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Q-Rung Probabilistic Dual Hesitant Fuzzy Sets and Their Application in Multi-Attribute Decision-Making

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Abstract: The probabilistic dual hesitant fuzzy sets (PDHFSs), which are able to consider multiple membership and non-membership degrees as well as their probabilistic information, provide decision experts a flexible manner to evaluate attribute values in complicated realistic multi-attribute decision-making (MADM) situations. However, recently developed MADM approaches on the basis of PDHFSs still have a number of shortcomings in both evaluation information expression and attribute values integration. Hence, our aim is to evade these drawbacks by proposing a new decision-making method. To realize this purpose, first of all a new fuzzy information representation manner is introduced, called q-rung probabilistic dual hesitant fuzzy sets (q-RPDHFSs), by capturing the probability of each element in q-rung dual hesitant fuzzy sets. The most attractive character of q-RPDHFSs is that they give decision experts incomparable degree of freedom so that attribute values of each alternative can be appropriately depicted. To make the utilization of q-RPDHFSs more convenient, we continue to introduce basic operational rules, comparison method and distance measure of q-RPDHFSs. When considering to integrate attribute values in q-rung probabilistic dual hesitant fuzzy MADM problems, we propose a series of novel operators based on the power average and Muirhead mean. As displayed in the main text, the new operators exhibit good performance and high efficiency in information fusion process. At last, a new MADM method with q-RPDHFSs and its main steps are demonstrated in detail. Its performance in resolving practical decision-making situations is studied by examples analysis.

Keywords: q-rung dual hesitant fuzzy sets; q-rung probabilistic dual hesitant fuzzy sets; power Muirhead mean; multi-attribute decision-making

1. Introduction

Multi-attribute decision-making (MADM), whose main purpose is to get the rank of candidates based on a set of principles and decision experts' opinions, is a promising research field, which has extensively gained many interests [1–5]. Nevertheless, decision makers (DMs) feel it is becoming increasingly difficult and complicated to evaluate the performance of all the possible alternatives and appropriately determine the most satisfactory one in MADM issues. One of the primary reasons is the extensive existence of uncertainty and indeterminacy in decision-making process. For the sake of more convenient expressions of evaluation information, a large number of researchers have made great efforts to probe theories and tools to assistant DMs. One of the most advantageous theories is hesitant fuzzy sets (HFSs) [6], which remind scholars and scientists the importance and necessity to consider both vagueness and DMs' hesitancy in one framework. The peculiarity and characteristics of HFS make it well-known and its applications in MADM approaches have soon been proven to be promising and potential [7–10]. Later on, scholars focused on extensions of the classical HFSs, and dual hesitant fuzzy set (DHFS) [11] is one of the most representative. The superiorities of DHFSs are reflected in two aspects, viz., they interpret fuzzy information from both positive and negative points, and efficiently represent DMs' high hesitancy. Soon afterwards, MADM methods based on aggregation operators (AOs) and information measures of DHFSs, as well as extensions of classical decision methods into dual hesitant fuzzy environment, have become an active research area.

Dual hesitant fuzzy MADM methods provide DMs convenient manners to choose a wise alternative, however, as discussed in many publications, they still have limitations and shortcomings. For example, Hao et al. [12] pointed out one defect of DHFSs in depicting fuzzy information. In the opinions of Hao et al. [12], each member in dual hesitant fuzzy element (DHFE) has the same importance, which is counterintuitive and inconsistent with actual situations to a certain degree. Actually, the importance or DMs' preferred intensity of each element in evaluation values should be counted and some similar researches have been done on the basis of this perspective. For instance, by considering the frequency or probability of each linguistic term in hesitant fuzzy linguistic term set, Pang et al. [13] introduced the probabilistic linguistic term sets. Analogously, Zhao et al. [14] proposed the probabilistic hesitant fuzzy sets by adding the probabilistic value of each membership degrees (MDs) in hesitant fuzzy elements. To evade the flaw of DHFSs, Hao et al. [12] continued to propose the probabilistic DHFSs (PDHFSs) by adding corresponding probability of each member in DHFEs. As an attractive extension of DHFSs, PDHFSs can describe DMs' assessment values more accurately and comprehensively, as they denote not only the MDs and non-membership degrees (NMDs), but also the corresponding probabilistic information. In Hao et al.'s [12] publication, authors investigated operations, comparison principle and AOs of PDHFSs as well as a novel decision method to facilitate their applications in realistic MADM problems. It is necessary to point out that Hao et al.'s [12] MADM method still have drawbacks, which limit its use in solving practical MADM issues and these flaws are still existing, although some improved decision approaches have been proposed. Generally speaking, the shortcomings of Hao et al.'s [12] method are two-folds. First, PDHFSs have drawbacks in presenting complex DMs' evaluation information and there exist many situations that cannot be adequately handled by PDHFSs. For instance, the restriction of PDHFSs is that the sum of MD and NMD should be less than one and if such sum is greater than one, then PDHFS is powerless. The second drawback is that the information integration methods proposed by Hao et al. [12] fail to handle complicated realistic situations, such as wherein attributes are correlated.

Based on the above analysis, our motivations and goals are to avert aforementioned shortcomings by proposing a novel MADM method. To this end, we first propose a new technique to overcome the drawback of PDHFSs in denoting fuzzy decision information. The q-rung dual hesitant fuzzy sets (q-RDHFSs) [15], as a new extension of Yager's [16] q-rung orthopair fuzzy sets (q-ROFSs), allow multiple MDs and NMDs, which is similar to DHFSs. However, q-RDHFSs are more powerful than DHFSs, as they inherit the remarkable advantage of q-ROFSs, i.e., permitting the sum of qth power of MD and qth power of NMD to be less than or equal to one. This character makes q-ROFSs and q-RDHFSs to be promising theories or tools, which has been widely noticed by scientists [17–29]. Therefore, aiming at the drawback of PDHFSs, we extend q-RDHFSs to q-rung probabilistic dual hesitant fuzzy sets (q-RPDHFSs) by taking the probability of each member in q-rung dual hesitant fuzzy element (q-RDHFE) into consideration. The q-RPDHFSs are parallel to PDHFS but are more powerful and useful, as they have a much laxer constraint, making the describable information space larger. Additionally, owing to the ability of denoting MDs, NMDs as well as their probabilities simultaneously, q-RPDHFSs also exhibit advantages over q-RDHFSs. To circumvent the second defect of Hao et al.'s [12] method, we provide a series of compound AOs of q-rung probabilistic dual hesitant fuzzy elements (q-RPDHFEs). Absorbing the advantages of power average operator [30] and Muirhead mean [31], the power Muirhead mean (PMM), originated by Li and her colleagues [32], has been proved to have flexibility and advantages in information fusion process [33–35]. Naturally, the characteristics of PMM motivate us to extend it to q-RPDHFSs to introduce some novel powerful hybrid AOs. Hence, we generalize PMM into q-rung probabilistic dual hesitant fuzzy environment to propose some new AOs for q-RPDHFEs. In this paper, we further illustrate why our AOs can overcome the second flaw of Hao et al.'s [12] method.

The novelties and contributions of this work are presented as follows. (1) A new information representation model, called *q*-RPDHFSs, was proposed. This contribution makes it easier and more convenient to depict DMs' complex and fuzzy assessment information in decision-making problems. (2) The operations, score function, accuracy function, comparison method and distance measure of *q*-RPDHFSs were presented and discussed. (3) Novel efficient AOs were put forward, which effectively aggregate integrate attribute values under *q*-RPDHFSs. (4) A new MADM method was developed to judge the best alternative in *q*-RPDHFSs. (5) Some actual MADM examples were provided to show the effectiveness of our new method. The structure of the rest of this paper is as follows. Section 2 recalls basic concepts. Section 3 proposes the *q*-RPDHFSs and introduces their related notions, such as operational rules, comparison method, distance measure, etc. Section 4 presents some AOs of *q*-RPDHFEs and discusses their properties. Section 5 presents a new MADM method under *q*-RPDHFSs. Section 6 conducts numerical experiments to show the performance of the new MADM method. Conclusions are provided in Section 7.

2. Preliminaries

In this section, the concepts of *q*-RDHFSs, PMM and power dual Muirhead mean (PDMM) operators are briefly reviewed, which are the theoretical basis of the proposed method. We extend *q*-RDHFSs to *q*-RPDHFSs by taking probabilities into consideration and develop some new AOs by applying PMM and PDMM to *q*-RPDHFSs in the following Sections 3 and 4, respectively.

2.1. q-Rung Dual Hesitant Fuzzy Sets

Definition 1 [15]. *Let X be a fixed set, a q-rung dual hesitant fuzzy set (q-RDHFS) A defined on X is given as follows*

$$A=\left\{\left\langle x,h(x),g(x)\right\rangle \middle| x\in X\right\}, \tag{1}$$

where $h(x)$ *and* $g(x)$ *are two sets of some values in the interval [0,1], denoting the possible MDs and NMDs, respectively. In addition,* $h(x)$ *and* $g(x)$ *should satisfy the following condition*

$$0\le\mu,\nu\le 1,\ \left(\mu^{+}\right)^{q}+\left(\nu^{+}\right)^{q}\le 1, \tag{2}$$

where $q\ge 1$ *, denoting the rung of the set A,* $\mu\in h(x)$ *,* $\nu\in g(x)$ *,* $\mu^{+}=\cup_{\mu\in h(x)}\max\{\mu\}$ *and* $\nu^{+}=\cup_{\nu\in g(x)}\max\{\nu\}$*. For convenience, the ordered pair* $e(x)=\left(h(x),g(x)\right)$ *is called a q-RPDHFE by Xu et al. [15], which can be denoted by* $e=(h,g)$ *for simplicity. It is noted that the rung q increases, the information space that the set A can describe increases.*

Xu et al. [15] proposed a method to rank any two *q*-RDHFEs.

Definition 2 [15]. *Let* $e=(h,g)$ *be a q-RDHFE, the score function of e is defined as*

$$S(e)=\left(\frac{1}{\#h}\sum_{\mu\in h}\mu\right)^{q}-\left(\frac{1}{\#g}\sum_{\nu\in g}\nu\right)^{q}, \tag{3}$$

and the accuracy function of e is defined as

$$H(e)=\left(\frac{1}{\#h}\sum_{\mu\in h}\mu\right)^{q}+\left(\frac{1}{\#g}\sum_{\nu\in g}\nu\right)^{q}, \tag{4}$$

where #h and #g denote the numbers of the elements in h and g respectively. For any two q-RDHFEs $e_1=(h_1,g_1)$ *and* $e_2=(h_2,g_2)$*, then*

(1) If $S(e_1) > S(e_2)$, *then* $e_1 > e_2$;

(2) If $S(e_1) = S(e_2)$, *then*

if $H(e_1) > H(e_2)$, *then* $e_1 > e_2$;

if $H(e_1) = H(e_2)$, *then* $e_1 = e_2$.

Xu et al. [15] also provided some operations of q-RDHFEs.

Definition 3 [15]. *Let* $e = (h, g)$, $e_1 = (h_1, g_1)$ *and* $e_2 = (h_2, g_2)$ *be any three q-RDHFEs and* λ *be a positive real number, then*

(1) $e_1 \oplus e_2 = \cup_{\mu_1 \in h_1, \nu_1 \in g_1, \mu_2 \in h_2, \nu_2 \in g_2} \left\{ \left\{ \left(\mu_1^q + \mu_2^q - \mu_1^q \mu_2^q \right)^{1/q} \right\}, \left\{ \nu_1 \nu_2 \right\} \right\}$;

(2) $e_1 \otimes e_2 = \cup_{\mu_1 \in h_1, \nu_1 \in g_1, \mu_2 \in h_2, \nu_2 \in g_2} \left\{ \left\{ \mu_1 \mu_2 \right\}, \left\{ \left(\nu_1^q + \nu_2^q - \nu_1^q \nu_2^q \right)^{1/q} \right\} \right\}$;

(3) $\lambda e = \cup_{\mu \in h, \nu \in g} \left\{ \left\{ \left(1 - \left(1 - \mu^q \right)^{\lambda} \right)^{1/q} \right\}, \left\{ \nu^{\lambda} \right\} \right\}$;

(4) $e^{\lambda} = \cup_{\mu \in h, \nu \in g} \left\{ \left\{ \mu^{\lambda} \right\}, \left\{ \left(1 - \left(1 - \nu^q \right)^{\lambda} \right)^{1/q} \right\} \right\}$.

2.2. Power Muirhead Mean Operators

Definition 4 [32]. *Let* $a_j \, (j = 1, 2, \ldots, n)$ *be a collection of crisp numbers and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. Then, the PMM is defined as follows:*

$$PMM^L(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(\frac{n\left(1 + T\left(a_{\vartheta(j)}\right)\right) a_{\vartheta(j)}}{\sum_{j=1}^{n} \left(1 + T\left(a_j\right)\right)} \right)^{l_j} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \tag{5}$$

where $T(a_j) = \sum_{i=1, i \neq j}^{n} Sup(a_i, a_j)$, $\vartheta(j) \, (j = 1, 2, \ldots, n)$ *represents any permutation of* $(1, 2, \ldots, n)$, L_n *denotes all possible permutations of* $(1, 2, \ldots, n)$, *n is the balancing coefficient, and* $Sup(a_i, a_j)$ *denotes the support for* a_i *from* a_j, *satisfying the following properties*

(1) $Sup(a_i, a_j) \in [0, 1]$;

(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$;

(3) If $d(a_i, a_j) < d(a_s, a_t)$, *then* $Sup(a_i, a_j) > Sup(a_j, a_i)$, *where* $d(a_i, a_j)$ *is the distance between* a_i *and* a_j.

Liu et al. [36] continued to introduce the power dual Muirhead mean (PDMM) operator, which is a combination of the power geometric (PG) [37] operator and the dual Muirhead mean (DMM) [38] operator.

Definition 5 [36]. *Let* $a_j \, (j = 1, 2, \ldots, n)$ *be a set of crisp numbers and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. Then, the PDMM is defined as follows:*

$$PDMM^{L}\left(a_1,a_2,...,a_n\right)=\frac{1}{\sum_{j=1}^{n}l_j}\left(\prod_{\theta\in S_n}\sum_{j=1}^{n}l_j a_{\vartheta(j)}^{\frac{n\left(1+T\left(a_{\vartheta(j)}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(a_j\right)\right)}}\right)^{\frac{1}{n!}}, \tag{6}$$

where $T\left(a_j\right)=\sum_{i=1,i\neq j}^{n}Sup\left(a_i,a_j\right)$, $\vartheta(j)(j=1,2,...,n)$ *represents any permutation of* $(1,2,...,n)$, L_n *denotes all possible permutations of* $(1,2,...,n)$, *n is the balancing coefficient, and* $Sup\left(a_i,a_j\right)$ *denotes the support for* a_i *from* a_j, *satisfying the properties in Definition 4.*

3. q-Rung Probabilistic Dual Hesitant Fuzzy Sets

In this section, we propose the concept of q-RPDHFSs. In order to do this, we first briefly introduce the motivations of proposing q-RPDHFSs and explain why we need q-RPDHFSs. Then the definition, operational rules, comparison method and distance measure of q-RPDHFSs are further introduced.

