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Article

Normal Toeplitz Operators on the Bergman Space

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Abstract: In this paper, we give a characterization of normality of Toeplitz operator T_φ on the Bergman space $A^2(\mathbb{D})$. First, we state basic properties for Toeplitz operator T_φ on $A^2(\mathbb{D})$. Next, we consider the normal Toeplitz operator T_φ on $A^2(\mathbb{D})$ in terms of harmonic symbols φ. Finally, we characterize the normal Toeplitz operators T_φ with non-harmonic symbols acting on $A^2(\mathbb{D})$.

Keywords: Toeplitz operator; normal operator; Bergman space

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1. Introduction

The purpose of this paper is to study the normality of Toeplitz operators acting on the Bergman space. Our interest is focused on Toeplitz operators with harmonic and non-harmonic symbols. Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *normal* if its self-commutator $[T^*, T] := T^*T - TT^* = 0$, where T^* denotes the adjoint of T. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$, dA be the normalized area measure on $\mathbb{D}$. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z).$$

The *Bergman space* $A^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$ and $L^\infty(\mathbb{D})$ be the space of bounded area measurable functions on $\mathbb{D}$. The *multiplication operator* M_ψ with symbol $\psi \in L^\infty(\mathbb{D})$ is defined by $M_\psi f = \psi f$ for $f \in A^2(\mathbb{D})$. For any $\varphi \in L^\infty(\mathbb{D})$, the *Toeplitz operator* T_φ on the Bergman space is defined by the formula

$$T_\varphi f = P(\varphi f)$$

for $f \in A^2(\mathbb{D})$ and P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. In this case, the function φ is called the *symbol* of T_φ. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. Recall that the Bergman space $A^2(\mathbb{D})$ is the space of analytic functions in $L^2(\mathbb{D}, dA)$ with the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 < \infty.$$

We will now consider the normality of Toeplitz operators on the Bergman space with various symbols. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space.

Let f, g are in $L^\infty(\mathbb{D})$ and $\alpha, \beta \in \mathbb{C}$, then we can easily check that

$$\text{(i)}\ T_{\alpha f+\beta g} = \alpha T_f + \beta T_g$$
$$\text{(ii)}\ T_f^* = T_{\overline{f}}$$
$$\text{(iii)}\ T_{\overline{f}} T_g = T_{\overline{f}g},\ \text{if } f \text{ or } g \text{ is analytic.}$$

Many authors in [1–6] studied intensively Normal operators and Toeplitz operators. It is a natural ask of when Toeplitz operator becomes normal. In 1963, A. Brown and P. Halmos [7] characterized normal Toeplitz operators on the Hardy space. This contains many of the fundamental results on the algebraic properties of Toeplitz operators. It has been central significance in operator theory. Thus we will focus primarily on normality of Toeplitz operators on the Bergman space.

Recently, the authors in [3] gave a properties of T_φ with non-harmonic symbols on the Bergman space. In view of this, first, we characterize the normal Toeplitz operator T_φ on $A^2(\mathbb{D})$ in terms of harmonic symbols φ. Next, we characterize the normal Toeplitz operators T_φ with non-harmonic symbols acting on $A^2(\mathbb{D})$.

The remainder of the paper is organized as follows. In Section 2, we study the normal Toeplitz operators T_φ on $A^2(\mathbb{D})$ with harmonic symbols φ. In Section 3, we consider the normal Toeplitz operators T_φ with non-harmonic symbols acting on $A^2(\mathbb{D})$ and their applications.

2. Toeplitz Operators with Harmonic Symbols

First, we study the normality of Toeplitz operators on $A^2(\mathbb{D})$ with the general symbol. In [2], the authors study the normality of Toeplitz operators on the weighted Bergman spaces as follows.

