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Relations between Generalized Bi-Periodic Fibonacci and Lucas Sequences

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Abstract: In this paper we consider a generalized bi-periodic Fibonacci $\{f_n\}$ and a generalized bi-periodic Lucas sequence $\{q_n\}$ which are respectively defined by $f_0 = 0, f_1 = 1, f_n = af_{n-1} + cf_{n-2}$ (n is even) or $f_n = bf_{n-1} + cf_{n-2}$ (n is odd), and $q_0 = 2d, q_1 = ad, q_n = bq_{n-1} + cq_{n-2}$ (n is even) or $q_n = af_{n-1} + cq_{n-2}$ (n is odd). We obtain various relations between these two sequences.

Keywords: generalized bi-periodic Fibonacci sequence; generalized bi-periodic Lucas sequence; Binet's formula

MSC: 11B37; 11B39

1. Introduction

As is well known, the Fibonacci sequence $\{F_n\}$ is generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial condition $F_0 = 0, F_1 = 1$, and the Lucas sequence $\{L_n\}$ satisfies the same recurrence relation with initial condition $L_0 = 2, L_1 = 1$. The Fibonacci and Lucas numbers possess many interesting properties and appear in a variety of application fields [1].

The Binet's formulas for $\{F_n\}$ and $\{L_n\}$ are respectively given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where α and β are roots of the equation $x^2 - x - 1 = 0$.

Many authors generalized the Fibonacci and Lucas sequences by changing initial conditions and/or recurrence relations. As a generalization of the Fibonacci sequence, Edson and Yayenie [2] introduced a bi-periodic Fibonacci sequence $\{p_n\}$ defined by

$$p_0 = 0, p_1 = 1, p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ bp_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \quad (1)$$

They obtained the Binet's formula and proved many interesting properties of $\{p_n\}$.

If $a = b = 1$, then $\{p_n\}$ becomes the Fibonacci sequence. For $a = b = 2$, $\{p_n\}$ denotes the Pell sequence. In addition, if $a = b = k$, then $\{p_n\}$ denotes the k -Fibonacci sequence defined in [3], etc.

On the other hand, Bilgici [4] generalized the Lucas sequence by introducing a bi-periodic Lucas sequence $\{l_n\}$ defined by

$$l_0 = 2, l_1 = a, l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2), \quad (2)$$

and proved many interesting relations between $\{p_n\}$ and $\{l_n\}$.

If $a = b = 1$, then $\{l_n\}$ becomes the Lucas sequence. For $a = b = 2$, $\{l_n\}$ denotes the Pell-Lucas sequence. If $a = b = k$, then $\{l_n\}$ becomes the k -Lucas sequence defined in [5].

In this paper we consider a generalized bi-periodic Fibonacci sequence $\{f_n\}$ and a generalized bi-periodic Lucas sequence $\{q_n\}$ which are generalizations of $\{p_n\}$ and $\{l_n\}$ respectively, and are defined as

$$f_0 = 0, f_1 = 1, f_n = \begin{cases} af_{n-1} + cf_{n-2}, & \text{if } n \text{ is even} \\ bf_{n-1} + cf_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2), \quad (3)$$

and

$$q_0 = 2d, q_1 = ad, q_n = \begin{cases} bq_{n-1} + cq_{n-2}, & \text{if } n \text{ is even} \\ aq_{n-1} + cq_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2), \quad (4)$$

where a, b, c and d are nonzero real numbers.

If $c = 1$, $\{f_n\}$ becomes $\{p_n\}$ defined in (1). If $a = b = 2$ and $c = k$, $\{f_n\}$ becomes the k -Pell sequence defined in [6]. If $a = b = 1$ and $c = 2$, $\{f_n\}$ becomes the Jacobsthal sequence $\{J_n\}$. If $a = b = k$ and $c = 2$, $\{f_n\}$ becomes the k -Jacobsthal sequence defined in [7]. If $a = b = s$ and $c = 2t$, $\{f_n\}$ becomes the (s, t) -Jacobsthal sequence defined in [8]. If $a = b = 6$ and $c = -1$, $\{f_n\}$ becomes the sequence of balancing numbers $\{B_n\}$ defined in [9]. If $c = -1$, $\{f_n\}$ becomes the sequence of bi-periodic balancing numbers defined in [10].

