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Weaker Conditions for the q-Steffensen Inequality and Some Related Generalizations

Ksenija Smoljak Kalamir

Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia; ksmoljak@ttf.hr

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Abstract: The aim of this paper is to study the q-Steffensen inequality and to prove some weaker conditions for this inequality in quantum calculus. Further, we prove q-analogues of some frequently used generalizations of Steffensen's inequality and obtain some refinements of q-Steffensen's inequality and its generalizations.

Keywords: Steffensen's inequality; q-integral; weaker conditions; generalizations; refinements

1. Introduction

First, let us recall some history and basic properties of quantum calculus. Quantum calculus can be described as a calculus without limits. In short form, we call it q-calculus. This type of calculus appears as a connection between mathematics and physics. Some of its applications in mathematics are in the following fields: number theory, combinatorics, orthogonal polynomials and basic hypergeometric functions. Further, it has applications in some sciences such as quantum theory, mechanics and the theory of relativity. Quantum calculus appears as a subfield of time scales calculus. More details about quantum calculus can be found in [1] and in the references cited in that book.

An analogue of the Riemann integral in q-calculus is the q-integral introduced by Thomae in [2] and Jackson in [3]. The purpose of this paper was to study Steffensen's inequality in q-calculus settings in order to prove weaker conditions for the function g in the q-Steffensen inequality and to obtain q-calculus analogues of some well known generalizations and refinements of Steffensen's inequality.

Let us recall some definitions and facts on q-derivatives and q-integrals needed for understanding this paper (for more details see [1]). Throughout this paper we assume that $q \in (0,1)$ is a real number.

The q-derivative is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

for an arbitrary function f. For a differentiable function f, $\lim_{q\to 1} D_q f(x) = \frac{df(x)}{dx}$.

Let f be a function defined on the interval $[0,b]$. The definite q-integral of the function f is defined by

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b).$$

For a function f defined on the interval $[a,b]$, the definite q-integral is defined by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

For an integrable function f defined on the interval $[0, b]$, we have

$$\lim_{q \to 1} \int_0^b f(x) d_q x = \int_0^b f(x) dx.$$

Note that the q-integral of a positive function does not have to be positive.

If a lower limit of integral has the special form $a = bq^k$, $k \in \mathbb{N}$, the definite q-integral reduces to the finite sum

$$\int_{bq^k}^b f(x) d_q x = (1-q) b \sum_{j=0}^{k-1} q^j f(bq^j). \tag{1}$$

This type of q-integral was studied by Gauchman in [4] and it is called the restricted definite q-integral.

In [4], Gauchman used the following notation:

$$c_j = bq^j, \text{ for } j \in \mathbb{N}_0, \qquad a = c_n = bq^n.$$

The restricted definite q-integral has no question about convergency since it is a finite sum.

For q-integration by parts we have the following formula:

$$\int_a^b f(x) D_q g(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_q f(x) d_q x. \tag{2}$$

Let us recall the definition of a q-increasing function (see [4]).

Definition 1. *The function f is called q-increasing (respectively, q-decreasing) on $[a, b]$ if $f(qx) \leq f(x)$ (respectively, $f(qx) \geq f(x)$) whenever $x \in [a, b]$ and $qx \in [a, b]$.*

In the following theorem we recall a criteria for a q-increasing function (see [4]).

Theorem 1. *The function f is q-increasing (respectively, q-decreasing) on $[a, b]$ if and only if $D_q f(x) \geq 0$ (respectively, $D_q f(x) \leq 0$) whenever $x \in [a, b]$ and $qx \in [a, b]$.*

Remark 1. *If the function f is increasing (respectively, decreasing), then it is also q-increasing (respectively, q-decreasing).*

The classical Steffensen inequality states (see [5]) :

Theorem 2. *Suppose that f is decreasing and g is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t) dt$. Then we have*

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t) g(t) dt \leq \int_a^{a+\lambda} f(t) dt. \tag{3}$$

The inequalities are reversed for increasing f.

Due to its importance in the theory of inequalities, Steffensen's inequality has been studied by many mathematicians and has many generalizations. For an extensive overview of its generalizations and refinements, see [6]. Some analogues of Equation (3) for the q-integral have been considered in papers [4,7,8]. Further, Steffensen's inequality for convex functions in quantum calculus is presented in paper [9].