Lemma 1 (Theorem 3.2 in [2])**.** *Let $\varphi = \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q$ with $n < m, p < q, n - m = q - p$ and for nonzero $\alpha, \beta, \gamma, \delta$ and $\varphi + \overline{\varphi}$ is a constant. Then T_φ is normal if and only if $\lambda \in \mathbb{T}$ and φ is one of exactly three types;*

(i) $\varphi = \alpha z^n - \lambda \overline{\alpha} \overline{z}^n$,
(ii) $\varphi = \alpha z^n + \beta z^m - \lambda(\overline{\alpha} \overline{z}^n + \overline{\beta} \overline{z}^m)$,
(iii) $\varphi = \beta z^m - \lambda \overline{\beta} \overline{z}^m$.

We need several auxiliary lemmas to prove the main theorem in this section. We begin with:

Lemma 2 ([8])**.** *For any $s, t \in \mathbb{N}$,*

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

The proof for the Lemma 3 follows the proof of Theorem 2.1 in [8].

Lemma 3. *For $0 \leq m \leq N$, we deduce that*

(i) $||\overline{z}^m \sum_{i=0}^{\infty} c_i z^i||^2 = \sum_{i=0}^{\infty} \frac{1}{i+m+1} |c_i|^2$,

(ii) $||P(\overline{z}^m \sum_{i=0}^{\infty} c_i z^i)||^2 = \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2} |c_i|^2$.

Next, we characterize the normality of Toeplitz operators with harmonic symbols on $A^2(\mathbb{D})$.

Theorem 1. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = \sum_{n=0}^{N} a_n z^n \quad \textit{and} \quad g(z) = \sum_{n=1}^{m} a_{-n} z^n$$

with $a_{-m}a_N \neq 0$. Then T_φ is normal on $A^2(\mathbb{D})$ if and only if $g(z) = e^{i\theta} f(z)$.

Proof. Observe that T_φ is normal if and only if $T_{\overline{f}f} + T_g T_{\overline{g}} = T_f T_{\overline{f}} + T_{\overline{g}g}$. First we show $N = m$. Without loss of generality, we assume that $N > m$. Then by acting z^m to the both side in the above relation, we have $T_{\overline{f}f} z^m = T_f T_{\overline{f}} z^m$, and so

$$P\Big(\sum_{n=0}^{N} \overline{a_n z}^n \sum_{n=0}^{N} a_n z^n z^m\Big) = \sum_{n=0}^{N} a_n z^n P\Big(\sum_{n=0}^{N} \overline{a_n z}^n z^m\Big).$$

Since

$$P\Big(\sum_{n=0}^{N} \overline{a_n z}^n \sum_{n=0}^{N} a_n z^n z^m\Big) = P\Big(\sum_{i=0}^{N} \sum_{j=\max\{0,i-m\}}^{N} \overline{a_i z}^i a_j z^{j+m}\Big)$$

and

$$\sum_{n=0}^{N} a_n z^n P\Big(\sum_{n=0}^{N} \overline{a_n z}^n z^m\Big) = \sum_{n=0}^{N} a_n z^n P\Big(\sum_{n=0}^{m} \overline{a_n z}^n z^m\Big),$$

for the coefficient of z^m, we deduce that

$$\sum_{j=0}^{N} \frac{m+1}{m+j+1} |a_j|^2 = \sum_{j=0}^{m} \frac{m-j+1}{m+1} |a_j|^2.$$

Moreover, since

$$\frac{m+1}{m+j+1} \geq \frac{m-j+1}{m+1}$$

for $1 \leq j \leq N$, we have $a_j = 0$ for all $1 \leq j \leq N$. This contradicts the assumption $a_N \neq 0$.

Next, we find the necessary and sufficient condition of normality. Since T_φ is normal if and only if $T_{\overline{f}f} - T_f T_{\overline{f}} = T_{\overline{g}g} - T_g T_{\overline{g}}$, for any $k \in \mathbb{N}$,

$$\begin{aligned}(T_{\overline{f}f} - T_f T_{\overline{f}}) z^k &= P\Big(\sum_{n=0}^{N} \overline{a_n z}^n \sum_{n=0}^{N} a_n z^n z^k\Big) - \sum_{n=0}^{N} a_n z^n P\Big(\sum_{n=0}^{N} \overline{a_n z}^n z^k\Big) \\ &= \sum_{i=0}^{\min\{j+k,N\}} \sum_{j=0}^{N} \frac{k+j-i+1}{k+j+1} \overline{a_i} a_j z^{j+k-i} - \sum_{j=0}^{N} a_j z^j \sum_{i=0}^{k} \frac{k-i+1}{k+1} \overline{a_i} z^{k-i}\end{aligned}$$