If $c = d = 1$, $\{q_n\}$ becomes $\{l_n\}$ defined in (2). If $a = b = 2$, $c = k$ and $d = 1$, $\{q_n\}$ becomes the k -Pell-Lucas sequence defined in [11]. If $a = b = 1$, $c = 2$ and $d = 1$, $\{q_n\}$ becomes the Jacobsthal-Lucas sequence $\{j_n\}$. If $a = b = k$, $c = 2$ and $d = 1$, $\{q_n\}$ becomes the k -Jacobsthal-Lucas sequence defined in [12]. If $a = b = s$, $c = 2t$ and $d = 1$, $\{q_n\}$ becomes the (s, t) -Jacobsthal-Lucas sequence defined in [8]. Setting $a = 2a$, $b = 2b$ and $c = d = 1$, $\{q_n\}$ becomes the bi-periodic Pell-Lucas sequence defined in [13]. If $c = 2$ and $d = 1$, $\{q_n\}$ becomes the bi-periodic Jacobsthal-Lucas sequence defined in [14]. If $a = b = 6$, $c = -1$ and $d = 1/2$, $\{q_n\}$ becomes the sequence of Lucas-balancing numbers $\{b_n\}$ defined in [15].

In the next section we derive various relations between $\{f_n\}$ and $\{q_n\}$. The relations obtained here not only include many existing ones as special cases but also give diverse new results.

2. Results

In this section we derive diverse relations between two sequences $\{f_n\}$ and $\{q_n\}$ respectively given in (3) and (4).

According to ([16] [Theorem 8]) and [17], the Binet's formulas for $\{f_n\}$ and $\{q_n\}$ are respectively given by

$$f_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \quad (5)$$

$$q_n = \frac{d}{(ab)^{\lfloor \frac{n}{2} \rfloor} b^{\zeta(n)}} (\alpha^n + \beta^n), \quad (6)$$

where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$, i.e., α and β are roots of the equation $x^2 - abx - abc = 0$, and $\zeta(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function such that $\zeta(n) = 0$ if n is even and $\zeta(n) = 1$ if n is odd.

From (5) and (6), we also have

$$f_{-n} = \frac{(-1)^{n+1}}{c^n} f_n \quad \text{and} \quad q_{-n} = \frac{(-1)^n}{c^n} q_n.$$

In the rest of this section, we will use the following properties without mentioning to prove our main results:

- (a) $\alpha + \beta = ab$.
- (b) $\alpha \cdot \beta = -abc$.

- (c) $\lfloor \frac{m+n}{2} \rfloor - \lfloor \frac{m}{2} \rfloor - \lfloor \frac{n}{2} \rfloor = \zeta(m)\zeta(n)$,
- (d) $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - \lfloor \frac{m+n}{2} \rfloor = n - \zeta(m)\zeta(n)$,
- (e) $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{m+n}{2} \rfloor = \zeta(m+1)\zeta(n)$,
- (f) $\lfloor \frac{m+n}{2} \rfloor - \lfloor \frac{m-1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor = \zeta(m+1)\zeta(n+1) + \zeta(n)$.
- (g) $\zeta(m)\zeta(n) + \zeta(m+1) - \zeta(m+n+1) = \zeta(m+1)\zeta(n)$.
- (h) $\zeta(m)\zeta(n) + \zeta(m+1) + \zeta(n+1) = 1 + \zeta(m+1)\zeta(n+1)$.
- (i) $\zeta(m)\zeta(m+n) = 1 - \zeta(m+1)\zeta(n+1)$.

