasi-metric from the exponential (or standard) fuzzy partial metric deduced from* p *using Theorem 2. This fact shows, in some sense, the consistence of the construction provided in Theorem 2 when comparing with the classical one.*

To finish this section, we will show that the topology induced by a fuzzy partial metric coincides with the topology induced by the fuzzy quasi-metric constructed from it by means of Theorem 2.

Proposition 1. *Let* $(\mathcal{X}, P, *)$ *be a fuzzy partial metric, where* $*$ *is a continuous Archimedean t-norm. Afterwards,* $\mathcal{T}_{Q_P} = \mathcal{T}_P$, *where* Q_P *is the fuzzy quasi-metric on* $\mathcal{X}$ *constructed from* P *given in Theorem 2.*

Proof. Let $(\mathcal{X}, P, *)$ be a fuzzy partial metric, where $*$ is a continuous Archimedean t-norm. Taking into account Remark 4.1 in [9], we have that, for each $\xi,\eta \in \mathcal{X}$, $0 < r < 1$ and $t > 0$, the open balls are defined, as follows:

$$B_P(\xi,r,t) = \{\eta \in \mathcal{X} : P(\xi,\xi,t) \rightarrow_* P(\xi,\eta,t) > 1 - r\}. \tag{25}$$

It ensures that $\eta \in B_P(\xi,r,t)$ if and only if $\eta \in B_{P_Q}(\xi,r,t)$. Indeed,

$$\eta \in B_P(\xi,r,t) \Leftrightarrow P(\xi,\xi,t) \rightarrow_* P(\xi,\eta,t) > 1 - r \Leftrightarrow Q_P(\xi,\eta,t) > 1 - r \Leftrightarrow \eta \in B_{P_Q}(\xi,r,t). \tag{26}$$

Hence, $\mathcal{T}_{Q_P} = \mathcal{T}_P$. □

4. From Fuzzy Quasi-Metrics to Fuzzy Partial Metrics

In this section, we tackle the conversely of the construction provided in Section 3, i.e., we establish a way to construct a fuzzy partial metric from a fuzzy quasi-metric. To achieve such a goal, we begin introducing a notion of fuzzy weighted quasi-metric adapting the classical notion of weighted quasi-metric to our fuzzy context. Besides, some axioms have been added in order to maintain the "essence" of the George and Veeramani's fuzzification.

Definition 7. *We will say that $(\mathcal{X}, Q, *, W)$ is a fuzzy weighted quasi-metric space, provided that $(\mathcal{X}, Q, *)$ is a fuzzy quasi-metric space and W is a fuzzy set on $\mathcal{X} \times (0, \infty)$, satisfying, for each $\xi, \eta \in \mathcal{X}$, $t > 0$, the following properties:*

(WGV0) $Q(\xi, \eta, t) * W(\xi, t) > 0$;
(WGV1) $Q(\xi, \eta, t) * W(\xi, t) = Q(\eta, \xi, t) * W(\eta, t)$.
(WGV2) *The assignment $W_\xi : (0, +\infty) \to (0, 1]$, given by $W_\xi(t) = W(\xi, t)$ for each $t > 0$, is a continuous function.*

In such a case, the fuzzy set W will be called the fuzzy weight function associated to the fuzzy quasi-metric space $(\mathcal{X}, Q, *)$.

Moreover, we will say that a fuzzy quasi-metric space $(\mathcal{X}, Q, *)$ is weightable if there exist a weight function $W : \mathcal{X} \times (0, \infty)$ satisfying axioms **(WGV0)–(WGV2)**.

After introducing the previous concept we provide, in the next two propositions, examples of fuzzy weighted quasi-metric spaces.