and

$$\begin{aligned}(T_{\overline{g}g} - T_g T_{\overline{g}}) z^k &= P\Big(\sum_{n=1}^{N} \overline{a_{-n} z}^n \sum_{n=1}^{N} a_{-n} z^n z^k\Big) - \sum_{n=1}^{N} a_{-n} z^n P\Big(\sum_{n=1}^{N} \overline{a_{-n} z}^n z^k\Big) \\ &= \sum_{i=1}^{\min\{j+k,N\}} \sum_{j=1}^{N} \frac{k+j-i+1}{k+j+1} \overline{a_{-i}} a_{-j} z^{j+k-i} - \sum_{j=1}^{N} a_{-j} z^j \sum_{i=1}^{k} \frac{k-i+1}{k+1} \overline{a_{-i}} z^{k-i}.\end{aligned}$$

By direct calculations of coefficients of z^{k+N-2}, we have

$$\begin{aligned}&\overline{a_0} a_{N-2} + \overline{a_1} a_{N-1} \frac{N+k-1}{N+k} + \overline{a_2} a_N \frac{N+k-1}{N+k+1} - \Big(\overline{a_0} a_{N-2} + \overline{a_1} a_{N-1} \frac{k}{k+1} + \overline{a_2} a_N \frac{k-1}{k+1}\Big) \\ &= \overline{a_{-1}} a_{-(N-1)} \frac{N+k-1}{N+k} + \overline{a_{-2}} a_{-N} \frac{N+k-1}{N+k+1} - \Big(\overline{a_{-1}} a_{-(N-1)} \frac{k}{k+1} + \overline{a_{-2}} a_{-N} \frac{k-1}{k+1}\Big)\end{aligned}$$

for any $k \in \mathbb{N}$. Therefore

$$\Big(\overline{a_1} a_{N-1} - \overline{a_{-1}} a_{-(N-1)}\Big)\Big(\frac{N+k-1}{N+k} - \frac{k}{k+1}\Big) + \Big(\overline{a_2} a_N - \overline{a_{-2}} a_{-N}\Big)\Big(\frac{N+k-1}{N+k+1} - \frac{k-1}{k+1}\Big) = 0$$

for any $k \in \mathbb{N}$. Since k is arbitrary, so that $\overline{a_1}a_{N-1} = \overline{a_{-1}}a_{-(N-1)}$ and $\overline{a_2}a_N = \overline{a_{-2}}a_{-N}$. By the similar arguments, we have

$$\overline{a_i}a_j = \overline{a_{-i}}a_{-j}$$

for all i, j with $1 \leq i, j \leq N$. Therefore,

$$a_{-j} = e^{i\theta}a_j$$

for all $1 \leq j \leq N$ and so $g(z) = e^{i\theta}f(z)$.

If $g(z) = e^{i\theta}f(z)$, then

$$T_\varphi^* T_\varphi = T_{\overline{f}+g}T_{\overline{g}+f} = T_{\overline{f}+e^{i\theta}f}T_{e^{-i\theta}\overline{f}+f} = T_{e^{-i\theta}\overline{f}+f}T_{\overline{f}+e^{i\theta}f} = T_\varphi T_\varphi^*.$$

Thus, T_φ is normal on $A^2(\mathbb{D})$. This completes the proof. □

3. Toeplitz Operators with Non-Harmonic Symbols

In this section, we study the normality of T_φ on $A^2(\mathbb{D})$ with the non-harmonic symbol φ. In this case, we cannot apply the tool as in Theorem 1 to φ, since it cannot be devided into analytic and co-analytic parts. Hence we should calculate the self-commutator of T_φ for non-harmonic symbol φ. For this, we discuss about the symbol φ of the form $\varphi(z) = a_{m,n}z^m\overline{z}^n$ with $a_{m,n} \in \mathbb{C}$.