Theorem 1. For any integer n , we have

$$d(f_{n+1} + cf_{n-1}) = q_n, \quad (7)$$

and

$$q_{n+1} + cq_{n-1} = (ab + 4c)df_n. \quad (8)$$

Proof. From (5) and (6), we have

$$\begin{aligned} \frac{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}{a^{\zeta(n)}} f_{n+1} + abc \frac{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}{a^{\zeta(n)}} f_{n-1} &= \frac{\alpha^{n+1} - \beta^{n+1} + abc(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\ &= \frac{\alpha^n(\alpha + \frac{abc}{\alpha}) - \beta^n(\beta + \frac{abc}{\beta})}{\alpha - \beta} \\ &= \alpha^n + \beta^n \\ &= \frac{(ab)^{\lfloor \frac{n}{2} \rfloor} b^{\zeta(n)}}{d} q_n, \end{aligned}$$

from which we obtain (7).

Similarly we can show that

$$\frac{(ab)^{\lfloor \frac{n+1}{2} \rfloor} b^{\zeta(n+1)}}{d} q_{n+1} + \frac{abc(ab)^{\lfloor \frac{n-1}{2} \rfloor} b^{\zeta(n-1)}}{d} q_{n-1} = \frac{ab(ab + 4c)(ab)^{\lfloor \frac{n}{2} \rfloor}}{a^{\zeta(n+1)}} f_n,$$

from which we obtain (8). $\square$

If $c = d = 1$, Theorem 1 reduces to ([4] [Theorem 3]).

If $a = b = d = 1$ and $c = 2$, then (7) and (8) respectively reduce to the known identities

$$J_{n+1} + 2J_{n-1} = j_n,$$

and

$$j_{n+1} + 2j_{n-1} = 9J_n.$$

If $a = b = 6$, $c = -1$ and $d = 1/2$, we obtain the new identities

$$B_{n+1} - B_{n-1} = 2b_n,$$

and

$$b_{n+1} - b_{n-1} = 16B_n.$$

Theorem 2. For any integer n , we have

$$d(f_{n+2} - c^2 f_{n-2}) = a^{\zeta(n+1)} b^{\zeta(n)} q_n, \quad (9)$$

and

$$q_{n+2} - c^2 q_{n-2} = a^{\zeta(n)} b^{\zeta(n+1)} (ab + 4c) d f_n. \quad (10)$$

Proof. From (5) and (6), we have

$$\begin{aligned} (\alpha - \beta)(f_{n+2} - c^2 f_{n-2}) &= \frac{a^{\zeta(n+3)}}{(ab)^{\lfloor \frac{n+2}{2} \rfloor}} (\alpha^{n+2} - \beta^{n+2}) - \frac{a^{\zeta(n-1)}}{(ab)^{\lfloor \frac{n-2}{2} \rfloor}} c^2 (\alpha^{n-2} - \beta^{n-2}) \\ &= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{n+2} - \beta^{n+2}}{ab} - abc^2 (\alpha^{n-2} - \beta^{n-2}) \right) \\ &= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{n+2} - a^2 b^2 c^2 \alpha^{n-2}}{ab} - \frac{\beta^{n+2} - a^2 b^2 c^2 \beta^{n-2}}{ab} \right) \\ &= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} (\alpha - \beta)(\alpha^n + \beta^n) \\ &= \frac{a^{\zeta(n+1)} b^{\zeta(n)}}{d} (\alpha - \beta) q_n, \end{aligned}$$

and we obtain (9).

The proof of (10) is similar, and is omitted. $\square$

For $a = b = 1$, $c = 2$ and $d = 1$, we obtain the known identities

$$J_{n+2} - 4J_{n-2} = j_n,$$

and

$$j_{n+2} - 4j_{n-2} = 9J_n.$$

For $a = b = 6$, $c = -1$ and $d = 1/2$, (9) and (10) respectively reduce to the new identities

$$B_{n+2} - B_{n-2} = 12b_n,$$

and

$$b_{n+2} - b_{n-2} = 96B_n.$$

Theorem 3. For any integers m and n , we have

$$\left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)} f_m q_n + \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n+1)} f_n q_m = 2d f_{m+n}, \quad (11)$$

and

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} q_m q_n + (ab + 4c) d^2 \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} f_m f_n = 2d q_{m+n}. \quad (12)$$