Let us recall the q-Steffensen inequality obtained by Gauchman in [4].

Theorem 3. *Suppose that $0 < q < 1$, $b > 0$, $n \in \mathbb{Z}^+$. Let $F, G\colon [a,b] \to \mathbb{R}$, where $a = bq^n$, be two functions such that F is q-decreasing and $0 \le G \le 1$ on $[a,b]$. Assume that $k, l \in \{0, 1, \ldots, n\}$ are such that*

$$b - c_l \le \int_a^b G(x) d_q x \le c_k - a, \quad \text{if } F \ge 0, \quad \text{on } [a,b]$$

and

$$c_k - a \le \int_a^b G(x) d_q x \le b - c_l, \quad \text{if } F \le 0, \quad \text{on } [a,b].$$

Then,

$$\int_{c_l}^b F(x) d_q x \le \int_a^b F(x) G(x) d_q x \le \int_a^{c_k} F(x) d_q x.$$

In [8] Rajković et al. improved Gauchman's result considering the q-integrals on $(0, b)$ when they are represented by infinite sums. In the following theorem we recall their result.

Theorem 4. *Let $0 < q < 1$, $b > 0$; $f(x)$ and $g(x)$ are both q-integrable functions on $[0,b]$; $f(x)$ is nonnegative and decreasing; and $0 \le g(x) \le 1$ for each $x \in [0,b]$ and $\lambda = \int_0^b g(x) d_q x$. Let $l, k \in \mathbb{N}_0$ be such that*

$$l = \lfloor \log_q (1 - \lambda/b) \rfloor, \quad k = \lfloor \log_q (\lambda/b) \rfloor.$$

Then

$$\int_{bq^l}^b f(x) d_q x \le \int_0^b f(x) g(x) d_q x \le \int_0^{bq^k} f(x) d_q x. \tag{4}$$

2. Weaker Conditions for q-Steffensen's Inequality

Instead of the condition $0 \le g \le 1$ in the classical Steffensen's inequality, Milovanović and Pečarić obtained weaker conditions for the function g in [10]. The aim of this section is to obtain weaker conditions for the function g in the q-Steffensen inequality Equation (4). Firstly, let us prove a lemma which will be used in the proof of weaker conditions.

Lemma 1. *Let $0 < q < 1$, $b > 0$. Let f, g be q-integrable functions on $[0,b]$ and $\lambda = \int_0^b g(x) d_q x$. Let $l, k \in \mathbb{N}_0$ be such that*

$$l = \lfloor \log_q (1 - \lambda/b) \rfloor, \quad k = \lfloor \log_q (\lambda/b) \rfloor. \tag{5}$$

Then the following inequalities hold

$$\int_0^{bq^k} f(x) d_q x - \int_0^b f(x) g(x) d_q x \ge \int_0^{bq^k} [f(x) - f(bq^k)][1 - g(x)] d_q x + \int_{bq^k}^b [f(bq^k) - f(x)] g(x) d_q x \tag{6}$$

and

$$\int_0^b f(x) g(x) d_q x - \int_{bq^l}^b f(x) d_q x \ge \int_0^{bq^l} [f(x) - f(bq^l)] g(x) d_q x + \int_{bq^l}^b [f(bq^l) - f(x)][1 - g(x)] d_q x. \tag{7}$$

Proof. Let us prove the identity Equation (6). We have

$$
\begin{aligned}
&\int_0^{bq^k} [f(x) - f(bq^k)][1 - g(x)]d_qx + \int_{bq^k}^{b} [f(bq^k) - f(x)]g(x)d_qx \\
&= \int_0^{bq^k} f(x)d_qx - \int_0^{bq^k} f(x)g(x)d_qx - f(bq^k)\int_0^{bq^k} d_qx + f(bq^k)\int_0^{bq^k} g(x)d_qx \\
&+ f(bq^k)\int_{bq^k}^{b} g(x)d_qx - \int_{bq^k}^{b} f(x)g(x)d_qx \\
&= \int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx + f(bq^k)\int_0^{b} g(x)d_qx - f(bq^k)(1-q)bq^k\sum_{j=0}^{\infty} q^j \\
&= \int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx + f(bq^k)\lambda - f(bq^k)bq^k.
\end{aligned}
$$

From Equation (5) we have $bq^k \geq \lambda$; hence

$$
\begin{aligned}
&\int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx + f(bq^k)\lambda - f(bq^k)bq^k \\
&\leq \int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx + f(bq^k)\lambda - f(bq^k k)\lambda \\
&= \int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx.
\end{aligned}
$$

Proof of the identity Equation (7) is similar. □

In the following theorems we give weaker conditions for q-Steffensen's inequality.

