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An Iteration Function Having Optimal Eighth-Order of Convergence for Multiple Roots and Local Convergence

Ramandeep Behl [1], Ioannis K. Argyros [2], Michael Argyros [3], Mehdi Salimi [4,5,*] and Arwa Jeza Alsolami [1]

1 Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; ramanbehl87@yahoo.in (R.B.); ramanbehl87@gmail.com (A.J.A.)
2 Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu
3 Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA; michael.argyros@cameron.edu
4 Center for Dynamics and Institute for Analysis, Department of Mathematics, Technische Universität Dresden, 01062 Dresden, Germany
5 Department of Mathematics, Simon Fraser University, 8888 University Dr., Burnaby, BC V5A 1S6, Canada
* Correspondence: mehdi.salimi@tu-dresden.de or mehdi_salimi@sfu.ca

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Abstract: In the study of dynamics of physical systems an important role is played by symmetry principles. As an example in classical physics, symmetry plays a role in quantum physics, turbulence and similar theoretical models. We end up having to deal with an equation whose solution we desire to be in a closed form. But obtaining a solution in such form is achieved only in special cases. Hence, we resort to iterative schemes. There is where the novelty of our study lies, as well as our motivation for writing it. We have a very limited literature with eighth-order convergent iteration functions that can handle multiple zeros $m \geq 1$. Therefore, we suggest an eighth-order scheme for multiple zeros having optimal convergence along with fast convergence and uncomplicated structure. We develop an extensive convergence study in the main theorem that illustrates eighth-order convergence of our scheme. Finally, the applicability and comparison was illustrated on real life problems, e.g., Van der Waal's equation of state, Chemical reactor with fractional conversion, continuous stirred reactor and multi-factor problems, etc., with existing schemes. These examples further show the superiority of our schemes over the earlier ones.

Keywords: nonlinear equations; Kung–Traub conjecture; multiple roots; optimal iterative methods; efficiency index

MSC: 65G99; 65H10; 65H10

1. Introduction

One of the problems of great significance and difficulty in the subject of computational mathematics is finding the multiple zeros for $f(x)$ ($f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ a sufficiently differentiable function in $\mathbb{D}$). It is difficult to obtain the exact solution in analytic form for such problems such that we can just say that it is almost fictitious. That is why in practice, we obtain an approximated and efficient solution up to any specific degree of accuracy by the means of an iterative procedure.

This is one of the main reasons that researchers have been making great efforts to develop iteration functions over the past few decades. Additionally, this accuracy also depends on some other facts such as: the considered iterative function, structure of the considered problem, initial guess with software like Maple, Fortran, MATLAB, Mathematica, etc. Further, the practitioners or researchers using these iterative schemes face many problems, like: choice of initial guess/approximation, slower convergence, derivative is zero about the root (in the case of derivative free multipoint schemes), divergence, oscillation, difficulty near the initial point, failure of the iterative method, etc., (for more details please see [1–5]).

In addition, there is not a single iteration function until now which is applicable to every problem. This is the main reason that there is an excessive amount of literature on the iteration functions for scalar equations. Here, we are concerned with the multiple zeros of the involved function in this study. Unfortunately, we have a small amount of literature belonging to higher-order iteration function in real equations that can handle multiple roots. The tough calculation work and more time considerations are the main reason behind this. Moreover, it is a more challenging task to construct an iterative procedure for multiple zeros as compared to simple ones.

Eighth-order multi-point schemes are faster and have a better efficiency index [6–11] if compared to fourth-order [12–21] and sixth-order [22,23] iteration functions. We mean that we can save computational time and cost by using them and obtain the estimate root within a small number of iterations as compared to the others. However, there are only few articles [24–27] discussing the eighth-order convergence for multiple roots. But, we know that there is always a scope in the research to obtain better approximation techniques with simple and compact body structure.

While keeping all these things in our mind, we not only present an eighth-order iteration scheme having optimal convergence for obtaining the multiple solutions of scalar equation which is better than the existing ones. Furthermore, our schemes achieve the minimum error among two consecutive iterations, minimum residual errors, and more balanced computational order of convergence when compared with existing ones of identical order of convergence. Moreover, we present a main theorem which demonstrates the eighth-order convergence provided multiplicity of roots is known. A practical exhibition of our proposed schemes to real life problems is also given.

We usually categorize schemes with local, semi-local and global converges. In local convergence, information about the solution is used to get determine a ball containing suitable (for convergence) initial points. In the semi-local convergence, convergence criteria are obtained using the initial point and the function involved. Finally, in the global convergence all solutions are sought and the ball of convergence usually coincides with the domain of the function. We are interested in local convergence, since in this case schemes are faster, the initial point is picked from the convergence ball and is close to the solution. However, we should mention that there is a plethora of global results, such as [28,29], to mention a few. Global results are more expensive, but return all roots in a given domain. The conditions (2) in our main Theorem 1 seem to be restrictive. But, they are very general and include many well studied schemes for special choices of the free parameters involved. In fact in Table 1, we present numerous such cases which satisfy the conditions (2) of Theorem 1. Our scheme applies to finding roots of functions not necessity of polynomial nature (see Examples 2, 3, 6 and 8).

In the rest of the examples (used to test the convergence criteria) polynomial clipping schemes may do better. However, we did not investigate this, since the main focus of our paper is in scheme (1). Another benefit of our local results is that we obtain estimates on $\|x_n - \xi\|$ not given in the aforementioned papers, so we know in advance the number of iterations needed to obtain a desired error tolerance.

Table 1. Some special cases of the proposed scheme (1).

Cases	$H(\nu)$	$G(\mu)$
Case-1	$\frac{m(\alpha-\beta+2\nu-2)}{\alpha-\beta}$	$m\Big[1+2\mu+(1-2\beta)\mu^2+2(\beta^2-2\beta-2)\mu^3\Big].$
Case-2	$\frac{m(\alpha-\beta+2\nu-2)}{\alpha-\beta}$	$\frac{m\left(2\beta^2\mu+\beta(2-4\mu^2)-(3\mu+1)^2\right)}{2\beta^2\mu+\beta(2-4\mu)-4\mu-1}$
Case-3	$a_1+\frac{a_2}{\nu}$, where, $a_1=-\frac{2m}{\alpha-\beta}, a_2=\frac{m(\alpha-\beta+2)}{\alpha-\beta}$	$m\Big[1+2\mu+(1-2\alpha)\mu^2+2(\alpha^2-2\alpha-2)\mu^3\Big]$
Case-4	$a_1+\frac{a_2}{\nu}$, where, $a_1=-\frac{2m}{\alpha-\beta}, a_2=\frac{m(\alpha-\beta+2)}{\alpha-\beta}$	$\frac{m\left(2\alpha^2\mu+\alpha(2-4\mu^2)-(3\mu+1)^2\right)}{2\alpha^2\mu+\alpha(2-4\mu)-4\mu-1}$
Case-5	$\frac{b_1}{\nu}+\frac{b_2}{1+\nu}$, where, $b_1=\frac{m(-\alpha+\beta-4)}{\alpha-\beta}$, $b_2=\frac{4m(\alpha-\beta+2)}{\alpha-\beta}$	$\frac{m}{4}\left(4+8\mu-2b_3\mu^2+b_4\mu^3\right)$ $b_3=\alpha^2-2\alpha(\beta-3)+\beta^2-2\beta-2,$ $b_4=3\alpha^3-5\alpha^2(\beta-2)+\alpha(\beta^2+4\beta-24)+\beta^3$ $-6\beta^2+8\beta-16$
Case-6	$\frac{m\left(\nu^2(3\alpha-3\beta+14)+\nu(3\alpha-3\beta-16)+2\right)}{3\nu(\nu+1)(\alpha-\beta)}$	$\frac{m\left(\mu^3\left(2\alpha(4\beta-7)+4\beta^2-28\beta-9\right)+\mu^2(-4\alpha-8\beta+27)+21\mu+6\right)}{3(\mu+1)(\mu+2)}$
Case-7	$\frac{m\left(\nu^2(\alpha-\beta+6)+\nu(\alpha-\beta-8)+2\right)}{\nu(\nu+1)(\alpha-\beta)}$	$\frac{m\left(\mu^3\left(-2\alpha^2+4\alpha\beta+2\beta^2-14\beta-3\right)+(9-4\beta)\mu^2+7\mu+2\right)}{(\mu+1)(\mu+2)}$

In all the above cases $\alpha \neq \beta$.