Lemma 4. *Let $\varphi(z) = a_{m,n}z^m\overline{z}^n$ with $a_{m,n} \in \mathbb{C}$. Then T_φ on $A^2(\mathbb{D})$ is normal if and only if $m = n$.*

Proof. For $\varphi(z) = a_{m,n}z^m\overline{z}^n$, T_φ is normal if and only if

$$\left\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*) \sum_{i=0}^{\infty} c_i z^i, \sum_{i=0}^{\infty} c_i z^i \right\rangle = 0$$

for all $c_i \in \mathbb{C}$. Using Lemmas 2 and 3, we have that

$$\begin{aligned}
&\|T_\varphi \sum_{i=0}^{\infty} c_i z^i\|^2 - \|T_\varphi^* \sum_{i=0}^{\infty} c_i z^i\|^2 \\
&= \|T_{a_{m,n}z^m\overline{z}^n} \sum_{i=0}^{\infty} c_i z^i\|^2 - \|T_{\overline{a}_{m,n}\overline{z}^m z^n} \sum_{i=0}^{\infty} c_i z^i\|^2 \\
&= \left\| P\left(a_{m,n}z^m\overline{z}^n \sum_{i=0}^{\infty} c_i z^i\right)\right\|^2 - \left\| P\left(\overline{a}_{m,n}\overline{z}^m z^n \sum_{i=0}^{\infty} c_i z^i\right)\right\|^2 \\
&= |a_{m,n}|^2 \sum_{i=\max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - |a_{m,n}|^2 \sum_{i=\max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 = 0.
\end{aligned}$$

Hence T_φ is normal if and only if

$$\sum_{i=\max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 = \sum_{i=\max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2$$

for all $c_i \in \mathbb{C}$. Since c_i's are arbitrary, we have that T_φ is normal if and only if $m = n$. This completes the proof. □

We now consider the normality of Toeplitz operators with two terms of non-harmonic symbols. The following theorem gives a general characterization of normal Toeplitz operators with the symbols is of the form $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ $(m \geq n \geq 0,\ t \geq s \geq 0)$ with some assumptions.

Theorem 2. *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ with $m \geq n \geq 0$, $t \geq s \geq 0$ and nonzeros $a, b \in \mathbb{C}$. Then T_φ is normal if and only if $\varphi(z)$ is of the form*

$$\varphi(z) = (a+b)|z|^{2m} \text{ or } \varphi(z) = az^m\overline{z}^n + bz^n\overline{z}^m \quad (|a| = |b|).$$

Proof. Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ with $m \geq n, t \geq s$. By similar arguments as in the proof of Lemma 4, T_φ is normal if and only if

$$\begin{aligned}
&\left\| T_\varphi \sum_{i=0}^{\infty} c_i z^i \right\|^2 - \left\| T_\varphi^* \sum_{i=0}^{\infty} c_i z^i \right\|^2 \\
&= \left\| T_{az^m\overline{z}^n} \sum_{i=0}^{\infty} c_i z^i + T_{bz^s\overline{z}^t} \sum_{i=0}^{\infty} c_i z^i \right\|^2 - \left\| T_{\overline{a}\overline{z}^m z^n} \sum_{i=0}^{\infty} c_i z^i + T_{\overline{b}\overline{z}^s z^t} \sum_{i=0}^{\infty} c_i z^i \right\|^2 \\
&= \left\| P\left(a\overline{z}^n \sum_{i=0}^{\infty} c_i z^{i+m} \right) + P\left(b\overline{z}^t \sum_{i=0}^{\infty} c_i z^{i+s} \right) \right\|^2 \\
&\quad - \left\| P\left(\overline{a}\overline{z}^m \sum_{i=0}^{\infty} c_i z^{i+n} \right) + P\left(\overline{b}\overline{z}^s \sum_{i=0}^{\infty} c_i z^{i+t} \right) \right\|^2 \\
&= |a|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2} |c_i|^2 + |b|^2 \sum_{i=t-s}^{\infty} \frac{s+i-t+1}{(s+i+1)^2} |c_i|^2 \\
&\quad - |a|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2} |c_i|^2 - |b|^2 \sum_{i=0}^{\infty} \frac{t+i-s+1}{(t+i+1)^2} |c_i|^2 = 0
\end{aligned} \tag{1}$$

for any $c_i \in \mathbb{C}$ $(i = 0, 1, 2, \ldots)$.

