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Definite Integral of Algebraic, Exponential and Hyperbolic Functions Expressed in Terms of Special Functions

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Abstract: While browsing through the famous book of Bierens de Haan, we came across a table with some very interesting integrals. These integrals also appeared in the book of Gradshteyn and Ryzhik. Derivation of these integrals are not listed in the current literature to best of our knowledge. The derivation of such integrals in the book of Gradshteyn and Ryzhik in terms of closed form solutions is pertinent. We evaluate several of these definite integrals of the form $\int_0^\infty \frac{(a+y)^k-(a-y)^k}{e^{by}-1}dy$, $\int_0^\infty \frac{(a+y)^k-(a-y)^k}{e^{by}+1}dy$, $\int_0^\infty \frac{(a+y)^k-(a-y)^k}{\sinh(by)}dy$ and $\int_0^\infty \frac{(a+y)^k+(a-y)^k}{\cosh(by)}dy$ in terms of a special function where k, a and b are arbitrary complex numbers.

Keywords: hyperbolic sine; hyperbolic cosine; algebraic function; definite integral; hankel contour; cauchy integral; gradshteyn and ryzhik; bierens de haan

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1. Introduction

We will derive integrals as indicated in the abstract in terms of special functions which by means of analytic continuation gives a greater range to the parameters in the integral. We also derive definite integrals which yield special cases. We also noticed errors in the formula cited in [1] and we were able to derive correct formula for these integrals as motivation for this work. Some special cases of these integrals have been reported in Gradshteyn and Ryzhik [2]. In 1867, David Bierens de Haan derived Table 87 in [1] when $a = 1$ in the integrals above and we will consider these as well. In our case the variables in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [3]. Generalized Cauchy's integral formula is given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw \tag{1}$$

where C will be defined below. This method involves using a form of Equation (1) then multiplying both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same. This method has been used by us in previous work, [3–7].

2. Integrals Involving the Product of Logarithmic Functions

2.1. Definite Integral of the Contour Integral

We use the method in [3]. Here the contour is similar to Figure 2 in [3] where the cut is along the positive imaginary axis and C is taken from positive infinity to the x-axis along the right side of the cut and zero distance from the cut, around the origin on a circle of zero radius and back to positive infinity on the left side of the cut, except we replace the vertical lines ± 1 by $\pm Re(b)$. In Generalized Cauchy's integral formula we replace y by $y + \log(a)$ and $-y + \log(a)$ then subtract these two equations, followed by multiplying both sides by $\frac{1}{\sinh(by)}$ to get

$$\frac{(y+\log(a))^k - (-y+\log(a))^k}{\Gamma(k+1)\sinh(by)} = \frac{1}{2\pi i}\int_C 2a^w w^{-k-1}\operatorname{csch}(by)\sinh(wy)dw \tag{2}$$

the logarithmic function is defined in Section (4.1) in [8]. We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{\Gamma(k+1)}\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{\sinh(by)}dy &= \frac{1}{\pi i}\int_0^\infty \int_C a^w w^{-k-1}\operatorname{csch}(by)\sinh(wy)dwdy \\
&= \frac{1}{\pi i}\int_C \left(\int_0^\infty \operatorname{csch}(by)\sinh(wy)dy\right)\frac{a^w dw}{w^{k+1}} \\
&= \frac{1}{2ib}\int_C a^w w^{-k-1}\tan\left(\frac{\pi w}{2b}\right)dw
\end{aligned} \tag{3}$$

from Equation (2.7.7.6) in [9] and the integral is valid for a, k and b complex and $-|Re(b)| < Re(w) < |Re(b)|$ and $Re(b) \neq 0$.

In a similar manner we can derive the second integral formula and multiplying by $\frac{1}{e^{by}-1}$ we get

$$\frac{(y+\log(a))^k - (-y+\log(a))^k}{\Gamma(k+1)(e^{by}-1)} = \frac{1}{2\pi i}\int_C \frac{2a^w w^{-k-1}\sinh(wy)}{e^{by}-1}dw \tag{4}$$

We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{\Gamma(k+1)}\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{e^{by}-1}dy &= \frac{1}{2\pi i}\int_0^\infty \int_C \frac{2a^w w^{-k-1}\sinh(wy)}{e^{by}-1}dwdy \\
&= \frac{1}{2\pi i}\int_C \left(\int_0^\infty \frac{2\sinh(wy)}{e^{by}-1}dy\right)\frac{a^w dw}{w^{k+1}} \\
&= \frac{1}{2\pi i}\int_C \left(a^w w^{-k-2} - \frac{a^w \pi w^{-k-1}\cot\left(\frac{\pi w}{b}\right)}{b}\right)dw
\end{aligned} \tag{5}$$

from Equation (2.3.3.12) in [9] and the integral is valid for $-|Re(b)| < Re(w) < |Re(b)|$ and $Re(b) > 0$.