2. Construction of Higher-Order Scheme

We develop an eighth-order scheme for multiple zeros with simple and compact body design. Therefore, we consider the new scheme in the following way:

$$
\begin{aligned}
y_\sigma &= x_\sigma - m\frac{f(x_\sigma)}{f'(x_\sigma)},\\
w_\sigma &= y_\sigma - \mu H(\nu)\frac{f(x_\sigma)}{f'(x_\sigma)},\\
x_{\sigma+1} &= w_\sigma - \kappa\mu\left(G(\mu)+\frac{m\kappa}{1-4\mu}\right)\frac{f(x_\sigma)}{f'(x_\sigma)},
\end{aligned}
\tag{1}
$$

where $\alpha,\ \beta$ are real numbers. In addition, two functions $H:\mathbb{C}\to\mathbb{C}$ and $G:\mathbb{C}\to\mathbb{C}$ are analytic in neighborhoods of (1) and (0) for $\nu=\frac{1+\alpha\mu}{1+\beta\mu}$, $\mu=\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)^{\frac{1}{m}}$, $\kappa=\left(\frac{f(w_\sigma)}{f(y_\sigma)}\right)^{\frac{1}{m}}$ with μ and κ multi-valued function. Suppose their principal analytic branches (see [30,31]) μ as a principal root given by $\mu=\exp\left[\frac{1}{m}\log\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)\right]$, with $\log\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)=\log\left|\frac{f(y_\sigma)}{f(x_\sigma)}\right|+i\mathrm{Arg}\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)$ for $-\pi<\mathrm{Arg}\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)\leq\pi$. The choice of $\mathrm{Arg}(z)$ for $z\in\mathbb{C}$ agrees with that of $\log z$ to be employed later in numerical experiments of section. We have in an analogous way $\mu=\left|\frac{f(y_\sigma)}{f(x_\sigma)}\right|^{\frac{1}{m}}.\exp\left[\frac{1}{m}\mathrm{Arg}\left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)\right]=O(e_\sigma)$, and $\kappa=\left|\frac{f(w_\sigma)}{f(y_\sigma)}\right|^{\frac{1}{m}}.\exp\left[\frac{1}{m}\mathrm{Arg}\left(\frac{f(w_\sigma)}{f(y_\sigma)}\right)\right]=O(e_\sigma)$.

In Theorem 1, we illustrate that the constructed scheme (1) attains maximum eighth-order of convergence for all $\alpha,\beta\in\mathbb{R}$ $(\alpha\neq\beta)$, without adopting any supplementary evaluation of function or its derivatives. Notice that the weight functions H and G play significant roles in the progress of the scheme (details can be found in Theorem 1).

Theorem 1. *Suppose ξ is a solution of multiplicity $m \geq 1$ of f. Consider that function $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ is analytic in $\mathbb{D}$ surrounding the required zero ξ. Then, the scheme given by (1) is of eighth-order convergence, provided*

$$\begin{aligned} &H(1) = m,\ H'(1) = \frac{2m}{\alpha - \beta}\ (\alpha \neq \beta),\ G(0) = m,\ G'(0) = 2m,\ G''(0) = H''(1)(\alpha-\beta)^2 + (2-4\beta)m, \\ &G'''(0) = (\alpha-\beta)^2\Big(H'''(1)(\alpha-\beta) - 6(\beta-1)H''(1)\Big) + 12m(\beta^2 - 2\beta - 2). \end{aligned} \tag{2}$$

Proof. Let us consider that $e_\sigma = x_\sigma - \xi$ and $c_k = \frac{m!}{(m-1+k)!}\frac{f^{m-1+k}(\xi)}{f^m(\xi)}$, $k = 2, 3, 4 \ldots, 8$ are the error in σth iteration and asymptotic error constant numbers, respectively. Now, we adopt Taylor's series expansions for the functions $f(x_\sigma)$ and $f'(x_\sigma)$ around $x = \xi$, which are given by

$$f(x_\sigma) = \frac{f^{(m)}(\xi)}{m!}e_\sigma^m\Big(1 + c_1e_\sigma + c_2e_\sigma^2 + c_3e_\sigma^3 + c_4e_\sigma^4 + c_5e_\sigma^5 + c_6e_\sigma^6 + c_7e_\sigma^7 + c_8e_\sigma^8 + O(e_\sigma^9)\Big) \tag{3}$$

and

$$\begin{aligned} f'(x_\sigma) = &\frac{f^m(\xi)}{m!}e_\sigma^{m-1}\Big(m + (m+1)c_1e_\sigma + (m+2)c_2e_\sigma^2 + (m+3)c_3e_\sigma^3 + (m+4)c_4e_\sigma^4 \\ &+ (m+5)c_5e_\sigma^5 + (m+6)c_6e_\sigma^6 + (m+7)c_7e_\sigma^7 + (m+8)c_8e_\sigma^8 + O(e_\sigma^9)\Big), \end{aligned} \tag{4}$$

respectively.

We have the following expression in view of expressions (3) and (4) from the scheme (1)

$$y_\sigma - \xi = \frac{c_1}{m}e_\sigma^2 + \frac{1}{m^2}(2mc_2 - (m+1)c_1^2)e_\sigma^3 + \sum_{i=0}^{4}\theta_i e_\sigma^{i+4} + O(e_\sigma^9), \tag{5}$$

where $\theta_i = \theta_i(m, c_1, c_2, \ldots, c_8)$, for example $\theta_0 = \frac{1}{m^3}\Big[3m^2c_3 + (m+1)^2c_1^3 - m(3m+4)c_1c_2\Big]$ and $\theta_1 = \frac{1}{m^4}\big[2c_2c_1^2m(2m^2+5m+3) - 2c_3c_1m^2(2m+3) - 2m^2\big(c_2^2(m+2) - 2c_4m\big) - c_1^4(m+1)^3\big]$, etc.

Expression (5) and Taylor Series expansion leads us to

$$\begin{aligned} f(y_\sigma) = &f^{(m)}(\xi)e_\sigma^{2m}\Bigg[\frac{\left(\frac{c_1}{m}\right)^m}{m!} + \frac{(2mc_2 - (m+1)c_1^2)\left(\frac{c_1}{m}\right)^m e_\sigma}{m!c_1} + \left(\frac{c_1}{m}\right)^{1+m}\frac{1}{2m!c_1^3}\{(3+3m \\ &+ 3m^2 + m^3)c_1^4 - 2m(2+3m+2m^2)c_1^2c_2 + 4(m-1)m^2c_2^2 + 6m^2c_1c_3\}e_\sigma^2 \\ &+ \sum_{i=0}^{4}\bar{\theta}_i e_\sigma^{i+3} + O(e_\sigma^8)\Bigg], \end{aligned} \tag{6}$$

where $\bar{\theta}_i = \bar{\theta}_i(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$.

We obtain the following expression from the expressions (3) and (6)

$$\mu = \frac{c_1e_\sigma}{m} + \frac{2mc_2 - (m+2)c_1^2}{m^2}e_\sigma^2 + \sum_{i=0}^{4}\bar{\bar{\theta}}_i e_\sigma^{i+3} + O(e_\sigma^8), \tag{7}$$

which in turn leads us to

$$\nu = \frac{\alpha\mu + 1}{\beta\mu + 1} = 1 + (\alpha - \beta)\sum_{k=1}^{8}\gamma_k e_\sigma^k + O(e_\sigma^9), \tag{8}$$

where $\bar{\bar{\theta}}_i = \bar{\bar{\theta}}_i(\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4)$ and $\gamma_k = \gamma_k(m, \alpha, \beta, c_1, c_2, \ldots, c_8)$, for example $\gamma_1 = \frac{c_1}{m}$, $\gamma_2 = \frac{1}{m^2}\left[2c_2m - c_1^2(\beta + m + 2)\right]$, $\gamma_3 = \frac{1}{2m^3}\left[(2\beta^2 + 8\beta + 2m^2 + (4\beta + 7)m + 7)c_1^3 + 6c_3m^2 - 2c_2c_1m(4\beta + 3m + 7)\right]$, etc.

Next, we set $\nu = 1 + \Omega$. Then, we expand the weight function $H(\nu)$ as:

$$H(\nu) = H(1) + H'(1)\Omega + \frac{1}{2!}H''(1)\Omega^2 + \frac{1}{3!}H'''(1)\Omega^3. \tag{9}$$

Adopting expressions (3)–(9) and the second substep of (1), we obtain

$$w_\sigma - \xi = -\frac{c_1\big(H(1) - m\big)}{m^2}e_\sigma^2 + \sum_{i=0}^{5} A_i e_\sigma^{i+3} + O(e_\sigma^9), \tag{10}$$

where $A_i = A_i(m, c_1, c_2, \ldots, c_8, \alpha, \beta, H(1), H'(1), H''(1), H'''(1))$. For example, the first coefficient is explicitly written as $A_0 = \frac{1}{m^3}\left[2c_2m(m - H(1)) - c_1^2\Big(m^2 + m - H(1)(m+3) + (\alpha - \beta)H'(1)\Big)\right]$ and we can also write other ones in the similar way.

By (10), we deduce at least third-order convergence, provided

$$H(1) = m. \tag{11}$$

By using expression (11) and $A_0 = 0$, we obtain

$$\frac{c_1^2\Big(H'(1)(\beta - \alpha) + 2m\Big)}{m^3} = 0, \tag{12}$$

which further yields to

$$H'(1) = \frac{2m}{\alpha - \beta}, \quad \alpha \neq \beta. \tag{13}$$

Hence, our scheme reaches at fourth-order of optimal convergence.