Case (1) If $m = n, t = s$, then the equality (1) holds, and so T_φ is normal.

Case (2) If $m > n, t > s$, set $c_0 = 1$ and $c_i = 0$ for $i \geq 1$, then

$$\frac{m-n+1}{(m+1)^2}|a|^2 = \frac{t-s+1}{(t+1)^2}|b|^2. \tag{2}$$

(i) If $m = n+1, t = s+1$, then we have

$$\frac{i+2}{(m+i+1)^2}|a|^2 + \frac{i}{(t+i)^2}|b|^2 = \frac{i}{(m+i)^2}|a|^2 + \frac{i+2}{(t+i+1)^2}|b|^2$$

for all $i \geq 1$. From the equality (2), we have

$$\frac{i+2}{(m+i+1)^2}\frac{(m+1)^2}{(t+1)^2} + \frac{i}{(t+i)^2} = \frac{i}{(m+i)^2}\frac{(m+1)^2}{(t+1)^2} + \frac{i+2}{(t+i+1)^2}$$

for all $i \geq 1$. Hence, for $i = 1$,

$$\frac{3}{(m+2)^2}\frac{(m+1)^2}{(t+1)^2} + \frac{1}{(t+1)^2} = \frac{1}{(m+1)^2}\frac{(m+1)^2}{(t+1)^2} + \frac{3}{(t+2)^2}$$

or equivalently,

$$\frac{(m+1)^2}{(t+1)^2}\left\{ \frac{3}{(m+2)^2} - \frac{1}{(m+1)^2} \right\} = \frac{3}{(t+2)^2} - \frac{1}{(t+1)^2}.$$

By a direct calculation, we have

$$m = t \text{ or } m = -\frac{3t+4}{2t+3}.$$

Since m and t are nonnegative, we have $m = t$ and so $n = s$.

(ii) Let $m > n+1, t = s+1$. Suppose that T_φ is normal. For a sufficiently large k,

$$\begin{aligned}
T_\varphi T_{\overline{\varphi}} z^k &= T_\varphi P(\overline{a}\overline{z}^m z^n z^k + \overline{b}\overline{z}^s z^{s+1} z^k) \\
&= T_\varphi\big(\overline{a}\frac{n+k-m+1}{n+k+1} z^{n+k-m} + \overline{b}\frac{k+2}{s+k+2} z^{k+1}\big) \\
&= |a|^2 \frac{(n+k-m+1)(k+1)}{(n+k+1)^2} z^k + \overline{a}b \frac{(n+k-m+1)(n+k-m)}{(n+k+1)(n+k-m+s+1)} z^{n+k-m-1} \\
&\quad + a\overline{b}\frac{(k+2)(m+k-n+2)}{(s+k+2)(m+k+2)} z^{m+k-n+1} + |b|^2 \frac{(k+1)(k+2)}{(s+k+2)^2} z^k,
\end{aligned}$$

and

$$\begin{aligned}
T_{\overline{\varphi}} T_\varphi z^k &= T_{\overline{\varphi}} P(a z^m \overline{z}^n z^k + b z^s \overline{z}^{s+1} z^k) \\
&= T_{\overline{\varphi}}\big(a\frac{m+k-n+1}{m+k+1} z^{m+k-n} + b\frac{k}{s+k+1} z^{k-1}\big) \\
&= |a|^2 \frac{(m+k-n+1)(k+1)}{(m+k+1)^2} z^k + \overline{a}b \frac{k(n+k-m)}{(s+k+1)(n+k)} z^{n+k-m-1} \\
&\quad + a\overline{b}\frac{(m+k-n+1)(m+k-n+2)}{(m+k+1)(m+k-n+s+2)} z^{m+k-n+1} + |b|^2 \frac{k(k+1)}{(s+k+1)^2} z^k.
\end{aligned}$$