3.1. Motivations of Proposing q-RPDHFSs

In actual MADM problems, it is highly necessary to comprehensively express DMs' evaluation information before determining the best alternatives. In other word, depicting DMs' evaluation values accurately and appropriately is a precondition, which makes the final decision consequences reliable and reasonable. As fuzziness and vagueness extensively exist in realistic decision-making issues, DMs usually express their assessment with the help of fuzzy sets. In addition, sometimes it is also needful to consider the probabilities of fuzzy values to more precisely denote attribute values provided by decision experts. We provide the following example to better demonstrate this phenomenon.

Example 1. *The library of a university plans to purchase a batch of books. The library invites three decision experts to evaluate the performance of a potential book vendor under the attribute "reputation". Each DM is required to use several values to denote the MDs and NMDs of his/her evaluation value. The assessment information provided by the three DMs is listed in Table 1.*

Table 1. The evaluation information provided by DMs in Example 1.

	Possible MDs	**Possible NMDs**
The first DM	0.4, 0.5, 0.6	0.1, 0.2, 0.3
The second DM	0.2, 0.3, 0.5	0.1, 0.2
The third DM	0.1, 0.4	0.2, 0.3, 0.5

If we integrate each DM's evaluation values in the form of DHFEs, then it can be denoted as {{0.1, 0.2, 0.3, 0.4, 0.5, 0.6}, {0.1, 0.2, 0.3, 0.5}}. However, it is noted that the multiple appearances of the MDs 0.4 and 0.5, and the NMDs 0.1, 0.2, and 0.3 are ignored, which implies that some fundamental information is lost. If we denote the group's overall evaluation value by PDHFE, then it can be expressed as {{0.1|0.125, 0.2|0.125, 0.3|0.125, 0.4|0.25, 0.5|0.25, 0.6|0.125}, {0.1|0.25, 0.2|0.375, 0.3|0.25, 0.5|0.125}}. It is noted that when using PDHFE to express the evaluation value of the group, not only each MD and NMD, but also their corresponding probabilistic information is taken into account, which indicates the superiority of PDHFE. This example reveals the advantage of PDHFSs over DHFS. Nevertheless, PDHFSs still have shortcomings. If the third DM would like to employ {0.4, 0.6} to denote his/her preferred MDs, then the overall evaluation value cannot be handled by PDHFSs as 0.6 + 0.5 = 1.1 > 1. This example reveals the shortcomings of PHFSs and PDHFSs is they fail to deal with situations in which the sum of MD and NMD is greater than one. Hence, to circumvent such drawback and more accurately describe groups' evaluation opinions, it is necessary to propose a new fuzzy information expression tool. Motivated by the q-RDHFSs, which have the character that the sum of qth power of MD and qth power of NMD is greater than one, we extend q-RDHFSs to q-RPDHFSs, which consider both the multiple MDs and NMDs, and their probabilistic information. The definition as well as some related notions of q-RPDHFSs are presented in the following subsections.

3.2. The Definition of q-RPDHFSs

Motivated by DHFSs, PDHFSs and q-RDHFS, we present the definition of q-RPDHFSs.

Definition 6. *Let X be a fixed set, a q-rung probabilistic dual hesitant fuzzy set (q-RPDHFS) D defined on X is given by the following mathematical symbol*

$$D=\left\{\left\langle x,h(x)\middle|p(x),g(x)\middle|t(x)\middle|x\in X\right\rangle\right\}, \tag{7}$$

where $h(x)|p(x)$ *and* $g(x)|t(x)$ *are two series of possible elements,* $h(x)$ *and* $g(x)$ *denote the possible MDs and NMDs of the element* $x\in X$ *to the set D, respectively.* $p(x)$ *and* $t(x)$ *are the probabilistic information for the MDs and NMDs, respectively. In addition, the elements* $h(x)$, $g(x)$, $p(x)$ *and* $t(x)$ *satisfying the following conditions:*

$$0\le\mu,\nu\le 1,\ 0\le\left(\mu^{+}\right)^{q}+\left(\nu^{+}\right)^{q}\le 1, \tag{8}$$

and

$$p_i\in[0,1],t_i\in[0,1],\sum_{i=1}^{\#h}p_i=1,\sum_{i=1}^{\#g}t_i=1, \tag{9}$$

where $\mu\in h(x)$, $\nu\in g(x)$, $\mu^{+}=\cup_{\mu\in h(x)}\max\{\mu\}$, $\nu^{+}=\cup_{\nu\in g(x)}\max\{\nu\}$, $p_i\in p(x)$ *and* $t_i\in t(x)$. *The symbols #h and #g represent the total numbers of elements in* $h(x)|p(x)$ *and* $g(x)|t(x)$, *respectively. For convenience,* $\alpha=\left(h(x)\middle|p(x),g(x)\middle|t(x)\right)$ *is called a q-rung probabilistic dual hesitant fuzzy element (q-RPDHFE), which can be denoted by* $\alpha=\left(h\middle|p,g\middle|t\right)$ *for simplicity.*

Remark 1. *Especially, if all the probability values are equal in p and t, then the q-RPDHFS reduces to the q-RDHFS. In addition, when q = 1, the q-RPDHFS reduces to PDHFS proposed by Hao et al. [12]. If q = 2, then the probabilistic dual Pythagorean hesitant fuzzy sets (PDPHFSs) are obtained. In other word, the PDHFSs and PDPHFSs are special cases of our proposed q-RPDHFSs and q-RPDHFS is a generalized form of PDHFS and PDPHFS.*

In Example 1, when the third DM uses {0.4, 0.6} to denote his/her preferred MDs, then the overall evaluation values of the group can be expressed as d = {{0.2|0.125, 0.3|0.125, 0.4|0.25, 0.5|0.25, 0.6|0.25}, {0.1|0.25, 0.2|0.375, 0.3|0.25, 0.5|0.125}}, which is a q-RPDHFE, as $0.6^2 + 0.5^2 = 0.61 < 1$. This example implies that the proposed q-RPDHFSs are more powerful and flexible and have a lager range of applications than PDHFSs. In addition, compared with the traditional q-RDHFSs, q-RPDHFSs can more comprehensively express DMs' evaluation opinions.

3.3. Basic Operational Rules of q-RPDHFEs

In this subsection, we introduce some basic operations of *q*-RPDHFEs and discuss their properties.

Definition 7. *Let* $\alpha_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$, $\alpha_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$ *and* $\alpha = \left(h \middle| p_h, g \middle| t_g\right)$ *be any three q-RPDHFEs, and* λ *be a possible real number, then*

(1) $\alpha_1 \oplus \alpha_2 = \bigcup_{\mu_1 \in h_1, \nu_1 \in g_1, \mu_2 \in h_2, \nu_2 \in g_2} \left\{ \left\{ \left(\mu_1^q + \mu_2^q - \mu_1^q \mu_2^q\right)^{1/q} \middle| p_{\mu_1} p_{\mu_2} \right\}, \left\{ \left(\nu_1 \nu_2\right) \middle| t_{\nu_1} t_{\nu_2} \right\} \right\}$;

(2) $\alpha_1 \otimes \alpha_2 = \bigcup_{\mu_1 \in h_1, \nu_1 \in g_1, \mu_2 \in h_2, \nu_2 \in g_2} \left\{ \left\{ \left(\mu_1 \mu_2\right) \mid p_{\mu_1} p_{\mu_2} \right\}, \left\{ \left(\nu_1^q + \nu_2^q - \nu_1^q \nu_2^q\right)^{1/q} \mid t_{\nu_1} t_{\nu_2} \right\} \right\}$;

(3) $\lambda \alpha = \bigcup_{\mu \in h, \nu \in g} \left\{ \left\{ \left(1 - \left(1 - \mu^q\right)^{\lambda}\right)^{1/q} \middle| p_\mu \right\}, \left\{ \nu^{\lambda} \middle| t_\nu \right\} \right\}$;

(4) $\alpha^{\lambda} = \bigcup_{\mu \in h, \nu \in g} \left\{ \left\{ \mu^{\lambda} \middle| p_\mu \right\}, \left\{ \left(1 - \left(1 - \nu^q\right)^{\lambda}\right)^{1/q} \middle| t_\nu \right\} \right\}$.

Example 2. *Let* $\alpha_1 = \{\{0.2|0.3, 0.5|0.7\}, \{0.3|0.6, 0.5|0.4\}\}$, $\alpha_2 = \{\{0.3|0.1, 0.8|0.9\}, \{0.5|0.8, 0.6|0.2\}\}$ *and* $\alpha = \{\{0.3|0.6, 0.8|0.4\}, \{0.5|0.5, 0.6|0.5\}\}$ *be three q-RPDHFEs (q = 3), then*

$$\alpha_1 \oplus \alpha_2 = \{\{0.83|0.63, 0.80|0.27, 0.53|0.07, 0.33|0.03\}, \{0.3|0.08, 0.25|0.32, 0.18|0.12, 0.15|0.48\}\},$$

$$\alpha_1 \otimes \alpha_2 = \{\{0.4|0.63, 0.16|0.27, 0.15|0.07, 0.06|0.03\}, \{0.68|0.08, 0.62|0.32, 0.62|0.12, 0.53|0.48\}\},$$

$$2\alpha = \{\{0.91|0.4, 0.38|0.6\}, \{0.36|0.5, 0.25|0.5\}\},$$

$$\alpha^2 = \{\{0.64|0.4, 0.09|0.6\}, \{0.73|0.5, 0.62|0.5\}\}.$$

Based on Definition 7, we can obtain the following theorem.

Theorem 1. *Let* $\alpha_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$, $\alpha_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$ *and* $\alpha_3 = \left(h_3 \middle| p_{h_3}, g_3 \middle| t_{g_3}\right)$ *be any three q-RPDHFEs, and* $\lambda, \lambda_1, \lambda_2 > 0$, *then*

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
(2) $\left(\alpha_1 \oplus \alpha_2\right) \oplus \alpha_3 = \alpha_1 \oplus \left(\alpha_2 \oplus \alpha_3\right)$;
(3) $\lambda\left(\alpha_1 \oplus \alpha_2\right) = \lambda \alpha_1 \oplus \lambda \alpha_2$;
(4) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
(5) $\left(\alpha_1 \otimes \alpha_2\right) \otimes \alpha_3 = \alpha_1 \otimes \left(\alpha_2 \otimes \alpha_3\right)$;
(6) $\left(\alpha_1 \otimes \alpha_2\right)^{\lambda} = \alpha_1^{\lambda} \otimes \alpha_2^{\lambda}$;
(7) $\alpha_1^{\lambda_1 + \lambda_2} = \alpha_1^{\lambda_1} \otimes \alpha_1^{\lambda_2}$.

Proof.

(1) $\alpha_1 \oplus \alpha_2 = \bigcup_{\mu_1 \in h_1, \nu_1 \in g_1, \mu_2 \in h_2, \nu_2 \in g_2} \left\{ \left\{ \left(\mu_1^q + \mu_2^q - \mu_1^q \mu_2^q\right)^{1/q} \middle| p_{\mu_1} p_{\mu_2} \right\}, \left\{ \left(\nu_1 \nu_2\right) \middle| t_{\nu_1} t_{\nu_2} \right\} \right\} = \alpha_2 \oplus \alpha_1$.

(2) $\left(\alpha_1 \oplus \alpha_2\right) \oplus \alpha_3 = \bigcup_{\substack{\mu_1 \in h_1, \nu_1 \in g_1, \\ \mu_2 \in h_2, \nu_2 \in g_2, \\ \mu_3 \in h_3, \nu_3 \in g_3}} \left\{ \left\{ \left(\mu_1^q + \mu_2^q + \mu_3^q - \mu_1^q \mu_2^q - \mu_1^q \mu_3^q - \mu_2^q \mu_3^q + \mu_1^q \mu_2^q \mu_3^q\right)^{1/q} \middle| p_{\mu_1} p_{\mu_2} p_{\mu_3} \right\}, \right.$

$\left. \left\{ \left(\nu_1 \nu_2 \nu_3\right) \mid t_{\nu_1} t_{\nu_2} t_{\nu_3} \right\} \right\} = \alpha_1 \oplus \left(\alpha_2 \oplus \alpha_3\right)$.

(3) $\lambda\left(\alpha_1 \oplus \alpha_2\right)=\cup_{\mu_1\in h_1,\nu_1\in g_1,\mu_2\in h_2,\nu_2\in g_2}\left\{\left\{\left(1-\left(1-\mu_1^q\right)^\lambda\left(1-\mu_2^q\right)^\lambda\right)^{1/q}\left|p_{\mu_1}p_{\mu_2}\right.\right\},\left\{\left(\nu_1\nu_2\right)^\lambda\left|t_{\nu_1}t_{\nu_2}\right.\right\}\right\}$

$=\cup_{\mu_1\in h_1,\nu_1\in g_1}\left\{\left\{\left(1-\left(1-\mu_1^q\right)^\lambda\right)^{1/q}\left|p_{\mu_1}\right.\right\},\left\{\left(\nu_1\right)^\lambda\left|t_{\nu_1}\right.\right\}\right\}\oplus\cup_{\mu_2\in h_2,\nu_2\in g_2}\left\{\left\{\left(1-\left(1-\mu_2^q\right)^\lambda\right)^{1/q}\left|p_{\mu_2}\right.\right\},\left\{\left(\nu_2\right)^\lambda\left|t_{\nu_2}\right.\right\}\right\}$

$=\lambda\alpha_2\oplus\lambda\alpha_1$.

(4) $\alpha_1\otimes\alpha_2=\cup_{\mu_1\in h_1,\nu_1\in g_1,\mu_2\in h_2,\nu_2\in g_2}\left\{\left\{\left(\mu_1\mu_2\right)\left|p_{\mu_1}p_{\mu_2}\right.\right\},\left\{\left(\nu_1^q+\nu_2^q-\nu_1^q\nu_2^q\right)^{1/q}\left|t_{\nu_1}t_{\nu_2}\right.\right\}\right\}=\alpha_2\otimes\alpha_1$.

(5) $\left(\alpha_1\otimes\alpha_2\right)\otimes\alpha_3=\cup_{\substack{\mu_1\in h_1,\nu_1\in g_1,\\ \mu_2\in h_2,\nu_2\in g_2,\\ \mu_3\in h_3,\nu_3\in g_3}}\left\{\left\{\left(\mu_1\mu_2\mu_3\right)\left|p_{\mu_1}p_{\mu_2}p_{\mu_3}\right.\right\},\right.$

$\left.\left\{\left(\nu_1^q+\nu_2^q+\nu_3^q-\nu_1^q\nu_2^q-\nu_1^q\nu_3^q-\nu_2^q\nu_3^q+\nu_1^q\nu_2^q\nu_3^q\right)^{1/q}\left|t_{\nu_1}t_{\nu_2}t_{\nu_3}\right.\right\}\right\}=\alpha_1\otimes\left(\alpha_2\otimes\alpha_3\right)$.