Next, by using (11) and (13) in (10), we have

$$w_\sigma - \xi = \left[\frac{\Big(m^2 - H''(1)(\alpha - \beta)^2 + (4\beta + 9)m\Big)c_1^3 - 2c_1c_2m^2}{2m^4}\right]e_\sigma^4 + \sum_{i=2}^{5} A_i e_\sigma^{i+3} + O(e_\sigma^9). \tag{14}$$

We obtain the following expression by adopting the Taylor series and (14)

$$\begin{aligned} f(w_\sigma) =& f^{(m)}(\xi)e_\sigma^{4m}\Bigg[\frac{2^{-m}\left(\frac{c_1^3\left(-H''(1)(\alpha-\beta)^2+m^2+(4\beta+9)m\right)-2c_1c_2m^2}{m^4}\right)^m}{m!} \\ &+ \sum_{i=1}^{5}\bar{A}_i e_\sigma^i + O(e_\sigma^6)\Bigg]. \end{aligned} \tag{15}$$

From the expressions (6) and (15), we further have

$$\kappa = \frac{c_1^2\Big(m^2 - H''(1)(\alpha - \beta)^2 + (4\beta + 9)m\Big) - 2c_2m^2}{2m^3}e_\sigma^2 + \sum_{i=1}^{5}\bar{\bar{A}}_i e_\sigma^{i+2} + O(e_\sigma^8). \tag{16}$$

The κ is of order e_σ^2 by (16). Hence, extending $G(\mu)$ about origin (0) up to third-order terms in the following way:

$$G(\mu) = G(0) + G'(0)\mu + \frac{1}{2!}G''(0)\mu^2 + \frac{1}{3!}G'''(0)\mu^3. \tag{17}$$

Inserting (3)–(17) into (1), we obtain

$$\begin{aligned} e_{\sigma+1} = {} & \frac{c_1\big(G(0)-m\big)\Big[c_1^2\Big(m^2 - H''(1)(\alpha-\beta)^2 + (4\beta+9)m\Big) - 2c_2m^2\Big]}{2m^5} e_\sigma^4 \\ & + \sum_{i=1}^{4} L_i e_\sigma^{i+4} + O(e_\sigma^9), \end{aligned} \tag{18}$$

where $L_i = L_i(\alpha, \beta, m, c_1, c_2, \ldots, c_8, H''(1), H'''(1), G'(0), G''(0), G'''(0))$.

Notice, we attain convergence order at least fifth, provided

$$G(0) = m. \tag{19}$$

We have the following expression by choosing $G(0) = m$ and $L_1 = 0$

$$-\frac{c_1^2\big(G'(0)-2m\big)\Big[c_1^2\Big(m^2 - H''(1)(\alpha-\beta)^2 + (4\beta+9)m\Big) - 2c_2m^2\Big]}{2m^6} = 0, \tag{20}$$

which further yield

$$G'(0) = 2m. \tag{21}$$

Again, we yield by inserting the value of $G(0)$ and $G'(0)$ into $L_2 = 0$

$$-\frac{c_1^3\Big[c_1^2\Big(m^2 - H''(1)(\alpha-\beta)^2 + (4\beta+9)m\Big) - 2c_2m^2\Big]\Big(G''(0) - H''(1)(\alpha-\beta)^2 + (4\beta-2)m\Big)}{4m^7} = 0, \tag{22}$$

which further gives

$$G''(0) = H''(1)(\alpha-\beta)^2 + (2-4\beta)m. \tag{23}$$

By using the expressions (19), (21) and (23) with $L_3 = 0$, we get

$$\begin{aligned} & -\frac{c_1^4\Big(c_1^2\big(-H''(1)(\alpha-\beta)^2 + m^2 + (4\beta+9)m\big) - 2c_2m^2\Big)}{12m^8} \\ & \times \Big(G'''(0) + (\alpha-\beta)^2\big(6(\beta-1)H''(1) + H'''(1)(\beta-\alpha)\big) - 12m(\beta^2-2\beta-2)\Big) = 0, \end{aligned} \tag{24}$$

which further provides

$$G'''(0) = (\alpha-\beta)^2\Big(H'''(1)(\alpha-\beta) - 6(\beta-1)H''(1)\Big) + 12m(\beta^2-2\beta-2). \tag{25}$$

The asymptotic error constant term is obtained if we insert (19), (21), (23) and (25) in (18). Then, we have

$$\begin{aligned} e_{\sigma+1} = {} & \frac{c_1\Big(c_1^2\big(m^2 - H''(1)(\alpha-\beta)^2 + (4\beta+9)m\big) - 2c_2m^2\Big)}{24m^9}\Big[c_1^4\Big\{(\alpha-\beta)^2\Big(3(6\beta^2-8\beta \\ & + 15)H''(1) - 2(3\beta-2)(\alpha-\beta)H'''(1)\Big) - m\Big(24\beta^3 - 48\beta^2 + 180\beta + 3H''(1)(\alpha-\beta)^2 \\ & + 433\Big) + 6(2\beta+1)m^2 + 7m^3\Big\} - 6c_2c_1^2m\Big(4m^2 - H''(1)(\alpha-\beta)^2 + (4\beta+2)m\Big) \\ & + 12c_3c_1m^3 + 12c_2^2m^3\Big]e_\sigma^8 + O(e_\sigma^9). \end{aligned} \tag{26}$$

Next, we want to demonstrate that our scheme (1) has optimal eighth-order of convergence. According to Kung–Traub conjecture [2], any iterative method without memory using n functional evaluations has maximum convergence order 2^{n-1}. If any method attains this maximum order of convergence it is known as an optimal method. Hence, our scheme (1) has an optimal convergence

$\Big($for all α, β (provided $\alpha \neq \beta$)$\Big)$ in the sense of Kung–Traub conjecture, since it uses only four functional evaluations $\Big($i.e., $f(x_n), f'(x_n), f(y_n)$, and $f(w_n)\Big)$ and attains maximum convergence order ($2^{4-1} = 8$). $\square$

3. Local Convergence

In order for us to provide the convergence of scheme (1), we first need to simplify it as

$$x_{\sigma+1} = x_\sigma - \mu_\sigma - \gamma \frac{f(x_\sigma)}{f'(x_\sigma)}, \tag{27}$$

where

$$\mu_\sigma = \mu \left[\frac{\gamma - m}{\mu} + H(v) + k\left(\epsilon(\mu) + \frac{mk}{1-4m}\right)\right] \frac{f(x_\sigma)}{f'(x_\sigma)}.$$

Other choices of γ and μ_σ lead to Newton's scheme ($\gamma = 1$, $\mu_\sigma = 0$), modified Newton's scheme ($\gamma = m$, $\mu_\sigma = 0$). That is why we study the convergence of (27) instead of (1) in this section.

The following standard auxiliary results on divided differences help the local convergence analysis of (27), see ([32] Section 2) for the next five lemmas.

Lemma 1. *Consider $\sigma + 1$ distinct arguments $w_0, w_1, \ldots, w_\sigma$ of a function f. Then, the divided differences $f[w_0, \ldots, w_\sigma]$ are*

$$\begin{aligned} f[w_0] &= f(w_0), \\ f[w_0, w_1] &= \frac{f(w_0) - f(w_1)}{w_0 - w_1}, \\ &\vdots \\ f[w_0, w_1, \ldots, w_\sigma] &= \frac{f[w_0, w_1, \ldots, w_{\sigma-1}] - f[w_0, w_1, \ldots, w_\sigma]}{w_0 - w_\sigma}. \end{aligned} \tag{28}$$

Moreover, provided say that f is σ-th differentiable, we have

$$f[w_0, w_1, \ldots\ldots, w_\sigma] = \frac{f^{(\sigma)}(w_0)}{\sigma!}, \tag{29}$$

although some w_i may be coincide.

Furthermore, $f[w_0, \ldots, w_\sigma]$ are symmetric with respect to $w_0, \ldots, w_\sigma$.

Lemma 2. *Let α be a zero with multiplicity m, and f has $(n+1)$-th derivative. Then,*

$$f(x) = f[w_0] + \sum_{i=1}^{\sigma} f[w_0, w_1, \ldots, w_\sigma] \prod_{j=0}^{i-1} (x - w_j) + f[w_0, w_1, \ldots, w_\sigma, x] \prod_{i=0}^{\sigma} (x - w_i), \tag{30}$$

holds for all x.

Lemma 3. *Assume the function f has $(m+1)$-th derivative and α is a zero with multiplicity m. Then,*

$$f(x) = f[\alpha, \overset{m-times}{\ldots\ldots}, \alpha, x](x - \alpha)^m \tag{31}$$

and

$$f'(x) = f[\alpha, \overset{m-times}{\ldots\ldots}, \alpha, x, x](x-\alpha)^m + m f[\alpha, \overset{m-times}{\ldots\ldots}, \alpha, x](x-\alpha)^{m-1}. \tag{32}$$

The next result is due to Genocchi.

Lemma 4. *Assume f has m-th derivative continuous, then*

$$f[w_0, w_1, \ldots, w_\sigma] = \int_0^1 \cdots \int_0^1 f^{(\sigma)}\Big(w_0 + \sum_{i=1}^{\sigma}(w_i - w_{i-1})\prod_{j=1}^{i}\tau_i\Big)\prod_{i=1}^{\sigma}(\tau_i^{n-i}d\tau_i). \tag{33}$$

Taylor's representation follows.

