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Linear Maps That Act Tridiagonally with Respect to Eigenbases of the Equitable Generators of $U_q(sl_2)$

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Abstract: Let $\mathcal{F}$ denote an algebraically closed field; let q be a nonzero scalar in $\mathcal{F}$ such that q is not a root of unity; let d be a nonnegative integer; and let X, Y, Z be the equitable generators of $U_q(sl_2)$ over $\mathcal{F}$. Let V denote a finite-dimensional irreducible $U_q(sl_2)$-module with dimension $d+1$, and let R denote the set of all linear maps from V to itself that act tridiagonally on the standard ordering of the eigenbases for each of X, Y, and Z. We show that R has dimension at most seven. Indeed, we show that the actions of $1, X, Y, Z, XY, YZ$, and ZX on V give a basis for R when $d \geq 3$.

Keywords: finite-dimensional $U_q(sl_2)$-modules; standard eigenbasis; Leonard pairs

MSC: 17B10; 17B37; 15A18

1. Introduction

We characterize the linear operators that act tridiagonally with respect to appropriately ordered eigenbases for all three equitable generators of $U_q(sl_2)$ acting on its finite-dimensional irreducible modules. To state the main result, we first recall the equitable presentation of $U_q(sl_2)$. Throughout this paper, let $\mathcal{F}$ denote an algebraically closed field, and let q be a nonzero scalar in $\mathcal{F}$ such that q is not a root of unity.

Lemma 1. *[Theorem 2.1] [1] The algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathcal{F}$-algebra with generators $X^{\pm 1}, Y, Z$ and the following relations:*

$$XX^{-1} = X^{-1}X = 1,$$

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \qquad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = 1, \qquad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.$$

We call $X^{\pm 1}, Y, Z$ the equitable generators for the quantum algebra $U_q(sl_2)$.

The equitable presentation of this algebra was introduced in [1], where its relationship to the usual presentation in terms of the Chevalley generators [2] is discussed. The equitable presentation has been studied in connection with tridiagonal pairs [3,4], Leonard pairs [5], the q-tetrahedron algebra [6–9], bidiagonal pairs [10], Q-polynomial distance-regular graphs [11–13], in Poisson algebras [14], and the universal Askey–Wilson algebra [15].

Other relevant references include [16–21].

Definition 1. *[Definition 5.2] [1] Let n_X, n_Y, n_Z denote the following elements of $U_q(sl_2)$:*

$$n_X = \frac{q(1-YZ)}{q-q^{-1}} = \frac{q^{-1}(1-ZY)}{q-q^{-1}},$$

$$n_Y = \frac{q(1-ZX)}{q-q^{-1}} = \frac{q^{-1}(1-XZ)}{q-q^{-1}},$$

$$n_Z = \frac{q(1-XY)}{q-q^{-1}} = \frac{q^{-1}(1-YX)}{q-q^{-1}}.$$

Definition 2. *Let V denote a vector space over $\mathcal{F}$ with dimension $d+1$. By a decomposition of V, we mean a sequence $\{V_i\}_{i=0}^{d}$ consisting of one-dimensional subspaces of V such that:*

$$V = V_0 + V_1 + \ldots + V_d, \qquad \textit{direct sum}.$$

Definition 3. *Let $\{V_i\}_{i=0}^{d}$ be a decomposition of V. For notational convenience, define $V_{-1} = 0$ and $V_{d+1} = 0$. For $A \in End(V)$, we say A raises $\{V_i\}_{i=0}^{d}$ whenever $AV_i = V_{i+1}$ for $0 \le i \le d$. We say A lowers $\{V_i\}_{i=0}^{d}$ whenever $AV_i = V_{i-1}$ for $0 \le i \le d$. An ordered pair A, B of elements in $End(V)$ is called LRwhenever there exists a decomposition of V that is lowered by A and raised by B. A three-tuple A, B, C of elements in $End(V)$ is called an LR triple whenever any two of A, B, C form an LR pair on V.*

Definition 4. *Let $0 \neq q \in \mathcal{F}$, $q^2 \neq 1$; an LR pair A, B on V is said to be the q-Weyl type whenever:*

$$\frac{qAB - q^{-1}BA}{q-q^{-1}} = I.$$

An LR triple A, B, C on V is said to be the q-Weyl type whenever the LR pairs A, B, B, C, and C, A all are the q-Weyl type.

Let A, B, C be an LR triple q-Weyl type on V. In [22], Nomura describes a family of linear maps that acts tridiagonally with respect to each of the (A, B), (B, C), and (C, A) decompositions for V.

The point of view of our work is quite different. To state the main result of this paper, we use the following definition.

Definition 5. *A square matrix is said to be tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A square matrix is said to be lower bidiagonal whenever each nonzero entry lies on either the diagonal or the subdiagonal; a square matrix is said to be upper bidiagonal whenever each nonzero entry lies on either the diagonal or the superdiagonal.*

Our main result is the following; an sl_2 analogue appears in [23].

Theorem 1. *Let V be a finite-dimensional $U_q(sl_2)$-module. Fix a linear map $\Psi : V \to V$. Then, the following are equivalent.*

- **(i)** *Ψ acts on V as a linear combination of one, X, Y, Z, XY, YZ, and ZX.*
- **(ii)** *All three of the matrices representing Ψ with respect to standard X-, Y-, and Z-eigenbases are tridiagonal.*
- **(iii)** *Any two of the matrices representing Ψ with respect to standard X-, Y-, and Z-eigenbases are tridiagonal.*

Moreover, one, X, Y, Z, XY, YZ, and ZX are linearly independent when $\dim V \geq 3$.