(6) $\left(\alpha_1\otimes\alpha_2\right)^\lambda=\cup_{\mu_1\in h_1,\nu_1\in g_1,\mu_2\in h_2,\nu_2\in g_2}\left\{\left\{\left(\mu_1\mu_2\right)^\lambda\left|p_{\mu_1}p_{\mu_2}\right.\right\},\left\{\left(1-\left(1-\nu_1^q\right)^\lambda\left(1-\nu_2^q\right)^\lambda\right)^{1/q}\left|t_{\nu_1}t_{\nu_2}\right.\right\}\right\}$

$=\cup_{\mu_1\in h_1,\nu_1\in g_1}\left\{\left\{\left(\mu_1\right)^\lambda\left|p_{\mu_1}\right.\right\},\left\{\left(1-\left(1-\nu_1^q\right)^\lambda\right)^{1/q}\left|t_{\nu_1}\right.\right\}\right\}\otimes\cup_{\mu_2\in h_2,\nu_2\in g_2}\left\{\left\{\left(\mu_2\right)^\lambda\left|p_{\mu_2}\right.\right\},\left\{\left(1-\left(1-\nu_2^q\right)^\lambda\right)^{1/q}\left|t_{\nu_2}\right.\right\}\right\}$

$=\alpha_1^\lambda\otimes\alpha_2^\lambda$.

(7) $\alpha_1^{\lambda_1+\lambda_2}=\cup_{\mu_1\in h_1,\nu_1\in g_1}\left\{\left\{\left(\mu_1\right)^{\lambda_1+\lambda_2}\left|p_{\mu_1}\right.\right\},\left\{\left(1-\left(1-\nu_1^q\right)^{\lambda_1+\lambda_2}\right)^{1/q}\left|t_{\nu_1}\right.\right\}\right\}$

$=\cup_{\mu_1\in h_1,\nu_1\in g_1}\left\{\left\{\left(\mu_1\right)^{\lambda_1}\left(\mu_1\right)^{\lambda_2}\left|p_{\mu_1}\right.\right\},\left\{\left(1-\left(1-\nu_1^q\right)^{\lambda_1}\left(1-\nu_1^q\right)^{\lambda_2}\right)^{1/q}\left|t_{\nu_1}\right.\right\}\right\}=\alpha_1^{\lambda_1}\otimes\alpha_1^{\lambda_2}$.

□

3.4. Comparison Method of q-RPDHFEs

Definition 8. *Let* $\alpha=\left(h|p,g|t\right)$ *be a q-RPDHFE, the score function of* α *is defined as*

$$S\left(\alpha\right)=\sum_{i=1\,\mu\in h}^{\#h}\mu_i^q p_{\mu_i}-\sum_{j=1\,\nu\in g}^{\#g}\nu_j^q t_{v_j}\ , \tag{10}$$

and the accuracy function of α *is defined as*

$$H\left(\alpha\right)=\sum_{i=1\,\mu\in h}^{\#h}\mu_i^q p_{\mu_i}+\sum_{j=1\,\nu\in g}^{\#g}\nu_j^q t_{v_j} \tag{11}$$

For any two q-RPDHFEs $\alpha_1=\left(h_1\left|p_{h_1},g_1\right|t_{g_1}\right)$ *and* $\alpha_2=\left(h_2\left|p_{h_2},g_2\right|t_{g_2}\right)$,

(1) If $S\left(\alpha_1\right)>S\left(\alpha_2\right)$, *then* $\alpha_1>\alpha_2$;

(2) If $S\left(\alpha_1\right)=S\left(\alpha_2\right)$, *then*

if $H\left(\alpha_1\right)>H\left(\alpha_2\right)$, *then* $\alpha_1>\alpha_2$;

if $H\left(\alpha_1\right)=H\left(\alpha_2\right)$, *then* $\alpha_1=\alpha_2$.

Example 3. *Let* α_1= *{{0.3|0.6, 0.5|0.4}, {0.2|0.3, 0.5|0.7}} and* α_2 = *{{0.3|0.1, 0.7|0.3, 0.8|0.6}, {0.4|0.6, 0.5|0.2, 0.6|0.2}} be two q-RPDHFEs (q = 2), then according to Definition 8, we have*

$$S\left(\alpha_1\right)=\left(0.3^2\times0.6+0.5^2\times0.4\right)-\left(0.2^2\times0.3+0.5^2\times0.7\right)=-0.033\,,$$

$$H\left(\alpha_1\right)=\left(0.3^2\times0.6+0.5^2\times0.4\right)+\left(0.2^2\times0.3+0.5^2\times0.7\right)=0.341\,,$$

$$S(\alpha_2) = \left(0.3^2 \times 0.1 + 0.7^2 \times 0.3 + 0.8^2 \times 0.6\right) - \left(0.4^2 \times 0.6 + 0.5^2 \times 0.2 + 0.6^2 \times 0.2\right) = 0.32$$

$$H(\alpha_2) = \left(0.3^2 \times 0.1 + 0.7^2 \times 0.3 + 0.8^2 \times 0.6\right) + \left(0.4^2 \times 0.6 + 0.5^2 \times 0.2 + 0.6^2 \times 0.2\right) = 0.758.$$

Hence, we can obtain $\alpha_2 > \alpha_1$.

3.5. Distance between Two q-RPDHFEs

In this subsection, we propose the distance between any two *q*-RPDHFEs and discuss its properties.

Definition 9. *Let* $\alpha_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$ *and* $\alpha_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$ *be any two q-RPDHFEs, then the distance measure between* α_1 *and* α_2 *is defined as*

$$d(\alpha_1, \alpha_2) = \frac{1}{\#h + \#g}\left(\sum_{i=1}^{\#h}\left|\left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}} - \left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}}\right| + \sum_{j=1}^{\#g}\left|\left(\nu_1^{\sigma(j)}\right)^q t_{\nu_1^{\sigma(j)}} - \left(\nu_2^{\sigma(j)}\right)^q t_{\nu_2^{\sigma(j)}}\right|\right), \quad (12)$$

where $\mu_1^{\sigma(i)} \in h_1, \nu_1^{\sigma(j)} \in g_1, \mu_2^{\sigma(i)} \in h_2, \nu_2^{\sigma(j)} \in g_2$. $\mu_1^{\sigma(i)}$ *and* $\mu_2^{\sigma(i)}$ *are the ith largest values of* h_1 *and* h_2, $\nu_1^{\sigma(j)}$ *and* $\nu_2^{\sigma(j)}$ *are the jth largest values of* g_1 *and* g_2. *The symbol #h denotes the number of values in* h_1 *and* h_2, *and #g represents the number of values in* g_1 *and* g_2.

Remark 2. *From Definition 9, we can find out that when calculating the distance between two q-RPDHFEs, they must have the same numbers of MDs and NMDs. However, this requirement cannot be always met. Hence, to operate correctly, the shorter q-RPDHFEs should be extended by adding some values until the numbers of the MDs and NMDs of the two q-RPDHFEs are equal. In the following, we present a principle to extend the short q-RPDHFEs. Let*

$$\alpha_1 = (h_1, g_1) = \left\{\left\{\mu_1^{\sigma(1)} \middle| p_{\mu_1^{\sigma(1)}}, \mu_1^{\sigma(2)} \middle| p_{\mu_1^{\sigma(2)}}, \ldots, \mu_1^{\sigma(\#h_1)} \middle| p_{\mu_1^{\sigma(\#h_1)}}\right\}, \left\{\nu_1^{\sigma(1)} \middle| t_{\nu_1^{\sigma(1)}}, \nu_1^{\sigma(2)} \middle| t_{\nu_1^{\sigma(2)}}, \ldots, \nu_1^{\sigma(\#g_1)} \middle| t_{\nu_1^{\sigma(\#g_1)}}\right\}\right\}, \quad (13)$$

and

$$\alpha_2 = (h_2, g_2) = \left\{\left\{\mu_2^{\sigma(1)} \middle| p_{\mu_2^{\sigma(1)}}, \mu_2^{\sigma(2)} \middle| p_{\mu_2^{\sigma(2)}}, \ldots, \mu_2^{\sigma(\#h_2)} \middle| p_{\mu_2^{\sigma(\#h_2)}}\right\}, \left\{\nu_2^{\sigma(1)} \middle| t_{\nu_2^{\sigma(1)}}, \nu_2^{\sigma(2)} \middle| t_{\nu_2^{\sigma(2)}}, \ldots, \nu_2^{\sigma(\#g_2)} \middle| t_{\nu_2^{\sigma(\#g_2)}}\right\}\right\} \quad (14)$$

If $\#h_1 < \#h_2$ *and* $\#g_2 < \#g_1$, *then we have two methods to extend* α_1 *and* α_2. *First, we assume DMs are optimistic to their evaluations, then we can extend* α_1 *and* α_2 *to*

$$\alpha_1' = (h_1', g_1) = \left\{\begin{array}{c}\left\{\mu_1^{\sigma(1)} \middle| p_{\mu_1^{\sigma(1)}}, \mu_1^{\sigma(2)} \middle| p_{\mu_1^{\sigma(2)}}, \ldots, \mu_1^{\sigma(\#h_1)} \middle| \dfrac{p_{\mu_1^{\sigma(\#h_1)}}}{\#h_2 - \#h_1 + 1}, \ldots, \mu_1^{\sigma(\#h_1)} \middle| \dfrac{p_{\mu_1^{\sigma(\#h_1)}}}{\#h_2 - \#h_1 + 1}\right\}, \\ \left\{\nu_1^{\sigma(1)} \middle| t_{\nu_1^{\sigma(1)}}, \nu_1^{\sigma(2)} \middle| t_{\nu_1^{\sigma(2)}}, \ldots, \nu_1^{\sigma(\#g_1)} \middle| t_{\nu_1^{\sigma(\#g_1)}}\right\}\end{array}\right\}, \quad (15)$$

and

$$\alpha_2' = (h_2, g_2') = \left\{\begin{array}{c}\left\{\mu_2^{\sigma(1)} \middle| p_{\mu_2^{\sigma(1)}}, \mu_2^{\sigma(2)} \middle| p_{\mu_2^{\sigma(2)}}, \ldots, \mu_2^{\sigma(\#h_2)} \middle| p_{\mu_2^{\sigma(\#h_2)}}\right\}, \\ \left\{\nu_2^{\sigma(1)} \middle| t_{\nu_2^{\sigma(1)}}, \nu_2^{\sigma(2)} \middle| t_{\nu_2^{\sigma(2)}}, \ldots, \nu_2^{\sigma(\#g_2)} \middle| \dfrac{t_{\nu_2^{\sigma(\#g_2)}}}{\#g_1 - \#g_2 + 1}, \ldots, \nu_2^{\sigma(\#g_2)} \middle| \dfrac{t_{\nu_2^{\sigma(\#g_2)}}}{\#g_1 - \#g_2 + 1}\right\}\end{array}\right\}, \quad (16)$$

respectively, where $\#h_1' = \#h_2$ *and* $\#g_2' = \#g_1$. *If DMs are pessimistic to their evaluations, then we can extend* α_1 *and* α_2 *to*

$$\alpha_1' = \left(h_1', g_1\right) = \left\{ \begin{array}{c} \left\{ \mu_1^{\sigma(1)} \left| \frac{p_{\mu_1^{\sigma(1)}}}{\# h_2 - \# h_1 + 1}, \ldots, \mu_1^{\sigma(1)} \right| \frac{p_{\mu_1^{\sigma(1)}}}{\# h_2 - \# h_1 + 1}, \mu_1^{\sigma(2)} \left| p_{\mu_1^{\sigma(2)}}, \ldots, \mu_1^{\sigma(\# h_1)} \right| p_{\mu_1^{\sigma(\# h_1)}} \right\}, \\ \left\{ \nu_1^{\sigma(1)} \left| t_{\nu_1^{\sigma(1)}}, \nu_1^{\sigma(2)} \right| t_{\nu_1^{\sigma(2)}}, \ldots, \nu_1^{\sigma(\# g_1)} \left| t_{\nu_1^{\sigma(\# g_1)}} \right. \right\} \end{array} \right\}, \tag{17}$$

and

$$\alpha_2' = \left(h_2, g_2'\right) = \left\{ \begin{array}{c} \left\{ \mu_2^{\sigma(1)} \left| p_{\mu_2^{\sigma(1)}}, \mu_2^{\sigma(2)} \right| p_{\mu_2^{\sigma(2)}}, \ldots, \mu_2^{\sigma(\# h_2)} \left| p_{\mu_2^{\sigma(\# h_2)}} \right. \right\}, \\ \left\{ \nu_2^{\sigma(1)} \left| \frac{t_{\nu_2^{\sigma(1)}}}{\# g_1 - \# g_2 + 1}, \ldots, \nu_2^{\sigma(1)} \right| \frac{t_{\nu_2^{\sigma(1)}}}{\# g_1 - \# g_2 + 1}, \nu_2^{\sigma(2)} \left| t_{\nu_2^{\sigma(2)}}, \ldots, \nu_2^{\sigma(\# g_2)} \right| t_{\nu_2^{\sigma(\# g_2)}} \right\} \end{array} \right\}, \tag{18}$$

respectively, where $\# h_1' = \# h_2$ *and* $\# g_2' = \# g_1$. *In addition, from Definition 8 we can find out that the score and accuracy values are invariable. In this paper, we assume DMs are optimistic to their evaluation values and we always take the first method to extend q-RPDHFEs. To better illustrate this method, we provide the following example. Let* $\alpha_1 = \{\{0.1|0.4, 0.2|0.6\}, \{0.6|0.7, 0.7|0.2, 0.9|0.1\}\}$ *and* $\alpha_2 = \{\{0.3|0.5, 0.4|0.2, 0.6|0.1, 0.8|0.2\}, \{0.7|0.4.0.8|0.6\}\}$ *be two q-RPDHFEs (q = 4), then we can extend* α_1 *and* α_2 *to* $\alpha_1' = \{\{0.1|0.4, 0.2|0.2, 0.2|0.2, 0.2|0.2\}, \{0.6|0.7, 0.7|0.2, 0.9|0.1\}\}$ *and* $\alpha_2' = \{\{0.3|0.5, 0.4|0.2, 0.6|0.1, 0.8|0.2\}, \{0.7|0.4.0.8|0.3, 0.8|0.3\}\}$. *Then, according to Equation (12), the distance between* α_1 *and* α_2 *is*

$$d(\alpha_1, \alpha_2) =$$

$$\frac{1}{4+3} \left(\begin{array}{c} \left|0.1^4 \times 0.4 - 0.3^4 \times 0.5\right| + \left|0.2^4 \times 0.2 - 0.4^4 \times 0.2\right| + \left|0.2^4 \times 0.2 - 0.6^4 \times 0.1\right| + \left|0.2^4 \times 0.2 - 0.8^4 \times 0.2\right| \\ \left|0.6^4 \times 0.7 - 0.7^4 \times 0.4\right| + \left|0.7^4 \times 0.2 - 0.8^4 \times 0.3\right| + \left|0.9^4 \times 0.1 - 0.8^4 \times 0.3\right| \end{array} \right) = 0.1384.$$

Theorem 2. *Let* α_1 *and* α_2 *be two q-RPDHFEs, then the distance between* α_1 *and* α_2 *satisfies the following conditions:*

(1) $0 \le d(\alpha_1, \alpha_2) \le 1$;

(2) $d(\alpha_1, \alpha_2) = d(\alpha_2, \alpha_1)$;

(3) $d(\alpha_1, \alpha_2) = 0$, *if and only if* $\alpha_1 = \alpha_2$.