Lemma 5. *Assume f is σ-times differentiable on $S(w_0, \varrho)$, $\varrho > 0$, and $f^{(\sigma)}$ is integrable from ξ to $x \in S(\xi, \varrho)$. Then, we yield*

$$\begin{aligned} f(x) &= f(\xi) + f'(\xi)(x-\xi) + \frac{1}{2}f''(\xi)(x-\xi)^2 + \cdots + \frac{1}{\sigma!}f^{(\sigma)}(\xi)(x-\xi)^\sigma \\ &+ \frac{1}{(\sigma-1)!}\int_0^1 \Big(f^{(\sigma)}(\xi + \tau(x-\varrho)) - f^{(\sigma)}(\xi)\Big)(x-\xi)^\sigma(1-\tau)^{\sigma-1}d\tau, \end{aligned} \tag{34}$$

$$\begin{aligned} f'(x) &= f'(\xi) + f''(\xi)(x-\xi) + \frac{1}{2}f'''(\xi)(x-\xi)^2 + \cdots + \frac{1}{(\sigma-1)!}f^{(\sigma)}(\xi)(x-\xi)^{\sigma-1} \\ &+ \frac{1}{(n-2)!}\int_0^1 \Big(f^{(\sigma)}(\xi + \tau(x-\xi)) - f^{(\sigma)}(\xi)\Big)(x-\xi)^{\sigma-1}(1-\tau)^{\sigma-2}d\tau, \end{aligned} \tag{35}$$

hold.

Set $A = [0, \infty)$, $B = (-\infty, \infty)$. Consider $\Psi_0 : A \to B$ to be non-decreasing, and continuous function with $\Psi_0(0) = 0$. Consider also functions $b_0, b : A \to B$ as

$$b_0(t) = (m-1)!(m-1)\int_0^1 \cdots \int_0^1 \Psi_0\Big(t\prod_{i=1}^{m}\tau_i\Big)\prod_{i=1}^{m}\tau_i^{m-i}d\tau_i,$$

$$b(t) = (m-1)!\int_0^1 \cdots \int_0^1 \Psi_0\Big(t\prod_{i=1}^{m-1}\tau_m\Big)\prod_{i=1}^{m-1}\tau_i^{m-i}d\tau_i + b_0(t).$$

Clearly b_0, b are non-decreasing, continuous with $b_0(0) = b(0) = 0$. Assume

$$b(t) \to \text{a positive real or } \infty \ \text{ as } \ t \to \infty. \tag{36}$$

Then $b(t) = 1$ has a minimal zero in $(0, \infty)$, say ϱ_0. Let $\lambda_1(t) = 1 - b(t)$. Consider $\Psi : [0, \varrho_0) \to A$ to be non-decreasing, continuous with $\Psi(0) = 0$. Define functions a, λ_0 and λ on $[0, \varrho_0)$ as

$$a(t) = (m-1)!\int_0^1 \cdots \int_0^1 \Psi\Big(t\prod_{i=1}^{m-1}\tau_i(1-\tau_m)\Big)\prod_{i=1}^{m}\tau_i^{m-i}d\tau_i d\tau_m,$$

$$\lambda_0(t) = m^{-1}a(t)t + b_0(t) + m^{-1}a(t)ct^{c_0} + b_0(t)ct^{c_0-1},$$

$$\lambda(t) = \frac{\lambda_0(t)}{\lambda_1(t)} - 1 \ \text{ for } \ c \geq 0, \ \text{ and } \ c_0 \geq 1.$$

By these definitions $\lambda(t) = -1$ and $\lambda(t) \to \infty$ with $t \to \varrho_0^-$. Then, let ϱ be the minimal zero of $\lambda(t) = 0$ in $(0, \varrho_0)$. We get

$$0 \leq b(t) < 1 \tag{37}$$

and

$$0 \leq \lambda(t) < 1 \tag{38}$$

for all $t \in [0, \varrho)$.

The conditions (H) shall be used:

(H_1) $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable m-times.*
(H_2) *f has a zero* α *with known multiplicity m.*
(H_3) $\Psi_0 : A \to B$ *is non-decreasing, continuous and* $\Psi_0(0) = 0$ *so that each* $x \in \Omega$ *satisfies*

$$\left\| f^{(m)}(\alpha)^{-1}\left(f^{(m)}(\alpha) - f^{(m)}(x)\right)\right\| \leq \Psi_0(\|\alpha - x\|).$$

Consider $\Omega_0 = \Omega \cap S(\alpha, \varrho_0)$ *with* ϱ_0 *given earlier.*
(H_4) $\Psi : [0, \varrho_0) \to B$ *is non-decreasing, continuous,* $\Psi(0) = 0$ *and for each* $x, y \in \Omega$ *satisfying*

$$\left\| f^{(m)}(\alpha)^{-1}\left(f^{(m)}(y) - f^{(m)}(x)\right)\right\| \leq \Psi(\|y - x\|).$$

(H_5) *Implication (36) holds.*
(H_6) $\bar{S}(\alpha, \varrho) \subseteq \Omega$
(H_7) $\|\mu_\sigma\| \leq c\|x_\sigma - \alpha\|^{c_0}$.

Theorem 2. *Assume conditions* (H), *and choose* $x_0 \in S(\alpha, \varrho) - \{\alpha\}$. *Then, sequence* $\{x_\sigma\} \subseteq S(\alpha, \varrho)$ *for all* $n \geq 0$, *and* $\lim_{\sigma \to \infty} x_\sigma = \alpha$.

Proof. We shall show that sequence

$$\delta_\sigma = x_\sigma - \alpha \tag{39}$$

is non-increasing and converges to zero. Using $\delta_\sigma = x_\sigma - \alpha$, scheme (1) for $\sigma = 0$, Lemma 3 and the following formulas:

$$h(x) = f[\alpha, \alpha, \overset{m-times}{\ldots\ldots}, \alpha, x], \;\; h_0(x) = f[\alpha, \alpha, \overset{m-times}{\ldots\ldots}\, \alpha, x, x], \tag{40}$$

$$f(x_0) = h(x_0)\delta_0^m, \tag{41}$$

and

$$f'(x_0) = [h_0(x_0)\delta_0 + mh(x_0)]\delta_0^{m-1}. \tag{42}$$

We can write

$$\delta_1 = \frac{h(\alpha)^{-1}N}{h(\alpha)^{-1}D}, \tag{43}$$

where

$$N = h_0(x_0)\delta^2 + [|m - \gamma|h(x_0) - h_0(x_0)\mu_0]\delta_0 - mh(x_0)\mu_0$$

and

$$D = h_0(x_0)\delta_0 + mh(x_0). \tag{44}$$

In view of the definition of divided differences, we have

$$h_0(x_0)\delta_0 = f[\alpha, \alpha, \overset{(m-1)-times}{\ldots\ldots}, \alpha, x_0, x_0] - h(x_0). \tag{45}$$

Then, we obtain from (29) and (45) that

$$\begin{aligned}
&\left\| 1 - (mh(\alpha))^{-1}[h_0(x_0)\delta_0 + mh(x_0)] \right\| \\
&= \left\| (mh(\alpha))^{-1}[h_0(x_0)\delta_0 + mh(x_0) - mg(\alpha)] \right\| \\
&= (m-1)\left\| f^{(m)}(\alpha)^{-1}\left(f[\alpha, \alpha, \overset{(m-1)-times}{\ldots\ldots}, \alpha, x_0, x_0] - h(\alpha) + (m-1)[h(x_0) - h(\alpha)]\right)\right\|.
\end{aligned} \tag{46}$$

We have, by Lemma 3

$$f[\alpha,\alpha,\overset{(m-1)-times}{\ldots},\alpha,x_0,x_0] = \int_0^1\cdots\int_0^1 f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i\Big)\prod_{i=1}^{m}(\tau_i^{m-1}d\tau_i), \tag{47}$$

$$h(x_0) = \int_0^1\cdots\int_0^1 f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i\Big)\prod_{i=1}^{m}(\tau_i^{m-1}d\tau_i), \tag{48}$$

$$h(\alpha) = \int_0^1\cdots\int_0^1 f^{(m)}(\alpha)\prod_{i=1}^{m}(\tau_i^{m-1}d\tau_i). \tag{49}$$

Substituting (46)–(49) using condition (H_3), $x_0\in S(\alpha,\varrho)$, and the definition of ϱ, we get

$$\begin{aligned}
&\|1-(mh(\alpha))^{-1}[h_0(x_0)\delta_0+mh(x_0)]\|\\
&=(m-1)!\Big\|\int_0^1\cdots\int_0^1 f^{(m)}(\alpha)^{-1}(f^{(m)}(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i)-f^{(m)}(\alpha))\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\\
&+(m-1)f^{(m)}(\alpha)^{-1}(f^{(m)}(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i)-f^{(m)}(\alpha))\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\Big\|\\
&\le(m-1)!\Big(\int_0^1\cdots\int_0^1\|f^{(m)}(\alpha)^{-1}(f^{(m)}(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i)-f^{(m)}(\alpha))\|\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\\
&+(m-1)\int_0^1\cdots\int_0^1\|f^{(m)}(\alpha)^{-1}(f^{(m)}(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i)-f^{(m)}(\alpha))\|\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\\
&\le(m-1)!\Big(\int_0^1\cdots\int_0^1\Psi(\|\delta_0\|\prod_{i=1}^{m-1}\tau_i)\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\\
&+(m-1)\int_0^1\cdots\int_0^1\Psi(\|\delta_0\|\prod_{i=1}^{m-1}\tau_i)\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\Big)\\
&\le b(\|\delta_0\|)<b(\varrho)<1.
\end{aligned} \tag{50}$$