2. Standard Eigenbases for $U_q(sl_2)$-Modules

In this section, we recall the finite-dimensional $U_q(sl_2)$-modules and some distinguished bases.

Lemma 2. *[Lemma 4.2] [1] For each nonnegative integer d and for $\epsilon \in \{1, -1\}$, there is an irreducible finite-dimensional $U_q(sl_2)$-module $V_{d,\epsilon}$ with basis $\{u_0, u_1, \ldots, u_d\}$ and action:*

$$
\begin{array}{rcll}
(\epsilon X - q^{d-2i} I)u_i & = & 0 & (0 \leq i \leq d), \\
(\epsilon Y - q^{2i-d} I)u_i & = & (q^{-d} - q^{2i+2-d})u_{i+1} & (0 \leq i \leq d-1), \\
(\epsilon Y - q^{d} I)u_d & = & 0, & \\
(\epsilon Z - q^{-d} I)u_0 & = & 0, & \\
(\epsilon Z - q^{2i-d} I)u_i & = & (q^{d} - q^{2i-2-d})u_{i-1} & (1 \leq i \leq d).
\end{array}
$$

The basis $\{u_0, u_1, \ldots, u_d\}$ is called the standard X-eigenbasis of $V_{d,\epsilon}$.

Since the module $V_{d,-1}$ can be treated similarly to $V_{d,1}$, we treat only the module $V_{d,1}$, and throughout this paper, we write V_d to mean $V_{d,1}$. For any vector space V, $\mathrm{End}(V)$ is the $\mathcal{F}$-algebra of all $\mathcal{F}$-linear transformations from V to itself.

Corollary 1. *With reference to Lemma 2, the actions of X, Y, Z, on V_d are each multiplicity free with eigenvalues $\{q^{d-2i}\}_{i=0}^{d}$. Moreover, each of X, Y, Z is invertible on V_d.*

Definition 6. *Let V_d be a finite-dimensional irreducible $U_q(sl_2)$-module; for $\tau \in \{X, Y, Z\}$, define the decomposition $[\tau]$ of V_d as follows. For $0 \leq i \leq d$ the ith component of $[\tau]$ is the eigenspace for τ with eigenvalue q^{d-2i}.*

Lemma 3. *With reference to Lemma 2, for $0 \leq i \leq d$, let $U_i = span\{u_i\}$, then $\{U_i\}_{i=0}^{d}$ is the $[X]$ decomposition of V_d.*

Proof. This is clear from Definition 6. □

Lemma 4. *[24] With reference to Definition 1, let V_d be a finite-dimensional irreducible $U_q(sl_2)$-module; the following hold:*

(i) $n_X^d V_d$ *is the eigenspace for Y (resp. Z) on V_d with eigenvalue q^{-d} (resp. q^d).*
(ii) $n_Y^d V_d$ *is the eigenspace for Z (resp. X) on V_d with eigenvalue q^{-d} (resp. q^d).*
(iii) $n_Z^d V_d$ *is the eigenspace for X (resp. Y) on V_d with eigenvalue q^{-d} (resp. q^d).*

Definition 7. *[24] Let V_d be a finite-dimensional irreducible $U_q(sl_2)$-module; for $\tau \in \{X, Y, Z\}$, a basis $\{v_i\}_{i=0}^{d}$ for V is said to be $[\tau]_{row}$ whenever the following hold:*

(i) *For $0 \leq i \leq d$, the vector v_i is contained in the component i of the decomposition $[\tau]$;*
(ii) $\Sigma_{i=0}^{d} v_i \in n_\tau^d V_d$.

Lemma 5. *With reference to Lemma 2, the basis $\mathbf{u} = \{u_0, u_1, \ldots, u_d\}$ is the $[X]_{row}$ basis for V_d.*

Proof. Note that by Lemma 2,

$$\begin{aligned}
Y(\Sigma_{i=0}^{d} u_i) &= Y(u_0 + u_1 + u_2 + \ldots + u_{d-1} + u_d) \\
&= Yu_0 + Yu_1 + Yu_2 + \ldots + Yu_{d-1} + Yu_d \\
&= q^{-d}u_0 + (q^{-d} - q^{2-d})u_1 + q^{2-d}u_1 + (q^{-d} - q^{4-d})u_2 \\
&+ q^{4-d}u_2 + (q^{-d} - q^{6-d})u_3 + \cdots + q^{d-2}u_{d-1} + (q^{-d} - q^{d})u_d + q^{d}\, u_d \\
&= q^{-d}(u_0 + u_1 + u_2 + \ldots + u_{d-1} + u_d) \\
&= q^{-d}(\Sigma_{i=0}^{d} u_i),
\end{aligned}$$

and:

$$\begin{aligned}
Z(\Sigma_{i=0}^{d} u_i) &= Z(u_0 + u_1 + u_2 + \ldots + u_{d-1} + u_d) \\
&= Zu_0 + Zu_1 + Zu_2 + \ldots + Zu_{d-1} + Zu_d \\
&= q^{-d}u_0 + q^{2-d}u_1 + (q^{d} - q^{-d})u_0 + q^{4-d}u_2 + (q^{d} - q^{2-d})u_1 \\
&+ q^{6-d}u_3 + (q^{d} - q^{4-d})u_2 + \cdots + q^{d-2}u_{d-1} + (q^{d} - q^{d-4})u_{d-2} \\
&+ q^{d}u_d + (q^{d} - q^{d-2})u_{d-1} \\
&= q^{d}(u_0 + u_1 + u_2 + \ldots + u_{d-1} + u_d) \\
&= q^{d}(\Sigma_{i=0}^{d} u_i).
\end{aligned}$$

Hence, by Lemma 4, $\Sigma_{i=0}^{d} u_i \in n_x^d V_d$. Moreover, note that from Lemma 3:

$$V_d = U_1 + U_2 + \ldots + U_d$$

and $u_i \in U_i$. Now, the result holds by Definition 7. □

Let $\{a_i\}_{i=0}^{d}$ and $\{b_i\}_{i=0}^{d}$ be two bases of the vector space V. By the transition matrix from $\{a_i\}_{i=0}^{d}$ to $\{b_i\}_{i=0}^{d}$, we mean the matrix $S \in Mat_{d+1}(\mathcal{F})$ such that $b_j = \Sigma_{i=0}^{d} S_{ij} a_i$ for $0 \le j \le d$.