Theorem 5. *Let $0 < q < 1$, $b > 0$. Let f and g be q-integrable functions on $[0, b]$ such that f is nonnegative and decreasing and $\lambda = \int_0^b g(x)d_qx$. Let $k \in \mathbb{N}_0$ be such that $k = \lfloor \log_q(\lambda/b) \rfloor$. If*

$$
\int_0^{qx} g(t)d_qt \leq qx \quad \text{and} \quad \int_{qx}^{b} g(t)d_qt \geq 0, \quad \text{for every } x \in [0, b],
$$

then

$$
\int_0^{b} f(x)g(x)d_qx \leq \int_0^{bq^k} f(x)d_qx.
$$

Proof. Applying the q-integration by parts (see Equation (2)), the inequality Equation (6) becomes

$$
\begin{aligned}
&\int_0^{bq^k} f(x)d_qx - \int_0^{b} f(x)g(x)d_qx \\
&\geq -\int_0^{bq^k} \left(\int_0^{qx} [1 - g(t)]d_qt\right) D_qf(x)d_qx - \int_{bq^k}^{b} \left(\int_{qx}^{b} g(t)d_qt\right) D_qf(x)d_qx.
\end{aligned}
$$

From Remark 1 we have that the decreasing function f is also q-decreasing; hence, from Theorem 1 we have $D_qf(x) \leq 0$. Thus, we conclude that we can replace the condition $0 \leq g \leq 1$ in q-Steffensen's inequality given by Equation (4) by the conditions

$$
\int_0^{qx} g(t)d_qt \leq qx \quad \text{for every } x \in [0, bq^k]
$$

and

$$
\int_{qx}^{b} g(t)d_qt \geq 0 \quad \text{for every } x \in [bq^k, b].
$$

□

Theorem 6. *Let $0 < q < 1$, $b > 0$. Let f and g be q-integrable functions on $[0, b]$ such that f is nonnegative and decreasing and $\lambda = \int_0^b g(x) d_q x$. Let $l \in \mathbb{N}_0$ be such that $l = \lfloor \log_q (1 - \lambda/b) \rfloor$. If*

$$\int_0^{qx} g(t) d_q t \geq 0 \quad \text{and} \quad \int_{qx}^b g(t) d_q t \leq b - qx, \quad \text{for every } x \in [0, b],$$

then

$$\int_{bq^l}^b f(x) d_q x \leq \int_0^b f(x) g(x) d_q x.$$

Proof. When applying the q-integration by parts, the inequality Equation (7) becomes

$$\begin{aligned} &\int_0^b f(x) g(x) d_q x - \int_{bq^l}^b f(x) d_q x \\ &\geq - \int_0^{bq^l} \left(\int_0^{qx} g(t) d_q t \right) D_q f(x) d_q x - \int_{bq^l}^b \left(\int_{qx}^b [1 - g(t)] d_q t \right) D_q f(x) d_q x. \end{aligned}$$

As in the proof of Theorem 5 we conclude that we can replace the condition $0 \leq g \leq 1$ in q-Steffensen's inequality with the conditions

$$\int_0^{qx} g(t) d_q t \geq 0 \quad \text{for every } x \in [0, bq^l]$$

and

$$\int_{qx}^b g(t) d_q t \leq b - qx \quad \text{for every } x \in [bq^l, b].$$

□

3. Generalizations of the q-Steffensen Inequality

The generalization of Steffensen's inequality given by Pečarić in [11] was extensively studied in many recent papers concerning Steffensen's inequality. Motivated by frequent use of this generalization we want to make a contribution by proving its q-analogue.