By a Banach result [33] and (50) then $h_0(x_0)\delta_0+mh(x_0)\neq 0$ and

$$\|(mh(\alpha)^{-1}h_0(x_0)\delta_0+mh(x_0))^{-1}\|\le\frac{1}{1-\beta(\|\delta_0\|)}<\frac{1}{1-\beta(\varrho)}. \tag{51}$$

Moreover, using (45), (47), (48) and (H_4), we have in turn that

$$\begin{aligned}
\Big\|(mh(\alpha))^{-1}h_0(x_0)\delta_0\Big\| &= (m-1)!\Big\|\int_0^1\cdots\int_0^1 f^{(m)}(\alpha)^{-1}\Big[f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m}\tau_i\Big)\\
&\quad-f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m}\tau_i\Big)\Big]\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\Big\|\\
&=(m-1)!\Big\|\int_0^1\cdots\int_0^1\Big\|f^{(m)}(\alpha)^{-1}\Big[f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m-1}\tau_i\Big)\\
&\quad-f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m}\tau_i\Big)\Big]\Big\|\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_i)\Big\|\\
&\le(m-1)!\int_0^1\cdots\int_0^1\Psi_0\Big(\|\delta_0\|\prod_{i=1}^{m-1}\tau_i(1-\tau_i)\Big)\prod_{i=1}^{m}(\tau_i^{m-i}d\tau_id\tau_m)\\
&=a(\|\delta_0\|)<a(\varrho)<1.
\end{aligned} \tag{52}$$

Furthermore, we have

$$
\begin{aligned}
\left\|h(\alpha)^{-1}h(x_0)\right\| &= \left\|h(\alpha)^{-1}\Big(h(x_0)-g(\alpha)\Big)\right\| \\
&= \left\|(m-1)!f^{(m)}(\alpha)^{-1}(m-1)\Big(h(x_0)-h(\alpha)\Big)\right\| \\
&= (m-1)(m-1)!\int_0^1\cdots\int_0^1 \left\|f^{(m)}(\alpha)^{-1}\Big[f^{(m)}\Big(\alpha+\delta_0\prod_{i=1}^{m}\tau_i\Big)-f^{(m)}(\alpha)\Big]\right\|\prod_{i=1}^{m}\tau_i^{m-i}d\tau_i \\
&\leq (m-1)(m-1)!\int_0^1\cdots\int_0^1 \Psi_0\left(|\delta_0|\prod_{i=1}^{m-1}\tau_i\right)\prod_{i=1}^{m}\left(\tau_i^{m-i}d\tau_i\right). \qquad (53)
\end{aligned}
$$

Using (50)–(53), we obtain that

$$\|\delta_1\| \leq d\|\delta_0\| < \|\delta_0\| < \varrho, \tag{54}$$

where $d=\lambda(|\delta_0|)\in[0,1)$, so $x_1\in S(\alpha,\varrho)$. By simply replacing x_0, x_1 by $x_\sigma, x_{\sigma+1}$, we get

$$\|x_{\sigma+1}-\alpha\| \leq d\|x_\sigma-\alpha\| < \varrho, \tag{55}$$

so $\lim_{n\to\infty} x_\sigma = \alpha$ and $x_{\sigma+1}\in S(\alpha,\varrho)$. $\square$

Concerning the uniqueness of the solution α, we have

Proposition 1. *Suppose that conditions (H) and*

$$\frac{m}{(s_2-s_1)^m}\int_{s_1}^{s_2}\Psi_0\left(\|t-s_1\|\right)\|s_2-t\|^{m-1}dt < 1 \tag{56}$$

for all s_1, t, s_2 with $0\leq s_1\leq t\leq s_2\leq\bar{\varrho}$ for some $\bar{\varrho}\geq\varrho$ hold. Then, the zero α is unique in $\Omega_1=\Omega\bigcup\bar{S}(\alpha,\bar{\varrho})$.

Proof. Assume that $\alpha^*\in\Omega_0$ solves equation $f(x)=0$ with $\alpha\neq\alpha^*$. Without loss of generality, assume $\alpha<\alpha^*$. We have

$$f(\alpha^*)-f(\alpha)=\frac{1}{(m-1)!}\int_\alpha^{\alpha^*}f^{(m)}(t)(\alpha^*-t)^{m-1}dt. \tag{57}$$

Using (H_3) and (55), we get in turn that

$$
\begin{aligned}
&\left\|1-\left(\frac{(\alpha^*-\alpha)^m}{m}f^{(m)}(\alpha)\right)^{-1}\int_\alpha^{\alpha^*}f^{(m)}(t)(\alpha^*-t)^{m-1}dt\right\| \\
=&\left\|\left(\frac{(\alpha^*-\alpha)^m}{m}f^{(m)}(\alpha)\right)^{-1}\int_\alpha^{\alpha^*}\left[f^{(m)}(t)-f^{(m)(\alpha)}\right](\alpha^*-t)^{m-1}dt\right\| \\
\leq&\frac{m}{(\alpha^*-\alpha)^m}\int_\alpha^{\alpha^*}\Psi_0(\|t-\alpha\|)\|\alpha^*-t\|^{m-1}dt<1,
\end{aligned}
$$

so $\left(\frac{(\alpha^*-\alpha)^m}{m}f^{(m)}(\alpha)\right)^{-1}\int_\alpha^{\alpha^*}f^{(m)}(t)(\alpha^*-t)^{m-1}dt$ is invertible, i.e., $\int_\alpha^{\alpha^*}f^{(m)}(t)(\alpha^*-t)^{m-1}dt$ exists. $\square$

4. Numerical Examples

Two numerical experiments demonstrate the local convergence results are given below:

Example 1. *Consider $\Omega=\left[\frac{1}{2},\frac{3}{2}\right]$ and a function f [32] on Ω, is given as*

$$f(x)=(x^{\frac{5}{2}}-1)^2. \tag{58}$$

We consider the case $\alpha = 1$, *and* $m = 2$. *We obtain by using (58)*

$$\begin{aligned} f(\alpha) &= 0, \\ f'(x) &= 5x^4 - 5x^{\frac{3}{2}}, \\ f'(\alpha) &= 0, \\ f''(x) &= 20x^3 - \frac{15}{2}x^{\frac{1}{2}} \end{aligned} \tag{59}$$

and

$$f''(\alpha) = \frac{25}{2}. \tag{60}$$

We are looking for L *so that* $\Psi(\|x-y\|) = L\|x-y\|$. *By (58), we get*

$$\begin{aligned} \|f''(x) - f''(y)\| &= \left\|20x^3 - \frac{15}{2}\sqrt{x} - 20y^3 + \frac{15}{2}\sqrt{y}\right\| \\ &\leq 20\left\|(x-y)(x^2+xy+y^2)\right\| + \frac{15}{2}\|\sqrt{x} - \sqrt{y}\| \\ &\leq \left(20\left\|(x^2+xy+y^2)\right\| + \frac{15}{2}\frac{1}{\|\sqrt{x}+\sqrt{y}\|}\right)\|x-y\|. \end{aligned} \tag{61}$$

We obtain for each $x, y \in \Omega$

$$\begin{aligned} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \leq \sqrt{x} + \sqrt{y} &\leq \frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} \\ \sqrt{2} \leq \sqrt{x} + \sqrt{y} &\leq \sqrt{6} \\ \frac{1}{\sqrt{6}} \leq \frac{1}{\sqrt{x}+\sqrt{y}} &\leq \frac{1}{\sqrt{2}} \end{aligned} \tag{62}$$

and

$$\left\|x^2 + xy + y^2\right\| \leq \frac{27}{4}. \tag{63}$$

By using (62) and (63) in (61), we get

$$\|f''(x) - f''(y)\| \leq \left(135 + \frac{15}{2\sqrt{2}}\right)\|x-y\|. \tag{64}$$

We obtain by adopting (60)–(64) in (H_3)

$$\left\|f''(\alpha)^{-1}\Big(f''(x) - f''(y)\Big)\right\| \leq \frac{2}{25}\left(135 + \frac{15}{2\sqrt{2}}\right)\|x-y\| \leq L\|x-y\|, \tag{65}$$

where $L = 11.224264$ *and*

$$\left\|f''(\alpha)^{-1}\Big(f''(x) - f''(y)\Big)\right\| \leq L\|x-y\|, \quad \forall \ x, y \in \Omega. \tag{66}$$