In a similar manner we can derive the third integral formula and multiplying by $\frac{1}{e^{by}+1}$ we get

$$\frac{(y+\log(a))^k - (-y+\log(a))^k}{\Gamma(k+1)(e^{by}+1)} = \frac{1}{2\pi i}\int_C \frac{2a^w w^{-k-1}\sinh(wy)}{e^{by}+1}dw \tag{6}$$

We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{\Gamma(k+1)}\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{e^{by}+1}dy &= \frac{1}{2\pi i}\int_0^\infty \int_C \frac{2a^w w^{-k-1}\sinh(wy)}{e^{by}+1}dwdy \\
&= \frac{1}{2\pi i}\int_C \left(\int_0^\infty \frac{2\sinh(wy)}{e^{by}+1}dy\right)\frac{a^w dw}{w^{k+1}} \\
&= \frac{1}{2\pi i}\int_C \left(-a^w w^{-k-2} + \frac{a^w \pi w^{-k-1}\csc\left(\frac{\pi w}{b}\right)}{b}\right)dw
\end{aligned} \tag{7}$$

from Equation (2.3.3.11) in [9] and the integral is valid for $-|Re(b)| < Re(w) < |Re(b)|$ and $Re(b) > 0$.

In a slightly different manner we can derive the fourth integral formula by adding the two equations and multiplying by $\frac{1}{\cosh(by)}$ we get

$$\frac{(y+\log(a))^k+(-y+\log(a))^k}{\Gamma(k+1)\cosh(by)} = \frac{1}{2\pi i}\int_C \frac{2a^w w^{-k-1}\cosh(wy)}{\cosh(by)}dw \tag{8}$$

We then take the definite integral over $y \in [0,\infty)$ of both sides to get

$$\begin{aligned}\frac{1}{\Gamma(k+1)}\int_0^\infty \frac{(y+\log(a))^k+(-y+\log(a))^k}{\cosh(by)}dy &= \frac{1}{2\pi i}\int_0^\infty\int_C \frac{2a^w w^{-k-1}\cosh(wy)}{\cosh(by)}dwdy \\ &= \frac{1}{2\pi i}\int_C\left(\int_0^\infty \frac{2\cosh(wy)}{\cosh(by)}dy\right)\frac{a^w dw}{w^{k+1}} \\ &= \frac{1}{2ib}\int_C a^w w^{-k-1}\sec\left(\frac{\pi w}{2b}\right)dw\end{aligned} \tag{9}$$

from Equation (1.7.7.1) in [9] and the integral is valid for $-|Re(b)| < Re(w) < |Re(b)|$ and $Re(b) \neq 0$.

2.2. Infinite Sum of the Contour Integral

Again, using the method in [3], replacing y with $2\pi i(p+1)/(2b)+\log(a)$ and multiplying both sides by $(-1)^{p+1}\left(\frac{2\pi i}{b}\right)$ to yield

$$(-1)^{p+1}\left(\frac{2\pi i}{b}\right)\frac{(2\pi i(p+1)/(2b)+\log(a))^k}{\Gamma(k+1)} = \frac{(-1)^{p+1}}{b}\int_C \frac{e^{w(2\pi i(p+1)/(2b)+\log(a))}}{w^{k+!}}a^w dw \tag{10}$$

followed by taking the infinite sum of both sides of Equation (10) with respect to p over $[0,\infty)$ to get

