The Italian Domination Numbers of Some Products of Directed Cycles

Kijung Kim

Department of Mathematics, Pusan National University, Busan 46241, Korea; knukkj@pusan.ac.kr

Received: 29 July 2020; Accepted: 31 August 2020; Published: 1 September 2020

Abstract: An Italian dominating function on a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \rightarrow \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. In this article, we determine the exact values of the Italian domination numbers of some products of directed cycles.

Keywords: Italian dominating function; Italian domination number; cartesian product; strong product

MSC: 05C69

1. Introduction and Preliminaries

Let $D = (V, A)$ be a finite simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. An arc joining v to w is denoted by $v \rightarrow w$. The maximum out-degree and maximum in-degree of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. The cartesian product of D_1 and D_2 is the digraph $D_1 \square D_2$ with vertex set $V_1 \times V_2$ and for two vertices (u_1, u_2) and (v_1, v_2) ,

$$(u_1, u_2) \rightarrow (v_1, v_2)$$

if one of the following holds: (i) $u_1 = v_1$ and $u_2 \rightarrow v_2$; (ii) $u_1 \rightarrow v_1$ and $u_2 = v_2$.

The strong product of D_1 and D_2 is the digraph $D_1 \otimes D_2$ with vertex set $V_1 \times V_2$ and for two vertices (u_1, u_2) and (v_1, v_2) ,

$$(u_1, u_2) \rightarrow (v_1, v_2)$$

if one of the following holds: (i) $u_1 \rightarrow v_1$ and $u_2 \rightarrow v_2$; (ii) $u_1 = v_1$ and $u_2 \rightarrow v_2$; (iii) $u_1 \rightarrow v_1$ and $u_2 = v_2$.

An Italian dominating function (IDF) on a digraph D is defined as a function $f : V(D) \rightarrow \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. In other words, we say that a vertex v for which $f(v) \in \{1, 2\}$ dominates itself, while a vertex v with $f(v) = 0$ is dominated by f if it has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. An Italian dominating function $f : V(D) \rightarrow \{0, 1, 2\}$ gives a partition $\{V_0, V_1, V_2\}$ of $V(D)$, where $V_i := \{x \in V(D) \mid f(x) = i\}$. The weight of an Italian dominating function f is the value $\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)$. The Italian domination number of a digraph D , denoted by $\gamma_I(D)$, is the minimum taken over the weights of all Italian dominating functions on D . A $\gamma_I(D)$ -function is an Italian dominating function on D with weight $\gamma_I(D)$.

The Italian dominating functions in graphs and digraphs have been studied in [1–7]. The authors of [2] introduce the concept of Italian domination and give bounds, relating the Italian domination number to some other domination parameters. The authors of [3] characterize the trees T with $\gamma(T) + 1 = \gamma_I(T)$ and also characterize the trees T with $\gamma_I(T) = 2\gamma(T)$. After that, there are some studies on the cartesian products of undirected cycles or undirected paths in [4,8–10]. Recently,

the author of [6] initiated the study of the Italian domination number in digraphs. In this article, we investigate the Italian domination numbers of $C_m \square C_n$ and $C_m \otimes C_n$.

The following results are useful to our study.

Proposition 1 ([6]). *Let D be a digraph of order n . Then $\gamma_I(D) \geq \lceil \frac{2n}{2+\Delta^+(D)} \rceil$.*

Proposition 2 ([6]). *Let D be a digraph of order n . Then $\gamma_I(D) \leq n$, and $\gamma_I(D) = n$ if and only if $\Delta^+(D), \Delta^-(D) \leq 1$.*

Proposition 3 ([6]). *If D is a directed path or a directed cycle of order n , then $\gamma_I(D) = n$.*

2. The Italian Domination Numbers of Some Products of Directed Cycles

In this section, we determine the exact values of $\gamma_I(C_m \square C_n)$ and $\gamma_I(C_m \otimes C_n)$.