Proposition 2. *Let $(\mathcal{X}, q, w)$ be a weighted quasi-metric space. Then, $(\mathcal{X}, Q_d, *_H, W_d)$ is a fuzzy weighted quasi-metric space, where*

$$Q_d(\xi, \eta, t) = \frac{t}{t + q(\xi, \eta)}, \text{ for each } \xi, \eta \in \mathcal{X}, t > 0 \tag{27}$$

$$W_d(\xi, t) = \frac{t}{t + w(\xi)}, \text{ for each } \xi, \eta \in \mathcal{X}, t > 0, \tag{28}$$

and $*_H$ is the Hamacher product t-norm.

Proof. On account of Example 3 (ii), we deduce that $(\mathcal{X}, Q_d, *_H)$ is a fuzzy quasi-metric space. Accordingly, we just need to show that W_d satisfies, for each $\xi, \eta \in \mathcal{X}$ and $t > 0$, axiom **(WGV1)**, since, by definition of W_d, it is not hard to check that **(WGV0)** and **(WGV2)** are held.

Let $\xi, \eta \in \mathcal{X}$ and $t \in> 0$. On the one hand,

$$\begin{gathered} Q_d(\xi, \eta, t) *_H W_d(\xi, t) = \frac{t}{t + q(\xi, \eta)} *_H \frac{t}{t + w(\xi)} = \\ = \frac{\frac{t}{t+q(\xi,\eta)} \cdot \frac{t}{t+w(\xi)}}{\frac{t}{t+q(\xi,\eta)} + \frac{t}{t+w(\xi)} - \frac{t}{t+q(\xi,\eta)} \cdot \frac{t}{t+w(\xi)}} = \frac{t}{t + q(\xi, \eta) + w(\xi)}. \end{gathered} \tag{29}$$

On the other hand,

$$\begin{gathered} Q_d(\eta, \xi, t) *_H W_d(\eta, t) = \frac{t}{t + q(\eta, \xi)} *_H \frac{t}{t + w(\eta)} = \\ = \frac{\frac{t}{t+q(\eta,\xi)} \cdot \frac{t}{t+w(\eta)}}{\frac{t}{t+q(\eta,\xi)} + \frac{t}{t+w(\eta)} - \frac{t}{t+q(\eta,\xi)} \cdot \frac{t}{t+w(\eta)}} = \frac{t}{t + q(\eta, \xi) + w(\eta)}. \end{gathered} \tag{30}$$

Because $(\mathcal{X}, q, w)$ is a weighted quasi-metric space, then $q(\xi,\eta) + w(\xi) = q(\eta,\xi) + w(\eta)$ and so $Q_d(\xi,\eta,t) *_H W_d(\xi,t) = Q_d(\eta,\xi,t) *_H W_d(\eta,t)$. □

Following similar arguments to the ones used in the preceding proof, one can show the next proposition.

Proposition 3. *Let $(\mathcal{X}, q, w)$ be a weighted quasi-metric space. Subsequently, $(\mathcal{X}, Q_e, *_P, W_e)$ is a fuzzy weighted quasi-metric space, where*

$$Q_e(\xi,\eta,t) = e^{-\frac{q(\xi,\eta)}{t}}, \text{ for each } \xi,\eta \in \mathcal{X}, t > 0, \tag{31}$$

$$W_e(\xi,t) = e^{-\frac{w(\xi)}{t}}, \text{ for each } \xi \in \mathcal{X}, t > 0, \tag{32}$$

and $*_P$ is the usual product t-norm.

On account of Definition 7, one can observe that W is defined on $\mathcal{X} \times (0,\infty)$ according to the George and Veeramani's context. The following theorem states a way to obtain a fuzzy partial metric from a fuzzy weighted quasi-metric.

Theorem 3. *Let $(\mathcal{X}, Q, *, W)$ be a fuzzy weighted quasi-metric space, where $*$ is a continuous Archimedean t-norm. afterwards, $(\mathcal{X}, P_Q, *)$ is a fuzzy partial metric space, where P_Q is the fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0,\infty)$, given by:*

$$P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * W(\xi,t), \text{ for each } \xi,\eta \in \mathcal{X}, t > 0. \tag{33}$$

Proof. We will show that every axiom of Definition 6 is satisfied, for each $\xi,\eta,\theta \in \mathcal{X}$ and $t > 0$.