Since T_φ is normal, we have that

$$\frac{(k+2)(m+k-n+2)}{(s+k+2)(m+k+2)} = \frac{(m+k-n+1)(m+k-n+2)}{(m+k+1)(m+k-n+s+2)}$$

for all sufficiently large k. By a direct calculation,

$$\begin{aligned}
&(m^2 - mn + ms + 7m - 3n + 3s + 8)k + 2(m+1)(m-n+s+2) \\
&\quad = (m^2 - mn + 2ms - ns + 3s + 7m - 4n + 8)k + (m-n+1)(s+2)(m+2).
\end{aligned}$$

Since k is arbitrary,

$$m^2 - mn + ms + 7m - 3n + 3s + 8 = m^2 - mn + 2ms - ns + 3s + 7m - 4n + 8$$

and

$$2(m+1)(m-n+s+2) = (m-n+1)(s+2)(m+2),$$

and so

$$ms - ns - n = 0 \text{ and } m^2 s - mns - 2n - 2ns + ms = 0.$$

By the first equality, $s = \frac{n}{m-n}$, and so $mn(m-n-1) = 0$. Therefore, $m = 0$ or $n = 0$ or $m = n+1$, a contradiction. Hence T_φ is not normal.

(iii) If $m > n+1, t > s+1$, set $c_1 = 1$ and $c_i = 0$ for $i \neq 1$, then by a similar argument as in (i), $m = t$.

By (i)–(iii), $m = t$ and so T_φ is normal if and only if

$$\begin{aligned}
&|a|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2} |c_i|^2 + |b|^2 \sum_{i=m-s}^{\infty} \frac{s+i-m+1}{(s+i+1)^2} |c_i|^2 \\
&\quad - |a|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2} |c_i|^2 - |b|^2 \sum_{i=0}^{\infty} \frac{m+i-s+1}{(m+i+1)^2} |c_i|^2 = 0
\end{aligned} \tag{3}$$

for any $c_i \in \mathbb{C}$ $(i = 0, 1, 2, \ldots)$. If $n \neq s$, let $n > s$ and from (3) with $c_1 = 1$ and $c_i = 0$ with $i \neq 1$. If $m - n = 1$, then

$$|a|^2 \frac{m-n+2}{(m+2)^2} - |a|^2 \frac{n-m+2}{(n+2)^2} - |b|^2 \frac{m-s+2}{(m+2)^2} = 0,$$

or equivalently,

$$|a|^2\frac{3}{(m+2)^2}-|a|^2\frac{1}{(m+1)^2}-|b|^2\frac{m-s+2}{(m+2)^2}=0.$$

By direct calculations with (2), we have

$$s=\frac{(m+1)(4m+5)}{2m+3}=2m+1+\frac{m+2}{2m+3}.$$

Therefore s is not a nonnegative integer. If $m-n>1$, then

$$|a|^2\frac{m-n+2}{(m+2)^2}-|b|^2\frac{m-s+2}{(m+2)^2}=0.$$

By (2), we have $n=s$, a contradiction. Therefore $n=s$ and so

$$\begin{aligned}&|a|^2\sum_{i=0}^{\infty}\frac{m+i-n+1}{(m+i+1)^2}|c_i|^2+|b|^2\sum_{i=m-n}^{\infty}\frac{n+i-m+1}{(n+i+1)^2}|c_i|^2\\&-|a|^2\sum_{i=m-n}^{\infty}\frac{n+i-m+1}{(n+i+1)^2}|c_i|^2-|b|^2\sum_{i=0}^{\infty}\frac{m+i-n+1}{(m+i+1)^2}|c_i|^2=0,\end{aligned}$$

or equivalently,

$$(|a|^2-|b|^2)\Big(\sum_{i=0}^{\infty}\frac{m+i-n+1}{(m+i+1)^2}|c_i|^2-\sum_{i=m-n}^{\infty}\frac{n+i-m+1}{(n+i+1)^2}|c_i|^2\Big)=0,$$

and hence we have the results. □

Corollary 1. *Consider the polynomial $\varphi(z)=z^m+\overline{z}^m$ with $m>0$. Then by Theorem 2, T_φ is normal.*