Proof.

(1) Since $0 \le \mu_1^{\sigma(i)}, \mu_2^{\sigma(i)}, p_{\mu_1^{\sigma(i)}}, p_{\mu_2^{\sigma(i)}} \le 1$, then we have $0 \le \left| \left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}} - \left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}} \right| \le 1$. Hence, we can further obtain $0 \le \sum_{i=1}^{\# h} \left| \left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}} - \left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}} \right| \le \# h$. Similarly, we can get $\sum_{j=1}^{\# g} \left| \left(\nu_1^{\sigma(j)}\right)^q t_{\nu_1^{\sigma(j)}} - \left(\nu_2^{\sigma(j)}\right)^q t_{\nu_2^{\sigma(j)}} \right|$. Therefore, we can derive $0 \le d(\alpha_1, \alpha_2) \le 1$.

(2) From Definition 9, we have

$$d(\alpha_1, \alpha_2) = \frac{1}{\# h + \# g} \left(\sum_{i=1}^{\# h} \left| \left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}} - \left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}} \right| + \sum_{j=1}^{\# g} \left| \left(\nu_1^{\sigma(j)}\right)^q t_{\nu_1^{\sigma(j)}} - \left(\nu_2^{\sigma(j)}\right)^q t_{\nu_2^{\sigma(j)}} \right| \right),$$

and

$$d(\alpha_2, \alpha_1) = \frac{1}{\# h + \# g} \left(\sum_{i=1}^{\# h} \left| \left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}} - \left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}} \right| + \sum_{j=1}^{\# g} \left| \left(\nu_2^{\sigma(j)}\right)^q t_{\nu_2^{\sigma(j)}} - \left(\nu_1^{\sigma(j)}\right)^q t_{\nu_1^{\sigma(j)}} \right| \right).$$

Hence, $d(\alpha_1,\alpha_2)=d(\alpha_2,\alpha_1)$.

(3) If $\alpha_1=\alpha_2$, then it is easy to get $d(\alpha_1,\alpha_2)=0$. If $d(\alpha_1,\alpha_2)=0$, then from Definition 9, we can obtain $\left(\mu_1^{\sigma(i)}\right)^q p_{\mu_1^{\sigma(i)}}=\left(\mu_2^{\sigma(i)}\right)^q p_{\mu_2^{\sigma(i)}}$ and $\left(\nu_1^{\sigma(j)}\right)^q t_{\nu_1^{\sigma(j)}}=\left(\nu_2^{\sigma(j)}\right)^q t_{\nu_2^{\sigma(j)}}$. From Definition 8, we can easily get $S(\alpha_1)=S(\alpha_2)$ and $H(\alpha_1)=H(\alpha_2)$. Thus, we can derive $\alpha_1=\alpha_2$. □

4. Some Aggregation Operators for *Q*-RPDHFEs and Their Properties

In this section, we extend PMM and PDMM to *q*-RPDHFSs and propose new AOs for *q*-RPDHFEs. We also investigate desirable properties of the proposed AOs.

4.1. The q-Rung Probabilistic Dual Hesitant Fuzzy Power Muirhead Mean (q-RPDHFPMM) Operator

Definition 10. *Let* $\alpha_j=\left(h_j\middle|p_{h_j},g_j\middle|t_{g_j}\right)(j=1,2,\dots,n)$ *be a collection of q-RPDHFEs, and* $L=(l_1,l_2,\cdots l_n)\in R^n$ *be a vector of parameters. Then, the q-rung probabilistic dual hesitant fuzzy power Muirhead mean (q-RPDHFPMM) operator is defined as follows*

$$q-RPDHFPMM^L(\alpha_1,\alpha_2,\cdots,\alpha_n)=\left(\frac{1}{n!}\underset{\vartheta\in S_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(\frac{n\left(1+T\left(\alpha_{\vartheta(j)}\right)\right)\alpha_{\vartheta(j)}}{\sum_{j=1}^{n}\left(1+T\left(\alpha_j\right)\right)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}, \tag{19}$$

where

$$T(\alpha_j)=\sum_{i=1,i\neq j}^{n}Sup(\alpha_i,\alpha_j),\ Sup(\alpha_i,\alpha_j)=1-d(\alpha_i,\alpha_j), \tag{20}$$

$\vartheta(j)(j=1,2,\dots,n)$ *represents any permutation of* $(1,2,\dots,n)$, S_n *denotes all possible permutations of* $(1,2,\dots,n)$, *n is the balancing coefficient, and* $Sup(\alpha_i,\alpha_j)$ *denotes the support for* α_i *from* α_j, *satisfying the following properties*

(1) $Sup(\alpha_i,\alpha_j)\in[0,1]$;

(2) $Sup(\alpha_i,\alpha_j)=Sup(\alpha_j,\alpha_i)$;

(3) If $d(\alpha_i,\alpha_j)\le d(\alpha_s,\alpha_t)$, *then* $Sup(\alpha_i,\alpha_j)\ge Sup(\alpha_j,\alpha_i)$, *where* $d(\alpha_i,\alpha_j)$ *is the distance between* α_i *and* α_j.

In order to simplify Equation (19), we assume

$$\delta_j=\frac{1+T(\alpha_j)}{\sum_{j=1}^{n}\left(1+T(\alpha_j)\right)}, \tag{21}$$

then Equation (19) can be written as

$$q-RPDHFPMM^L(\alpha_1,\alpha_2,\cdots,\alpha_n)=\left(\frac{1}{n!}\underset{\vartheta\in S_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}, \tag{22}$$

where $0\le\delta_j\le1$ and $\sum_{j=1}^{n}\delta_j$.

Theorem 3. *Let* $\alpha_j = \left(h_j \middle| p_{h_j}, g_j \middle| t_{g_j}\right)(j = 1, 2, \ldots, n)$ *be a collection of q-RPDHFEs, and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. The aggregated value using the q-RPDHFPMM operator is still a q-RPDHFE and*

$$q - RPDHFPMM^L(\alpha_1, \alpha_2, \cdots, \alpha_n) =$$

$$\bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(\left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{l_j} \right)^{1/n!} \right)^{1/q} \right)^{1 \Big/ \sum_{j=1}^{n} l_j} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\mu_{\vartheta(j)}} \right\}, \right. \tag{23}$$

$$\left. \left\{ \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}} \right)^{l_i} \right)^{1/n!} \right)^{1 \Big/ \sum_{i=1}^{n} l_i} \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} t_{\nu_{\vartheta(j)}} \right\} \right\}$$

Proof. According to Definition 7, we get

$$n\delta_{\vartheta(j)}\alpha_{\vartheta(j)} = \bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{1/q} \middle| p_{\mu_{\vartheta(j)}} \right\}, \left\{ \left(\nu_{\vartheta(j)} \right)^{n\delta_{\vartheta(j)}} \middle| t_{\nu_{\vartheta(j)}} \right\} \right\},$$

then,

$$\left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)} \right)^{l_j} = \bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{l_i/q} \mid p_{\mu_{\vartheta(j)}} \right\}, \left\{ \left(1 - \left(1 - \nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}} \right)^{l_i} \right)^{1/q} \middle| t_{\nu_{\vartheta(j)}} \right. \right.$$

Therefore,

$$\overset{n}{\underset{j=1}{\otimes}} \left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)} \right)^{l_j} =$$

$$\bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{l_j/q} \middle| \prod_{j=1}^{n} p_{\mu_{\vartheta(j)}} \right\}, \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\vartheta(j)}^{n\delta_{\vartheta(j)}} \right)^{l_j} \right)^{1/q} \middle| \prod_{j=1}^{n} t_{\nu_{\vartheta(j)}} \right\} \right\},$$

and

$$\underset{\vartheta \in S_n}{\oplus} \overset{n}{\underset{j=1}{\otimes}} \left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)} \right)^{l_j} = \bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{l_j} \right) \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\mu} \right. \right.$$

$$\left. \left\{ \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}} \right)^{l_j} \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} t_{\nu_{\vartheta(j)}} \right\} \right\}.$$

Thus,

$$\left(\frac{1}{n!} \underset{\vartheta \in S_n}{\oplus} \overset{n}{\underset{j=1}{\otimes}} \left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)} \right)^{l_j} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}} =$$

$$\bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\delta_{\vartheta(j)}} \right)^{l_j} \right)^{1/n!} \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\nu_{\vartheta(j)}} \right\}, \right.$$

$$\left\{\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}}\right)^{l_i}\right)^{1/q}\right)^{1/n!}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right.\right\}\right\}$$

Finally, we can obtain

$$\left(\frac{1}{n!}\underset{\vartheta\in S_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}=$$

$$\bigcup_{\mu_{\vartheta(j)}\in h_{\vartheta(j)},\nu_{\vartheta(j)}\in g_{\vartheta(j)}}\left\{\left\{\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(i)}^{q}\right)^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/q}\right)^{1/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\right.\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}l_j}\right)^{1/q}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right.\right\}\right\}.$$

□

In addition, the q-RPDHFPMM operator has the property of boundedness.

Theorem 4. *(Boundedness) Let* $\alpha_j=\left(h_j\left|p_{h_j},g_j\right|t_{g_j}\right)(j=1,2,\ldots,n)$ *be a collection of q-RPDHFEs, if*

$$\alpha^{-}=\left\{\left\{\min\left\{h_{\alpha_s}\right\}\middle|\min\left\{p_{h_{\alpha_s}}\right\}\right\},\left\{\max\left\{g_{\alpha_s}\right\}\middle|\max\left\{t_{g_{\alpha_s}}\right\}\right\}\right\},$$

and

$$\alpha^{+}=\left\{\left\{\max\left\{h_{\alpha_s}\right\}\middle|\max\left\{p_{h_{\alpha_s}}\right\}\right\},\left\{\min\left\{g_{\alpha_s}\right\}\middle|\min\left\{t_{g_{\alpha_s}}\right\}\right\}\right\},$$

then

$$q-RPDHFPMM^{L}\left(\alpha^{-}\right)\le q-RPDHFPMM^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)\le q-RPDHFPMM^{L}\left(\alpha^{+}\right).\qquad(24)$$

Proof. For each element in the q-RPDHFE, we have $\min\left\{h_{\alpha_s}\right\}\le\mu_{\alpha_s}$ and $\nu_{\alpha_s}\le\max\left\{g_{\alpha_s}\right\}$. Then

$$\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(i)}^{q}\right)^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/q}\right)^{1/\sum_{j=1}^{n}l_j}$$

$$\ge\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\min\left\{h_{\alpha_s}\right\}\right)^{q}\right)^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/q}\right)^{1/\sum_{j=1}^{n}l_j}=\min\left\{h_{\alpha_s}\right\}$$

and

$$\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{qn\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\right)^{1/q}\leq s$$

$$\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\max\left\{g_{\alpha_s}\right\}\right)^{qn\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\right)^{1/q}\leq \max\left\{g_{\alpha_s}\right\}$$

.

For the probabilities, it is easy to get $\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\geq\prod\prod\left\{\min\left\{p_{h_{\alpha_s}}\right\}\right\}$ and $\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\leq\prod\prod\left\{\max\left\{p_{h_{\alpha_s}}\right\}\right\}$. In addition, according to Theorem 3, we have

$$q-RPDHFPMM^{L}\left(\alpha^{-}\right)=$$

$$\bigcup\left\{\left\{\left(\left(1-\prod\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min\left\{h_{\alpha_s}\right\}\right)^{n\delta_j}\right)^{l_j}\right)^{1/n!}\right)^{1/q}\right)^{1\Big/\sum_{j=1}^{n}l_j}\middle|\prod\prod\min\left\{p_{h_{\alpha_s}}\right\}\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\prod\left(1-\prod_{j=1}^{n}\left(1-\left(\max\left\{g_{\alpha_s}\right\}\right)^{qn\delta_j}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\right)^{1/q}\middle|\prod\prod\max\left\{t_{g_{\alpha_s}}\right\}\right\}\right\}$$

According to the score function, we have

$$q-RPDHFPMM^{L}\left(\alpha^{-}\right)\leq q-RPDHFPMM^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right).$$

Similarly, we have

$$q-RPDHFPMM^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)\leq q-RPDHFPMM^{L}\left(\alpha^{+}\right),$$

and so that the proof of Theorem 4 is completed. □

From Definition 10, we can find out that the proposed *q*-RPDHFPMM operator is a generalized AO. Hence, it is interesting and necessary to study the special cases of the *q*-RPDHFPMM operator with respect to its contained parameters, which are presented as follows.