For all integers k and for all nonnegative integers n, m, write:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \qquad [n]! = [1][2]\cdots[n], \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[n-m]![m]!} & \text{if } n \ge m, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8. *Let P and Q denote $(d+1) \times (d+1)$ matrices with entries P_{ij} and Q_{ij}, respectively, where:*

$$P_{ij} = (-1)^j q^{(j-d)(i-1)} \begin{bmatrix} i \\ d-j \end{bmatrix} \qquad (0 \le i, j \le d),$$

$$Q_{ij} = (-1)^j q^{j(d-i-1)} \begin{bmatrix} d-i \\ j \end{bmatrix} \qquad (0 \le i, j \le d).$$

Theorem 2. *[Theorem 16.3, Lemma 16.6] [24] With reference to Definition 8, let V_d be a finite-dimensional irreducible $U_q(sl_2)$-module, and let $[X]_{row}$, $[Y]_{row}$, $[Z]_{row}$ be bases for V_d, then:*

(i) *The matrix P is a transition matrix from $[X]_{row}$ to $[Y]_{row}$.*

(ii) *The matrix Q is a transition matrix from $[X]_{row}$ to $[Z]_{row}$.*

Lemma 6. *With reference to Lemma 2 and Definition 8,*

(i) *Let $v_j = \sum_{i=d-j}^{d} P_{ij} u_i$ $(0 \le j \le d)$, then $\mathbf{v} = \{v_0, v_1, \ldots, v_d\}$ is the $[Y]_{row}$ basis for V_d.*

(ii) *Let $w_j = \sum_{i=0}^{d-j} Q_{ij} u_i$ $(0 \le j \le d)$, then $\mathbf{w} = \{w_0, w_1, \ldots, w_d\}$ is the $[Z]_{row}$ basis for V_d.*

Proof. By Lemma 5, the basis $\mathbf{u} = \{u_0, u_1, \ldots, u_d\}$ is the $[X]_{row}$ basis for V_d. Hence, the results hold by Theorem 2. □

Theorem 3. *[Theorem 10.12] [24] With reference to Lemma 2, let* $\mathbf{v} = \{v_0, v_1, \ldots, v_d\}$*, and let* $\mathbf{w} = \{w_0, w_1, \ldots, w_d\}$ *be as in Lemma 6, then:*

$$\begin{array}{rcll}
(Y - q^{d-2i}I)v_i & = & 0 & (0 \le i \le d), \\
(Z - q^{2i-d}I)v_i & = & (q^{-d} - q^{2i+2-d})v_{i+1} & (0 \le i \le d-1), \\
(Z - q^{d}I)v_d & = & 0, & \\
(X - q^{-d}I)v_0 & = & 0, & \\
(X - q^{2i-d}I)v_i & = & (q^{d} - q^{2i-2-d})v_{i-1} & (1 \le i \le d). \\
(Z - q^{d-2i}I)w_i & = & 0 & (0 \le i \le d), \\
(X - q^{2i-d}I)w_i & = & (q^{-d} - q^{2i+2-d})w_{i+1} & (0 \le i \le d-1), \\
(X - q^{d}I)w_d & = & 0, & \\
(Y - q^{-d}I)w_0 & = & 0, & \\
(Y - q^{2i-d}I)w_i & = & (q^{d} - q^{2i-2-d})w_{i-1} & (1 \le i \le d).
\end{array}$$

In view of Theorem 3, the bases $[Y]_{row}$ and $[Z]_{row}$ are called the standard Y- and Z-eigenbases of V_d, respectively.

Let $[T]_{\mathcal{B}}$ denote the matrix representing a linear operator T with respect to an ordered basis $\mathcal{B}$. We say that T acts upper bidiagonally, lower bidiagonally, or tridiagonally on $\mathcal{B}$ when the matrix $[T]_{\mathcal{B}}$ has the stated shape.

Lemma 7. *With reference to Lemma 2 and Theorem 3,*

$$[X]_{\mathbf{u}} = [Y]_{\mathbf{v}} = [Z]_{\mathbf{w}},$$

$$[X]_{\mathbf{v}} = [Y]_{\mathbf{w}} = [Z]_{\mathbf{u}},$$

$$[X]_{\mathbf{w}} = [Y]_{\mathbf{u}} = [Z]_{\mathbf{v}}.$$

Proof. This is clear from Theorem 3. □

Lemma 8. *With reference to Theorem 3, for all* $s \in U_q(sl_2)$*,*

$$[s]_{\mathbf{u}}P = P[s]_{\mathbf{v}}, \quad [s]_{\mathbf{v}}P = P[s]_{\mathbf{w}}, \quad [s]_{\mathbf{w}}P = P[s]_{\mathbf{u}}.$$

Proof. By Theorem 2 and elementary linear algebra, for $s \in \{X, Y, Z\}$, $[s]_{\mathbf{u}}P = P[s]_{\mathbf{v}}$. Now, from Lemma 7, $[Z]_{\mathbf{u}}P = P[Z]_{\mathbf{v}}$ gives $[X]_{\mathbf{v}}P = P[X]_{\mathbf{w}}$ and $[Y]_{\mathbf{u}}P = P[Y]_{\mathbf{v}}$ gives $[X]_{\mathbf{w}}P = P[X]_{\mathbf{u}}$. Similarly, we can prove the result for Y and Z. Since these formulas hold on generators, they must hold for all $s \in U_q(sl_2)$. □

Lemma 9. *[Lemmas 16.5 and 16.6] [24]* $P^3 = q^{d(d-1)}I$, $PQ = (-1)^d I$.