First, we consider $C_m \square C_n$. We denote the vertex set of a directed cycle C_m by $\{1, 2, \dots, m\}$, and assume that $i \rightarrow i+1$ is an arc of C_m . For every vertex $(i, j) \in V(C_m \square C_n)$, the first and second components are considered modulo m and n , respectively. For each $1 \leq k \leq n$, we denote by C_m^k the subdigraph of $C_m \square C_n$ induced by the set $\{(j, k) \mid 1 \leq j \leq m\}$. Note that C_m^k is isomorphic to C_m . Let f be a $\gamma_I(C_m \square C_n)$ -function and set $a_k = \sum_{x \in V(C_m^k)} f(x)$. Then $\gamma_I(C_m \square C_n) = \sum_{k=1}^n a_k$. It is easy to see that $C_m \square C_n$ is isomorphic to $C_n \square C_m$. So, $\gamma_I(C_m \square C_n) = \gamma_I(C_n \square C_m)$.

Theorem 1. *If $m = 2r$ and $n = 2s$ for some positive integers r, s , then $\gamma_I(C_m \square C_n) = \frac{mn}{2}$.*

Proof. Define $f : V(C_m \square C_n) \rightarrow \{0, 1, 2\}$ by

$$f((2i-1, 2j-1)) = f((2i, 2j)) = 1$$

for each $1 \leq i \leq r$ and $1 \leq j \leq s$, and

$$f((x, y)) = 0$$

otherwise. It is easy to see that f is an IDF of $C_m \square C_n$ with weight $\frac{mn}{2}$ and so $\gamma_I(C_m \square C_n) \leq \frac{mn}{2}$. Since $\Delta^+(D) = 2$, it follows from Proposition 1 that $\gamma_I(C_m \square C_n) \geq \frac{mn}{2}$. Thus, we have $\gamma_I(C_m \square C_n) = \frac{mn}{2}$. $\square$

Theorem 2. *For an odd integer $n \geq 3$, $\gamma_I(C_2 \square C_n) = n + 1$.*

Proof. Define $f : V(C_2 \square C_n) \rightarrow \{0, 1, 2\}$ by

$$f((1, 2j-1)) = 1$$

for each $1 \leq j \leq \frac{n+1}{2}$,

$$f((2, 2j)) = 1$$

for each $1 \leq j \leq \frac{n-1}{2}$,

$$f((2, n)) = 1$$

and

$$f((x, y)) = 0$$

otherwise. It is easy to see that f is an IDF of $C_2 \square C_n$ with weight $n + 1$ and so $\gamma_I(C_2 \square C_n) \leq n + 1$.

Now we claim that $\gamma_I(C_2 \square C_n) \geq n + 1$. Suppose to the contrary that $\gamma_I(C_2 \square C_n) \leq n$. Let f be a $\gamma_I(C_2 \square C_n)$ -function. If $a_k = 0$ for some k , assume without loss of generality $k = 3$, then $f((1, 3)) =$

$f((2,3)) = 0$. To dominate the vertices $(1,3)$ and $(2,3)$, we must have $f((1,2)) = f((2,2)) = 2$. Define $g : V(C_2 \square C_n) \rightarrow \{0,1,2\}$ by

$$g((1,2)) = g((2,1)) = g((2,3)) = 1, g((2,2)) = 0$$

and

$$g((x,y)) = f((x,y))$$

otherwise. Then g is an IDF of $C_2 \square C_n$ with a weight less than $\omega(f)$, which is a contradiction. Thus, $a_k \geq 1$ for each k . By assumption, $a_k = 1$ for each k . Without loss of generality, we assume that $f((1,2)) = 1$. To dominate $(2,2)$, we must have $f((2,1)) = 1$. Since $a_3 = 1$ and $f((2,2)) = 0$, we have $f((2,3)) = 1$. By repeating this process, we obtain $f((1,2i)) = 1$ for each $1 \leq i \leq \frac{n-1}{2}$, $f((2,2i-1)) = 1$ for $1 \leq i \leq \frac{n+1}{2}$ and $f((x,y)) = 0$ otherwise. However, the vertex $(1,1)$ is not dominated, and this is a contradiction. Thus we have $\gamma_I(C_2 \square C_n) \geq n+1$. This completes the proof. $\square$

Theorem 3. For an integer $n \geq 3$, $\gamma_I(C_3 \square C_n) = 2n$.