Case 1. *If* $L=(1,0,0,\ldots,0)$*, then the q-RPDHFPMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power average (q-RPDHFPA) operator, i.e.,*

$$q-RPDHFPMM^{(1,0,0,\ldots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$=\bigcup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\left(\left(1-\prod_{j=1}^{n}\left(1-\mu_j^q\right)^{\delta_j}\right)^{1/q}\right)\middle|\prod_{j=1}^{n}p_{\mu_j}\right\},\left\{\prod_{j=1}^{n}\nu_j^{\delta_j}\middle|\prod_{j=1}^{n}t_{\nu_j}\right\}\right\}\tag{25}$$

$$=\bigoplus_{j=1}^{n}\delta_j\alpha_j=q-RPDHFPA\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

In this case, if $Sup\left(\alpha_i,\alpha_j\right)=s>0$ for all $i,j=1,2,\ldots,n\left(i\neq j\right)$, then the *q*-RPDHFPMM operator reduces to the *q*-rung probabilistic dual hesitant fuzzy average (*q*-RPDHFA) operator i.e.,

$$q-RPDHFPMM^{(1,0,0,\ldots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$=\bigcup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\left(\left(1-\prod_{j=1}^{n}\left(1-\mu_j^q\right)^{1/n}\right)^{1/q}\right)\middle|\prod_{j=1}^{n}p_{\mu_j}\right\},\left\{\prod_{j=1}^{n}\nu_j^{1/n}\middle|\prod_{j=1}^{n}t_{\nu_j}\right\}\right\} \tag{26}$$

$$=\frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}\alpha_j=q-RPDHFA\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

Case 2. *If* $L=\left(1,1,0,0,\ldots,0\right)$, *then the q-RPDHFPMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power Bonferroni mean (q-RPDHFPBM) operator, i.e.,*

$$q-RPDHFPMM^{(1,1,0,0,\ldots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$=\bigcup_{\mu_i\in h_i,\mu_j\in h_j,\nu_i\in g_i,\nu_j\in g_j}\left\{\left\{\left(\left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-\left(1-\mu_i^q\right)^{n\delta_i}\right)\left(1-\left(1-\mu_j^q\right)^{n\delta_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{q}}\right)^{1/2}\middle|\prod_{i,j=1,i\neq j}^{n}p\right.\right.$$

$$\left.\left.\left\{\left(1-\left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-\nu_i^{qn\delta_i}\right)\left(1-\nu_j^{qn\delta_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)^{1/q}\middle|\prod_{i,j=1,i\neq j}^{n}t_{\nu_i}t_{\nu_j}\right\}\right\} \tag{27}$$

$$=\left(\frac{1}{n\left(n-1\right)}\overset{n}{\underset{\substack{i,j=1\\i\neq j}}{\oplus}}\left(\left(n\delta_i\alpha_i\right)\otimes\left(n\delta_j\alpha_j\right)\right)\right)^{\frac{1}{2}}=q-RPDHFPBM^{1,1}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

In this case, if $Sup\left(\alpha_i,\alpha_j\right)=s>0$ for $i,j=1,2,\ldots,n\left(i\neq j\right)$, then the *q*-RPDHFPMM operator reduces to the *q*-rung probabilistic dual hesitant fuzzy Bonferroni mean (*q*-RPDHFBM) operator, i.e.,

$$q-RPDHFPMM^{(1,1,0,0,\ldots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$\bigcup_{\mu_i\in h_i,\mu_j\in h_j,\nu_i\in g_i,\nu_j\in g_j}\left\{\left\{\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(\mu_i\mu_j\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{1/2q}\middle|\prod_{i,j=1,i\neq j}^{n}p_{\mu_i}p_{\mu_j}\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(\nu_i^q+\nu_j^q-\nu_i^q\nu_j^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right)^{1/q}\middle|\prod_{i,j=1,i\neq j}^{n}t_{\nu_i}t_{\nu_j}\right\}\right\} \tag{28}$$

$$=\left(\frac{1}{n\left(n-1\right)}\overset{n}{\underset{\substack{i,j=1\\i\neq j}}{\oplus}}\left(\alpha_i\otimes\alpha_j\right)\right)^{\frac{1}{2}}=q-RPDHFBM^{1,1}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

Case 3. *If* $L=(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})$, *then q-RPDHFPMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power Maclaurin symmetric mean (q-RPDHFPMSM) operator, i.e.,*

$$q-RPDHFPMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

$$=\bigcup_{\mu_{i_j}\in h_{i_j},\nu_{i_j}\in g_{i_j}}\left\{\left\{\left(\left(1-\left(\prod_{1<i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^{q}\right)^{n\delta_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{q}}\right)^{\frac{1}{k}}\middle|\prod_{1<i_1<i_2<\ldots<i_k\le n}\prod_{j=1}^{k}p_{\mu_{i_j}}\right\}\right.,$$

$$\left.\left\{\left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\nu_{i_j}^{n\delta_{i_j}}\right)^{q}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}}\middle|\prod_{1\le i_1<i_2<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\nu_{i_j}}\right\}\right\} \tag{29}$$

$$=\left(\frac{1}{C_n^k}\underset{1\le i_1<i_2<\ldots<i_k\le n}{\oplus}\overset{n}{\underset{j=1}{\otimes}}\left(n\delta_{i_j}\alpha_{i_j}\right)\right)^{1/k}=q-RPDHFPMSM^{(k)}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

In this case, if $Sup(\alpha_i,\alpha_j)=t\,(t>0)$ for $i,j=1,2,\ldots,n\,(i\ne j)$, then the q-RPDHFPMM operator reduces to the q-rung probabilistic dual hesitant fuzzy Maclaurin symmetric mean (q-RPDHFMSM) operator, i.e.,

$$q-RPDHFPMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\cdots,\alpha_n)=$$

$$\bigcup_{\mu_{i_j}\in h_{i_j},\nu_{i_j}\in g_{i_j}}\left\{\left\{\left(\left(\left(1-\left(\prod_{1<i_1<i_2<\ldots<i_k\le n}\left(1-\left(\prod_{j=1}^{k}\mu_{i_j}\right)^{q}\right)\right)^{1/C_n^k}\right)^{1/q}\right)^{1/k}\middle|\prod_{1<i_1<i_2<\ldots<i_k\le n}\prod_{j=1}^{k}p_{\mu_{i_j}}\right\}\right.,$$

$$\left.\left\{\left(1-\left(1-\left(\prod_{1<i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\nu_{i_j}^{q}\right)\right)\right)^{1/C_n^k}\right)^{1/k}\right)^{1/q}\middle|\prod_{1\le i_1<i_2<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\nu_{i_j}}\right\}\right\} \tag{30}$$

$$=\left(\frac{1}{C_n^k}\underset{1\le i_1<i_2<\ldots<i_k\le n}{\oplus}\overset{n}{\underset{j=1}{\otimes}}\alpha_{i_j}\right)^{1/k}=q-RPDHFMSM^{(k)}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

Case 4. *If* $L=(1,1,\ldots,1)$ *or* $L=(1/n,1/n,\ldots,1/n)$, *then the q-RPDHFPMM operator reduces to the following form*

$$q-RPDHFPMM^{(1,1,\ldots,1)\ or\ (1/n,1/n,\ldots,1/n)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\overset{n}{\underset{j=1}{\otimes}}\left(n\delta_j\alpha_j\right)^{1/n}$$

$$=\bigcup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\left(\prod_{j=1}^{n}\left(1-\left(1-\mu_j^{q}\right)^{n\delta_j}\right)\right)^{1/qn}\middle|\prod_{j=1}^{n}p_{\mu_j}\right\},\left\{\left(1-\left(\prod_{j=1}^{n}\left(1-\nu_j^{qn\delta_j}\right)\right)^{1/n}\right)^{1/q}\middle|\prod_{j=1}^{n}t_{\nu_j}\right\}\right\} \tag{31}$$

In this case, if $Sup(\alpha_i,\alpha_j)=t(t>0)$ for $i,j=1,2,\ldots,n(i\neq j)$, then q-RPDHFPMM operator reduces to the q-rung probabilistic dual hesitant fuzzy geometric (q-RPDHFG) operator, i.e.,

$$q-RPDHFPMM^{(1,1,\ldots,1)\ or\ (1/n,1/n,\ldots,1/n)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=$$

$$=\cup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\prod_{j=1}^{n}\mu_j^{1/n}\left|\prod_{j=1}^{n}p_{\mu_j}\right.\right\},\left\{\left(1-\prod_{j=1}^{n}\left(1-\nu_j^q\right)^{1/n}\right)^{1/q}\left|\prod_{j=1}^{n}t_{\nu_j}\right.\right\}\right\} \tag{32}$$

$$=\otimes_{j=1}^{n}\alpha_j^{1/n}=q-RPDHFG(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

Case 5. *If q = 2, then the q-RPDHFPMM operator reduces to the following form probabilistic dual Pythagorean hesitant fuzzy power Muirhead mean (PDPHFPMM) operator, i.e.,*

$$q-RPDHFPMM_{q=2}^{L}(\alpha_1,\alpha_2,\cdots,\alpha_n)=$$

$$\cup_{\mu_{\vartheta(j)}\in h_{\vartheta(j)},\nu_{\vartheta(j)}\in g_{\vartheta(j)}}\left\{\left\{\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^2\right)^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/2}\right)^{1\Big/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\right.\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{2n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\right)^{1/2}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right.\right\}\right\} \tag{33}$$

$$=\left(\frac{1}{n!}\oplus_{\vartheta\in S_n}\otimes_{j=1}^{n}\left(n\delta_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}=PDPHFPMM^{L}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

Case 6. *If q = 1, then the q-RPDHFPMM reduces to the probabilistic dual hesitant fuzzy power Muirhead mean (PDHFPMM) operator i.e.,*

$$q-RPDHFPMM_{q=1}^{L}(\alpha_1,\alpha_2,\cdots,\alpha_n)=$$

$$=\cup_{\mu_{\vartheta(j)}\in h_{\vartheta(j)},\nu_{\vartheta(j)}\in g_{\vartheta(j)}}\left\{\left\{\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}\right)^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\right.\right\},\right.$$

$$\left.\left\{1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{n\delta_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1\Big/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right.\right\}\right\} \tag{34}$$

$$=\left(\frac{1}{n!}\oplus_{\vartheta\in S_n}\otimes_{j=1}^{n}\alpha_{\vartheta(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}=PDHFMM^{L}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

4.2. The q-Rung Probabilistic Dual Hesitant Fuzzy Power Weighted Muirhead Mean (q-RPDHFPWMM) Operator

Definition 11. *Let* $\alpha_j = \left(h_j \left| p_{h_j}, g_j \right| t_{g_j}\right) (j = 1,2,...,n)$ *be a collection of q-RPDHFEs,* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters and* $w = (w_1, w_2, ..., w_n)^T$ *be the corresponding weight vector, satisfying* $0 \le w_j \le 1$ *and* $\sum_{j=1}^{n} w_j = 1$ *. The q-rung probabilistic dual hesitant fuzzy power weighted Muirhead mean (q-RPDHFPWMM) operator is expressed as*

$$q-RPDHFPWMM^L(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\frac{1}{n!} \underset{\vartheta \in S_n}{\oplus} \overset{n}{\underset{j=1}{\otimes}} \left(n \frac{w_{\vartheta(j)}\left(1 + T\left(\alpha_{\vartheta(j)}\right)\right)}{\sum_{j=1}^{n} w_j \left(1 + T\left(\alpha_j\right)\right)} \alpha_{\vartheta(j)} \right)^{l_j} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \tag{35}$$

where

$$T(\alpha_j) = \sum_{i=1, i \ne j}^{n} Sup(\alpha_i, \alpha_j), Sup(\alpha_i, \alpha_j) = 1 - d(\alpha_i, \alpha_j) \tag{36}$$

$d(\alpha_i, \alpha_j)$ *is the distance between* α_i *and* α_j, $\vartheta(j) (j = 1,2,...,n)$ *represents any permutation of* $(1,2,...,n)$ *,* S_n *denotes all possible permutations of* $(1,2,...,n)$, *n is the balancing coefficient, and* $Sup(\alpha_i, \alpha_j)$ *denotes the support for* α_i *from* α_j*, satisfying the properties in Definition 10. Similarly, let*

$$\xi_j = \frac{w_j \left(1 + T\left(\alpha_j\right)\right)}{\sum_{j=1}^{n} w_j \left(1 + T\left(\alpha_j\right)\right)}, \tag{37}$$

then Equation (35) can be written as

$$q-RPDHFPWMM^L(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\frac{1}{n!} \underset{\vartheta \in S_n}{\oplus} \overset{n}{\underset{j=1}{\otimes}} \left(n \xi_{\vartheta(j)} \alpha_{\vartheta(j)} \right)^{l_j} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \tag{38}$$

where $0 \le \xi_j \le 1$ *and* $\sum_{j=1}^{n} \xi_j = 1$.

Theorem 5. *Let* $\alpha_j = \left(h_j \left| p_{h_j}, g_j \right| t_{g_j}\right) (j = 1,2,...,n)$ *be a collection of q-RPDHFEs, and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. The aggregated value using q-RPDHFPWMM operator is still a q-RPDHFE and*

$$\begin{aligned} & q-RPDHFPWMM^L(\alpha_1, \alpha_2, \cdots, \alpha_n) = \\ & \cup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(\left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^{q} \right)^{n\xi_{\vartheta(j)}} \right)^{l_j} \right)^{1/n!} \right)^{1/q} \right)^{1 \big/ \sum_{j=1}^{n} l_j} \middle| \prod_{\vartheta \in S_n} \prod_{i=1}^{n} p_{\mu_{\vartheta(j)}} \right\}, \right. \\ & \left. \left\{ \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\vartheta(j)}^{qn\xi_{\vartheta(j)}} \right)^{l_i} \right)^{1/n!} \right)^{1 \big/ \sum_{i=1}^{n} l_i} \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} t_{\nu_{\vartheta(j)}} \right\} \right\} \end{aligned} \tag{39}$$

The proof of Theorem 5 is similar to that of Theorem 3, which is mitted here. In addition, it is easy to prove that the *q*-RPDHFPWMM operator has the property of boundedness, but does not have the properties of monotonicity and idempotency.

4.3. The q-Rung Probabilistic Dual Hesitant Fuzzy Power Dual Muirhead Mean (q-RPDHFPDMM) Operator

Definition 12. *Let* $\alpha_j = \left(h_j \middle| p_{h_j}, g_j \middle| t_{g_j}\right)(j = 1,2,...,n)$ *be a collection of q-RPDHFEs, and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. Then q-rung probabilistic dual hesitant fuzzy power dual Muirhead mean (q-RPDHFPDMM) operator is expressed*

$$q - RPDHFPDMM^L(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{\sum_{j=1}^n l_j}\left(\underset{\vartheta \in S_n}{\otimes} \overset{n}{\underset{j=1}{\oplus}} l_j \alpha_{\vartheta(j)}^{n\left(1+T\left(\alpha_{\vartheta(j)}\right)\right)/\sum_{j=1}^n\left(1+T\left(\alpha_j\right)\right)}\right)^{\frac{1}{n!}}, \tag{40}$$

where

$$T(\alpha_j) = \sum_{t=1, t \neq j}^n Sup(\alpha_t, \alpha_j) \quad Sup(\alpha_i, \alpha_j) = 1 - d(\alpha_i, \alpha_j), \tag{16}$$

and $\vartheta(j)(j = 1,2,...,n)$ *is any permutation of (1, 2, …, n),* S_n *is the collection of all permutations of (1, 2, …, n), and n is the balancing coefficient.* $d(\alpha_i, \alpha_j)$ *is the distance between* α_i *and* α_j*, and* $Sup(\alpha_i, \alpha_j)$ *denotes the support for* α_i *from* α_j*, satisfying the properties presented in Definition 10. To simplify Equation (40), we denote*

$$\tau_j = \frac{1 + T(\alpha_j)}{\sum_{j=1}^n \left(1 + T(\alpha_j)\right)}, \tag{17}$$

then (40) can be written as

$$q - RPDHFPDMM^L(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{\sum_{j=1}^n l_j}\left(\underset{\vartheta \in S_n}{\otimes} \overset{n}{\underset{j=1}{\oplus}} l_j \alpha_{\vartheta(j)}^{n\tau_{\vartheta(j)}}\right)^{\frac{1}{n!}}, \tag{18}$$

where $0 \le \tau_j \le 1$ *and* $\sum_{j=1}^n \tau_j = 1$.