$$\begin{aligned}\frac{(2\pi)^{k+1}i^{k+1}b^{-k-1}}{\Gamma(k+1)}&\left(-\zeta\left(-k,\frac{\pi-ib\log(a)}{2\pi}\right)+\zeta\left(-k,\frac{2\pi-ib\log(a)}{2\pi}\right)\right) \\ &= \frac{1}{b}\sum_{p=0}^\infty\int_C \frac{e^{w\left(\frac{2\pi i(p+1)}{(2b)}+\log(a)\right)}}{(-1)^{-p-1}w^{k+1}}a^w dw \\ &= \frac{1}{b}\int_C\sum_{p=0}^\infty \frac{e^{w\left(\frac{2\pi i(p+1)}{(2b)}+\log(a)\right)}}{(-1)^{-p-1}w^{k+1}}a^w dw \\ &= \frac{1}{2ib}\int_C\left(-ia^w w^{-k-1}\right)dw \\ &+\frac{1}{2ib}\int_C\left(a^w w^{-k-1}\tan\left(\frac{\pi w}{2b}\right)\right)dw\end{aligned} \tag{11}$$

from (1.232.1) in [2] where $\tanh(ix) = i\tan(x)$ from (4.5.9) in [8] and $Im(w) > 0$ for the convergence of the sum and if the $Re(k) < 0$ then the argument of the sum over p cannot be zero for some value of p. We use (9.521) in [2] where $\zeta(s,u)$ is the Hurwitz Zeta function.

Similarly, using the method in [3], replacing y with $2\pi i(p+1)/b+\log(a)$ and multiplying both sides by $\left(\frac{2\pi i}{b}\right)$ to yield

$$\left(\frac{2\pi i}{b}\right)\frac{(2\pi i(p+1)/b+\log(a))^k}{\Gamma(k+1)} = \frac{1}{b}\int_C \frac{e^{w(2\pi i(p+1)/b+\log(a))}}{w^{k+1}}a^w dw \tag{12}$$

followed by taking the infinite sum of both sides of Equation (12) with respect to p over $[0, \infty)$ to get

$$\begin{aligned} \frac{(2\pi)^{k+1} i^{k+1} b^{-k-1}}{\Gamma(k+1)} \zeta\left(-k, \frac{2\pi - ib\log(a)}{2\pi}\right) &= \frac{1}{b} \sum_{p=0}^{\infty} \int_C \frac{e^{w(2\pi i(p+1)/b+\log(a))}}{w^{k+1}} a^w dw \\ &= \frac{1}{b} \int_C \sum_{p=0}^{\infty} \frac{e^{w(2\pi i(p+1)/b+\log(a))}}{w^{k+1}} a^w dw \\ &= \frac{1}{2ib} \int_C \left(-ia^w w^{-k-1} - a^w w^{-k-1} \cot\left(\frac{\pi w}{b}\right)\right) dw \end{aligned} \tag{13}$$

from Equation (1.232.1) in [2] where $\operatorname{ctnh}(iw) = -i\cot(w)$, $\tan(w + \pi/2) = -\cot(w)$ and

$$\operatorname{ctnh}(z) = -1 - 2\sum_{k=0}^{\infty} e^{2z(k+1)} \tag{14}$$

where $Re(z) < 0$, which implies that $Im(w) > 0$.

Similarly, using the method in [3], replacing y with $\pi i(2p+1)/b + \log(a)$ and multiplying both sides by $-\left(\frac{2\pi i}{b}\right)$ to yield

$$-\left(\frac{2\pi i}{b}\right) \frac{(\pi i(2p+1)/b + \log(a))^k}{\Gamma(k+1)} = -\frac{1}{b} \int_C \frac{e^{w(\pi i(2p+1)/b+\log(a))}}{w^{k+1}} a^w dw \tag{15}$$

followed by taking the infinite sum of both sides of Equation (15) with respect to p over $[0, \infty)$ to get

$$\begin{aligned} \frac{(2\pi)^{k+1} i^{k-1} b^{-k-1}}{\Gamma(k+1)} \zeta\left(-k, \frac{\pi - ib\log(a)}{2\pi}\right) &= -\frac{1}{b} \sum_{p=0}^{\infty} \int_C \frac{e^{w(\pi i(2p+1)/b+\log(a))}}{w^{k+1}} a^w dw \\ &= -\frac{1}{b} \int_C \sum_{p=0}^{\infty} \frac{e^{w(\pi i(2p+1)/b+\log(a))}}{w^{k+1}} a^w dw \\ &= \frac{1}{2ib} \int_C a^w w^{-k-1} \csc\left(\frac{\pi w}{b}\right) dw \end{aligned} \tag{16}$$