The matrix representation of the Toeplitz operator T_φ with non-harmonic symbol φ has the following form:

Lemma 5. *Suppose that $\varphi(z)=a_{m,n}z^m\overline{z}^n+e^{i\theta}a_{n,m}z^n\overline{z}^m$ with $0\le n\le m$ and $\theta\in[0,2\pi)$. Then the Toeplitz matrix with symbol φ given by*

$$T_\varphi=\begin{pmatrix}0&\cdots&\frac{1}{m+1}e^{i\theta}a_{n,m}&0&0&\cdots\\\vdots&0&\cdots&\frac{2}{m+2}a_{n.m}&0&\ddots\\\frac{m-n+1}{m+1}a_{m,n}&\vdots&0&\ddots&\ddots&\ddots\\0&\frac{m-n+2}{m+2}a_{m,n}&\vdots&\ddots&\ddots&\ddots\\0&0&\ddots&\ddots&\ddots&\ddots\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots\end{pmatrix}$$

where $\frac{1}{m+1}e^{i\theta}a_{n,m}$ is a $(1,m-n)$-th entry and $\frac{m-n+1}{m+1}a_{m,n}$ is a $(m-n,1)$-th entry.

We will give a necessary and sufficient condition that the sum of two normal Toeplitz operator with non-harmonic symbol is again a normal Toeplitz operator.

Theorem 3. *For $a,b\in\mathbb{C}$, let $\varphi(z)=a\varphi_1(z)+b\varphi_2(z)$ be of the form*

$$\varphi_1(z)=z^m\overline{z}^n+e^{i\theta_1}z^n\overline{z}^m,\ \varphi_2(z)=z^s\overline{z}^t+e^{i\theta_2}z^t\overline{z}^s$$

where $\theta_1, \theta_2 \in [0, 2\pi)$, $m \geq n \geq 0$ and $t \geq s \geq 0$. Then T_φ is normal if and only if either $\overline{a}be^{i\theta_1} = a\overline{b}e^{i\theta_2}$ or $\varphi(z) = a(1+e^{i\theta_1})|z|^{2m} + b(1+e^{i\theta_2})|z|^{2s}$.

Proof. Let $\varphi(z) = a\varphi_1(z) + b\varphi_2(z)$ then T_φ is normal if and only if

$$T_{a\varphi_1}T_{\overline{b\varphi_2}} + T_{b\varphi_2}T_{\overline{a\varphi_1}} = T_{\overline{a\varphi_1}}T_{b\varphi_2} + T_{\overline{b\varphi_2}}T_{a\varphi_1},$$

or equivalently,

$$a\overline{b}e^{i\theta_2}T_{\varphi_1}T_{\varphi_2} + \overline{a}be^{i\theta_1}T_{\varphi_2}T_{\varphi_1} = \overline{a}be^{i\theta_1}T_{\varphi_1}T_{\varphi_2} + a\overline{b}e^{i\theta_2}T_{\varphi_2}T_{\varphi_1},$$

we have

$$(\overline{a}be^{i\theta_1} - a\overline{b}e^{i\theta_2})(T_{\varphi_1}T_{\varphi_2} - T_{\varphi_2}T_{\varphi_1}) = 0.$$

Therefore $T_{\varphi_1}T_{\varphi_2} = T_{\varphi_2}T_{\varphi_1}$ if and only if

$$\begin{aligned}
&\frac{m-n+1}{m+1}\frac{s+m-n-t+1}{s+m-n+1}z^{s+m-n-t} + e^{i\theta_1}\frac{m-n+1}{m+1}\frac{t+m-n-s+1}{t+m-n+1}z^{m-n+t-s} \\
&= \frac{s-t+1}{s+1}\frac{s+m-t-n+1}{s+m-t+1}z^{s+m-n-t} + e^{i\theta_2}\frac{s-t+1}{s+1}\frac{s+n-t-m+1}{s+n-t+1}z^{s+n-m-t}.
\end{aligned}$$