Theorem 6. *Let* $\alpha_j = \left(h_j \middle| p_{h_j}, g_j \middle| t_{g_j}\right)(j = 1,2,...,n)$ *be a collection of q-RPDHFEs, and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. The aggregated value using q-RPDHFPDMM operator is still a q-RPDHFE and*

$$q - RPDHFPDMM^L(\alpha_1, \alpha_2, ..., \alpha_n) =$$

$$\cup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^n \left(1 - \left(\mu_{\vartheta(j)}^{n\tau_{\vartheta(j)}} \right)^q \right)^{l_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n l_j}} \right)^{1/q} \middle| \prod_{\vartheta \in S_n} \prod_{j=1}^n p_{\mu_{\vartheta(j)}} \right\}, \right. \tag{19}$$

$$\left\{\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\vartheta(j)}^{q}\right)^{n\tau_{\vartheta(j)}}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right\}\right\}$$

The proof of Theorem 6 is similar to that of Theorem 3, which is mitted here. In addition, it is easy to prove that the q-RPDHFPDMM operator has the property of boundedness, but does not have the properties of monotonicity and idempotency.

In the followings, we discuss some special cases of the q-RPDHFPDMM operator with respect to its contained parameters.

Case 7. *If* $L=(1,0,0,\dots,0)$*, then the q-RPDHFPDMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power geometric (q-RPDHFPG) operator, i.e.,*

$$q-RPDHFPDMM^{(1,0,\cdots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$=\bigcup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\prod_{j=1}^{n}\mu_j^{\tau_j}\left|\prod_{j=1}^{n}p_{\mu_j}\right.\right\},\left\{\left(1-\prod_{j=1}^{n}\left(1-v_j^{q}\right)^{\tau_j}\right)^{1/q}\left|\prod_{j=1}^{n}t_{\nu_j}\right.\right\}\right\} \tag{20}$$

$$=\overset{n}{\underset{j=1}{\otimes}}\alpha_j^{\tau_j}=q-RPDHFPG\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

In this case, if $\sup\left(\alpha_i,\alpha_j\right)=s>0$ for all $i,j=1,2,\dots,n(i\neq j)$, then the q-RPDHFPDMM operator reduces to the q-RPDHFG operator, which is shown as Equation (32).

Case 8. *If* $L=(1,1,0,0,\dots,0)$*, then the q-RPDHFPDMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power geometric Bonferroni mean (q-RPDHFPGBM) operator, i.e.,*

$$q-RPDHFPDMM^{(1,1,0,0,\dots,0)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$\bigcup_{\mu_i\in h_i,\mu_j\in h_j,\nu_i\in g_i,\nu_j\in g_j}\left\{\left\{\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\mu_i^{nq\tau_i}\right)\left(1-\mu_j^{nq\tau_j}\right)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{qn(n-1)}}\left|\prod_{i,j=1,i\neq j}^{n}p_{\mu_i}p_{\mu_j}\right.\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-\nu_i^{q}\right)^{n\tau_i}\right)\left(1-\left(1-\nu_j^{q}\right)^{n\tau_j}\right)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{q}}\left|\prod_{i,j=1,i\neq j}^{n}t_{\nu_i}t_{\nu_j}\right.\right\}\right\} \tag{21}$$

$$=\frac{1}{2}\left(\overset{n}{\underset{\substack{i,j=1\\i\neq j}}{\otimes}}\left(\alpha_i^{n\tau_i}\oplus\alpha_j^{n\tau_j}\right)\right)^{\frac{1}{n(n-1)}}=q-RPDHFPGBM^{1,1}\left(\alpha_1,\alpha_2,\dots,\alpha_n\right)$$

In this case, if $Sup\left(\alpha_i,\alpha_j\right)=s>0$ for $i,j=1,2,\dots,n(i\neq j)$, then the q-RPDHFPDMM operator reduces to the q-rung probabilistic dual hesitant fuzzy geometric Bonferroni mean (q-RPDHFGBM) operator, i.e.,

$$q-RPDHFPDMM^{(1,1,0,0,\ldots,0)}(\alpha_1,\alpha_2,\cdots,\alpha_n)=$$

$$\bigcup_{\mu_i\in h_i,\mu_j\in h_j,\nu_i\in g_i,\nu_j\in g_j}\left\{\left\{\left(\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\mu_i^q\right)\left(1-\mu_j^q\right)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{qn(n-1)}}\middle|\prod_{i,j=1,i\neq j}^{n}p_{\mu_i}p_{\mu_j}\right\},\right.$$

$$\left.\left\{\left(1-\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\nu_i^q\nu_j^q\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{q}}\middle|\prod_{i,j=1,i\neq j}^{n}t_{\nu_i}t_{\nu_j}\right\}\right\} \tag{22}$$

$$=\frac{1}{2}\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(\alpha_i\oplus\alpha_j\right)\right)^{\frac{1}{n(n-1)}}=q-RPDHFGBM^{1,1}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

Case 9. *If* $L=(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})$*, then q-RPDHFPDMM operator reduces to the q-rung probabilistic dual hesitant fuzzy power dual Maclaurin symmetric mean (q-RPDHFPDMSM) operator, i.e.,*

$$q-RPDHFPDMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

$$=\bigcup_{\mu_{i_j}\in h_{i_j},\nu_{i_j}\in g_{i_j}}\left\{\left\{\left(\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\mu_{i_j}^{qn\tau_{i_j}}\right)\right)^{1/C_n^k}\right)^{1/C_n^k}\right)^{1/q}\middle|\prod_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}p_{\mu_{i_j}}\right\},\right.$$

$$\left.\left\{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\nu_{i_j}^q\right)^{n\tau_{i_j}}\right)\right)^{1/C_n^k}\right)^{1/qk}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\nu_{i_j}}\right\}\right\} \tag{23}$$

$$=\frac{1}{k}\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\overset{k}{\underset{j=1}{\oplus}}\alpha_{i_j}^{n\tau_{i_j}}\right)^{1/C_n^k}\right)=q-RPDHFPDMSM^{(k)}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

In this case, if $Sup(\alpha_i,\alpha_j)=s>0$ for $i,j=1,2,\ldots,n(i\neq j)$, then the *q*-RPDHFPDMM operator reduces to the *q*-rung probabilistic dual hesitant fuzzy dual Maclaurin symmetric mean (*q*-RPDHFDMSM) operator, i.e.,

$$q-RPDHFPDMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

$$=\bigcup_{\mu_{i_j}\in h_{i_j},\nu_{i_j}\in g_{i_j}}\left\{\left\{\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\mu_{i_j}^{q}\right)\right)^{1/C_n^k}\right)^{1/C_n^k}\right)^{1/q}\middle|\prod_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}p_{\mu_{i_j}}\right\},\right. \tag{24}$$

$$\left\{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\nu_{i_j}^{q}\right)^{1/C_n^k}\right)^{1/qk}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\nu_{i_j}}\right\}\right\}$$

$$=\frac{1}{k}\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\overset{k}{\underset{j=1}{\oplus}}\alpha_{i_j}\right)^{1/C_n^k}\right)=q-RPDHFDMSM^{(k)}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

Case 10. *If* $L=(1,1,\ldots,1)$ *or* $L=(1/n,1/n,\ldots,1/n)$, *then the q-RPDHFPDMM operator reduces to the following form*

$$q-RPDHFPDMM^{(1/n,1/n,\cdots,1/n)or(1,1,\cdots,1)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}\alpha_j^{n\tau_j}$$

$$=\cup_{\mu_j\in h_j,\nu_j\in g_j}\left\{\left\{\left(1-\left(\prod_{j=1}^{n}\left(1-\mu_j^{qn\tau_j}\right)\right)^{1/n}\right)^{1/q}\middle|\prod_{j=1}^{n}p_{\mu_j}\right\},\left\{\left(\prod_{j=1}^{n}\left(1-\left(1-\nu_j^{q}\right)^{n\tau_j}\right)\right)^{1/qn}\middle|\prod_{j=1}^{n}t_{\nu_j}\right\}\right\} \quad (25)$$

In this case, if $Sup\left(\alpha_i,\alpha_j\right)=t\left(t>0\right)$ for $i,j=1,2,\ldots,n\left(i\ne j\right)$, then q-RPDHFPDMM operator reduces to the q-RPDHFA operator, which is shown as Equation (25).

Case 11. *If q = 2, then the q-RPDHFPDMM operator reduces to the probabilistic dual Pythagorean hesitant fuzzy power dual Muirhead mean (PDPHFPDMM) operator, i.e.,*

$$q-RPDHFPDMM_{q=2}^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=$$

$$=\cup_{\mu_{\vartheta(j)}\in h_{\vartheta(j)},\nu_{\vartheta(j)}\in g_{\vartheta(j)}}\left\{\left\{\left(1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\vartheta(j)}^{n\tau_{\vartheta(j)}}\right)^2\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/2}\middle|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\right\},\right.$$

$$\left.\left\{\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\vartheta(j)}^{2}\right)^{n\tau_{\vartheta(j)}}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/2}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\middle|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right\}\right\} \quad (26)$$

$$=\frac{1}{\sum_{j=1}^{n}l_j}\left(\underset{\vartheta\in S_n}{\otimes}\overset{n}{\underset{j=1}{\oplus}}l_j\alpha_{\vartheta(j)}^{n\tau_{\vartheta(j)}}\right)^{\frac{1}{n!}}=PDPHFPDMM^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

Case 12. *If q = 1, then the q-RPDHFPDMM reduces to the probabilistic dual hesitant fuzzy power dual Muirhead mean (PDHFPDMM) operator, i.e.,*

$$q-RPDHFPDMM_{q=1}^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)= \quad (27)$$

$$=\bigcup_{\mu_{\vartheta(j)}\in h_{\vartheta(j)},\nu_{\vartheta(j)}\in g_{\vartheta(j)}}\left\{\left\{\left[1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\vartheta(j)}^{n\tau_{\vartheta(j)}}\right)\right)^{l_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\mu_{\vartheta(j)}}\right.\right\}\right.,$$

$$\left.\left\{\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\vartheta(j)}\right)^{n\tau_{\vartheta(j)}}\right)^{l_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}l_j}\left|\prod_{\vartheta\in S_n}\prod_{j=1}^{n}t_{\nu_{\vartheta(j)}}\right.\right\}\right\}$$

$$=\frac{1}{\sum_{j=1}^{n}l_j}\left(\underset{\vartheta\in S_n}{\otimes}\overset{n}{\underset{j=1}{\oplus}}l_j\alpha_{\vartheta(j)}^{n\tau_{\vartheta(j)}}\right)^{\frac{1}{n!}}=PDHFPDMM^{L}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$$

4.4. The q-Rung Probabilistic Dual Hesitant Fuzzy Power Weighted Dual Muirhead Mean (q-RPDHFPWDMM) Operator

Definition 13. *Let* $\alpha_j=\left(h_j\left|p_{h_j},g_j\right|t_{g_j}\right)(j=1,2,\ldots,n)$ *be a collection of q-RPDHFEs,* $L=\left(l_1,l_2,\cdots l_n\right)\in R^n$ *be a vector of parameters and* $w=\left(w_1,w_2,\ldots,w_n\right)^T$ *be the corresponding weight vector, satisfying that* $0\le w_j\le 1$ *and* $\sum_{j=1}^{n}w_j=1$. *The q-rung probabilistic dual hesitant fuzzy power weighted dual Muirhead mean (q-RPDHFPWDMM) operator is expressed as*

$$q-RPDHFPWDMM^{L}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\frac{1}{\sum_{j=1}^{n}l_j}\left(\underset{\vartheta\in S_n}{\otimes}\overset{n}{\underset{j=1}{\oplus}}l_j\alpha_{\vartheta(j)}^{nw_{\vartheta(j)}\left(1+T\left(\alpha_{\vartheta(j)}\right)\right)/\sum_{j=1}^{n}w_j\left(1+T\left(\alpha_j\right)\right)}\right)^{\frac{1}{n!}},\quad(28)$$

where

$$T\left(\alpha_j\right)=\sum_{t=1,t\ne j}^{n}Sup\left(\alpha_t,\alpha_j\right)\quad Sup\left(\alpha_i,\alpha_j\right)=1-d\left(\alpha_i,\alpha_j\right),\quad(29)$$

and $\vartheta(j)(j=1,2,\ldots,n)$ *is any permutation of (1, 2, …, n),* S_n *is the collection of all permutations of (1, 2, …, n), and n is the balancing coefficient.* $d\left(\alpha_i,\alpha_j\right)$ *is the distance between* α_i *and* α_j, *and* $Sup\left(\alpha_i,\alpha_j\right)$ *denotes the support for* α_i *from* α_j, *satisfying the properties presented in Definition 10. Similarly, we assume*

$$\eta_j=\frac{w_j\left(1+T\left(\alpha_j\right)\right)}{\sum_{j=1}^{n}w_j\left(1+T\left(\alpha_j\right)\right)},\quad(30)$$

thus (53) can be written as

$$q-RPDHFPWDMM^{L}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\frac{1}{\sum_{j=1}^{n}l_j}\left(\underset{\vartheta\in S_n}{\otimes}\overset{n}{\underset{j=1}{\oplus}}l_j\alpha_{\vartheta(j)}^{n\eta_{\vartheta(j)}}\right)^{\frac{1}{n!}},\quad(31)$$

where $0\le\eta_j\le 1$ *and* $\sum_{j=1}^{n}\eta_j=1$.

Theorem 7. *Let* $\alpha_j = \left(h_j \left| p_{h_j}, g_j \right| t_{g_j} \right) (j = 1, 2, \ldots, n)$ *be a collection of q-RPDHFEs, and* $L = (l_1, l_2, \cdots l_n) \in R^n$ *be a vector of parameters. The aggregated value using q-RPDHFPWDMM operator is still a q-RPDHFE and*

$$
\begin{aligned}
&q - RPDHFPWDMM^L (\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&\bigcup_{\mu_{\vartheta(j)} \in h_{\vartheta(j)}, \nu_{\vartheta(j)} \in g_{\vartheta(j)}} \left\{ \left\{ \left(\left(1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{\vartheta(j)}^{n\eta_{\vartheta(j)}} \right)^q \right)^{l_j} \right) \right)^{1/n!} \right)^{1 \big/ \sum_{j=1}^{n} l_j} \right)^{1/q} \left| \prod_{\vartheta \in S_n} \prod_{i=1}^{n} p_{\mu_{\vartheta(j)}} \right. \right\}, \right. \\
&\left. \left\{ \left(\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \nu_{\vartheta(j)}^q \right)^{n\eta_{\vartheta(j)}} \right)^{l_j} \right) \right)^{1/n!} \right)^{1/q} \right)^{1 \big/ \sum_{j=1}^{n} l_j} \left| \prod_{\vartheta \in S_n} \prod_{j=1}^{n} t_{\nu_{\vartheta(j)}} \right. \right\} \right\}
\end{aligned}
\tag{32}
$$

The proof of Theorem 7 is similar to that of Theorem 3, which is mitted here. In addition, it is easy to prove that the q-RPDHFPWDMM operator has the property of boundedness, but does not have the properties of monotonicity and idempotency.