Proof. From (5) and (6), we have

$$\begin{aligned} \frac{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} b^{\zeta(n)}}{da^{\zeta(m+1)}} f_m q_n + \frac{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} b^{\zeta(m)}}{da^{\zeta(n+1)}} f_n q_m &= \frac{2(\alpha^{m+n} - \beta^{m+n})}{\alpha - \beta} \\ &= \frac{2(ab)^{\lfloor \frac{m+n}{2} \rfloor}}{a^{\zeta(m+n+1)}} f_{m+n}, \end{aligned}$$

or

$$\frac{a^{\zeta(m+n+1)-\zeta(m+1)} b^{\zeta(n)}}{(ab)^{\zeta(m)\zeta(n)}} f_m q_n + \frac{a^{\zeta(m+n+1)-\zeta(n+1)} b^{\zeta(m)}}{(ab)^{\zeta(m)\zeta(n)}} f_n q_m = 2d f_{m+n}.$$

Since

$$\frac{a^{\zeta(m+n+1)-\zeta(m+1)}b^{\zeta(n)}}{(ab)^{\zeta(m)\zeta(n)}} = \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)},$$

we obtain (11).

Also we have

$$\begin{aligned} \frac{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} b^{\zeta(m)+\zeta(n)}}{d^2} q_m q_n + \frac{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} (\alpha - \beta)^2}{a^{\zeta(m+1)+\zeta(n+1)}} f_m f_n &= 2(\alpha^{m+n} + \beta^{m+n}) \\ &= \frac{2(ab)^{\lfloor \frac{m+n}{2} \rfloor} b^{\zeta(m+n)}}{d} q_{m+n}, \end{aligned}$$

or

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} q_m q_n + \frac{ab(ab+4c)d^2}{a^{\zeta(m+1)+\zeta(n+1)}(ab)^{\zeta(m)\zeta(n)}b^{\zeta(m+n)}} f_m f_n = 2d q_{m+n}.$$

Since

$$\frac{ab}{a^{\zeta(m+1)+\zeta(n+1)}(ab)^{\zeta(m)\zeta(n)}b^{\zeta(m+n)}} = \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)},$$

then we obtain (12). $\square$

If $c = d = 1$, then (11) reduces to the second identity in ([4] [Theorem 5]). Also, using the same notation as in [4], the first identity in ([4] [Theorem 5]) should be corrected as

$$l_{m+n} = \frac{1}{2} \left[(ab+4) \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} q_m q_n + \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} l_m l_n \right].$$

If $a = b = d = 1$ and $c = 2$, then (11) and (12) respectively reduce to the known identities

$$J_m J_n + J_n J_m = 2J_{m+n},$$

and

$$j_m j_n + 9J_m J_n = 2j_{m+n}.$$

If $a = b = 6$, $c = -1$ and $d = 1/2$, then (11) reduces to the known identity

$$B_m b_n + B_n b_m = B_{m+n},$$

and (12) reduces to the new identity

$$b_m b_n + 8B_m B_n = b_{m+n}.$$

Replacing n by $-n$ in Theorem 3, we obtain the following result.

Corollary 1. For any integers m and n , we have

$$\left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)} f_m q_n - \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n+1)} f_n q_m = (-1)^n 2c^n d f_{m-n}, \quad (13)$$

and

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} q_m q_n - (ab+4c)d^2 \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} f_m f_n = (-1)^n 2c^n d q_{m-n}. \quad (14)$$

If $c = d = 1$, then (13) reduces to the first identity in ([4] [Corollary 3]). Also, using the same notation as in [4], the second identity in ([4] [Corollary 3]) should be corrected as

$$l_{m-n} = \frac{(-1)^n}{2} \left[\left(\frac{b}{a} \right)^{\zeta(m)\zeta(n)} l_m l_n - (ab + 4c) d^2 \left(\frac{b}{a} \right)^{\zeta(m+1)\zeta(n+1)} q_m q_n \right].$$

If $a = b = d = 1$ and $c = 2$, (13) and (14) respectively reduce to the known identities