3. Linear Combinations of $1, X, Y, Z, Xy, Yz, Zx$

In this section, we define the linear transformation A and describe the action of A on the bases $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ given in the previous section, which we will use later to prove Theorem 1 and some special cases of this theorem.

Lemma 10. *With reference to Lemma 2,*

$$
\begin{aligned}
XYu_i &= u_i + (q^{-2(i+1)} - 1)u_{i+1} && (0 \le i \le d-1),\\
XYu_d &= u_d,\\
ZXu_i &= u_i + q^{2(d-i)}(1 - q^{2(i-d-1)})u_{i-1} && (1 \le i \le d),\\
ZXu_0 &= u_0,\\
YZu_0 &= q^{-2d}u_0 + q^{-2d}(1-q^2)u_1,\\
YZu_d &= q^{2d-2}(1-q^{-2})u_{d-1} + (q^{2d} + (1-q^{2d})(1-q^{-2})u_d,\\
YZu_i &= q^{2(i-1)}(1-q^{2(i-1-d)})u_{i-1}\\
&+ (q^{2(2i-d)} + (1-q^{2i})(1-q^{2(i-d-1)}))u_i\\
&+ q^{2(i-d)}(1-q^{2(i+1)})u_{i+1} && (1 \le i \le d-1).
\end{aligned}
$$

Proof. Performroutine calculations using the action of X, Y, and Z on the basis $\mathbf{u}$ in Lemma 2. □

Lemma 11. *With reference to Lemma 2 and Theorem 3,*

$$
\begin{aligned}
[XY]_{\mathbf{u}} &= [YZ]_{\mathbf{v}} = [ZX]_{\mathbf{w}},\\
[XY]_{\mathbf{v}} &= [YZ]_{\mathbf{w}} = [ZX]_{\mathbf{u}},\\
[XY]_{\mathbf{w}} &= [YZ]_{\mathbf{u}} = [ZX]_{\mathbf{v}}.
\end{aligned}
$$

Proof. By elementary linear algebra and Lemma 7,

$$[XY]_{\mathbf{u}} = [X]_{\mathbf{u}}[Y]_{\mathbf{u}} = [Y]_{\mathbf{v}}[Z]_{\mathbf{v}} = [YZ]_{\mathbf{v}},$$

$$[XY]_{\mathbf{u}} = [X]_{\mathbf{u}}[Y]_{\mathbf{u}} = [Z]_{\mathbf{w}}[X]_{\mathbf{w}} = [ZX]_{\mathbf{w}}.$$

Similarly, we can prove the other results. □

Definition 9. *Let d be a nonnegative integer, and consider V_d. Let $A \in \mathrm{End}(V_d)$ denote any linear combination of $\{1, X, Y, Z, XY, YZ, ZX\}$. Write:*

$$A = a_I 1 + a_x X + a_y Y + a_z Z + a_{xy} XY + a_{yz} YZ + a_{zx} ZX.$$

Lemma 12. *With reference to Lemma 2 and Definition 9, the action of A on the basis $\mathbf{u}$ is given by:*

$$
\begin{aligned}
Au_0 &= \alpha_0 u_0 + \beta_1 u_1,\\
Au_i &= \gamma_{i-1}u_{i-1} + \alpha_i u_i + \beta_{i+1}u_{i+1} && (1 \le i \le d-1),\\
Au_d &= \gamma_{d-1}u_{d-1} + \alpha_d u_d,
\end{aligned}
$$

where:

$$
\begin{aligned}
\alpha_i &= a_I + q^{d-2i}a_x + q^{2i-d}(a_y + a_z) + a_{xy} + a_{zx}\\
&\quad + (q^{2(2i-d)} + (1-q^{2(i-d-1)})(1-q^{2i}))a_{yz} && (0 \le i \le d),\\
\gamma_i &= (1-q^{2(i-d)})(q^d a_z + q^{2i}a_{yz} + q^{2(d-i-1)}a_{zx}) && (1 \le i \le d),\\
\beta_i &= (1-q^{2i})(q^{-d}a_y + q^{-2i}a_{xy} + q^{2(i-d-1)}a_{yz}) && (0 \le i \le d-1).
\end{aligned}
$$

Proof. Use the actions of X, Y, and Z from Lemma 2 and the actions of XY, YZ, and ZX from Lemma 10 on the basis $\mathbf{u}$ to get the result. □

Lemma 13. *With reference to Theorem 3 and Definition 9, the action of A on the basis* $\mathbf{v}$ *is given by:*

$$\begin{aligned} Av_0 &= \alpha'_0 v_0 + \beta'_1 v_1, \\ Av_i &= \gamma'_{i-1} v_{i-1} + \alpha'_i v_i + \beta'_{i+1} v_{i+1} \qquad (1 \le i \le d-1), \\ Av_d &= \gamma'_{d-1} v_{d-1} + \alpha'_d v_d, \end{aligned}$$

where:

$$\begin{aligned} \alpha'_i &= a_I + q^{d-2i} a_y + q^{2i-d}(a_z + a_x) + a_{yz} + a_{xy} \\ &\quad + (q^{2(2i-d)} + (1 - q^{2(i-d-1)})(1 - q^{2i})) a_{zx} \qquad (0 \le i \le d), \\ \gamma'_i &= (1 - q^{2(i-d)})(q^d a_x + q^{2i} a_{zx} + q^{2(d-i-1)} a_{xy}) \qquad (1 \le i \le d), \\ \beta'_i &= (1 - q^{2i})(q^{-d} a_z + q^{-2i} a_{yz} + q^{2(i-d-1)} a_{zx}) \qquad (0 \le i \le d-1). \end{aligned}$$