Proof. When $n = 3r$ for some positive integer r , define $f_0 : V(C_3 \square C_n) \rightarrow \{0,1,2\}$ by

$$f_0((1,3j+1)) = f_0((2,3j+1)) = 1$$

for each $0 \leq j \leq n-1$,

$$f_0((2,3j+2)) = f_0((3,3j+2)) = 1$$

for each $0 \leq j \leq n-1$,

$$f_0((1,3j+3)) = f_0((3,3j+3)) = 1$$

for each $0 \leq j \leq n-1$ and

$$f_0((x,y)) = 0$$

otherwise.

When $n = 3r+1$ for some positive integer r , define $f_1 : V(C_3 \square C_n) \rightarrow \{0,1,2\}$ by

$$f_1((2,n)) = f_1((3,n)) = 1$$

and

$$f_1((x,y)) = f_0((x,y))$$

otherwise.

When $n = 3r+2$ for some positive integer r , define $f_2 : V(C_3 \square C_n) \rightarrow \{0,1,2\}$ by

$$f_2((1,n-1)) = f_2((2,n-1)) = f_2((1,n)) = f_2((3,n)) = 1$$

and

$$f_2((x,y)) = f_0((x,y))$$

otherwise. It is easy to see that f_i ($i = 0,1,2$) is an IDF of $C_3 \square C_n$ with weight $2n$ and so $\gamma_I(C_3 \square C_n) \leq 2n$.

Now we prove that $\gamma_I(C_3 \square C_n) \geq 2n$. Let f be a $\gamma_I(C_3 \square C_n)$ -function.

Claim 1. $a_k \geq 1$ for each $1 \leq k \leq n$.

Suppose to the contrary that $a_k = 0$ for some k , say $k = n$. To dominate $(1, n)$, $(2, n)$ and $(3, n)$, we must have $f((1, n-1)) = f((2, n-1)) = f((3, n-1)) = 2$. However, the function g defined by

$$g((1, n-1)) = g((2, n-1)) = g((3, n-1)) = 1,$$

$$g((1, n)) = g((2, n)) = 1$$

and

$$g((x, y)) = f((x, y))$$

otherwise, is an IDF of $C_3 \square C_n$ with a weight less than $\omega(f)$. This is a contradiction. $\square$

We choose a $\gamma_I(C_3 \square C_n)$ -function h so that the size of $M_h := \{k \mid a_k = 1\}$ is as small as possible.

Claim 2. $|M_h| = 0$.

Suppose to the contrary that $|M_h| \geq 1$. Without loss of generality, assume that $a_n = 1$ and $h((1, n)) = 1$. To dominate $(2, n)$ and $(3, n)$, we must have $h((2, n-1)) = 1$ and $h((3, n-1)) = 2$. If $n = 3$, then clearly $a_1 \geq 2$ and so $\gamma_I(C_3 \square C_3) \geq 6$. However, when $n = 3$, the previously defined function f_0 is a $\gamma_I(C_3 \square C_3)$ -function such that $\omega(f_0) = 6$ and $|M_{f_0}| = 0$. This contradicts the choice of h . From now on, assume $n \geq 4$. We divide our consideration into the following two cases.

Case 1. $a_{n-2} = 1$.