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n},$$

and

$$j_m j_n - 9 J_m J_n = (-1)^n 2^{n+1} j_{m-n}.$$

For $a = b = 6$, $c = -1$ and $d = 1/2$, (13) leads to the known identity

$$B_m b_n - B_n b_m = B_{m-n},$$

and (14) leads to the new identity

$$b_m b_n - 8 B_m B_n = b_{m-n}.$$

Theorem 4. For any integers m and n , we have

$$\left(\frac{b}{a} \right)^{\zeta(m)\zeta(n)} f_{m+1} q_n + c \left(\frac{b}{a} \right)^{\zeta(m+1)\zeta(n+1)} f_m q_{n-1} = q_{m+n}. \quad (15)$$

Proof. From (5) and (6), we have

$$\begin{aligned} & \frac{(ab)^{\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} b^{\zeta(n)}}{d a^{\zeta(m)}} f_{m+1} q_n + \frac{abc(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor} b^{\zeta(n+1)}}{d a^{\zeta(m+1)}} f_m q_{n-1} \\ &= \frac{(\alpha^{m+1} - \beta^{m+1})(\alpha^n + \beta^n) + abc(\alpha^m - \beta^m)(\alpha^{n-1} + \beta^{n-1})}{\alpha - \beta} \\ &= \frac{\alpha^{m+n}(\alpha + \frac{abc}{\alpha}) - \beta^{m+n}(\beta + \frac{abc}{\beta}) - \alpha^{n-1}\beta^m(\alpha\beta + abc) + \alpha^m\beta^{n-1}(\alpha\beta + abc)}{\alpha - \beta} \\ &= \alpha^{m+n} + \beta^{m+n} \\ &= \frac{(ab)^{\lfloor \frac{m+n}{2} \rfloor} b^{\zeta(m+n)}}{d} q_{m+n}, \end{aligned}$$

or

$$\frac{(ab)^{\zeta(m)\zeta(n+1)} b^{\zeta(n)}}{a^{\zeta(m)} b^{\zeta(m+n)}} f_{m+1} q_n + \frac{abcb^{\zeta(n+1)}}{a^{\zeta(m+1)} b^{\zeta(m+n)} (ab)^{\zeta(m)+\zeta(m+1)\zeta(n+1)}} f_m q_{n-1} = q_{m+n}.$$

We can easily show that

$$\frac{(ab)^{\zeta(m)\zeta(n+1)} b^{\zeta(n)}}{a^{\zeta(m)} b^{\zeta(m+n)}} = \left(\frac{b}{a} \right)^{\zeta(m)\zeta(n)},$$

and

$$\frac{ab b^{\zeta(n+1)}}{a^{\zeta(m+1)} b^{\zeta(m+n)} (ab)^{\zeta(m)+\zeta(m+1)\zeta(n+1)}} = \left(\frac{b}{a} \right)^{\zeta(m+1)\zeta(n+1)}.$$

Hence we obtain (15). $\square$

Theorem 4 is a generalization of ([4] [Theorem 8]).

If $a = b = d = 1$ and $c = 2$, then (15) reduces to the known identity

$$J_{m+1}j_n + J_m j_{n-1} = j_{m+n}.$$

For $a = b = 6$, $c = -1$ and $d = 1/2$, we obtain the new identity

$$B_{m+1}b_n - B_m b_{n-1} = b_{m+1}.$$

Replacing n by $-n$ in Theorem 4, we obtain the following result.

Corollary 2. For any integers m and n , we have

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m+1}q_n - \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} f_m q_{n+1} = (-1)^n c^n q_{m-n}. \quad (16)$$