(PGV1) Let $\xi,\eta \in \mathcal{X}$ and $t > 0$. On the one hand, since W is a fuzzy weight function, axiom **(WGV0)** ensures that $P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * W(\xi,t) > 0$. On the other hand, $P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * W(\xi,t) \leq Q(\xi,\xi,t) * W(\xi,t) = P(\xi,\xi,t)$. Thus, $0 < P_Q(\xi,\eta,t) \leq P_Q(\xi,\xi,t)$

(PGV2) Obviously, $\xi = \eta$ implies $P_Q(\xi,\xi,t) = P_Q(\xi,\eta,t) = P_Q(\eta,\eta,t)$.

Now, suppose that $P_Q(\xi,\xi,t) = P_Q(\xi,\eta,t) = P_Q(\eta,\eta,t)$ for some $\xi,\eta \in \mathcal{X}$, $t > 0$. Afterwards, on the one hand,

$$W(\xi,t) = Q(\xi,\xi,t) * W(\xi,t) = P_Q(\xi,\xi,t) = P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * W(\xi,t). \tag{34}$$

On the other hand,

$$W(\eta,t) = Q(\eta,\eta,t) * W(\eta,t) = P_Q(\eta,\eta,t) = P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * W(\xi,t). \tag{35}$$

Besides, because W is a fuzzy weight function, axiom **(WGV1)** ensures that $Q(\xi,\eta,t) * W(\xi,t) = Q(\eta,\xi,t) * W(\eta,t)$. So, $W(\eta,t) = Q(\eta,\xi,t) * W(\eta,t)$.

Because $*$ is an Archimedean t-norm and, $Q(\xi,\eta,t) > 0$ and $Q(\eta,\xi,t) > 0$, then $Q(\xi,\eta,t) = Q(\eta,\xi,t) = 1$. Thus, axiom **(QGV2)** implies $\xi = \eta$.

(PGV3) Let $\xi,\eta \in \mathcal{X}$. Because W is a fuzzy weight function, by axiom **(WGV1)**, we have that

$$P_Q(\xi,\eta,t) = Q(\xi,\eta,t) * w(\eta,t) = Q(\eta,\xi,t) * w(\eta,t) = P_Q(\eta,\xi,t).$$

(PGV4) Let $\xi,\eta,\theta \in \mathcal{X}$ and $t,s > 0$. We will see that the following holds:

$P_Q(\xi,\xi,t+s) \rightarrow_* P_Q(\xi,\theta,t+s) \geq \left(P_Q(\xi,\xi,t) \rightarrow_* P_Q(\xi,\eta,t)\right) * \left(P_Q(\eta,\eta,s) \rightarrow_* P_Q(\eta,\theta,s)\right)$.

To show it, we claim that $P_Q(u,u,r) \rightarrow_* P_Q(u,v,r) = Q(u,v,r)$, for each $u,v \in \mathcal{X}$ and $r > 0$.

Fix $u, v \in \mathcal{X}$ and $r > 0$. First, since $*$ is a continuous Archimedean t-norm, there exists an additive generator f_* of $*$. Subsequently, using equality (6) and taking into account that $\left(f_* \circ f_*^{(-1)}\right)(a) = \left(f_*^{(-1)} \circ f_*\right)(a) = a$ for each $a \in [0, f_+(0)]$, we have that