Similarly, we find an upper bound in the form of $\Psi_0(\|x-y\|) \leq L_0\|x-y\|$ *for* $\|f''(x)-f''(\alpha)\|$. *In view of (58), we have*

$$\begin{aligned}\|f''(x)-f''(\alpha)\| &= \left\|20x^3 - \frac{15}{2}\sqrt{x} - 20\alpha^3 + \frac{15}{2}\sqrt{\alpha}\right\| \\ &\leq 20\left\|(x-\alpha)(x^2+x\alpha+\alpha^2)\right\| + \frac{15}{2}\left\|\sqrt{x}-\sqrt{\alpha}\right\| \\ &\leq \left(20\left\|x^2+x\alpha+\alpha^2\right\| + \frac{15}{2}\frac{1}{\|\sqrt{x}+\sqrt{\alpha}\|}\right)\|x-\alpha\|. \end{aligned} \tag{67}$$

Then, we get for all $x \in \Omega$

$$\begin{aligned} 1+\frac{1}{\sqrt{5}} &\leq \sqrt{x}+\sqrt{\alpha} \leq \frac{\sqrt{3}}{\sqrt{2}}+1 \\ \frac{\sqrt{2}+1}{2} &\leq \sqrt{x}+\sqrt{\alpha} \leq \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}} &\leq \frac{1}{\sqrt{x}+\sqrt{\alpha}} \leq \frac{\sqrt{2}}{\sqrt{2}+1} \end{aligned} \tag{68}$$

and

$$||x^2+x\alpha+\alpha^2\| \leq \frac{19}{4}. \tag{69}$$

Furthermore, we obtain by using (68) and (69) in (67),

$$\|f''(x)-f''(\alpha)\| \leq \left(95+\frac{15}{\sqrt{2}(\sqrt{5}+1)}\right)\|x-\alpha\|. \tag{70}$$

We have by using (60) and (68) in (H_3)

$$\left\|f''(\alpha)^{-1}\Big(f''(x)-f''(\alpha)\Big)\right\| \leq \frac{2}{25}\left(95+\frac{15}{\sqrt{2}(\sqrt{5}+1)}\right)\|x-\alpha\| \leq L_0\|x-\alpha\|, \tag{71}$$

where $L_0 = 7.951471$ *and*

$$\left\|f''(\alpha)^{-1}\Big(f''(x)-f''(\alpha)\Big)\right\| \leq L_0\|x-y\|, \quad \forall\ x \in \Omega. \tag{72}$$

Therefore, we get $b_0(t) = \frac{7}{12}L_0 t$, $b(t) = \frac{13}{12}L_0 t$, $\lambda_1(t) = 1-b(t) = 1-\frac{13}{12}L_0 t$, $a(t) = \frac{1}{6}Lt$. *For* $C_0 = C = 1$, *we obtain*

$$\lambda_0(t) = \frac{1}{6}Lt^2 + \frac{7}{6}L_0 t$$

and

$$\lambda(t) = \frac{2(Lt^2+7L_0 t)}{12-13L_0 t} - 1 = 0.$$

The values of parameters are

$$\varrho_0 = 0.116089 \quad \textit{and} \quad \varrho = 0.0555717.$$

Example 2. *Consider function* f *on* $\Omega = \mathbb{R}$ *as follows:*

$$f(x) = \int_0^x G(x)dx, \tag{73}$$

with

$$G(x) = \int_0^x \left(1 + x\sin\frac{\pi}{x}\right)dx. \tag{74}$$

We show $\alpha = 0$ is a zero of f with $m = 2$. By (73) and (74), we have $f(\alpha) = 0$, $f'(x) = G(x)$,

$$f''(x) = \begin{cases} 1 + x \sin \frac{\pi}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Hence, we get $f'(\alpha) = 0$ and $f''(\alpha) = 1$. Hence, we conclude $m = 2$. For all $x, y \in \Omega$, we can obtain that

$$\left\| f''(\alpha)^{-1} \Big(f''(x) - f''(\alpha) \Big) \right\| = \left\| x \sin \frac{\pi}{x} \right\| \leq \|x - \alpha\| \tag{75}$$

and

$$\left\| f''(\alpha)^{-1} \Big(f''(x) - f''(y) \Big) \right\| = \left\| x \sin \frac{\pi}{x} - y \sin \frac{\pi}{y} \right\| \leq \|x - y\|. \tag{76}$$

Then, we have that $\Psi_0(t) = L_0 t$, $\Psi(t) = Lt$, where $L_0 = L = 1$. The values of parameters are

$$\varrho_0 = 0.923077 \quad \textit{and} \quad \varrho = 0.430703.$$

Some Special Studies

Next, we specialize functions H and G. The resulting choices satisfy the conditions of Theorem 1. The parameters α and β are arbitrary but $\alpha \neq \beta$.

5. Numerical Experimentation

We specialize α and β to conduct specific numerical calculations. More precisely, we use case-1 for $\left(\alpha = \frac{1}{2},\ \beta = -\frac{3}{2}\right)$, case-2 for $(\alpha = 0,\ \beta = -2)$ and case-7 for $(\alpha = 0,\ \beta = -2)$ in scheme (1), known by $PM1$, $PM2$ and $PM3$, respectively. We choose four real life problems having multiple and simple zeros and two standard academic problems with multiple zeros and can be found in examples (3)–(8).

We consider several existing schemes of order six and eight (optimal). Firstly, we compare our schemes with a sixth-order iteration functions given by Geum et al. [23], in particular, choose 5YD, defined as

$$\begin{aligned} y_\sigma &= x_\sigma - m \frac{f(x_\sigma)}{f'(x_\sigma)}, \ m \geq 1, \\ w_\sigma &= x_\sigma - m \left[\frac{(u_\sigma - 2)\,(2u_\sigma - 1)}{(u_\sigma - 1)\,(5u_\sigma - 2)} \right] \frac{f(x_\sigma)}{f'(x_\sigma)}, \\ x_{\sigma+1} &= x_\sigma - m \left[\frac{(u_\sigma - 2)\,(2u_\sigma - 1)}{(5u_\sigma - 2)\,(u_\sigma + v_\sigma - 1)} \right] \frac{f(x_\sigma)}{f'(x_\sigma)}, \end{aligned} \tag{77}$$

where $u_\sigma = \left(\frac{f(y_\sigma)}{f(x_\sigma)} \right)^{\frac{1}{m}}$ and $v_\sigma = \left(\frac{f(w_\sigma)}{f(x_\sigma)} \right)^{\frac{1}{m}}$. We denote this scheme by (GM) for computational work.

In addition, we demonstrate the same with an optimal eighth-order iteration function developed by Behl et al. [26], which is given by

$$\begin{aligned} y_\sigma &= x_\sigma - m \frac{f(x_\sigma)}{f'(x_\sigma)}, \\ w_\sigma &= y_\sigma - m u_\sigma \frac{f'(x_\sigma)}{f'(x_\sigma)} \left[\frac{1 + \beta u_\sigma}{(\beta - 2) u_\sigma + 1} \right], \\ x_{\sigma+1} &= w_\sigma - u_\sigma v_\sigma \frac{f(x_\sigma)}{f'(x_\sigma)} \left[\frac{1}{2} m \big\{ (2v_\sigma + 1)\big(4(\beta^2 - 6\beta + 6) u_\sigma^3 + (10 - 4\beta) u_\sigma^2 + 4u_\sigma + 1\big) + 1 \big\} \right] \end{aligned} \tag{78}$$

where $u_\sigma = \left(\frac{f(y_\sigma)}{f(x_\sigma)} \right)^{\frac{1}{m}}$ and $v_\sigma = \left(\frac{f(w_\sigma)}{f(y_\sigma)} \right)^{\frac{1}{m}}$. We shall call this scheme (BM). This (BM) scheme is called by (78) in [26] and claimed to be the best scheme among all other family members.

Moreover, a comparison is given with optimal eighth-order iterative schemes constructed in [27]. Consider the specializations

$$\begin{aligned} y_\sigma &= x_\sigma - m\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ w_\sigma &= y_\sigma - m u_\sigma (6u_\sigma^3 - u_\sigma^2 + 2u_\sigma + 1)\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ x_{\sigma+1} &= w_\sigma - m u_\sigma v_\sigma (1+2u_\sigma)(1+v_\sigma)\left(\frac{2w_\sigma + 1}{A_2 P_0}\right)\frac{f(x_\sigma)}{f'(x_\sigma)} \end{aligned} \tag{79}$$

and

$$\begin{aligned} y_\sigma &= x_\sigma - m\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ w_\sigma &= y_\sigma - m u_\sigma \left(\frac{1 - 5u_\sigma^2 + 8u_\sigma^3}{1 - 2u_\sigma}\right)\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ x_{\sigma+1} &= w_\sigma - m u_\sigma v_\sigma (1+2u_\sigma)(1+v_\sigma)\left(\frac{3w_\sigma + 1}{A_2 P_0 (1 + w_\sigma)}\right)\frac{f(x_\sigma)}{f'(x_\sigma)}, \end{aligned} \tag{80}$$

with $u_\sigma = \left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)^{\frac{1}{m}}$, $v_\sigma = \left(\frac{f(w_\sigma)}{f(y_\sigma)}\right)^{\frac{1}{m}}$, $w_\sigma = \left(\frac{f(w_\sigma)}{f(x_\sigma)}\right)^{\frac{1}{m}}$, with $A_2 = P_0 = 1$. Both the schemes (79) and (80) are standing as $(FM1)$ and $(FM2)$, respectively.