Proof. The actions of X, Y, and Z on the basis $\mathbf{v}$ are given in Theorem 3, and by Lemma 11, the actions of XY, YZ, and ZX on the basis $\mathbf{v}$ have the same coefficients of the actions of ZX, XY, and YZ on the basis $\mathbf{u}$, respectively, which appear in Lemma 10. Now, expand the actions of these components on the basis $\mathbf{v}$ to get the result. □

Lemma 14. *With reference to Theorem 3 and Definition 9, the action of A on the basis* $\mathbf{w}$ *is given by:*

$$\begin{aligned} Aw_0 &= \alpha^*_0 w_0 + \beta^*_1 w_1, \\ Aw_i &= \gamma^*_{i-1} w_{i-1} + \alpha^*_i w_i + \beta^*_{i+1} w_{i+1} \qquad (1 \le i \le d-1), \\ Aw_d &= \gamma^*_{d-1} w_{d-1} + \alpha^*_d w_d, \end{aligned}$$

where:

$$\begin{aligned} \alpha^*_i &= a_I + q^{d-2i} a_z + q^{2i-d}(a_x + a_y) + a_{zx} + a_{yz} \\ &\quad + (q^{2(2i-d)} + (1 - q^{2(i-d-1)})(1 - q^{2i})) a_{xy} \qquad (0 \le i \le d), \\ \gamma^*_i &= (1 - q^{2(i-d)})(q^d a_y + q^{2i} a_{xy} + q^{2(d-i-1)} a_{yz}) \qquad (1 \le i \le d), \\ \beta^*_i &= (1 - q^{2i})(q^{-d} a_x + q^{-2i} a_{zx} + q^{2(i-d-1)} a_{xy}) \qquad (0 \le i \le d-1). \end{aligned}$$

Proof. The actions of X, Y, and Z on the basis $\mathbf{w}$ are given in Theorem 3, and by Lemma 11, the actions of XY, YZ, and ZX on the basis $\mathbf{w}$ have the same coefficients of the actions of YZ, ZX, and XY on the basis $\mathbf{u}$, respectively, which appear in Lemma 10. Now, expand the actions of these components on the basis $\mathbf{w}$ to get the result. □

Corollary 2. *With reference to Definition 9, the matrices representing A with respect to the $[X]_{row}$, $[Y]_{row}$, and $[Z]_{row}$ bases for V_d are tridiagonal.*

Proof. This is clear from Lemmas 12–14. □

Note that Lemmas 12–14 give that (i) implies (ii) in Theorem 1.

Routine calculations using Lemmas 12–14 and taking the advantage of the symmetry of the actions of X, Y, Z on the bases $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ give expressions for a_I, a_x, a_y, a_z, a_{xy}, a_{yz}, a_{zx}, which appear in the following corollaries.

Corollary 3. *With reference to Lemma 12, assume $d \ge 2$. Then:*

$$
\begin{aligned}
a_{I} &= \left(q^{-4} - q^{3-2d}(q-q^{-1}) + \frac{[4]}{[2]}\right) a_{yz} + \frac{[2]\beta_1 - q\gamma_{d-1} - \beta_2/[2]}{(q-q^{-1})} \\
&\quad - \frac{q^{-d}\alpha_0 - q^d\alpha_d}{q^d - q^{-d}} - \frac{q^{2d-2}\beta_1 + q^{2-2d}\gamma_{d-1}}{(q-q^{-1})^2} + \frac{q^{2d-1}\beta_2 + q^{1-2d}\gamma_{d-2}}{[2](q-q^{-1})^2}, \\
a_{x} &= -(q^{d-4} + q^{2-d})a_{yz} \\
&\quad + \frac{\alpha_0 - \alpha_d}{(q^d - q^{-d})} + \frac{q^{d-2}\beta_1 + q^{2-d}\gamma_{d-1}}{(q-q^{-1})^2} + \frac{q^{d-1}\beta_2 + q^{1-d}\gamma_{d-2}}{[2](q-q^{-1})^2}, \\
a_{y} &= -q^{1-d}(q+q^{-1})a_{yz} + \frac{q^{d-2}[2]\beta_1 - q^{d-1}\beta_2}{[2](q-q^{-1})^2}, \\
a_{z} &= -q^{d-3}(q+q^{-1})a_{yz} + \frac{q^{2-d}[2]\gamma_{d-1} - q^{1-d}\gamma_{d-2}}{[2](q-q^{-1})^2}, \\
a_{xy} &= q^{4-2d}a_{yz} + \frac{q^2[2]\beta_1 - q\beta_2}{[2](q-q^{-1})^2}, \\
a_{zx} &= q^{2d-4}a_{yz} + \frac{[2]\gamma_{d-1} - q\gamma_{d-2}}{[2](q-q^{-1})^2}.
\end{aligned}
$$

If $d \geq 3$, then:

$$
a_{yz} = \frac{-q^{2d-4}\left(\beta_1 - \beta_2 + \beta_3/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2} = \frac{q^{6-2d}\left(\gamma_{d-1} - \gamma_{d-2} + \gamma_{d-3}/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2}.
$$