Theorem 7. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that h is positive, f is nonnegative, f/h is decreasing and $0 \leq g \leq 1$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} h(x) d_q x \geq \int_0^b h(x) g(x) d_q x. \tag{8}$$

Then

$$\int_0^b f(x) g(x) d_q x \leq \int_0^{bq^k} f(x) d_q x. \tag{9}$$

Proof. Let us consider the inequality Equation (9); we have

$$\begin{aligned} DIFF &= \int_0^{bq^k} f(x) d_q x - \int_0^b f(x) g(x) d_q x \\ &= \int_0^{bq^k} \frac{h(x) f(x)}{h(x)} (1 - g(x)) d_q x - \int_{bq^k}^b f(x) g(x) d_q x. \end{aligned}$$

We have

$$\int_0^{bq^k} \frac{h(x) f(x)}{h(x)} (1 - g(x)) d_q x = (1 - q) bq^k \sum_{j=0}^{\infty} q^j \frac{h(bq^{k+j}) f(bq^{k+j})}{h(bq^{k+j})} (1 - g(bq^{k+j})),$$

where we used the definition of the definite q-integral. For a decreasing function f/h we have that

$$\frac{f(bq^{k+j})}{h(bq^{k+j})} \geq \frac{f(bq^k)}{h(bq^k)}, \quad \text{for } j \in \mathbb{N}_0,$$

so

$$\begin{aligned}
&(1-q)bq^k \sum_{j=0}^{\infty} q^j \frac{h(bq^{k+j}) f(bq^{k+j})}{h(bq^{k+j})} (1 - g(bq^{k+j})) \\
&\qquad \geq (1-q)bq^k \frac{f(bq^k)}{h(bq^k)} \sum_{j=0}^{\infty} q^j h(bq^{k+j})(1 - g(bq^{k+j})) \\
&\qquad = \frac{f(bq^k)}{h(bq^k)} \int_0^{bq^k} h(x) d_q x - \frac{f(bq^k)}{h(bq^k)} \int_0^{bq^k} h(x) g(x) d_q x.
\end{aligned}$$

Now using the condition Equation (8) and the assumptions that f is nonnegative and h is positive we have

$$\begin{aligned}
&\frac{f(bq^k)}{h(bq^k)} \int_0^{bq^k} h(x) d_q x - \frac{f(bq^k)}{h(bq^k)} \int_0^{bq^k} h(x) g(x) d_q x \\
&\qquad \geq \frac{f(bq^k)}{h(bq^k)} \int_0^{b} h(x) g(x) d_q x - \frac{f(bq^k)}{h(bq^k)} \int_0^{bq^k} h(x) g(x) d_q x \\
&\qquad = \frac{f(bq^k)}{h(bq^k)} \int_{bq^k}^{b} h(x) g(x) d_q x.
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
DIFF &\geq \frac{f(bq^k)}{h(bq^k)} \int_{bq^k}^{b} h(x) g(x) d_q x - \int_{bq^k}^{b} f(x) g(x) d_q x \\
&= \int_{bq^k}^{b} \left(\frac{f(bq^k)}{h(bq^k)} - \frac{f(x)}{h(x)} \right) h(x) g(x) d_q x.
\end{aligned}$$

According to Equation (1) we have that

$$\begin{aligned}
&\int_{bq^k}^{b} \left(\frac{f(bq^k)}{h(bq^k)} - \frac{f(x)}{h(x)} \right) h(x) g(x) d_q x \\
&\qquad = (1-q)b \sum_{j=0}^{k-1} q^j \left(\frac{f(bq^k)}{h(bq^k)} - \frac{f(bq^j)}{h(bq^j)} \right) h(bq^j) g(bq^j) \geq 0,
\end{aligned}$$

since

$$\frac{f(bq^k)}{h(bq^k)} \geq \frac{f(bq^j)}{h(bq^j)}, \quad j = 0, \ldots, k-1,$$

h is positive, g is nonnegative, $0 < q < 1$ and $b > 0$.
Hence, $DIFF \geq 0$, which completes the proof. □

Theorem 8. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that h is positive, f is nonnegative, f/h is decreasing and $0 \leq g \leq 1$ on $[0, b]$. Let $l \in \mathbb{N}_0$ be such that*

$$\int_{bq^l}^{b} h(x) d_q x \leq \int_0^{b} h(x) g(x) d_q x.$$

Then

$$\int_{bq^l}^{b} f(x) d_q x \leq \int_0^b f(x) g(x) d_q x.$$

Proof. Similar to the proof of Theorem 7. □

In the following theorems we give generalizations of q-Steffensen's inequality by using the restricted definite q-integral studied by Gauchman in [4].