Consider in contrast with another family of eighth-order schemes presented by Behl et al. [24]. We choose the following expression

$$\begin{aligned} y_\sigma &= x_\sigma - m\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ w_\sigma &= x_\sigma - m u_\sigma \left(1 + 2u_\sigma\right)\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ x_{\sigma+1} &= w_\sigma - \frac{u_\sigma w_\sigma}{1 - w_\sigma}\left(\frac{m\left(u_\sigma\left(8v_\sigma + 6\right) + 9u_\sigma^2 + 2v_\sigma + 1\right)}{4u_\sigma + 1}\right)\frac{f(x_\sigma)}{f'(x_\sigma)} \end{aligned} \tag{81}$$

and

$$\begin{aligned} y_\sigma &= x_\sigma - m\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ w_\sigma &= y_\sigma - m u_\sigma \left(1 + 2u_\sigma\right)\frac{f(x_\sigma)}{f'(x_\sigma)}, \\ x_{\sigma+1} &= w_\sigma - \frac{u_\sigma w_\sigma}{1 - w_\sigma}\left(4u_4^3 - u_4^2 - 2u_4 - 2v_4 - 1\right)\frac{f(x_\sigma)}{f'(x_\sigma)}, \end{aligned} \tag{82}$$

where $u_\sigma = \left(\frac{f(y_\sigma)}{f(x_\sigma)}\right)^{\frac{1}{m}}$, $v_\sigma = \left(\frac{f(w_\sigma)}{f(y_\sigma)}\right)^{\frac{1}{m}}$, $w_\sigma = \left(\frac{f(w_\sigma)}{f(x_\sigma)}\right)^{\frac{1}{m}}$. We denote these schemes (81) and (82) by $(RM1)$ and $(RM2)$, respectively.

In Tables 2 and 3, we report our findings using many significant digits (minimum 5000 significant digits) in order to minimize the errors. Due to the limited paper space, we depicted the value up to specific number of significant digits. We adopted *Mathematica* 11 with multiple precision arithmetic for calculating the required values. In the Tables 2 and 3, $a(\pm b)$ stands for $a \times 10^{(\pm b)}$.

Example 3. ***Chemical reactor with fractional conversion***

We assume the expression (see [34]), given by

$$f_1(x) = \frac{x}{1-x} - 5\log\left(\frac{0.4(1-x)}{0.4 - 0.5x}\right) + 4.45977, \tag{83}$$

Here, x serve as a fractional conversion of particular species B in the chemical reactor. If we yield either $x < 0$ or $x > 1$ then these values have no physical description. Therefore, x is bounded in $[0,1]$ and our needed zero of (83) is $\xi = 0.757396246253753879459641 3$. In addition, the function f_1 is not defined for $x \in [0.8,1]$ that is very near to the required zero. Moreover, some other properties that related to f_1 are discussed in details in [34] that make the solution more tough. We have to be very careful while choosing the initial approximation for this function because the derivative tends to zero for $x \in [0,0.5]$ and an infeasible zero for $x = 1.098$. Keeping all these problems in our mind, we assume $x_0 = 0.76$ as the starting point for f_1.

On the basis of obtained results in Tables 2 and 3, we conclude that our scheme $(PM2)$ has the minimum error difference between two iterations and residual error among all the other mentioned schemes in the case of Example 3.

Example 4. ***Continuous stirred tank reactor (CSTR)***

Here, we assume an isothermal continuous stirred tank reactor (CSTR) problem. Let us consider that components M_1 and M_2 stand for feed rates to the reactors A_1 and $A_2 - A_1$, respectively. Then, we obtain the following reaction scheme in the reactor (for more details see [35]):

$$\begin{aligned} M_1 + M_2 &\to B_1 \\ B_1 + M_2 &\to C_1 \\ C_1 + M_2 &\to D_1 \\ C_1 + M_2 &\to E_1 \end{aligned}$$

Douglas [36] studied the above model, when he was designing a simple model for feedback control systems. He converted the above model in to the following mathematical expression:

$$R_{C_1} \frac{2.98(x+2.25)}{(x+1.45)(x+2.85)^2(x+4.35)} = -1, \tag{84}$$

with R_{C_1} as the gain of proportional controller. The expression (84) is balanced for the negative real values of values of R_{C_1}. In particular, by choosing $R_{C_1} = 0$, we yield

$$f_2(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{85}$$

the zeros of function f_2 are known as the poles of the open-loop transfer function. The function f_2 has 4 zeros $\xi = -1.45, -2.85, -2.85, -4.35$. But, our desired ones is $\xi = -2.85$ with multiplicity $m = 2$. We assume $x_0 = -2.7$ as the starting point for f_2.

The results obtained from Tables 2 and 3 conclude that all the schemes behave similarly to each other in terms of the difference between two iterations, residual error and computational order of convergence in the Example 4.

Example 5. ***Van der Waals equation of state***

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT$$

describes the nature of a real gas comprising two gases, namely α_1 and α_2, when we introduce the ideal gas equations. For calculating the volume V of gases, we need the solution of the preceding expression in terms of the remaining constants

$$PV^3 - (na_2 P + nRT)V^2 + \alpha_1 n^2 V - \alpha_1 \alpha_2 n^2 = 0.$$

For choosing the particular values of gases α_1 and α_2, we can easily obtain the values for n, P and T. Then, we yield

$$f_3(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675.$$

The function f_3 has 3 zeros and among them $\xi = 1.75$ is a multiple zero of multiplicity and $m = 2$ and $\xi = 1.72$ is a simple zero. We choose the starting guess $x_0 = 0.76$ for the required zero $\xi = 1.75$ in f_3.

We conclude on the basis of obtained results in Tables 2 and 3 that our scheme's $PM1$ and $PM2$ have the minimum error difference between two iterations and residual error among all the other mentioned schemes for the Example 5.

Example 6. ***Multi-factor problem***

The unwanted RF disruption that occurs in high power microwave equipment which is working under vacuum conditions is called multi-factor [37]. For instance, multi-factor can be found inside the parallel plate wave guide. An electric field exists with an electric potential difference that originates from the movement of electrons between two sheets or plates. We can find an interesting case when we are studying trajectories of the electrons that reach the plate having a zero of multiplicity $m = 2$. The mathematical formation of the trajectory of an electron between two parallel sheets that have some air gap is given by

$$y(t) \quad = \quad y_0 + \left[v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right](t - t_0) + e\frac{E_0}{m\omega^2}\Big(\cos(\omega t + \alpha) - \cos(\omega t_0 + \alpha)\Big) \tag{86}$$

where m and e are the mass and charge of the electron at rest, $E_0 \sin(\omega t + \alpha)$ is the RF electric field between plates and y_0 and v_0 are the position and velocity of the electron at time t_0. By choosing some particular values in (86), we have:

$$f_4(x) = x + \cos(x) - \frac{\pi}{2} \tag{87}$$

with the zero $\xi = \frac{\pi}{2}$ of multiplicity 3. For the function f_4, we assume the initial guess as $x_0 = 1.6$.

On the basis of the results obtained in Tables 2 and 3, we conclude that our methods $RM1$ and $RM2$ have the minimum error difference between two iterations and residual error among all the other mentioned schemes in the case of Example 6.

Example 7. *Now, we study a polynomial equation [3], describes as follows:*

$$f_5(x) = ((x-1)^3 - 1)^{100}. \tag{88}$$

Function f_5 having $\xi = 2$ a multiple zero of multiplicity $m = 100$. We choose the starting point $x_0 = 2.1$ for f_5.

From Tables 2 and 3, we deduced that the minimum error difference between two iterations and residual errors among all the other mentioned schemes belongs to our scheme $PM3$ in the case of Example 7.

Example 8. *Finally, we introduced the function*

$$f_6(x) = \left(1 - \sqrt{1 - x^2} + x + \cos\left(\frac{\pi x}{2}\right)\right)^3. \tag{89}$$

Function f_6 having a multiple zero $\xi = -0.7285840464448267167123331$ of multiplicity $m = 3$. We assume $x_0 = -0.6$ as starting guess for f_6.

We conclude on the basis of obtained results in Tables 2 and 3 that our scheme $PM2$ has the minimum error difference between two iterations and residual error among all the other mentioned schemes in the case of Example 7.

Table 2. Errors between iterations ($|x_{\sigma+1} - x_{\sigma}|$) among different iteration functions.