Corollary 4. *With reference to Lemma 13, assume $d \geq 2$. Then:*

$$
\begin{aligned}
a_{I} &= \left(q^{-4} - q^{3-2d}(q-q^{-1}) + \frac{[4]}{[2]}\right) a_{zx} + \frac{[2]\beta'_1 - q\gamma'_{d-1} - \beta'_2/[2]}{(q-q^{-1})} \\
&\quad - \frac{q^{-d}\alpha'_0 - q^d\alpha'_d}{q^d - q^{-d}} - \frac{q^{2d-2}\beta'_1 + q^{2-2d}\gamma'_{d-1}}{(q-q^{-1})^2} + \frac{q^{2d-1}\beta'_2 + q^{1-2d}\gamma'_{d-2}}{[2](q-q^{-1})^2}, \\
a_{y} &= -(q^{d-4} + q^{2-d})a_{zx} \\
&\quad + \frac{\alpha'_0 - \alpha'_d}{(q^d - q^{-d})} + \frac{q^{d-2}\beta'_1 + q^{2-d}\gamma'_{d-1}}{(q-q^{-1})^2} + \frac{q^{d-1}\beta'_2 + q^{1-d}\gamma'_{d-2}}{[2](q-q^{-1})^2}, \\
a_{z} &= -q^{1-d}(q+q^{-1})a_{zx} + \frac{q^{d-2}[2]\beta'_1 - q^{d-1}\beta'_2}{[2](q-q^{-1})^2}, \\
a_{x} &= -q^{d-3}(q+q^{-1})a_{zx} + \frac{q^{2-d}[2]\gamma'_{d-1} - q^{1-d}\gamma'_{d-2}}{[2](q-q^{-1})^2}, \\
a_{yz} &= q^{4-2d}a_{zx} + \frac{q^2[2]\beta'_1 - q\beta'_2}{[2](q-q^{-1})^2}, \\
a_{xy} &= q^{2d-4}a_{zx} + \frac{[2]\gamma'_{d-1} - q\gamma'_{d-2}}{[2](q-q^{-1})^2}.
\end{aligned}
$$

If $d \geq 3$, then:

$$
a_{zx} = \frac{-q^{2d-4}\left(\beta'_1 - \beta'_2 + \beta'_3/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2} = \frac{q^{6-2d}\left(\gamma'_{d-1} - \gamma'_{d-2} + \gamma'_{d-3}/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2}.
$$

Corollary 5. *With reference to Lemma 14, assume $d \geq 2$. Then:*

$$\begin{aligned}
a_{I} &= \left(q^{-4} - q^{3-2d}(q-q^{-1}) + \frac{[4]}{[2]}\right) a_{xy} + \frac{[2]\beta_1^* - q\gamma_{d-1}^* - \beta_2^*/[2]}{(q-q^{-1})} \\
&\quad - \frac{q^{-d}\alpha_0^* - q^d\alpha_d^*}{q^d - q^{-d}} - \frac{q^{2d-2}\beta_1^* + q^{2-2d}\gamma_{d-1}^*}{(q-q^{-1})^2} + \frac{q^{2d-1}\beta_2^* + q^{1-2d}\gamma_{d-2}^*}{[2](q-q^{-1})^2}, \\
a_{z} &= -(q^{d-4} + q^{2-d})a_{xy} \\
&\quad + \frac{\alpha_0^* - \alpha_d^*}{(q^d - q^{-d})} + \frac{q^{d-2}\beta_1^* + q^{2-d}\gamma_{d-1}^*}{(q-q^{-1})^2} + \frac{q^{d-1}\beta_2^* + q^{1-d}\gamma_{d-2}^*}{[2](q-q^{-1})^2}, \\
a_{x} &= -q^{1-d}(q+q^{-1})a_{xy} + \frac{q^{d-2}[2]\beta_1^* - q^{d-1}\beta_2^*}{[2](q-q^{-1})^2}, \\
a_{y} &= -q^{d-3}(q+q^{-1})a_{xy} + \frac{q^{2-d}[2]\gamma_{d-1}^* - q^{1-d}\gamma_{d-2}^*}{[2](q-q^{-1})^2}, \\
a_{zx} &= q^{4-2d}a_{xy} + \frac{q^2[2]\beta_1^* - q\beta_2^*}{[2](q-q^{-1})^2}, \\
a_{yz} &= q^{2d-4}a_{xy} + \frac{[2]\gamma_{d-1}^* - q\gamma_{d-2}^*}{[2](q-q^{-1})^2}.
\end{aligned}$$

If $d \geq 3$, then:

$$a_{xy} = \frac{-q^{2d-4}\left(\beta_1^* - \beta_2^* + \beta_3^*/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2} = \frac{q^{6-2d}\left(\gamma_{d-1}^* - \gamma_{d-2}^* + \gamma_{d-3}^*/[3]\right)}{(q^2-q^{-2})(q-q^{-1})^2}.$$

4. Tridiagonal Operators

In this section, we prove that (iii) implies (i) in Theorem 1.