$$\begin{aligned} P_Q(u,u,r) \rightarrow_* P_Q(u,v,r) = W(u,r) \rightarrow_* Q(u,v,r) * W(u,r) = \\ = f_*^{(-1)}(\max\{f_*(Q(u,v,r) * W(u,r)) - f_*(W(u,r)), 0\}) = \\ = f_*^{(-1)}(f_*(Q(u,v,r) * W(u,r)) - f_*(W(u,r))) = \\ = f_*^{(-1)}\left(f_*\left(f_*^{(-1)}(f_*(Q(u,v,r)) + f_*(W(u,r)))\right) - f_*(W(u,r))\right) = \\ = f_*^{(-1)}\left((f_* \circ f_*^{(-1)})(f_*(Q(u,v,r)) + f_*(W(u,r))) - f_*(W(u,r))\right) = \\ = f_*^{(-1)}(f_*(Q(u,v,r)) + f_*(W(u,r)) - f_*(W(u,r))) = \\ = f_*^{(-1)}(f_*(Q(u,v,r))) = f_*^{(-1)}(f_*(Q(u,v,r)) = Q(u,v,r). \end{aligned} \tag{36}$$

Observe that $f_*(Q(u,v,r)) + f_*(W(u,r)) < f_*(0)$ since $Q(u,v,r) * W(u,r) > 0$. Indeed, if we suppose that $f_*(Q(u,v,r)) + f_*(W(u,r)) \geq f_*(0)$ then $Q(u,v,r) * W(u,r) = f_*^{(-1)}(f_*(Q(u,v,r)) + f_*(W(u,r))) \leq f_*^{(-1)}(f_*(0)) = 0$, a contradiction.

Therefore, $P_Q(u,u,r) \rightarrow_* P_Q(u,v,r) = Q(u,v,r)$.

Afterwards,

$$\begin{aligned} P_Q(\xi,\xi,t+s) \rightarrow_* P_Q(\xi,\theta,t+s) = Q(\xi,\theta,t+s) \geq Q(\xi,\eta,t) * Q(\eta,\theta,s) = \\ = \left(P_Q(\xi,\xi,t) \rightarrow_* P_Q(\xi,\eta,t)\right) * \left(P_Q(\eta,\eta,s) \rightarrow_* P_Q(\eta,\theta,s)\right). \end{aligned} \tag{37}$$

(PGV5) The function $(P_Q)_{\xi,\eta}(t) = Q(\xi,\eta,t) * W(\xi,t)$ is continuous because of the continuity of both $Q_{\xi,\eta}(t) = Q(\xi,\eta,t)$ and $W_\xi(t) = W(\xi,t)$, and the continuity of the t-norm $*$.

Hence, $(\mathcal{X}, P_Q, *)$ is a fuzzy partial metric space. □

In the next example we will show that the assumption on the t-norm, which has to be Archimedean, cannot be removed in Theorem 3. For that purpose, we introduce the following lemma.

Lemma 1. *Let $(\mathcal{X}, M, *)$ be a fuzzy metric space, where $*$ is a continuous integral t-norm (i.e., $a * b = 0$ if and only if $\min\{a,b\} = 0$). Then, for every (fixed) $k \in [0,1]$, $(\mathcal{X}, Q, *, W_k)$ is a fuzzy weighted quasi-metric space, where $Q(\xi,\eta,t) = M(\xi,\eta,t)$ and $W_k(\xi,t) = k$.*

Proof. Let $(\mathcal{X}, M, *)$ be a fuzzy metric space, where $*$ is a continuous integral t-norm, and let $k \in [0,1]$. Obviously, every $(\mathcal{X}, M, *)$ is a fuzzy quasi-metric space. Accordingly, we need to prove that $W(\xi,t) = k$ is a fuzzy weight function.

(WGV0) Suppose that $Q(\xi,\eta,t) * W_k(\xi,t) = 0$ for some $\xi,\eta \in \mathcal{X}$ and $t > 0$. Because $*$ is integral, our assumption implies that $Q(\xi,\eta,t) = 0$ or $W_k(\xi,t) = 0$, which is a contradiction. Hence, $Q(\xi,\eta,t) * W_k(\xi,t) > 0$.

(WGV1) Let $\xi,\eta \in \mathcal{X}$ and $t > 0$. By axiom **(GV3)**, we have that $Q(\xi,\eta,t) = Q(\eta,\xi,t)$, so $Q(\xi,\eta,t) * W_k(\xi,t) = Q(\xi,\eta,t) * k = Q(\eta,\xi,t) * k = Q(\eta,\xi,t) * W_k(\eta,t)$.