from Equation (1.232.3) in [2] Similarly, using the method in [3], replacing y with $\pi i(2p+1)/(2b) + \log(a)$ and multiply both sides by $(-1)^p \left(\frac{2\pi i}{b}\right)$ to yield

$$(-1)^p \left(\frac{2\pi i}{b}\right) \frac{(\pi i(2p+1)/(2b) + \log(a))^k}{\Gamma(k+1)} = \frac{(-1)^p}{b} \int_C \frac{e^{w(\pi i(2p+1)/(2b)+\log(a))}}{w^{k+1}} a^w dw \tag{17}$$

followed by taking the infinite sum of both sides of Equation (17) with respect to p over $[0, \infty)$ to get

$$\begin{aligned} &\frac{(2\pi)^{k+1} i^k b^{-k-1}}{\Gamma(k+1)} \left(\zeta\left(-k, \frac{\pi - 2ib\log(a)}{4\pi}\right) - \zeta\left(-k, \frac{3\pi - 2ib\log(a)}{2\pi}\right)\right) \\ &= \frac{(-1)^p}{b} \sum_{p=0}^{\infty} \int_C \frac{e^{w\left(\frac{\pi i(2p+1)}{2b}+\log(a)\right)}}{w^{k+1}} a^w dw \\ &= \frac{1}{b} \int_C \sum_{p=0}^{\infty} \frac{e^{w\left(\frac{\pi i(2p+1)}{(2b)}+\log(a)\right)}}{(-1)^{-p} w^{k+1}} a^w dw \\ &= \frac{1}{2ib} \int_C a^w w^{-k-1} \sec\left(\frac{\pi w}{2b}\right) dw \end{aligned} \tag{18}$$

from Equation (1.232.2) in [2].

To obtain the first contour integral in the last line of Equations (5) and (7) we use the Cauchy formula by replacing y by $\log(a)$, k by $k+1$, and multiplying both sides by -1 and simplifying we get

$$-\frac{\log^{k+1}(a)}{(k+1)!} = -\frac{1}{2\pi i}\int_C a^w w^{-2-k} dw \tag{19}$$

To obtain the first contour integral in the last line in Equations (11) and (13) we use the Cauchy formula by replacing y by $\log(a)$ and multiplying both sides by $\frac{\pi}{ib}$ and simplifying we get

$$\frac{\pi \log^k(a)}{ib\Gamma(k+1)} = -\frac{1}{2b}\int_C a^w w^{-1-k} dw \tag{20}$$

Since the right hand-side of Equation (3) is equal to the addition of (11) and (20), we can equate the left hand-sides and simplify to get

$$\begin{aligned}
&\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{\sinh(by)} dy \\
&= (2\pi)^{k+1} i^{k+1} b^{-k-1} \left(-\zeta\left(-k, \frac{\pi - ib\log(a)}{2\pi}\right) + \zeta\left(-k, \frac{2\pi - ib\log(a)}{2\pi}\right)\right) + \frac{i\pi \log^k(a)}{b}
\end{aligned} \tag{21}$$

where $Re(b) \neq 0$ and k and a are general complex numbers.

We can write down an equivalent formula for the corresponding Hurwitz Zeta function for the second integral using Equations (5), (13), (19) and (20),

$$\begin{aligned}
\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{e^{by}-1} dy = (2\pi)^{k+1} i^{k+1} b^{-k-1} \zeta\left(-k, \frac{2\pi - ib\log(a)}{2\pi}\right) \\
+ \frac{\log^{k+1}(a)}{k+1} + \frac{i\pi \log^k(a)}{b}
\end{aligned} \tag{22}$$

where $Re(b) > 0$ and k and a are general complex numbers.

We can write down an equivalent formula for the corresponding Hurwitz Zeta function for the second integral using Equations (7), (16) and (19),

$$\begin{aligned}
\int_0^\infty \frac{(y+\log(a))^k - (-y+\log(a))^k}{e^{by}+1} dy = (2\pi)^{k+1} i^{k-1} b^{-k-1} \zeta\left(-k, \frac{\pi - ib\log(a)}{2\pi}\right) \\
- \frac{\log^{k+1}(a)}{k+1}
\end{aligned} \tag{23}$$

where $Re(b) > 0$ and k and a are general complex numbers.