5. A Novel MADM Approach Based on *Q*-RPDHFEs

This section gives a MADM method under q-RPDHFSs on the basis of the aforementioned AOs. We assume the alternative set is denoted as $X = \{x_1, x_2, \ldots, x_m\}$, and the attribute set is denoted as $G = \{G_1, G_2, \ldots, G_n\}$. The weight vector of attributes is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$. When evaluating the performance of alternative $x_i \, (i = 1, 2, \ldots, m)$ under attribute $G_j \, (j = 1, 2, \ldots, n)$, each DM provides his/her preferred MDs and NMDs and based on DMs' preferred degrees and the probabilistic values, the overall evaluation value can be denoted by $\alpha_{ij} = (h_{ij} \mid p_{ij}, g_{ij} \mid q_{ij})$, which is a q-RPDHFE. Finally, a q-rung probabilistic dual hesitant fuzzy matrix can be obtained, which can be denoted as $A = (\alpha_{ij})_{m \times n}$. Based on the proposed AOs, we put forward a new MADM method, which consists of the following steps

Step 1. Normalize the decision matrix. In real MADM problems, attributes can be generally divided into two types: benefit attribute and cost attribute. Therefore, the decision matrix should be normalized in the following method

$$
\alpha_{ij} = \begin{cases} (h_{ij} \mid p_{ij}, g_{ij} \mid t_{ij}) & G_j \in I_1 \\ (g_{ij} \mid t_{ij}, h_{ij} \mid p_{ij}) & G_j \in I_2 \end{cases},
\tag{33}
$$

where I_1 and I_2 represent the benefit-type attribute and the cost-type attribute respectively.

Step 2. Calculate the support $Sup(\alpha_{ij}, \alpha_{is})$ by

$$
Sup(\alpha_{ij}, \alpha_{is}) = 1 - d(\alpha_{ij}, \alpha_{is}) (i = 1, 2, \ldots, m; j, s = 1, 2, \ldots, n; j \ne s),
\tag{34}
$$

where $d(\alpha_{ij}, \alpha_{is})$ is the distance between the two q-PRDHFEs α_{ij} and α_{is}.

Step 3. Compute the overall supports $T(\alpha_{ij})$ by

$$T\left(\alpha_{ij}\right)=\sum_{j=1;j\neq s}^{n} Sup\left(\alpha_{ij},\alpha_{is}\right). \tag{35}$$

Step 4. Compute the power weight ξ_{ij} associated with the q-PRDHFE α_{ij} by

$$\xi_{ij}=\frac{w_j\left(1+T\left(\alpha_{ij}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\alpha_{ij}\right)\right)}. \tag{36}$$

Step 5. Utilize the q-RPDHFPWMM operator

$$\alpha_i = q-RPDHFPWMM^L\left(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{in}\right), \tag{37}$$

or the q-RPDHFPWDMM operator

$$\alpha_i = q-RPDHFPWDMM^L\left(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{in}\right), \tag{38}$$

to determine the collective overall preference value $\alpha_i\left(i=1,2,\ldots,m\right)$ of alternatives $x_i\left(i=1,2,\ldots,m\right)$.

Step 6. According to Definition 8, calculate the score function $S\left(\alpha_i\right)$ and accuracy function $H\left(\alpha_i\right)$ of the overall preference value $\alpha_i\left(i=1,2,\ldots,m\right)$.

Step 7. Order the alternatives $\left\{x_1,x_2,\ldots,x_m\right\}$ and select the optimal alternative(s).

6. Numerical Example

In recent years, more and more enterprises have shown their interests in the issue of investment evaluation in order to achieve long-term stable development. To help them grasp investment opportunities and assess investment project properly, we apply the q-RPDHFS theory to the investment evaluation process and demonstrate the validity of the newly proposed approach, and details are presented in Example 4.

Example 4. *After preliminary analysis, four possible investment alternatives are taken into account, they are denoted by* $\left\{x_1,x_2,x_3,x_4\right\}$. *In this paper, we consider three commonly used attributes in investment evaluation decision: (1)* G_1 *the quality of product and service; (2)* G_2 *social and environmental impacts; (3)* G_3 *economic benefits. The weight vector of the attributes is* $w=\left(0.3,0.2,0.5\right)^T$. *The DMs are required to use the q-RPDHFEs to assess the four alternatives' performance from three aspects respectively. The decision matrix* $A=\left[\alpha_{ij}\right]_{4\times 3}$ *is shown in Table 2.*

Table 2. The decision matrix A given by the domain expert.

	G_1	G_2	G_3
x_1	$\{\{0.7\|0.2,0.6\|0.2,0.5\|0.6\},\{0.2\|1\}\}$	$\{\{0.7\|1\},\{0.5\|1\}\}$	$\{\{0.2\|1\},\{0.2\|1\}\}$
x_2	$\{\{0.1\|1\},\{0.4\|1\}\}$	$\{\{0.3\|1\},\{0.7\|1\}\}$	$\{\{0.7\|1\},\{0.3\|0.5,0.2\|0.5\}\}$
x_3	$\{\{0.6\|1\},\{0.5\|1\}\}$	$\{\{0.6\|1\},\{0.2\|1\}\}$	$\{\{0.1\|1\},\{0.7\|1\}\}$
x_4	$\{\{0.05\|0.7,0.2\|0.3\},\{0.5\|1\}\}$	$\{\{0.3\|1\},\{0.6\|0.5,0.5\|0.5\}\}$	$\{\{0.8\|1\},\{0.5\|1\}\}$

6.1. The Procedure of Choosing the Optimal Alternative

Step 1. Because all the attributes are benefit type, there is no need to standardize the original decision matrix.

Step 2. Compute the support for α_{is} from $\alpha_{if}\left(i=1,2,3,4;s,f=1,2,3;s\neq f\right)$, which can be denoted by Sup_f^s for convenience, and we can obtain

$$Sup_2^1=Sup_1^2=\left(0.9317,0.8475,0.9415,0.9825\right)$$
$$Sup_3^1=Sup_1^3=\left(0.9553,0.8705,0.7835,0.8302\right)$$
$$Sup_3^2=Sup_2^3=\left(0.7740,0.7862,0.7250,0.8232\right)$$

Step 3. Calculate the $T\left(\alpha_{ij}\right)$ according to Equation (54) and we have

$$T=\begin{bmatrix}1.8870 & 1.7057 & 1.7293\\ 1.7180 & 1.6337 & 1.6567\\ 1.7250 & 1.6665 & 1.5085\\ 1.8127 & 1.8057 & 1.6533\end{bmatrix}$$

Step 4. Compute the power weights ξ_{ij} of the q-RPDHFEs α_{ij} and we have

$$\xi=\begin{bmatrix}0.3125 & 0.1952 & 0.4923\\ 0.3053 & 0.1972 & 0.4974\\ 0.3138 & 0.2047 & 0.4815\\ 0.3089 & 0.2054 & 0.4851\end{bmatrix}$$

Step 5. Employ the q-RPDHFPWMM operator (q = 3 and $L=\left(1,1,1\right)$) to aggregate attribute values and we can obtain the comprehensive evaluation values of alternatives. As the aggregation results are very complicated, we omit them here.

Step 6. Calculate the scores $S\left(\alpha_i\right)\left(i=1,2,3,4\right)$ of alternatives base on Definition 8 and we can obtain

$$S\left(\alpha_1\right)=-0.0373\quad S\left(\alpha_2\right)=-0.2261\quad S\left(\alpha_3\right)=-0.1033\quad S\left(\alpha_4\right)=-0.1563$$

Step 7. Therefore, the ranking result is $x_1\succ x_3\succ x_4\succ x_2$. So x_1 is the best investment alternative.

In Step 5, if we use the q-RPDHFPWDMM operator to aggregate attributes, then the score values of alternatives are (q = 3 and $L=\left(1,1,1\right)$)

$$S\left(\alpha_1\right)=0.2632\quad S\left(\alpha_2\right)=0.0421\quad S\left(\alpha_3\right)=0.1607\quad S\left(\alpha_4\right)=0.0481$$

Therefore, the ranking result is $x_1\succ x_3\succ x_4\succ x_2$ and the best investment alternative is also x_1.

6.2. Sensitivity Analysis

In this subsection, we conduct sensitivity analysis by studying the impact of *L* and *q* on the score values and the ranking orders of alternatives.

6.2.1. The Effect of the Parameter Vector *L*

We assign different parameter vectors to *L* in the *q*-RPDHFPWMM and *q*-RPDHFPWDMM operator, and present the score values of alternatives and ranking orders in Tables 3 and 4.

As seen from Table 3, different scores and ranking orders are obtained with different parameter vectors *L* in *q*-RPDHFPWMM. For convenience, we employ the symbol $n_l\left(n_l=1,2,3\right)$ to denote the number of related parameters in parameter vector *L*. When $n_l=1$, along with the increase of the value in *L*, the score values of alternatives also increase. In addition, when $n_l=1$ the ranking order of alternatives is different from others (when $n_l=2,3$). This is because when $n_l=1$, our method does not consider the interrelationship among attributes. When $n_l=2,3$, the interrelationship among attributes is taken into account. Moreover, we can find out that when $n_l=2$, the ranking results is different from that when $n_l=3$. This is because when $n_l=2$, the interrelationship among any two attributes is considered and when $n_l=3$, the interrelationship among all the three attributes is reflected. As seen from Table 4, we can also find the similar phenomena. However, in the *q*-RPDHFPWDMM, when $n_l=1$, the score values become smaller with the increase of the value in *L*, which is opposite to the property of the *q*-RPDHFPWMM.

Table 3. The score values and ranking orders with different *L* in the *q*-RPDHFPWMM.

L	**Score Values** $S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	**Ranking Results**
$L=(1,0,0)$	0.1275	0.1493	0.0032	0.1664	$x_4 \succ x_2 \succ x_1 \succ x_3$
$L=(2,0,0)$	0.1552	0.2401	0.0289	0.2792	$x_4 \succ x_2 \succ x_1 \succ x_3$
$L=(5,0,0)$	0.1922	0.3426	0.0586	0.4061	$x_4 \succ x_2 \succ x_1 \succ x_3$
$L=(10,0,0)$	0.2159	0.3922	0.0806	0.4759	$x_4 \succ x_2 \succ x_1 \succ x_3$
$L=(1,1,0)$	0.0674	−0.1077	−0.0342	−0.1035	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(1,2,0)$	0.0922	−0.0581	−0.0125	−0.0515	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(1,3,0)$	0.1130	0.0030	0.0034	0.0161	$x_1 \succ x_4 \succ x_3 \succ x_2$
$L=(2,2,0)$	0.1015	−0.0794	−0.0040	−0.0842	$x_1 \succ x_3 \succ x_2 \succ x_4$
$L=(1,1,1)$	−0.0373	−0.2261	−0.1033	−0.1563	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(2,2,2)$	−0.0353	−0.2261	−0.1033	−0.1507	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(3,3,3)$	−0.0335	−0.2261	−0.1033	−0.1493	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(1,2,3)$	0.0034	−0.1781	−0.0719	−0.1261	$x_1 \succ x_3 \succ x_4 \succ x_2$

Table 4. The score values and ranking orders with different *L* in the *q*-RPDHFPWDMM.

L	**Score Values** $S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	**Ranking Results**
$L=(1,0,0)$	0.0115	−0.0709	−0.2017	−0.1002	$x_1 \succ x_2 \succ x_4 \succ x_3$
$L=(2,0,0)$	−0.0057	−0.0999	−0.2659	−0.1064	$x_1 \succ x_2 \succ x_4 \succ x_3$
$L=(5,0,0)$	−0.0317	−0.1468	−0.3547	−0.1223	$x_1 \succ x_4 \succ x_2 \succ x_3$

$L=(10,0,0)$	−0.0466	−0.1721	−0.3997	−0.1381	$x_1 \succ x_4 \succ x_2 \succ x_3$
$L=(1,1,0)$	0.2047	0.0199	0.0657	0.0107	$x_1 \succ x_3 \succ x_2 \succ x_4$
$L=(1,2,0)$	0.1814	0.0020	0.0171	−0.0010	$x_1 \succ x_3 \succ x_2 \succ x_4$
$L=(1,3,0)$	0.1514	−0.0218	−0.0383	−0.0162	$x_1 \succ x_4 \succ x_2 \succ x_3$
$L=(2,2,0)$	0.1893	0.0093	0.0234	0.0028	$x_1 \succ x_3 \succ x_2 \succ x_4$
$L=(1,1,1)$	0.2632	0.0421	0.1607	0.0481	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(2,2,2)$	0.2632	0.0394	0.1607	0.0467	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(3,3,3)$	0.2611	0.0374	0.1607	0.0454	$x_1 \succ x_3 \succ x_4 \succ x_2$
$L=(1,2,3)$	0.2363	0.0257	0.1089	0.0329	$x_1 \succ x_3 \succ x_4 \succ x_2$

6.2.2. Sensitivity Analysis of q

We assign different q to solve the same MADM problem and the decision results by using the q-RPDHFPWMM and q-RPDHFPWDMM operators are presented in Tables 5 and 6, respectively.

Table 5. Effect of the parameter q on the score values and ranking results utilizing the q-RPDHFPWMM.

Parameter *q*	**Score Values**				**Ranking Results**
	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
$q = 3$	−0.0373	−0.2261	−0.1033	−0.1563	$x_1 \succ x_3 \succ x_4 \succ x_2$
$q = 5$	−0.0112	−0.0276	−0.1158	−0.0593	$x_1 \succ x_3 \succ x_2 \succ x_4$
$q = 7$	−0.0086	−0.0738	−0.0098	−0.0279	$x_1 \succ x_3 \succ x_4 \succ x_2$
$q = 10$	−0.0040	−0.0370	−0.0022	−0.0091	$x_3 \succ x_1 \succ x_4 \succ x_2$

Table 6. Effect of the parameter q on the score values and ranking results utilizing the q-RPDHFPWDMM.