(WGV2) Obviously, for each $\xi \in \mathcal{X}$ the assignment $(W_k)_\xi$ is a continuous function on $(0,\infty)$, since $(W_k)_\xi(t) = k$ for each $t > 0$.

□

Now, the previous lemma allows for us to introduce the announced (counter) example.

Example 7. *Let $(\mathcal{X}, d_u)$ be the metric space, where $\mathcal{X} = [0,1]$ and d_u is the usual metric of $\mathbb{R}$ restricted to $[0,1]$.*

Consider the stantard fuzzy metric $(\mathcal{X}, M_{d_u}, *_M)$ deduced from d_u, where $*_M$ is the minimum t-norm (see [25]) and

$$M_{d_u}(\xi,\eta,t) = \frac{t}{t + d_u(\xi,\eta)}, \text{ for each } \xi,\eta \in \mathcal{X}, t > 0. \tag{38}$$

Then, since $*_M$ is an integral t-norm $(\mathcal{X}, Q, *_M, W_{\frac{1}{2}})$ is a fuzzy weighted quasi-metric space by Lemma 1, where $Q(\xi,\eta,t) = M_{d_u}(\xi,\eta,t)$ for each $\xi,\eta \in \mathcal{X}, t > 0$. Let $\xi = 1$, $\eta = 0.9$ and $t = 10$. We have that

$$Q(1,0.9,10) = Q(0.9,1,10) = \frac{10}{10 + |1 - 0.9|} = \frac{10}{10.1} \approx 0.99. \tag{39}$$

Hence, we have that

$$Q(1,0.9,10) *_M W_{\frac{1}{2}}(1,10) = Q(0.9,1,10) *_M W_{\frac{1}{2}}(0.9,10) = \min\{0.99, 0.5\} = 0.5, \tag{40}$$

$$Q(1,1,10) *_M W_{\frac{1}{2}}(1,10) = \min\{1, 0.5\} = 0.5 \tag{41}$$

$$Q(0.9,0.9,10) *_M W_{\frac{1}{2}}(0.9,10) = \min\{1, 0.5\} = 0.5. \tag{42}$$

Therefore,

$$Q(1,0.9,10) *_M W_{\frac{1}{2}}(1,10) = Q(1,1,10) * w(1,10) = Q(0.9,0.9,10) *_M W_{\frac{1}{2}}(0.9). \tag{43}$$

*If we define $P_Q(\xi,\eta,t) = Q(\xi,\eta,t) *_M W_{\frac{1}{2}}(\xi,t)$, then P_Q does not fulfill axiom **(PGV2)**. Indeed, as it has been shown, $P_Q(1,0.9,10) = P_Q(1,1,10) = P_Q(0.9,0.9,10)$ but $1 \neq 0.9$.*

In the following example, we introduce two fuzzy partial metrics using Proposition 2 and 3 and Theorem 3.

Example 8. *Let $(\mathcal{X}, q, w)$ be a weighted quasi-metric space. Following the Matthews' construction, we have that p_q is a partial metric on $\mathcal{X}$, where $p_q(\xi,\eta) = q(\xi,\eta) + w(\xi)$ for each $\xi,\eta \in \mathcal{X}$.*

(i) By Proposition 2, $(\mathcal{X}, Q_d, *_H, W_d)$ is a fuzzy weighted quasi-metric space, where

$$Q_d(\xi,\eta,t) = \frac{t}{t + q(\xi,\eta)}, \text{ for each } \xi,\eta \in \mathcal{X}, t > 0, \tag{44}$$

$$W_d(\xi,t) = \frac{t}{t + w(\xi)}, \text{ for each } \xi,\eta \in \mathcal{X}, t > 0, \tag{45}$$

and $*_H$ is the Hamacher product t-norm. Since $*_H$ is a continuous Archimedean t-norm then, by Theorem 3, we have that $(\mathcal{X}, P_{Q_d}, *_H)$ is a fuzzy partial metric space, where P_{Q_d} is given by

$$P_{Q_d}(\xi,\eta,t) = Q_d(\xi,\eta,t) *_H W_d(\xi,t), \tag{46}$$

for each $\xi,\eta \in \mathcal{X}, t > 0$.