We can write down an equivalent formula for the corresponding Hurwitz Zeta function for the second integral using Equations (9) and (18),

$$\begin{aligned}
&\int_0^\infty \frac{(y+\log(a))^k + (-y+\log(a))^k}{\cosh(by)} dy \\
&= (2\pi)^{k+1} i^k b^{-k-1} \left(\zeta\left(-k, \frac{\pi - 2ib\log(a)}{4\pi}\right) - \zeta\left(-k, \frac{3\pi - 2ib\log(a)}{4\pi}\right)\right)
\end{aligned} \tag{24}$$

where $Re(b) \neq 0$ and k and a are general complex numbers.

3. Derivation of Integrals

In this section, we derive the entries of Table 87 [1] in terms of the Hurwitz zeta and zeta functions which will analytically continue these integrals. The integrals in the Table can be achieved by using the Equation (12.11.17) in [10] given by

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$$

for $n \geq 0$ where $B_k(a)$ is the Bernoulli polynomial.

3.1. Using Equation (23) and Setting $a = e^i$, $b = \pi$ and Replacing k with $2k$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k} - (1-iy)^{2k}}{i(e^{\pi y}+1)} dy = 2^{2k+1}\zeta(-2k) + \frac{1}{2k+1} \tag{25}$$

from Equation (9.534) in [2]. This result is Equation (1) Table 87 in [1] where this result is in error.

3.2. Using Equation (23) and Setting $a = e^i$, $b = \pi$ and Replacing k with $2k-1$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k-1} - (1-iy)^{2k-1}}{i(e^{\pi y}+1)} dy = 2^{2k}\zeta(1-2k) + \frac{1}{2k} \tag{26}$$

from Equation (9.534) in [2]. This result is Equation (2) Table 87 in [1] where this result is in error.

3.3. Using Equation (22) and Setting $a = e^i$, $b = \pi$ and Replacing k with $2k-1$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k-1} - (1-iy)^{2k-1}}{i(e^{\pi y}-1)} dy = -2^{2k}\zeta\left(1-2k, \frac{3}{2}\right) - \frac{1}{2k} - 1 \tag{27}$$

from Equation (9.534) in [2]. This result is Equation (3) Table 87 in [1] where this result is in error.

3.4. Using Equation (22) and Setting $a = e^i$, $b = 2\pi$ and Replacing k with $2k$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k} - (1-iy)^{2k}}{i(e^{2\pi y}-1)} dy = -(-1+\zeta(-2k)) - \frac{1}{2k+1} - \frac{1}{2} \tag{28}$$

from Equation (9.534) in [2]. This result is Equation (4) Table 87 in [1] where this result is in error.

3.5. Using Equation (22) and Setting $a = e^i$, $b = 2\pi$ and Replacing k with $2k-1$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k-1} - (1-iy)^{2k-1}}{i(e^{2\pi y}-1)} dy = -(-1+\zeta(1-2k)) - \frac{1}{2k} - \frac{1}{2} \tag{29}$$

from Equation (9.534) in [2]. This result is Equation (5) Table 87 in [1] where this result is in error.

3.6. Using Equation (24) and Setting $a = e^i$, $b = \pi/2$ and Replacing k with $2k-1$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k-1} + (1-iy)^{2k-1}}{\cosh(\frac{\pi y}{2})} dy = 2^{4k}(-\zeta(1-2k) + (-1+2^{1-2k})\zeta(1-2k)) \tag{30}$$

from Equation (9.534) in [2]. This result is Equation (6) Table 87 in [1] where this result is in error.

3.7. Using Equation (21) and Setting $a = e^i$, $b = \pi/2$ and Replacing k with 2k We Get

$$\int_0^\infty \frac{(1+iy)^{2k}-(1-iy)^{2k}}{i\sinh(\frac{\pi y}{2})}dy = -2+2^{4k}\zeta\left(-2k,\frac{3}{4}\right)-2^{4k}\zeta\left(-2k,\frac{5}{4}\right) \tag{31}$$

This result is Equation (7) Table 87 in [1] where this result is in error.

3.8. Using Equation (21) and Setting $a = e^i$, $b = \pi/2$ and Replacing k with $2k-1$ We Get

$$\int_0^\infty \frac{(1+iy)^{2k-1}-(1-iy)^{2k-1}}{i\sinh(\frac{\pi y}{2})}dy = -\left(2+2^{4k}\left(-\zeta\left(1-2k,\frac{3}{4}\right)-2^{4k}\zeta\left(1-2k,\frac{5}{4}\right)\right)\right) \tag{32}$$

This result is Equation (8) Table 87 in [1] where this result is in error.