If $a = b = d = 1$ and $c = 2$, then (16) reduces to the known identity

$$J_{m+1}j_n - J_m j_{n+1} = (-1)^n 2^n j_{m-n}.$$

If $a = b = 6$, $c = -1$ and $d = 1/2$, then (16) reduces to the new identity

$$B_{m+1}b_n - B_m b_{n+1} = b_{m-n}.$$

Theorem 5. For any integer n , we have

$$\frac{b}{ad} q_{2n+1} f_{2n} = f_{4n+1} - c^{2n}, \quad (17)$$

and

$$\frac{b}{ad} q_{2n+1} f_{2n+2} = f_{4n+3} - c^{2n+1}. \quad (18)$$

Proof. From (5) and (6), we have

$$\begin{aligned} \frac{(ab)^{\lfloor \frac{2n+1}{2} \rfloor + \lfloor \frac{2n}{2} \rfloor} b^{\zeta(2n+1)}}{a^{\zeta(2n+1)} d} q_{2n+1} f_{2n} &= \frac{(ab)^{2n} b}{ad} q_{2n+1} f_{2n} \\ &= \frac{\alpha^{4n+1} - \beta^{4n+1}}{\alpha - \beta} - (abc)^{2n} \\ &= \frac{(ab)^{\lfloor \frac{4n+1}{2} \rfloor}}{a^{\zeta(4n+2)}} f_{4n+1} - (abc)^{2n} \\ &= (ab)^{2n} f_{4n+1} - (abc)^{2n}, \end{aligned}$$

from which we obtain (17).

The proof of (18) is similar, and is omitted. $\square$

Theorem 5 is an extension of ([4] [Theorem 12]).

For $a = b = d = 1$ and $c = 2$, (17) and (18) respectively reduce to the known identities

$$j_{2n+1}J_{2n} = J_{4n+1} - 4^n,$$

and

$$j_{2n+1}J_{2n+2} = J_{4n+3} - 2 \cdot 4^n.$$

If $a = b = 6, c = -1$ and $d = 1/2$, then (17) and (18) respectively reduce to the known identities

$$2b_{2n+1}B_{2n} = B_{4n+1} - 1,$$

and

$$2b_{2n+1}B_{2n+2} = B_{4n+3} + 1.$$

Various identities given in Theorem 6 below can be proved similarly, and details are omitted.

Theorem 6. For any integers m and n , we have

$$df_{m+n} + (-1)^n c^n f_{m-n} = \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)} f_m q_n, \quad (19)$$

$$d\left(q_{m+n} + (-1)^n c^n q_{m-n}\right) = \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} q_m q_n, \quad (20)$$

$$d\left(f_{m+n} - (-1)^n c^n f_{m-n}\right) = \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n+1)} f_n q_m, \quad (21)$$

$$q_{m+n} - (-1)^n c^n q_{m-n} = \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} (ab + 4c) d f_m f_n, \quad (22)$$

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} q_m f_{m(n-1)} = f_{mn} + (-1)^m c^m f_{m(n-2)}, \quad (23)$$

$$q_n q_{n+1} = d q_{2n+1} + (-1)^n a c d^2, \quad (24)$$

$$c\left(\frac{b}{a}\right)^{\zeta(n)} q_n^2 + \left(\frac{b}{a}\right)^{\zeta(n+1)} q_{n+1}^2 = c d q_{2n} + d q_{2n+2}, \quad (25)$$

$$(ab + 4c) d \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n+1)} f_m f_n = q_{m+n} - (-1)^n c^n q_{m-n}, \quad (26)$$

$$c f_{2n+1} = \left(\frac{b}{a}\right)^{\zeta(n)} f_{n+1} q_{n+2} - c f_{n+2} q_n + (-1)^n c^n (ab - c), \quad (27)$$

$$\left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)} f_m q_{m+n} = d f_{2m+n} - (-1)^m c^m d f_n, \quad (28)$$

$$\left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m+n} q_n = d f_{m+2n} + (-1)^m c^n d f_m. \quad (29)$$

Consider the sequence $\{g_n\}$ defined by

$$g_0 = 2, \quad g_1 = b, \quad g_n = \begin{cases} a g_{n-1} + c g_{n-2}, & \text{if } n \text{ is even} \\ b g_{n-1} + c g_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \quad (30)$$