Then, for each $\xi,\eta \in \mathcal{X}, t > 0$, we have that

$$Q_d(\xi,\eta,t) *_H W_d(\xi,t) = \frac{\frac{t}{t+q(\xi,\eta)} \cdot \frac{t}{t+w(\xi)}}{\frac{t}{t+q(\xi,\eta)} + \frac{t}{t+w(\xi)} - \frac{t}{t+q(\xi,\eta)} \cdot \frac{t}{t+w(\xi)}} = \frac{t}{t + q(\xi,\eta) + w(\xi)} = \frac{t}{t + p_q(\xi,\eta)}. \tag{47}$$

Thus, $P_{Q_d}(\xi,\eta,t) = \frac{t}{t+p_q(\xi,\eta)}$, for each $\xi,\eta \in \mathcal{X}, t > 0$.

(ii) *By Proposition 3, $(\mathcal{X}, Q_e, *_P, W_e)$ is a fuzzy weighted quasi-metric space, where*

$$Q_e(\xi, \eta, t) = e^{-\frac{q(\xi,\eta)}{t}}, \text{ for each } \xi, \eta \in \mathcal{X}, t > 0, \tag{48}$$

$$W_e(\xi, t) = e^{-\frac{w(\xi)}{t}}, \text{ for each } \xi \in \mathcal{X}, t > 0, \tag{49}$$

and $*_P$ is the usual product t-norm. Since $*_P$ is a continuous Archimedean t-norm then, by Theorem 2, we have that $(\mathcal{X}, P_{Q_e}, *_P)$ is a fuzzy partial metric space, where P_{Q_e} is given by

$$P_{Q_e}(\xi, \eta, t) = Q_e(\xi, \eta, t) *_P W_e(\xi, t), \tag{50}$$

for each $\xi, \eta \in \mathcal{X}, t > 0$.

Then, for each $\xi, \eta \in \mathcal{X}, t > 0$, we have that

$$Q_e(\xi, \eta, t) *_P W_e(\xi, t) = e^{-\frac{q(\xi,\eta)}{t}} \cdot e^{-\frac{w(\xi)}{t}} = e^{-\frac{q(\xi,\eta)+w(\xi)}{t}} = e^{-\frac{p_q(\xi,\eta)}{t}}. \tag{51}$$

Thus, $P_{Q_e}(\xi, \eta, t) = e^{-\frac{p_q(\xi,\eta)}{t}}$, for each $\xi, \eta \in \mathcal{X}, t > 0$.

Remark 3. *Again, the previous example shows the consistence of the construction provided in Theorem 3 comparing with the classical one. Indeed, we obtain the same fuzzy partial metric, both if we construct the exponential (or standard) fuzzy partial metric deduced from p_q and if we construct the fuzzy partial metric from the exponential (or standard) fuzzy quasi-metric deduced from q while using Theorem 3.*

Moreover, the next proposition shows that the topology induced by a fuzzy weighted quasi-metric coincides with the topology induced by the fuzzy partial metric constructed from it applying Theorem 3.