$f(x)$	σ	GM	BM	$FM1$	$FM2$	$RM1$	$RM2$	$PM1$	$PM2$	$PM3$
$f_1(x)$	1	1.8 (−10)	5.1 (−12)	5.1 (−11)	7.7 (−11)	8.0 (−12)	1.4 (−11)	9.4 (−13)	1.3 (−14)	8.4 (−13)
	2	1.7 (−53)	1.2 (−81)	1.6 (−72)	5.9 (−71)	1.4 (−79)	9.4 (−78)	5.8 (−88)	4.3 (−105)	7.8 (−89)
	3	1.3 (−311)	1.5 (−638)	1.5 (−564)	7.3 (−552)	1.2 (−621)	4.7 (−607)	1.3 (−689)	7.4 (−829)	4.0 (−697)
	ρ	6.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
$f_2(x)$	1	9.5 (−3)	2.0 (−2)	2.0 (−2)	2.0 (−2)	2.7 (−4)	2.7 (−4)	2.0 (−2)	2.0 (−2)	2.0 (−2)
	2	8.1 (−16)	4.2 (−18)	5.2 (−18)	5.2 (−18)	9.1 (−14)	9.1 (−14)	4.2 (−18)	4.2 (−18)	4.2 (−18)
	3	3.9 (−94)	3.1 (−143)	1.9 (−142)	1.7 (−142)	3.4 (−42)	3.4 (−42)	3.0 (−143)	3.0 (−143)	3.0 (−143)
	ρ	5.9929	7.9858	7.9846	7.9847	3.0005	3.0005	7.9861	7.9862	7.9862
$f_3(x)$	1	3.9 (−4)	2.6 (−4)	3.9 (−4)	4.1 (−4)	2.6 (−4)	2.7 (−4)	2.9 (−5)	3.3 (−5)	4.3 (−5)
	2	1.0 (−14)	3.6 (−19)	5.2 (−17)	9.8 (−17)	1.4 (−19)	1.1 (−18)	1.1 (−27)	2.4 (−27)	1.2 (−25)
	3	3.9 (−78)	6.1 (−138)	5.9 (−120)	1.2 (−117)	1.0 (−141)	6.1 (−134)	3.3 (−207)	7.5 (−207)	5.3 (−190)
	ρ	5.9975	7.9977	7.9945	7.9941	8.0026	7.9971	7.9996	7.9995	7.9996
$f_4(x)$	1	2.5 (−6)	4.3 (−6)	4.3 (−6)	4.3 (−6)	1.4 (−10)	1.4 (−10)	4.3 (−6)	4.3 (−6)	4.3 (−6)
	2	1.5 (−18)	1.4 (−30)	1.4 (−30)	1.4 (−30)	3.8 (−52)	3.8 (−52)	1.4 (−30)	1.4 (−30)	1.4 (−30)
	3	3.7 (−55)	5.9 (−153)	5.9 (−153)	5.9 (−153)	5.3 (−260)	5.3 (−260)	5.9 (−153)	5.9 (−153)	5.9 (−153)
	ρ	3.0000	5.0000	5.0000	5.000	5.0000	5.0000	5.0000	5.0000	5.0000
$f_5(x)$	1	2.0 (−7)	9.5 (−8)	4.8 (−7)	6.5 (−7)	6.3 (−8)	1.9 (−7)	2.3 (−8)	1.5 (−8)	2.9 (−8)
	2	1.8 (−41)	1.6 (−55)	5.7 (−49)	8.4 (−48)	4.2 (−57)	8.0 (−53)	2.6 (−59)	1.7 (−15)	7.0 (−60)
	3	1.0 (−245)	1.3 (−437)	2.2 (−384)	6.6 (−375)	5.9 (−169)	9.6 (−416)	3.2 (−454)	1.9 (−118)	7.5 (−473)
	ρ	6.0000	8.0000	8.0000	8.0000	2.2745	8.0000	8.0000	14.862	8.0000
$f_6(x)$	1	3.5 (−6)	1.7 (−7)	2.4 (−7)	2.4 (−7)	9.3 (−8)	9.7 (−8)	1.2 (−7)	1.1 (−7)	1.2 (−7)
	2	1.2 (−32)	4.4 (−53)	2.0 (−51)	2.5 (−51)	3.0 (−55)	5.8 (−55)	1.2 (−54)	2.6 (−55)	1.0 (−54)
	3	1.8 (−191)	9.4 (−418)	5.3 (−404)	3.6 (−403)	3.1 (−435)	1.0 (−432)	8.7 (−431)	2.8 (−436)	4.0 (−431)
	ρ	6.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000

Table 3. Contrast on the ground of residual errors (i.e., $|f(x_{\sigma})|$).

$f(x)$	σ	GM	BM	$FM1$	$FM2$	$RM1$	$RM2$	$PM1$	$PM2$	$PM3$
$f_1(x)$	1	1.4 (−8)	4.1 (−10)	4.1 (−9)	6.1 (−9)	6.4 (−10)	1.1 (−9)	7.5 (−11)	1.0 (−12)	6.7 (−11)
	2	1.4 (−51)	9.9 (−80)	1.3 (−70)	4.7 (−69)	1.1 (−77)	7.5 (−76)	4.7 (−86)	3.4 (−103)	6.2 (−87)
	3	1.0 (−309)	1.2 (−636)	1.2 (−562)	5.8 (−550)	9.7 (−620)	3.8 (−605)	1.0 (−687)	5.9 (−827)	3.5 (−695)
$f_2(x)$	1	1.9 (−4)	8.0 (−4)	8.5 (−4)	8.5 (−4)	1.5 (−7)	1.5 (−7)	8.0 (−4)	8.0 (−4)	8.0 (−4)
	2	1.4 (−30)	3.7 (−35)	5.7 (−35)	5.6 (−35)	1.7 (−26)	1.7 (−26)	3.7 (−35)	3.7 (−35)	3.7 (−35)
	3	3.2 (−187)	2.0 (−285)	7.3 (−284)	6.3 (−284)	2.5 (−83)	2.5 (−83)	1.9 (−285)	1.9 (−285)	1.9 (−285)
$f_3(x)$	1	4.6 (−9)	2.0 (−9)	4.6 (−9)	5.1 (−9)	2.0 (−9)	2.3 (−9)	2.5 (−11)	3.2 (−11)	5.6 (−11)
	2	3.2 (−30)	4.0 (−39)	8.0 (−35)	2.9 (−34)	5.9 (−40)	3.4 (−38)	3.3 (−56)	1.7 (−55)	4.5 (−52)
	3	4.6 (−157)	1.1 (−276)	1.1 (−240)	4.3 (−236)	3.1 (−284)	1.2 (−268)	3.3 (−415)	1.7 (−410)	8.4 (−381)

Table 3. *Cont.*

$f(x)$	σ	GM	BM	$FM1$	$FM2$	$RM1$	$RM2$	$PM1$	$PM2$	$PM3$
$f_4(x)$	1	2.6 (−18)	1.3 (−17)	1.3 (−17)	1.3 (−17)	4.7 (−31)	4.7 (−31)	1.3 (−17)	1.3 (−17)	1.3 (−17)
	2	6.2 (−55)	5.0 (−91)	5.0 (−91)	5.0 (−91)	9.1 (−156)	9.1 (−156)	5.0 (−91)	5.0 (−91)	5.0 (−91)
	3	8.4 (−165)	3.5 (−458)	3.5 (−458)	3.5 (−458)	2.4 (−779)	2.4 (−779)	3.5 (−458)	3.5 (−458)	3.5 (−458)
$f_5(x)$	1	1.1 (−622)	2.2 (−655)	4.4 (−585)	5.3 (−572)	3.9 (−673)	3.1 (−626)	1.3 (−709)	3.7 (−736)	1.6 (−706)
	2	9.7 (−4027)	1.4 (−5431)	1.2 (−4777)	8.7 (−4661)	11. (−5590)	9.1 (−5163)	6.7 (−5376)	5.3 (−1429)	1.2 (−5868)
	3	5.4 (−24,451)	4.1 (−43,641)	2.7 (−38,318)	5.1 (−37,371)	1.1 (−16,775)	7.8 (−41,455)	6.1 (−41,287)	5.9 (−11,726)	1.1 (−47,165)
$f_6(x)$	1	1.1 (−6)	1.3 (−20)	3.5 (−20)	3.7 (−20)	2.1 (−21)	2.3 (−21)	4.8 (−21)	3.5 (−21)	4.2 (−21)
	2	4.3 (−96)	2.2 (−157)	2.1 (−152)	4.2 (−152)	6.7 (−164)	5.1 (−163)	4.3 (−162)	4.7 (−164)	2.9 (−162)
	3	1.5 (−572)	2.1 (−1251)	3.8 (−1210)	1.2 (−1207)	7.5 (−1304)	2.5 (−1296)	1.7 (−1290)	5.4 (−1307)	1.6 (−1291)

6. Conclusions

We developed a new 8th-order iteration function having optimal eight-order convergence for multiple zeros of a univariate function with faster convergence and simple and compact body structure. The present scheme is based on weight functions that play a fruitful role in the establishment of 8th-order convergence. In addition, we presented local convergence analysis showing 8th-order of convergence. Each member of our scheme is optimal as stated in the conjecture by Kung–Traub. Moreover, we can obtain several new specializations by adopting weight functions in the suggested scheme (1). Minimum residual errors, minimum errors among two consecutive iterations and balanced ρ were identified with our schemes while comparing to the existing ones on real problems like continuous stirred tank reactor, chemical conversion, multi factor problem, Vander Waal's equation of state, etc. Based on the obtained results, we deduce that our schemes are more efficient and useful than the earlier ones.

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