Lemma 15. *Assume $d \geq 2$. Fix $\Psi \in \mathrm{End}(V_d)$. Let $\mathbf{u} = \{u_0, u_1, \ldots, u_d\}$ and $\mathbf{v} = \{v_0, v_1, \ldots, v_d\}$ be $[X]_{row}$ and $[Y]_{row}$ bases for V_d, respectively. Assume that Ψ acts tridiagonally on $\mathbf{u}$ and $\mathbf{v}$, so for some α_i, β_i, γ_i, α'_i, β'_i, $\gamma'_i \in \mathcal{F}$:*

$$\begin{aligned}
\Psi u_0 &= \alpha_0 u_0 + \beta_1 u_1, & \Psi v_0 &= \alpha'_0 v_0 + \beta'_1 v_1, \\
\Psi u_i &= \gamma_{i-1}u_{i-1} + \alpha_i u_i + \beta_{i+1}u_{i+1}, & \Psi v_i &= \gamma'_{i-1}v_{i-1} + \alpha'_i v_i + \beta'_{i+1}v_{i+1} \\
&& &\qquad (1 \leq i \leq d-1), \\
\Psi u_d &= \gamma_{d-1}u_{d-1} + \alpha_d u_d, & \Psi v_d &= \gamma'_{d-1}v_{d-1} + \alpha'_d v_d.
\end{aligned}$$

(i) *α_i, β_i, γ_i, α'_i, β'_i, γ'_i, and a_I, a_x, a_y, a_z, a_{xy}, a_{yz}, a_{zx} are related by Lemmas 12 and 13.*

(ii) *Ψ acts on V_d as $A = a_I 1 + a_x X + a_y Y + a_z Z + a_{xy}XY + a_{yz}YZ + a_{zx}ZX$.*

(iii) *Ψ acts tridiagonally on any $[Z]_{row}$ basis.*

Proof. By Lemma 8, $[\Psi]_{\mathbf{u}}P - P[\Psi]_{\mathbf{v}} = 0$. Take $\beta_0 = \beta'_0 = \gamma_d = \gamma'_d = 0$. For $0 \leq i, j \leq d$, matrix multiplication gives the (i,j)-entry:

$$P_{ij}\,(\alpha_i - \alpha'_j) + P_{i-1j}\,\beta_i + P_{i+1j}\,\gamma_i - P_{ij-1}\,\gamma'_{j-1} - P_{ij+1}\,\beta'_{j+1} = 0. \tag{1}$$

It is routine to verify that the parameters given in terms of a_I, a_x, a_y, a_z, a_{xy}, a_{yz}, and a_{zx} by Lemmas 12 and 13 are a solution. we now show that there is a unique solution in the parameters α_0, α_d, β_1, β_2, γ_{d-1}, γ_{d-2}, and γ_{d-3}.

Respectively taking i to be $d-j-1$, $d-j$, and $d-j+1$ in (1) gives:

$$
\begin{aligned}
\beta'_k &= -q^{2(k-d)}\gamma_{d-k} \qquad (1 \le k \le d),\\
\alpha'_k &= \alpha_{d-k} + q^{k-d}[d-k+1]\gamma_{d-k} + q^{d-k-1}[d-k]\beta'_{k+1} \qquad (0 \le k \le d),\\
\gamma'_k &= q^{d-k-1}[d-k](\alpha'_{k+1} - \alpha_{d-k}) - q^{2(d-k-1)}\beta_{d-k} - \frac{[d-k][d-k+1]}{[2]}\gamma_{d-k}\\
&- \frac{q^{2(d-k-1)}[d-k-1][d-k]}{[2]}\beta'_{k+2} \qquad (0 \le k \le d-1).
\end{aligned}
$$

Substituting the above into (1) at $j = d, d-1$, and $d-2$ gives respectively:

$$
\begin{aligned}
\alpha_k + \beta_k + \gamma_k &= q^{1-k}[k](\alpha_1 + \beta_1 + \gamma_1)\\
&+ (1 - q^{1-k}[k])(\alpha_0 + \gamma_0) \qquad (2 \le k \le d),
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
q[k]\alpha_k + q^2[k-1]\beta_k + [k+1]\gamma_k &= (q^k - q[k] - \frac{(q^3 - q[2])[k-1][k]}{q^{k-1}[2]})\gamma_0\\
&+(1 - q^{2-k}[k-1])[k](q\alpha_1 + [2]\gamma_1)\\
&+\frac{[k-1][k]}{q^{k-2}[2]}(q[2]\alpha_2 + q^2\beta_2 + [3]\gamma_2) \qquad (3 \le k \le d),
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&q^2[k-1][k]\alpha_k + q^2[k-2][k-1]\beta_k + [k][k+1]\gamma_k\\
&= -(q^{4-k}(q-[2])[k-2][k-1][k] + [2][k](q[k-1] - q^{k-1}))\gamma_1\\
&+(1 - q^{3-k}[k-2][k-1][k](q^2\alpha_2 + [3]\gamma_2)\\
&+\frac{[k-2][k-1][k]}{q^{k-3}}(q^2\alpha_3 + q^4\frac{\beta_3}{[3]} + \frac{[4]\gamma_3}{[2]}) \qquad (4 \le k \le d).
\end{aligned} \tag{4}
$$

For $4 \le k \le d$, the system of equations in α_k, β_k, γ_k has a coefficient matrix with determinant $(q^2+1)q^{3(1-k)} \neq 0$, so these parameters are uniquely determined by those with a lower index. Use (2) at $k = 2, 3$ and (3) at $k = 3$ to solve for α_2, α_3, and β_3. Since $\gamma_d = 0$, (2), (3), and (4) at $k = d$ can be solved for α_1, γ_3, and β_d in term of α_0, α_d, β_1, β_2, γ_0, γ_1, and γ_2. We may use (2) at $k = d-1, d-2$, and $d-3$ to replace γ_0, γ_1, and γ_2 with γ_{d-1}, γ_{d-2}, and γ_{d-3} to get a unique solution in terms of α_0, α_d, β_1, β_2, γ_{d-1}, γ_{d-2}, and γ_{d-3}. Now, (i) follows.