Proposition 4. *Let $(\mathcal{X}, Q, *, W)$ be a fuzzy weighted quasi-metric space, where $*$ is a continuous Archimedean t-norm. Then, $\mathcal{T}_{P_Q} = \mathcal{T}_Q$, where P_Q is the fuzzy partial metric on $\mathcal{X}$ constructed from Q given in Theorem 3.*

Proof. Let $(\mathcal{X}, Q, *, W)$ be a fuzzy quasi-metric space, where $*$is a continuous Archimedean t-norm. On the one hand, for each $\xi \in \mathcal{X}, 0 < r < 1$ and $t > 0$, we have that

$$B_Q(\xi, r, t) = \{\eta \in \mathcal{X} : Q(\xi, \eta, t) > 1 - r\}. \tag{52}$$

On the other hand, by Proposition 1 and Remark 4.1 in [9] we have that

$$B_{P_Q}(\xi, r, t) = \{\eta \in \mathcal{X} : P_Q(\xi, \xi, t) \rightarrow_* P_Q(\xi, \eta, t) > 1 - r\}, \tag{53}$$

for each $\xi \in \mathcal{X}, 0 < r < 1$ and $t > 0$.

Moreover, in the demonstration of Theorem 3, $P_Q(\xi, \xi, t) \rightarrow_* P_Q(\xi, \eta, t) = Q(\xi, \eta, t)$. Thus, it is obvious that, for each $\xi \in \mathcal{X}, 0 < r < 1$ and $t > 0$, $\eta \in B_Q(\xi, r, t)$ if and only if $\eta \in B_{P_Q}(\xi, r, t)$. Hence, $T_{P_Q} = \mathcal{T}_Q$. □

To finish this section, we tackle a question related with the construction given in Theorem 2. In such a theorem, we provide a way of obtaining a fuzzy quasi-metric from a fuzzy partial one. It is based on the results given by Matthews in [1] for the classical case. Taking into account that, in the construction of Matthews, the obtained quasi-metric from a partial one turns out to be weightable, we wonder if it is so in the fuzzy context. The next theorem affirmatively answers such a question.

Theorem 4. *Let $(\mathcal{X}, P, *)$ be a fuzzy partial metric space, where $*$ is a continuous Archimedean t-norm. Then, $(\mathcal{X}, Q_P, *, W_P)$ is a fuzzy weighted quasi-metric space, where*

$$Q_P(\xi, \eta, t) = P(\xi, \xi, t) \to_* P(\xi, \eta, t) \text{ for each } \xi, \eta \in \mathcal{X}, t > 0, \tag{54}$$

and

$$W_P(\xi, t) = P(\xi, \xi, t) \text{ for each } \xi \in \mathcal{X}, t > 0. \tag{55}$$

Proof. Let $(\mathcal{X}, P, *)$ be a fuzzy partial metric space, where $*$ is a continuous Archimedean t-norm. Theorem 2 ensures that $(\mathcal{X}, Q_P, *)$ is a fuzzy quasi-metric space. So, we just need to show that W_P satisfies, for each $\xi, \eta \in \mathcal{X}$ and $t > 0$, axioms **(WGV0)–(WGV2)**.

First, observe that $*$ is a continuous Archimedean t-norm, so there exists a continuous additive generator f_* of $*$. Now, fix $\xi, \eta \in \mathcal{X}$ and $t > 0$:

(WGV0) $Q_P(\xi, \eta, t) * W_P(\xi, t) = (P(\xi, \xi, t) \to_* P(\xi, \eta, t)) * P(\xi, \xi, t)$. By definition of additive generator and taking into account Formula (6), since $P(\xi, \xi, t) \geq P(\xi, \eta, t) > 0$ by axiom **(PGV1)**, we have that

$$\begin{aligned} &(P(\xi, \xi, t) \to_* P(\xi, \eta, t)) * P(\xi, \xi, t) = \\ &= f_*^{(-1)}\left(f_*(P(\xi, \xi, t) \to_* P(\xi, \eta, t)) + f_*(P(\xi, \xi, t))\right) = \\ &= f_*^{(-1)}\left(f_*\left(f^{(-1)}\left(f_*(P(\xi, \eta, t)) - f_*(P(\xi, \xi, t))\right)\right) + f_*(P(\xi, \xi, t)\right) = \\ &= f_*^{(-1)}\left(f_*(P(\xi, \eta, t)) - f_*(P(\xi, \xi, t)) + f_*(P(\xi, \xi, t))\right) = \\ &= f_*^{(-1)}\left(f_*(P(\xi, \eta, t))\right) = P(\xi, \eta, t). \end{aligned} \tag{56}$$

Hence, $Q_P(\xi, \eta, t) * W_P(\xi, t) = (P(\xi, \xi, t) \to_* P(\xi, \eta, t)) * P(\xi, \xi, t) = P(\xi, \eta, t) > 0$.