By the same argument as above, we have $a_{n-3} \geq 3$. So $a_{n-3} + a_{n-2} + a_{n-1} + a_n \geq 8$. If $n = 4$, then the previously defined function f_1 induces a contradiction. Suppose $n \geq 5$. Since $a_{n-4} \geq 1$ by Claim 1, $h((i, n-4)) = 1$ or 2 for some $i \in \{1, 2, 3\}$. Without loss of generality, we may assume $h((1, n-4)) = 1$ or 2 . Define $t : V(C_3 \square C_n) \rightarrow \{0, 1, 2\}$ by

$$t((1, n-3)) = t((2, n-2)) = t((1, n-1)) = t((2, n)) = 0,$$

$$t((2, n-3)) = t((1, n-2)) = t((2, n-1)) = t((1, n)) = 1,$$

$$t((3, n-3)) = t((3, n-2)) = t((3, n-1)) = t((3, n)) = 1$$

and

$$t((x, y)) = h((x, y))$$

otherwise. Then it is easy to see that t is an IDF of $C_3 \square C_n$ such that $|M_t| < |M_h|$. This contradicts the choice of h .

Case 2. $a_{n-2} \geq 2$.

Now $a_{n-2} + a_{n-1} + a_n \geq 6$. Since $a_{n-3} \geq 1$ by Claim 1, $h((i, n-3)) = 1$ or 2 for some $i \in \{1, 2, 3\}$. Without a loss of generality, we may assume $h((1, n-3)) = 1$ or 2 . Define $t : V(C_3 \square C_n) \rightarrow \{0, 1, 2\}$ by

$$t((1, n-2)) = t((2, n-1)) = t((3, n)) = 0,$$

$$t((2, n-2)) = t((1, n-1)) = t((1, n)) = 1,$$

$$t((3, n-2)) = t((3, n-1)) = t((2, n)) = 1$$

and

$$t((x, y)) = h((x, y))$$

otherwise. Then it is easy to see that t is an IDF of $C_3 \square C_n$ such that $|M_t| < |M_h|$. This contradicts the choice of h . $\square$

By Claims 1 and 2, we have $\gamma_I(C_3 \square C_n) \geq 2n$. This completes the proof. $\square$

Next, we consider $C_m \otimes C_n$. We denote the vertex set of a directed cycle C_m by $\{1, 2, \dots, m\}$, and assume that $i \rightarrow i + 1$ is an arc of C_m . For every vertex $(i, j) \in V(C_m \otimes C_n)$, the first and second components are considered modulo m and n , respectively. For each $1 \leq k \leq n$, we denote by C_m^k the subdigraph of $C_m \otimes C_n$ induced by the set $\{(j, k) \mid 1 \leq j \leq m\}$. Note that C_m^k is isomorphic to C_m . Let f be a $\gamma_I(C_m \otimes C_n)$ -function and set $a_k = \sum_{x \in V(C_m^k)} f(x)$. Then $\gamma_I(C_m \otimes C_n) = \sum_{k=1}^n a_k$.

Lemma 1. For positive integers $m, n \geq 2$, $\gamma_I(C_m \otimes C_n) \geq \lceil \frac{mn}{2} \rceil$.

Proof. Note that the vertices of C_m^k are dominated by vertices of C_m^{k-1} or C_m^k . It suffices to verify that $\sum_{k=1}^n a_k \geq \lceil \frac{mn}{2} \rceil$. In order to do so, we claim $a_k + a_{k+1} \geq m$ for each k . First of all, we assume that $a_{k+1} = 0$. Then, to dominate $(i, k + 1)$ for each $1 \leq i \leq m$, we must have

$$f((i-1, k)) + f((i, k)) \geq 2.$$

Then, $2a_k = \sum_{i=1}^m (f((i-1, k)) + f((i, k))) \geq 2m$ and hence $a_k + a_{k+1} \geq m$. If $a_{k+1} = t > 0$, then there will be at least $m - t$ vertices in V_0 that will be dominated only by vertices of C_m^k . This fact induces $a_k \geq m - t$ and so $a_k + a_{k+1} \geq m$. Therefore, we have

$$2\gamma_I(C_m \otimes C_n) = 2 \sum_{k=1}^n a_k = \sum_{k=1}^n (a_k + a_{k+1}) \geq nm.$$

This completes the proof. $\square$

Theorem 4. For positive integers $m, n \geq 2$, $\gamma_I(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

Proof. We divide our consideration into the following two cases.

Case 1. m or n is even.

Since $C_m \otimes C_n$ is isomorphic to $C_n \otimes C_m$, we may assume that $n = 2s$ for some positive integers s .