Theorem 9. *Suppose that $0 < q < 1$, $b > 0$, $n \in \mathbb{Z}^+$. Let the functions $f, g, h\colon [a,b] \to \mathbb{R}$, where $a = bq^n$, be such that h is positive, f is nonnegative, f/h is q-decreasing and $0 \leq g \leq 1$ on $[a,b]$. Assume that $k \in \mathbb{N}_0$ is such that*

$$\int_a^b h(x) g(x) d_q x \leq \int_a^{c_k} h(x) d_q x.$$

Then

$$\int_a^b f(x) g(x) d_q x \leq \int_a^{c_k} f(x) d_q x.$$

Proof. Similar to the proof of Theorem 7. □

Theorem 10. *Suppose that $0 < q < 1$, $b > 0$, $n \in \mathbb{Z}^+$. Let the functions $f, g, h\colon [a,b] \to \mathbb{R}$, where $a = bq^n$, be such that h is positive, f is nonnegative, f/h is q-decreasing and $0 \leq g \leq 1$ on $[a,b]$. Assume that $l \in \mathbb{N}_0$ is such that*

$$\int_{c_l}^b h(x) d_q x \leq \int_a^b h(x) g(x) d_q x.$$

Then

$$\int_{c_l}^b f(x) d_q x \leq \int_a^b f(x) g(x) d_q x.$$

Proof. Similar to the proof of Theorem 7. □

Remark 2. *We can obtain q-Steffensen's inequality given in Theorem 4 by taking $h \equiv 1$ in Theorems 7 and 8. Additionally, we can obtain q-Steffensen's inequality given in Theorem 3 by taking $h \equiv 1$ in Theorems 9 and 10.*

Since Mercer's generalization of Steffensen's inequality (see [12]) was incorrect, as stated, several papers concerning its corrected version were published. For details see pp. 56–57 in [6]. Here we will give a q-analogue of this type of generalization similar to the one obtained in [13] which follows from Theorems 7 and 8.

Theorem 11. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0,b]$ such that f is decreasing and nonnegative and $0 \leq g \leq h$ on $[0,b]$. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} h(x) d_q x \geq \int_0^b g(x) d_q x. \tag{10}$$

Then

$$\int_0^b f(x) g(x) d_q x \leq \int_0^{bq^k} f(x) h(x) d_q x. \tag{11}$$

Proof. Taking $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 7. □

Theorem 12. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leq g \leq h$ on $[0, b]$. Let $l \in \mathbb{N}_0$ be such that*

$$\int_{bq^l}^{b} h(x) d_q x \leq \int_0^b g(x) d_q x. \tag{12}$$

Then

$$\int_{bq^l}^{b} f(x)h(x) d_q x \leq \int_0^b f(x)g(x) d_q x. \tag{13}$$

Proof. Taking $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 8. □

The following lemma will be needed to obtain refinements of inequalities given in Theorems 11 and 12.

Lemma 2. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$. Let $k, l \in \mathbb{N}_0$ be such that Equations (10) and (12) hold. Then the following inequalities hold*

$$\int_0^b f(t)g(t) d_q t \leq \int_0^{bq^k} (f(t)h(t) - [f(t) - f(bq^k)][h(t) - g(t)]) d_q t + \int_{bq^k}^{b} [f(t) - f(bq^k)] g(t) d_q t \tag{14}$$

and

$$\int_0^b f(t)g(t) d_q t \geq \int_{bq^l}^{b} (f(t)h(t) - [f(t) - f(bq^l)][h(t) - g(t)]) d_q t + \int_0^{bq^l} [f(t) - f(bq^l)] g(t) d_q t. \tag{15}$$