4. Generalizations and Table of Integrals

In this section we summarized the integrals evaluated in this work in the form of a table (see Table 1).

Table 1. Table of Integrals.

$Q(y)$	$\int_0^\infty Q(y)dy$
$\frac{(1+iy)^{2k}-(1-iy)^{2k}}{i(e^{\pi y}+1)}$	$2^{2k+1}\zeta(-2k)+\frac{1}{2k+1}$
$\frac{(1+iy)^{2k-1}-(1-iy)^{2k-1}}{i(e^{\pi y}+1)}$	$4^k\zeta(1-2k)+\frac{1}{2k}$
$\frac{(1+iy)^{2k-1}-(1-iy)^{2k-1}}{i(e^{\pi y}-1)}$	$-2^{2k}\zeta\left(1-2k,\frac{3}{2}\right)-\frac{1}{2k}-1$
$\frac{(1+iy)^{2k}-(1-iy)^{2k}}{i(e^{2\pi y}-1)}$	$-(-1+\zeta(-2k))-\frac{1}{2k+1}-\frac{1}{2}$
$\frac{(1+iy)^{2k-1}-(1-iy)^{2k-1}}{i(e^{2\pi y}-1)}$	$-(-1+\zeta(1-2k))-\frac{1}{2k}-\frac{1}{2}$
$\frac{(1+iy)^{2k-1}+(1-iy)^{2k-1}}{\cosh(\frac{\pi y}{2})}$	$2^{4k}(-\zeta(1-2k)+(-1+2^{1-2k})\zeta(1-2k))$
$\frac{(1+iy)^{2k}-(1-iy)^{2k}}{i\sinh(\frac{\pi y}{2})}$	$-2+2^{2+4k}\zeta\left(-2k,\frac{3}{4}\right)-2^{2+4k}\zeta\left(-2k,\frac{5}{4}\right)$
$\frac{(1+iy)^{2k-1}-(1-iy)^{2k-1}}{i\sinh(\frac{\pi y}{2})}$	$\left(2+16^k\left(-\zeta\left(1-2k,\frac{3}{4}\right)-2^{2+4k}\zeta\left(1-2k,\frac{5}{4}\right)\right)\right)$

5. Special Cases of the Definite Integrals

In this ection we will look at a few of the integrals derived in this work and evaluate special cases in terms of the log-gamma function and π.

5.1. When a Is Replaced by e^{mi}

We take the first derivative of Equation (24) with respect to k then set $k = 0$, factorize and simplify the the log terms to get

$$
\begin{aligned}
&\int_0^\infty \log\left(m^2+y^2\right)\operatorname{sech}(by)dy \\
&= \frac{\pi}{b}\left(\log\left(\frac{2i\pi}{b}\right) - 2\log(-3\pi - 2ib\log(e^{mi})) + 2\log(-\pi - 2ib\log(e^{mi}))\right) \\
&\quad + \frac{\pi i}{b}\left(2\log\left[\frac{\Gamma\left(-\frac{\pi+2ib\log(e^{mi})}{4\pi}\right)}{\Gamma\left(-\frac{3\pi+2ib\log(e^{mi})}{4\pi}\right)}\right]\right) \\
&= \frac{\pi}{b}\left(\log\left(\frac{2\pi}{b}\right) + \log\left[\left(\frac{\Gamma\left(\frac{3\pi+2bm}{4\pi}\right)}{\Gamma\left(\frac{\pi+2bm}{4\pi}\right)}\right)^2\right]\right)
\end{aligned}
\tag{33}
$$

from (3.10) in [11] where $Re(b) > 0$. This is the same as (4.373.1) in [2].

5.1.1. When $m = 1$ and $b = \pi/2$

Using Equation (33) we get

$$
\int_0^\infty \log\left(1+y^2\right)\operatorname{sech}\left(\frac{\pi y}{2}\right)dy = 2\log\left(\frac{4}{\pi}\right)
\tag{34}
$$

This result is listed as (4.373.2) in [2].