Define $f : V(C_m \otimes C_n) \rightarrow \{0, 1, 2\}$ by

$$f((i, 2j-1)) = 1$$

for each $1 \leq i \leq m$ and $1 \leq j \leq s$, and

$$f((x, y)) = 0$$

otherwise. It is easy to see that f is an IDF of $C_m \otimes C_n$ with weight $\frac{mn}{2}$ and so $\gamma_I(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil$. Thus, it follows from Lemma 1 that $\gamma_I(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

Case 2. $m = 2r + 1, n = 2s + 1$ for some positive integers r, s .

Define $f : V(C_m \otimes C_n) \rightarrow \{0, 1, 2\}$ by

$$f((2i+1, 2j+1)) = 1$$

for each $0 \leq i \leq r$ and $0 \leq j \leq s$,

$$f((2i, 2j)) = 1$$

for each $1 \leq i \leq r$ and $1 \leq j \leq s$ and

$$f((x, y)) = 0$$

otherwise. It is easy to see that f is an IDF of $C_m \otimes C_n$ with weight $(r+1)(s+1) + rs$ and so $\gamma_I(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil$. Thus, it follows from Lemma 1 that $\gamma_I(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$. $\square$

3. Conclusions

In this article, we determined the exact values of $\gamma_I(C_2 \square C_l)$, $\gamma_I(C_3 \square C_l)$ and $\gamma_I(C_m \square C_n)$ for an integer l and even integers m, n . The other cases are still open. We conclude by giving a conjecture.

Conjecture 1. For an odd integer n , $\gamma_I(C_4 \square C_n) = 2n + 2$.

Acknowledgments: The author would like to thank anonymous referees for their valuable comments.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (2020R111A1A01055403).

Conflicts of Interest: The author declares no conflict of interest.

References

1. Fan, W.; Ye, A.; Miao, F.; Shao, Z.; Samodivkin, V.; Sheikholeslami, S.M. Outer-Independent Italian Domination in Graphs. *IEEE Access* **2019**, *7*, 22756–22762. [\[CrossRef\]](#)
2. Chellali, M.; Haynes, T.W.; Hedetniemi, S.T.; McRae, A.A. Roman $\{2\}$ -domination. *Discret. Appl. Math.* **2016**, *204*, 22–28. [\[CrossRef\]](#)
3. Henning, M.A.; Klostermeyer, W.F. Italian domination in trees. *Discret. Appl. Math.* **2017**, *217*, 557–564. [\[CrossRef\]](#)
4. Gao, H.; Wang, P.; Liu, E.; Yang, Y. More results on Italian domination in $C_n \square C_m$. *Mathematics* **2020**, *8*, 465. [\[CrossRef\]](#)
5. Gao, H.; Xi, C.; Li, K.; Zhang, Q.; Yang, Y. The Italian domination numbers of generalized Petersen graphs $P(n, 3)$. *Mathematics* **2019**, *7*, 714. [\[CrossRef\]](#)
6. Volkmann, L. Italian domination in digraphs. *J. Combin. Math. Combin. Comput.* to appear.
7. Volkmann, L. The Italian domatic number of a digraph. *Commun. Comb. Optim.* **2019**, *4*, 61–70.
8. Stepień, Z.; Szymaszkiewicz, A.; Szymaszkiewicz, L.; Zwierzchowski, M. 2-Rainbow domination number of $C_n \square C_5$. *Discret. Appl. Math.* **2014**, *107*, 113–116. [\[CrossRef\]](#)
9. Gao, H.; Xu, T.T.; Yang, Y.S. Bagging approach for Italian domination in $C_n \square P_m$. *IEEE Access* **2019**, *7*, 105224–105234. [\[CrossRef\]](#)
10. Li, Z.P.; Shao, Z.H.; Xu, J. Weak $\{2\}$ -domination number of Cartesian products of cycles. *J. Comb. Optim.* **2018**, *35*, 75–85. [\[CrossRef\]](#)

© 2020 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).