Proof. Let us prove the identity Equation (14). We have

$$\begin{aligned}
&\int_0^{bq^k} (f(t)h(t) - [f(t) - f(bq^k)][h(t) - g(t)]) d_q t + \int_{bq^k}^{b} [f(t) - f(bq^k)] g(t) d_q t \\
&= \int_0^{bq^k} f(t)h(t) d_q t - \int_0^{bq^k} f(t)h(t) d_q t + \int_0^{bq^k} f(t)g(t) d_q t + f(bq^k) \int_0^{bq^k} h(t) d_q t \\
&- f(bq^k) \int_0^{bq^k} g(t) d_q t + \int_{bq^k}^{b} f(t)g(t) d_q t - f(bq^k) \int_{bq^k}^{b} g(t) d_q t \\
&= \int_0^b f(t)g(t) d_q t + f(bq^k) \int_0^{bq^k} h(t) d_q t - f(bq^k) \int_0^b g(t) d_q t \\
&\geq \int_0^b f(t)g(t) d_q t,
\end{aligned}$$

since Equation (10) holds and f is nonnegative.

Proof of the identity Equation (15) is similar so we omit the details. □

Motivated by refinement of Mercer's result given by Wu and Srivastava in [14], we obtain refinements of inequalities Equation (11) and (13).

Theorem 13. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leq g \leq h$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that Equation (10) holds. Then the following inequality is valid*

$$\int_0^b f(x)g(x) d_q x \leq \int_0^{bq^k} (f(x)h(x) - [f(x) - f(bq^k)][h(x) - g(x)]) d_q x \leq \int_0^{bq^k} f(x)h(x) d_q x. \tag{16}$$

Proof. For a decreasing function f on the interval $[0, b]$, we get $f(bq^j) \leq f(bq^k)$, $j = 0, 1, \dots, k-1$ and $f(bq^{k+j}) \geq f(bq^k)$, $j \in \mathbb{N}_0$. Then

$$\int_{bq^k}^{b} [f(t) - f(bq^k)] g(t) d_q t = (1-q) b \sum_{j=0}^{k-1} q^j [f(bq^j) - f(bq^k)] g(bq^j) \leq 0$$

and

$$\int_0^{bq^k} [f(t) - f(bq^k)][h(t) - g(t)] d_q t = (1-q) bq^k \sum_{j=0}^{\infty} q^j [f(bq^{k+j}) - f(bq^k)][h(bq^{k+j}) - g(bq^{k+j})] \geq 0.$$

Using Equation (14) and the above inequalities we obtain Equation (16). □

Theorem 14. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leq g \leq h$ on $[0, b]$. Let $l \in \mathbb{N}_0$ be such that Equation (12) holds. Then the following inequality is valid*

$$\int_0^b f(x) g(x) d_q x \geq \int_{bq^l}^{b} (f(x) h(x) - [f(x) - f(bq^l)][h(x) - g(x)]) d_q x \geq \int_{bq^l}^{b} f(x) h(x) d_q x. \tag{17}$$

Proof. For a decreasing function f on the interval $[0, b]$, we get $f(bq^j) \leq f(bq^l)$, $j = 0, 1, \dots, l-1$ and $f(bq^{l+j}) \geq f(bq^l)$, $j \in \mathbb{N}_0$. Then

$$\int_0^{bq^l} [f(t) - f(bq^l)] g(t) d_q t = (1-q) bq^l \sum_{j=0}^{\infty} q^j [f(bq^{l+j}) - f(bq^l)] g(bq^{l+j}) \geq 0$$

and

$$\int_{bq^l}^{b} [f(t) - f(bq^l)][h(t) - g(t)] d_q t = (1-q) b \sum_{j=0}^{l-1} q^j [f(bq^j) - f(bq^l)][h(bq^j) - g(bq^j)] \leq 0.$$

Using Equation (15) and the above inequalities we obtain Equation (17). □

In the following corollaries we give refinements of q-Steffensen's inequality by taking $h \equiv 1$ in Theorems 13 and 14.