If $m-n = s-t$ then

$$\frac{m-n+1}{m+1}\frac{2(m-n)+1}{s+m-n+1}z^{2(m-n)} = \frac{m-n+1}{s+1}\frac{2(m-n)+1}{s+m-t+1}z^{2(m-n)},$$

or equivalently,

$$(m+1)(s+m-n+1) = (s+1)(s+m-t+1).$$

By a direct calculation, $m = n$ and $s = t$, and so

$$\varphi(z) = a(1+e^{i\theta_1})|z|^{2m} + b(1+e^{i\theta_2})|z|^{2s}.$$

This completes the proof. □

Corollary 2. *Let $\varphi(z) = \overline{az}^m + \overline{bz}^N + cz^m + dz^N$. Then T_φ is normal on $A^2(\mathbb{D})$ if and only if $|b| = |d|$ and $\overline{c}d = \overline{a}b$.*

Example 1. *Let $\varphi(z) = a(z\overline{z}^2 + e^{i\theta_1}z^2\overline{z}) + b(z\overline{z}^3 + e^{i\theta_2}z^3\overline{z})$. It follows from Theorem 3, T_φ is normal if and only if $\varphi(z)$ is of the form*

$$\varphi(z) = 1(z\overline{z}^2 + \overline{z}z^2) + (1+i)(z\overline{z}^3 + i\overline{z}z^3).$$

We thus have:

Corollary 3. *Let $\varphi(z) = a(z^m\overline{z}^n + e^{i\theta}z^n\overline{z}^m)$ with $m > n$ and nonzero $a \in \mathbb{C}$. Then T_{φ^2} is normal if and only if $\theta = \frac{k}{2}\pi$ $(k = \mathbb{Z})$.*

Proof. Let $\varphi(z) = a(z^m\overline{z}^n + e^{i\theta}z^n\overline{z}^m)$ then

$$\varphi^2(z) = a^2(z^{2m}\overline{z}^{2n} + e^{2i\theta}\overline{z}^{2m}z^{2n}) + 2a^2e^{i\theta}|z|^{2(m+n)}.$$

Put

$$\begin{aligned}
\varphi_1(z) &= a^2(z^{2m}\overline{z}^{2n} + e^{2i\theta}\overline{z}^{2m}z^{2n}), \\
\varphi_2(z) &= 2a^2e^{i\theta}|z|^{2(m+n)} = a^2e^{i\theta}(z^{m+n}\overline{z}^{m+n} + z^{m+n}\overline{z}^{m+n}).
\end{aligned}$$

Then by Theorem 3, we have $e^{4i\theta} = 1$. □

As some applications of Theorems 2 and 3, we get the following result.

Corollary 4. *Let*

$$\varphi(z) = \sum_{i=1}^{n} a_i \varphi_i(z)$$

where $\varphi_i(z) = z^{m_i}\overline{z}^{n_i} + e^{i\theta_i} z^{n_i}\overline{z}^{m_i}$ *with nonzero* m_i *and* n_i*, and* $\theta_i \in \mathbb{R}$*. If* $a_i\overline{a_j}e^{i\theta_j} - \overline{a_i}a_j e^{i\theta_i}$ *for all* i, j*, then* T_φ *is normal.*

Proof. Let $\varphi(z) = \sum_{i=1}^{n} a_i\varphi_i(z)$ where $\varphi_i(z) = z^{m_i}\overline{z}^{n_i} + e^{i\theta_i} z^{n_i}\overline{z}^{m_i}$ with nonzero m_i and n_i, and $\theta_i \in \mathbb{R}$. Then by the similar arguments as in Theorem 3, T_φ is normal if and only if

$$\sum_{i,j=1, i\le j}^{n} (a_i\overline{a_j}e^{i\theta_j} - \overline{a_i}a_j e^{i\theta_i})(T_{\varphi_i}T_{\varphi_j} - T_{\varphi_j}T_{\varphi_i}) = 0. \tag{4}$$

For any i, j, if $a_i\overline{a_j}e^{i\theta_j} - \overline{a_i}a_j e^{i\theta_i}$, then the relation (4) hold and so T_φ is normal. This completes the proof. □

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