(WGV1) As it was exposed above, $Q_P(\xi, \eta, t) * W_P(\xi, t) = P(\xi, \eta, t)$. Analogously, $Q_P(\eta, \xi, t) * W_P(\eta, t) = P(\eta, \xi, t)$. By axiom **(PGV3)**, we have that $P(\xi, \eta, t) = P(\eta, \xi, t)$. Therefore, $Q_P(\xi, \eta, t) * W_P(\xi, t) = P(\xi, \eta, t) = P(\eta, \xi, t) = Q_P(\eta, \xi, t) * W_P(\eta, t)$.

(WGV2) By axiom **(PGV5)**, we deduce that the assignment $P_{\xi,\xi} : [0, \infty] \to [0, 1]$ is a continuous function. Thus, because $(W_P)_\xi(t) = P_{\xi,\xi}(t)$ for each $t \in [0, \infty]$ then, the assignment $(W_P)_\xi : (0, \infty) \to (0, 1]$ is a continuous function.

Hence, $(\mathcal{X}, Q_P, *, W_P)$ is a fuzzy weighted quasi-metric space. □

5. Discussion

The aim of the present paper is to adapt to the fuzzy context the duality relationship between quasi-metrics and partial metrics established by Matthews in [1]. Concretely, we have focused in the notions of fuzzy quasi-metric, as given by Gregori and Romaguera in [18], and fuzzy partial metric, as given by Gregori et al. in [9], both based on the concept of fuzzy metric due to George and Veeramani. In this frame, we study the duality relationship between fuzzy quasi-metrics and fuzzy partial metrics whenever both are defined for a continuous Archimedean t-norm. Concretely, it is sought for a method to construct a fuzzy quasi-metric from a fuzzy partial one, and vice versa. Such a study is carried out with the goal that the topologies are preserved.

6. Conclusions and Future Work

In this paper, we have established a duality relationship between a concept of fuzzy quasi-metric introduced in [18] and a concept of fuzzy partial metric introduced in [9]. Concretely, on the one hand, it has provided a method to obtain a fuzzy quasi-metric from a partial one. On the other hand, after introducing the notion of fuzzy weighted quasi-metrics, a way to construct a fuzzy partial metric from a fuzzy weighted quasi-metric has been presented. These two constructions have been obtained by imposing on the continuous t-norm to be Archimedean. Moreover, both of the methods have been implemented with the aim to preserve the topology. In addition, by means of different examples, it has been illustrated the consistence of them with respect to the classical constructions.

The study provided in this paper opens some possible lines of research to continue it. On the one hand, in this paper, the relationship between quasi-metrics and partial ones in the fuzzy context using the George and Veeramani's approach has been studied. Accordingly, on account that both in [18] and [9] were defined fuzzy versions of quasi-metric and partial metric, respectively, following the Kramosil and Michalek approach, it is interesting to try to retrieve the results provided here in such a context. On the other hand, in [26] it is generalized the duality relationship between quasi-metrics and partial ones introduced by Matthews. Accordingly, it is an interesting issue to explore such a generalization in the fuzzy context, both in the Georege and Veramani's sense and in the Kramosil and Michalek's one. Finally, both fuzzy quasi-metrics and fuzzy partial metrics generalize the concept of fuzzy metric, but there exist fuzzy quasi-metrics, which are not fuzzy partial metric and vice versa. So, it remains to define a notion that encompasses both fuzzy quasi-metrics and fuzzy partial metrics based on the concept of quasi-partial metric introduced in [27] or the concept of partial quasi metric given in [28].

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