Corollary 1. *Let $0 < q < 1$, $b > 0$. Let f and g be q-integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leq g \leq 1$ on $[0, b]$ and $\lambda = \int_0^b g(x) d_q x$. Let $k \in \mathbb{N}_0$ be such that*

$$k = \lfloor \log_q(\lambda / b) \rfloor.$$

Then the following inequality is valid

$$\int_0^b f(x) g(x) d_q x \leq \int_0^{bq^k} (f(x) - [f(x) - f(bq^k)][1 - g(x)]) d_q x \leq \int_0^{bq^k} f(x) d_q x.$$

Corollary 2. *Let $0 < q < 1$, $b > 0$. Let f and g be q-integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leq g \leq 1$ on $[0, b]$ and $\lambda = \int_0^b g(x) d_q x$. Let $l \in \mathbb{N}_0$ be such that*

$$l = \lfloor \log_q(1 - \lambda / b) \rfloor.$$

Then the following inequality is valid

$$\int_0^b f(x)g(x)d_qx \geq \int_{bq^l}^b (f(x) - [f(x) - f(bq^l)][1 - g(x)])d_qx \geq \int_{bq^l}^b f(x)d_qx.$$

In [12] Steffensen's inequality is generalized in a one more way. Here we give a q-analogue of that type of generalization. As we show, it is equivalent to generalizations obtained in Theorems 7 and 8.

Theorem 15. *Let $0 < q < 1$, $b > 0$. Let f, g, h and k be q-integrable functions on $[0, b]$ such that k is positive, f is nonnegative, f/k is decreasing and $0 \leq g \leq h$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} h(x)k(x)d_qx \geq \int_0^b g(x)k(x)d_qx.$$

Then

$$\int_0^b f(x)g(x)d_qx \leq \int_0^{bq^k} f(x)h(x)d_qx.$$

Theorem 16. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that k is positive, f is nonnegative, f/k is decreasing and $0 \leq g \leq h$ on $[0, b]$. Let $l \in \mathbb{N}_0$ be such that*

$$\int_{bq^l}^b h(x)k(x)d_qx \leq \int_0^b g(x)k(x)d_qx.$$

Then

$$\int_{bq^l}^b f(x)h(x)d_qx \leq \int_0^b f(x)g(x)d_qx.$$

Remark 3. *Theorems 7 and 15 are equivalent. We can obtain Theorem 7 by taking $h \equiv 1$ in Theorem 15. Oppositely, we can obtain Theorem 15 by taking $h \mapsto hk$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 7.*

Similarly, we obtain that Theorems 8 and 16 are equivalent.

Remark 4. *Taking $k \equiv 1$ in Theorems 15 and 16 we obtain q-generalizations obtained in Theorems 11 and 12.*

4. Concluding Remarks

Similar as in Theorems 9 and 10 we can obtain other generalizations and refinements given in Theorems 11–16 for the restricted definite q-integral but here we omit the details.

In the classical Steffensen inequality there is no assumption on nonnegativity of the function f, but in its q-analogue the nonnegativity assumption is necessary since k and l have to be from $\mathbb{N}_0$. In [15] Pečarić and Smoljak Kalamir studied this type of results for the parameter λ in the classical Steffensen inequality. If the function f is additionally nonnegative, in [15] we have weaker conditions on the parameter λ in Steffensen's inequality and its generalizations.

Without the assumption that the function f is nonnegative, the inequality Equation (8) in Theorem 7 should be changed with the equality

$$\int_0^{bq^k} h(x)d_qx = \int_0^b h(x)g(x)d_qx, \tag{18}$$

but then k might not be from $\mathbb{N}_0$ as requested.

Further, Theorem 7 still holds if the function f is negative. We should only reverse the inequality in Equation (8). Hence, we have the following.

Theorem 17. *Let $0 < q < 1$, $b > 0$. Let f, g and h be q-integrable functions on $[0, b]$ such that h is positive, f is negative, f/h is decreasing and $0 \leq g \leq 1$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} h(x) d_q x = \int_0^b h(x) g(x) d_q x. \tag{19}$$

Then

$$\int_0^b f(x) g(x) d_q x \leq \int_0^{bq^k} f(x) d_q x. \tag{20}$$

Using the same reasoning, the other results given in Section 3 also hold if the function f is negative. We only have to assume that the reverse inequalities for $k, l \in \mathbb{N}_0$ hold in the related theorems.

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