5.1.2. When $m = -1$ and $b = \pi$

Using Equation (33) and simplifying the integral we get

$$
\int_0^\infty \log\left(1+y^2\right)\operatorname{sech}(\pi y)dy = \log\left[\frac{2\Gamma^2\left(\frac{1}{4}\right)}{\Gamma^2\left(-\frac{1}{4}\right)}\right]
\tag{35}
$$

5.2. When a Is Replaced by e^{ni}

We take the first derivative of Equation (21) with respect to k then set $k = 0$ to get

$$
\begin{aligned}
\int_0^\infty \log\left(\frac{ni+y}{ni-y}\right)\operatorname{csch}(by)dy &= \frac{\pi i}{b}\left(-\log\left(\frac{8i\pi}{b}\right) + \log(\log(e^{ni})) - 2\log\left(-\frac{ib\log(e^{ni})}{2}\right)\right) \\
&\quad + \frac{\pi i}{b}\left(2\log(-\pi - ib\log(e^{ni})) + 2\log\left[\frac{\Gamma\left(-\frac{\pi+ib\log(e^{ni})}{2\pi}\right)}{\Gamma\left(-\frac{ib\log(e^{ni})}{2\pi}\right)}\right]\right) \\
&= \frac{\pi i}{b}\left(\log\left(\frac{bn}{8\pi}\right) + \log\left[\frac{4\Gamma^2\left(\frac{\pi+bn}{2\pi}\right)}{\Gamma^2\left(\frac{2\pi+bn}{2\pi}\right)}\right]\right)
\end{aligned}
\tag{36}
$$

from (3.10) in [11] where $Re(b) > 0$.

5.2.1. When $n = 1$ and $b = \pi$

Using Equation (36) and simplifying the integral we get

$$\int_0^\infty \log\left(\frac{i+y}{i-y}\right) \operatorname{csch}(\pi y) dy = -i \log\left(\frac{\pi}{2}\right) \tag{37}$$

5.2.2. When a is replaced by e^{qi}

Using Equation (21) and using L'Hospital's rule when $k \to 1$ we get

$$\int_0^\infty \frac{y \operatorname{csch}(by)}{q^2+y^2} dy = \frac{1}{2}\left(\frac{\pi}{qb} - \psi^{(0)}\left(\frac{qb}{2\pi}+1\right) + \psi^{(0)}\left(\frac{qb+\pi}{2\pi}\right)\right) \tag{38}$$

where q, b are general complex numbers, $\psi_m(z) = (-1)^{m+1} m! \zeta(m+1, z)$.

6. Summary

In this article we derived the definite logarithmic algebraic functions in terms of the Hurwitz Zeta function. We were able to produce a closed form solution for integrals tabled in Bierens de Haan [1,2] not previously derived. We will be looking at other integrals using this contour integral method for future work. The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram.

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References

1. Bierens de Haan, D. *Nouvelles Tables D'intégrales Définies*; Engels, P., Ed.; Leide: Amsterdam, The Netherlands, 1867.
2. Erdéyli, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G.; Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: Cambridge, MA, USA, 2000.
3. Reynolds, R.; Stauffer, A. A Method for Evaluating Definite Integrals in Terms of Special Functions with Examples. *Int. Math. Forum* **2020**, *15*, 235–244. [CrossRef]
4. Reynolds, R.; Stauffer, A. Definite Integral of Arctangent and Polylogarithmic Functions Expressed as a Series. *Mathematics* **2019**, *7*, 1099. [CrossRef]
5. Reynolds, R.; Stauffer, A. A Definite Integral Involving the Logarithmic Function in Terms of the Lerch Function. *Mathematics* **2019**, *7*, 1148. [CrossRef]
6. Reynolds, R.; Stauffer, A. Derivation of Logarithmic and Logarithmic Hyperbolic Tangent Integrals Expressed in Terms of Special Functions. *Mathematics* **2020**, *8*, 687. [CrossRef]
7. Reynolds, R.; Stauffer, A. Definite integrals involving product of logarithmic functions and logarithm of square root functions expressed in terms of special functions. *AIMS Math.* **2020**, *5*, 5724. [CrossRef]
8. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed.; Dover Publications: New York, NY, USA; Dover, UK, 1982.
9. Oberhettinger, F. *Tables of Fourier Transforms and Fourier Transforms of Distributions*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 1990.

10. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA, 1995.
11. Junesang, C.; Srivastava, H.M. A family of log-gamma integrals and associated results. *J. Math. Anal. Appl.* **2005**, *303*, 436–449.