For (ii), expand the actions of X, Y, and Z on the basis $\mathbf{u}$ to verify that the action of A agrees with the action of Ψ on the basis $\mathbf{u}$. Now, (iii) follows from (ii) since the operators in the sum act tridiagonally on any $[Z]_{row}$ basis by Theorem 3. □

We can use the same argument to prove the result in Lemma 15 when Ψ acts tridiagonally on the bases $\mathbf{v}$ and $\mathbf{w}$ or when Ψ acts tridiagonally on the bases $\mathbf{w}$ and $\mathbf{u}$.

5. Main Results

In this section, we prove the main result of this paper, and then, we give some special cases.

Theorem 4. *Let V be a finite-dimensional $U_q(sl_2)$-module. Fix a linear map $\Psi : V \to V$. Then, the following are equivalent.*

(i) *Ψ acts on V as a linear combination of one, X, Y, Z, XY, YZ, and ZX.*

(ii) *All three of the matrices representing Ψ with respect to standard X-, Y-, and Z-eigenbases are tridiagonal.*

(iii) *Any two of the matrices representing* Ψ *with respect to standard* X-, Y-, *and* Z-*eigenbases are tridiagonal.*

Moreover, one, X, Y, Z, XY, YZ, *and* ZX *are linearly independent when* $\dim V \geq 3$.

Proof. The theorem holds for $d < 2$ since (i), (ii), and (iii) do. Suppose $d \geq 2$. Corollary 2 and Lemma 15 show that (i) implies (ii). Clearly, (ii) implies (iii). Finally, (iii) implies (i) by Lemma 15. Thus, the result holds. □

Corollary 6. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$. *Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, Y, Z, *and* YZ.
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

- **(a)** Ψ *acts tridiagonally on a* $[X]_{row}$ *basis.*
- **(b)** Ψ *acts lower bidiagonally on a* $[Y]_{row}$ *basis.*
- **(c)** Ψ *acts upper bidiagonally on a* $[Z]_{row}$ *basis.*

Proof. The actions of Y, Z, and YZ are described in Lemma 2, and Theorem 3 shows that (a), (b), and (c) hold when Ψ acts as a linear combination of I, Y, Z, and YZ. If all three of (a), (b), and (c) hold, then clearly so do any two of them. Finally, suppose any two of (a), (b), (c) hold, so one of (b) and (c) holds. Then, the matrices representing Ψ with respect to standard X-, Y-, and Z-bases are all tridiagonal, so Lemma 15 gives that Ψ acts on V_d as $A = a_I 1 + a_x X + a_y Y + a_z Z + a_{xy} XY + a_{yz} YZ + a_{zx} ZX$ with the coefficients as described in that lemma. Now, (b) implies that all $\gamma_i' = 0$, so $a_x = a_{zx} = a_{xy} = 0$ by Corollary 4. Similarly, (c) implies that all $\beta_i^* = 0$, so $a_x = a_{zx} = a_{xy} = 0$ by Corollary 5. Thus, Ψ acts on V_d as a linear combination of I, Y, Z, and YZ. □

Corollary 7. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$. *Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, Z, X, *and* ZX.
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

- **(a)** Ψ *acts tridiagonally on a* $[Y]_{row}$ *basis.*
- **(b)** Ψ *acts lower bidiagonally on a* $[Z]_{row}$ *basis.*
- **(c)** Ψ *acts upper bidiagonally on a* $[X]_{row}$ *basis.*

Proof. This is similar to the proof of Corollary 6. □

Corollary 8. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$. *Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, X, Y, *and* XY.
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

- **(a)** Ψ *acts tridiagonally on a* $[Z]_{row}$ *basis.*
- **(b)** Ψ *acts lower bidiagonally on a* $[X]_{row}$ *basis.*
- **(c)** Ψ *acts upper bidiagonally on a* $[Y]_{row}$ *basis.*

Proof. This is similar to the proof of Corollary 6. □

Corollary 9. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$*. Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, X*.*
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

(a) Ψ *acts diagonally on a* $[X]_{row}$ *basis.*
(b) Ψ *acts upper bidiagonally on a* $[Y]_{row}$ *basis.*
(c) Ψ *acts lower bidiagonally on a* $[Z]_{row}$ *basis.*

Proof. We argue as in the proof of Corollary 6. The result is clear if (a) holds. Note that (b) implies that all $\beta_i' = 0$, so $a_z = a_{zx} = a_{yz} = 0$ by Corollary 4, and (c) implies that all $\gamma_i^* = 0$, so $a_y = a_{yz} = a_{xy} = 0$ by Corollary 5. Thus, Ψ acts on V_d as a linear combination of I, X. □

Corollary 10. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$*. Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, Y*.*
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

(a) Ψ *acts diagonally on a* $[Y]_{row}$ *basis.*
(b) Ψ *acts upper bidiagonally on a* $[Z]_{row}$ *basis.*
(c) Ψ *acts lower bidiagonally on a* $[X]_{row}$ *basis.*

Proof. This is similar to the proof of Corollary 9. □

Corollary 11. *With reference to Lemma 2 and Theorem 3, let* $\Psi \in \mathrm{End}(V_d)$*. Then, the following* (i)–(iii) *are equivalent.*

(i) Ψ *acts on* V_d *as a linear combination of* I, Z*.*
(ii) *All three of* (a), (b), (c) *below hold.*
(iii) *Any two of* (a), (b), (c) *below hold.*

(a) Ψ *acts diagonally on a* $[Z]_{row}$ *basis.*
(b) Ψ *acts upper bidiagonally on a* $[X]_{row}$ *basis.*
(c) Ψ *acts lower bidiagonally on a* $[Y]_{row}$ *basis.*

Proof. This is similar to the proof of Corollary 9. □

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