Parameter *q*	**Score Values**				**Ranking Results**
	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
$q = 3$	0.2632	0.0421	0.1607	0.0481	$x_1 \succ x_3 \succ x_4 \succ x_2$
$q = 5$	0.1421	0.0131	0.0750	0.0347	$x_1 \succ x_3 \succ x_4 \succ x_2$
$q = 7$	0.0849	0.0053	0.0432	0.0228	$x_1 \succ x_3 \succ x_4 \succ x_2$
$q = 10$	0.0408	0.0017	0.0180	0.0013	$x_1 \succ x_3 \succ x_2 \succ x_4$

From Tables 5 and 6, we can see that the scores of the overall values are different by assigning different values of parameter q in the q-RPDHFPWMM and q-RPDHFPWDMM operator. However, no matter what the parameter q is, x_1 and x_3 are the best and second-best alternatives, respectively. In addition, as the parameter q increases, the score values of alternatives become smaller. Hence, how to choose a proper value of q is important for the decision results. In reference 15, Xu et al. gave a principle for choosing an appropriate value of q for dealing with MADM problems based on q-RDHFSs, i.e., the value of q should be taken as the smallest integer that can make the sum of qth power of maximum element in membership degree set and qth power of maximum value in non-membership degree set no larger than one. As q-RPDHFSs are an extension of the q-RDHFSs, we can use the same principle for determining the value of q, when handling MADM problems under q-RPDHFSs. For instance, a group of DMs use $\alpha = \{\{0.1|0.2, 0.5|0.5, 0.8|0.3\}, \{0.4|0.6, 0.7|0.4\}\}$, which is a q-RPDHFE to denote their evaluation value, then as $0.8^2 + 0.7^2 = 1.13 > 1$ and $0.8^3 + 0.7^3 = 0.855 < 1$, the value of q can be chosen as 3.

6.3. Validity Test

To further illustrate the correctness and effectiveness of our proposed method, we employ existing MADM method and our new decision-making method to solve the same example and analyze the decision results. Here, we compare our method with that proposed by Hao et al.'s [12] based on the probabilistic dual hesitant fuzzy weighted average (PDHFWA) operator. It is noted that Hao et al.'s [12] method employs PDHFSs to describe DMs' evaluation information. In addition, our method can also deal with decision-making problems where DMs' evaluation values are in the form of q-RPDHFEs. To make the decision results comparative, we modify Example 2 by employing PDHFSs to describe attribute values.

Example 5. *In this example, DMs utilize PDHFEs to denote their evaluation values and the new decision matrix is presented in Table 7. The weight vector of attributes is still* $w=(0.3,0.2,0.5)^T$.

We use Hao et al.'s [12] and our MADM methods to solve Example 5 and present the decision results in Table 8.

Table 7. The probabilistic dual hesitant fuzzy decision matrix.

	G_1	G_2	G_3
x_1	$\{\{0.7\|0.2,0.6\|0.2,0.5\|0.6\},\{0.2\|1\}\}$	$\{\{0.7\|1\},\{0.2\|1\}\}$	$\{\{0.2\|1\},\{0.2\|1\}\}$
x_2	$\{\{0.1\|1\},\{0.4\|1\}\}$	$\{\{0.3\|1\},\{0.7\|1\}\}$	$\{\{0.7\|1\},\{0.3\|0.5,0.2\|0.5\}\}$
x_3	$\{\{0.3\|1\},\{0.5\|1\}\}$	$\{\{0.6\|1\},\{0.2\|1\}\}$	$\{\{0.1\|1\},\{0.7\|1\}\}$
x_4	$\{\{0.05\|0.7,0.2\|0.3\},\{0.5\|1\}\}$	$\{\{0.3\|1\},\{0.6\|0.5,0.4\|0.5\}\}$	$\{\{0.8\|1\},\{0.1\|1\}\}$

Table 8. The score values and ranking results of Example 5 by utilizing different methods.

Method	**Score Values**				**Ranking Results**
	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
Hao et al.'s [12] method based on the PDHFWA	0.2526	0.1540	−0.2022	0.3732	$x_4 \succ x_1 \succ x_2 \succ x_3$
Our method based on the q-RPDHFPWMM ($q=1$, $L=(1,0,0)$)	0.2574	0.1535	−0.2095	0.3497	$x_4 \succ x_1 \succ x_2 \succ x_3$

As we can see from Table 8, the ranking order produced by Hao et al.'s [12] MADM method is the same as that obtained by our method, which illustrates the validity of our method.

6.4. Advantages and Superiorities Analysis

We try to investigate the advantages of our method and in order to do this, we use our method and some existing MADM methods to solve numerical examples and conduct comparative analysis. These methods include Hao et al.'s [12] method based on the PDHFWA and Xu et al.'s [15] under q-RDHFSs.

6.4.1. Its Efficiency in Reducing the Negative Influence of DMs' Unduly High or Low Evaluation Values

It is mentioned that as our method is based on the PA operator, it has the ability of reducing the negative influence of DMs' extreme evaluation values on the results. Hence, the decision result derived by our method is more reasonable and reliable. To illustrate this characteristic, we provide the following example.

Example 6. *In real MADM problems, DMs may be prejudiced over some attributes of certain alternatives and they may express unduly high or low evaluation values. For instance, in Example 6, a few DMs have prejudice over G_3 of x_4 and give low value of MD. So the evaluation value* α_{43} *changes from* $\{\{0.8|1\},\{0.1|1\}\}$ *to* $\{\{0.8|0.5,0.1|0.5\},\{0.1|1\}\}$ *and the new decision matrix is shown in Table 9.*

Table 9. The decision matrix of Example 6.

	G_1	G_2	G_3
x_1	$\{\{0.7\|0.2,0.6\|0.2,0.5\|0.6\},\{0.2\|1\}\}$	$\{\{0.7\|1\},\{0.2\|1\}\}$	$\{\{0.2\|1\},\{0.2\|1\}\}$
x_2	$\{\{0.1\|1\},\{0.4\|1\}\}$	$\{\{0.3\|1\},\{0.7\|1\}\}$	$\{\{0.7\|1\},\{0.3\|0.5,0.2\|0.5\}\}$
x_3	$\{\{0.3\|1\},\{0.5\|1\}\}$	$\{\{0.6\|1\},\{0.2\|1\}\}$	$\{\{0.1\|1\},\{0.7\|1\}\}$
x_4	$\{\{0.05\|0.7,0.2\|0.3\},\{0.5\|1\}\}$	$\{\{0.3\|1\},\{0.6\|0.5,0.4\|0.5\}\}$	$\{\{0.8\|0.5,0.1\|0.5\},\{0.1\|1\}\}$

Then we use Hao et al.'s [12] decision-making method based on the PDHFWA operator and our method based on the *q*-RPDHFPWMM operator to solve Example 6 and present the decision results in Table 10.

Table 10. The score values and ranking results utilizing different methods in Example 6.

Method	**Score Values**				**Ranking Results**
	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
Hao et al.'s [12] method based on the PDHFWA	0.2526	0.1540	−0.2022	0.1468	$x_1 \succ x_2 \succ x_4 \succ x_3$
Our method based on the *q*-RPDHFPWMM ($q=1$, $L=(1,0,0)$)	0.2574	0.1535	−0.2095	0.1597	$x_1 \succ x_4 \succ x_2 \succ x_3$

As we can see from Table 10, the ranking result derived by Hao et al.'s [12] is changed from $x_4 \succ x_1 \succ x_2 \succ x_3$ to $x_1 \succ x_2 \succ x_4 \succ x_3$, but ours is $x_1 \succ x_4 \succ x_2 \succ x_3$. Further, due to the bias of some DMs, the ranking of x_4 dropped from the first to the third in Hao et al.'s [12] method, but only to the second in our method. This is a good example of the strength of our approach for reducing the negative effect of extreme values on the result.

6.4.2. The Ability of Flexibly Capturing the Interrelationship among Attributes

It is well-known that in most real MADM problems, there exists interrelationship among attributes. Additionally, such kind of interrelationship usually exists among multiple attributes. In order word, usually multiple attributes are correlated. Hence, to obtain reasonable decision results, it is necessary to take the interrelationship among multiple attributes into account. For example, in Example 4, when $L=(1,1,0)$, then our method takes the interrelationship between any two attributes into consideration. When $L=(1,1,1)$, then the interrelationship among all the three attributes is captured. If there is indeed no interrelationship between attributes, then we can set $L=(1,0,0)$. Hao et al.'s [12] method is based on the simple weighted average operator, which fails to handle MADM problems where attributes are dependent. This character reveals that Hao et al.'s [12] method is insufficient or inadequate to handle most real MADM problems. Hence, our method is more powerful and flexible than Hao et al.'s [12].

6.4.3. More Liberty that it Provides for DMs

The proposed q-RPDHFS is an extension of q-RDHFS. Hence, q-RPDHFS takes the constraint of q-RDHFS, i.e., the sum of qth power of MD and qth power of NMD is less than or equal to one. Compared with the PDHFS, the constraint of q-RPDHFS is much laxer. As mentioned in Section 3, PDHFS is only a special case of q-RPDHFS. When DMs employ PDHFSs to express their evaluation values, some important decision information may be lost, which may further result in unreasonable decision results. However, our method gives DMs enough freedom to comprehensively provide their evaluation information. To better illustrate this advantage, we use Example 4 to elaborate. In Table 2, DMs' assessments are expressed as q-RPDHFEs. Hao et al.'s [12] method based on PDHFS fails to handle Example 4, for the sum of MD and NMD is greater than one in some assessment values such as $\alpha_{12} = \{\{0.7|1\},\{0.5|1\}\}$, $\alpha_{31} = \{\{0.6|1\},\{0.5|1\}\}$ and $\alpha_{43} = \{\{0.8|1\},\{0.5|1\}\}$. However, our method based on q-RPDHFPWMM or q-RPDHFPWDMM is still effective and the ranking order is $x_1 \succ x_3 \succ x_4 \succ x_2$, as mentioned in Section 6.1.

6.4.4. The Ability of Taking the Probabilistic Information of DMs' Evaluation Values into Account

Our method is based on q-RPDHFS, which can be regarded as an extension of the classical q-RDHFS. We generalize q-RDHFS to q-RPDHFS by taking the probabilistic information of each MD and NMD into account. Hence, our method can also solve the MADM problems under q-RDHFSs. In this subsection, we attempt to illustrate the advantage of our proposed method over that developed by Xu et al. [15]. In order to do this, we provide the following example.

Example 7. *In Example 4, if the probabilistic information of evaluation values is ignored, then a new original decision matrix is obtained, which is shown in Table 11. It is noted the decision matrix is a q-rung dual hesitant fuzzy decision matrix. Then, we apply Xu et al.'s [15] MADM method to solve this example and present the results in Table 12 (without loss of generality, we assume q = 3 and s = t = 1). As mentioned above, our method can also deal with MADM problems wherein attribute values are provided in the form of q-RDHFEs. Hence, we use our method to solve Example 7 and the decision results are also presented in Table 12.*

As seen in Table 12, the method introduced by Xu et al. [15] and our proposed method produce the same ranking order $x_1 \succ x_3 \succ x_4 \succ x_2$. However, when the probabilistic information of DMs' evaluation values is considered in our proposed method, then a different ranking order is obtained, i.e., $x_1 \succ x_3 \succ x_2 \succ x_4$. This example demonstrates the powerfulness and flexibility of our proposed method. First, similar to Xu et al.'s [15] method, our MADM method can also solve decision-making problems when DMs use q-RDHFSs to express their evaluations. Second, our method can consider probabilistic information of DMs' evaluation values. Actually, as mentioned in Introduction, it is usually necessary to consider the probabilistic information of the corresponding evaluation values in order to comprehensively depict DMs' assessments. Hence, our proposed method is more powerful than Xu et al.'s [15] approach.

Table 11. The decision matrix based on q-RDHFEs.

	G_1	G_2	G_3
x_1	$\{\{0.7,0.6,0.5\},\{0.2\}\}$	$\{\{0.7\},\{0.5\}\}$	$\{\{0.2\},\{0.2\}\}$
x_2	$\{\{0.1\},\{0.4\}\}$	$\{\{0.3\},\{0.7\}\}$	$\{\{0.7\},\{0.3,0.2\}\}$
x_3	$\{\{0.6\},\{0.5\}\}$	$\{\{0.6\},\{0.2\}\}$	$\{\{0.1\},\{0.7\}\}$
x_4	$\{\{0.05,0.2\},\{0.5\}\}$	$\{\{0.3\},\{0.6,0.5\}\}$	$\{\{0.8\},\{0.5\}\}$

Table 12. The score values and ranking results utilizing Xu et al.'s [15] and our methods.

Method	Score Values				Ranking Results
	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
Xu et al.'s [15] method based on the q-RDHFWHM	0.1559	−0.0072	0.1282	0.1230	$x_1 \succ x_3 \succ x_4 \succ x_2$
Our method based on the q-RPDHFPWMM (without probabilities)	0.0797	−0.1077	−0.0342	−0.1010	$x_1 \succ x_3 \succ x_4 \succ x_2$
Our method based on the q-RPDHFPWMM (with probabilities)	0.0674	−0.1077	−0.0342	−0.1085	$x_1 \succ x_3 \succ x_2 \succ x_4$

7. Conclusions

This paper demonstrated a novel MADM method, which can be used to solve practical decision-making problems effectively. The main contributions of this paper are three-fold. Firstly, we proposed a novel tool, called q-RPDHFSs to more accurately and effectively depict DMs' complicated evaluation information. Compared with q-RDHFSs, our proposed q-RPDHFSs more effectively deal with DMs' fuzzy judgements as they not only describe the MD and NMD, but also depict their corresponding probabilistic information. Compared with the PDHFSs, the q-RPDHFSs are more powerful as they provide DMs more freedom to express their evaluation values. Due to this characteristic, in the framework of q-RPDHFSs, DMs can fully express their evaluations, which leads to less information loss. Secondly, a series of AOs of q-RPDHFEs were developed, which are useful to aggregate attribute values given in the form of q-rung probabilistic dual hesitant fuzzy information. The advantages of superiorities of our proposed AOs are obvious, as they not only reduce the negative effect of DMs' unduly high or low evaluation values on the final decision results, but also reflect the interrelationship among any numbers of attributes. Thirdly, a new MADM method was originated to help DMs to choose the optimal alternatives. Through numerical examples, the effectiveness of our method has been clearly illustrated. By comparative analysis, the advantages of our method are that it not only provides DMs great freedom to express their decision information, but also produces reasonable and reliable decision results. These characteristics make our method more suitable to deal with MADM problems in actual life.

In the further, we plan to continue our research from three aspect. Firstly, we shall study new applications of our decision-making method in more practical MADM problems, such as selection real estate investment [39], medicine selection [40], best research topic selection [41], evaluation of outsourcing for information systems [42], etc. Secondly, we will study more AOs of q-RPDHFEs and propose corresponding MADM methods. Thirdly, we shall continue to investigate extensions of q-RPDHFSs, such as interval-valued q-RPDHFSs, complex q-RPDHFSs, complex interval-valued q-RPDHFSs, etc.

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