Recently Tan and Leung [18] studied the properties of the bi-periodic Fibonacci and Lucas sequences, and stated that the following identities can be obtained using matrix method in [19]:

$$f_m g_n + f_n g_m = 2 \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m+n}, \quad (31)$$

$$g_m g_n + \frac{b(ab + 4c)}{a} f_m f_n = 2 \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} g_{m+n}, \quad (32)$$

$$f_m g_n - f_n g_m = (-1)^n 2c^n \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m-n}, \quad (33)$$

$$g_m g_n - \frac{b(ab + 4c)}{a} f_m f_n = (-1)^n 2c^n \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} g_{m-n}, \quad (34)$$

$$f_{m+n} + (-1)^n c^n f_{m-n} = \left(\frac{a}{b}\right)^{\zeta(m)\zeta(n)} f_m g_n, \quad (35)$$

$$g_{m+n} + (-1)^n c^n g_{m-n} = \left(\frac{a}{b}\right)^{\zeta(m)\zeta(n)} g_m g_n. \quad (36)$$

In fact more general identities than (31)–(36) can be obtained using the results of this paper. To show this, consider the sequence $\{\hat{g}_n\}$ defined by

$$\hat{g}_0 = 2d, \hat{g}_1 = bd, \hat{g}_n = \begin{cases} a\hat{g}_{n-1} + c\hat{g}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{g}_{n-1} + c\hat{g}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \quad (37)$$

Since $\{\hat{g}_n\}$ is obtained from (4) by exchanging the roles of a and b , the Binet's formula is given by

$$\hat{g}_n = \frac{d}{(ab)^{\lfloor \frac{n}{2} \rfloor} a^{\zeta(n)}} (\alpha^n + \beta^n), \quad (38)$$

where α and β are as defined in (6), and we have

$$q_n = \left(\frac{a}{b}\right)^{\zeta(n)} \hat{g}_n. \quad (39)$$

Now, inserting (39) into (11), we have

$$\begin{aligned} \left(\frac{b}{a}\right)^{\zeta(m+1)\zeta(n)-\zeta(n)} f_m \hat{g}_n + \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n+1)-\zeta(m)} f_n \hat{g}_m &= \left(\frac{b}{a}\right)^{-\zeta(m)\zeta(n)} (f_m \hat{g}_n + f_n \hat{g}_m) \\ &= 2d f_{m+n}, \end{aligned}$$

and so

$$f_m \hat{g}_n + f_n \hat{g}_m = 2d \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m+n}. \quad (40)$$

Similarly, from (12)–(14), (19) and (20), we obtain the following identities:

$$\hat{g}_m \hat{g}_n + \frac{bd^2(ab+4c)}{a} f_m f_n = 2d \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} \hat{g}_{m+n}, \quad (41)$$

$$f_m \hat{g}_n - f_n \hat{g}_m = (-1)^n 2c^n d \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} f_{m-n}, \quad (42)$$

$$\hat{g}_m \hat{g}_n - \frac{bd^2(ab+4c)}{a} f_m f_n = (-1)^n 2c^n d \left(\frac{b}{a}\right)^{\zeta(m)\zeta(n)} \hat{g}_{m-n}, \quad (43)$$

$$d f_{m+n} + (-1)^n c^n f_{m-n} = \left(\frac{a}{b}\right)^{\zeta(m)\zeta(n)} f_m \hat{g}_n, \quad (44)$$

$$d (\hat{g}_{m+n} + (-1)^n c^n \hat{g}_{m-n}) = \left(\frac{a}{b}\right)^{\zeta(m)\zeta(n)} \hat{g}_m \hat{g}_n. \quad (45)$$

If $d = 1$, then $\hat{g}_n = g_n$ and (40)–(45) respectively reduce to (31)–(36).

3. Discussion

In this paper we considered two kinds of general sequences, i.e., generalized bi-periodic Fibonacci sequence and generalized bi-periodic Lucas sequence. The generalized bi-periodic Fibonacci sequence includes many sequences as special cases. For example, the Fibonacci, Pell and Jacobsthal sequences and the sequence of balancing numbers are particular examples. Also the generalized bi-periodic Lucas sequence include the Lucas, Pell-Lucas and Jacobsthal-Lucas sequences and the sequence of Lucas-balancing numbers as special cases.

We derived many interesting relations between these two sequences. The relations obtained in the paper not only include many existing one as special cases but also give diverse new results.

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