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Bases of $G - V$ Intuitionistic Fuzzy Matroids

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Abstract: The purpose of this paper is to study intuitionistic fuzzy bases ($IFBs$) and the intuitive structure of a $G - V$ IFM. Firstly, the intuitionistic fuzzy basis (IFB) of a $G - V$ IFM is defined; then the h-range and properties of an IFB are presented and a necessary and sufficient condition of a closed $G - V$ IFM is studied. Especially, a necessary and sufficient condition of judging an IFB is presented and the intuitive tree structure of a closed $G - V$ IFM is proposed and its properties are discussed.

Keywords: matroid; bases; fuzzy matroid; intuitionistic fuzzy matroid; intuitionistic fuzzy bases

1. Introduction

Whitney's 1935 article laid the groundwork for the field of combinatorial geometries and matroid [1]. Matroid theory has been widely applied to combinatorial mathematics, combinatorial optimization and group theory [2–8]. Based on the fuzzy set theory proposed by Zadeh in 1965 [9], matroid theory has been generalized to various forms related to fuzzy sets. Shi [10,11] proposed the (L, M)-fuzzy matroid based on latticevalued fuzzy set theory and studied the base axioms of fuzzitying matroids [12–14]. Hsueh presented a fuzzification of matroids which extends the independence axioms of matroids [15]. Al-Hawary introduced a method to the fuzzifying of matroids which is called fuzzy C-matroids [16,17]. In 1988, Goetschel and Voxman proposed an important fuzzy matroid (briefly, $G - V$ fuzzy matroid) in [18]. They further studied some important concepts and their properties, such as the fuzzy bases and the fuzzy rank function [19–22]. Following them, some scholars studied the axioms, the connectedness and the structure of $G - V$ fuzzy matroid, etc. [23–25].

The intuitionistic fuzzy set (IFS), introduced by Atanassov originally in 1983 [26] and made widely accessible in 1986 [27], is a generalization of Zadeh's fuzzy set. An IFS of each element is an ordered pair which is called an intuitionistic fuzzy value (IFV) and each IFV is characterized by a membership degree, a nonmembership degree and a hesitancy degree. From then on, many scholars were attracted to study the IFS and obtained a lot of valuable results. For ranking the $IFSs$, Hong and Choi proposed the accuracy function in 2000 [28] and Szmidt and Kacprzyk proposed a similarity function of $IFSs$ in 2004 [29]. Based on the accuracy function and the similarity function, Zhang and Xu introduced a new method for ranking $IFSs$ in 2012 [30]. In 2013, Rangasamy et al. proposed a method by ranking to be done using the scores and accuracy for finding the shortest hyperpath in an intuitionistic fuzzy weighted hypergraph [31]. Some other scholars studied the aggregation operators and fuzzy clustering of $IFSs$ [32–34]. After decades of effort from scholars, the relevant achievements of intuitionistic fuzzy theory became very rich. In 1999, Atanassov completed his first monograph which discussed the concept and operators of $IFSs$, the interval valued $IFSs$, some other extensions of $IFSs$, the elements of $IFSs$ and the applications of $IFSs$ [35]. There are some other scholars' results worthy of learning and researching; see [36–38]. In 2017, Li and Yi proposed an intuitionistic fuzzy matroid based on matroids and intuitionistic fuzzy sets [39]. In [40], Li et al. extended $G-V$ fuzzy matroids and introduced a $G-V$ intuitionistic fuzzy matroid and studied the induced matroid sequence and the rank function. In this paper, based on the literature [19,25,40], we study the bases and the structure of a $G-V$ intuitionistic fuzzy matroid (briefly, $G-V$ IFM), which are actually generalizations of some conclusions of $G-V$ fuzzy matroid.

This paper is arranged as follows. Some basic definitions and results are introduced in Section 2. The $IFBs$ of a $G-V$ IFM are studied in Section 3. The judgment of an IFB is investigated in Section 4. Finally, we propose the tree structure of a closed $G-V$ IFM and study its properties in Section 5.

2. Preliminaries and Notations

We introduce some basic and useful concepts related to matroid theory here; see [41,42]. Firstly, we introduce the concept of the matroid.

Definition 1. *Let I be a nonempty family of subsets of a finite set E and satisfy:*

1. $\emptyset \in I$.
2. *If $X \in I$, and $Y \subset X$, then $Y \in I$.*
3. *If $X, Y \in I$, and $|Y| > |X|$, then there exists an $x \in Y \backslash X$ such that $X \cup \{x\} \in I$.*

Then the pair $M = (E, I)$ is called a matroid (or a crisp matroid). For any $A \subseteq E$, if $A \in I$, then A is called an independent set; otherwise A is called a dependent set.

In matroid theory, rank function and its submodularity are very important. They are defined as follows.

Definition 2. *Let $P(E)$ be the power set of finite set E and $M = (E, I)$ be a matroid. R is called rank function of M, where $R : P(E) \to \{0, 1, 2, \cdots, |E|\}$ is a mapping and is defined as follows:*

$$R(X)=max\{|Y| \,|\, Y \subseteq X, \text{ and } Y \in I\}.$$

From the definition of R, the following properties can be easily obtained.

1. If $X \subseteq Y$, then $R(X) \leq R(Y)$;
2. $R(X) \leq |X|$ for any $X \in P(E)$;
3. If $X \in I$, then $R(X) = |X|$,

where $X, Y \in P(E)$.

Definition 3. *Let $\sigma : P(E) \to [0, \infty)$ be a mapping, where $P(E)$ is the power set of finite set E. σ is called submodular if*

$$\sigma(X) + \sigma(Y) \geq \sigma(X \cap Y) + \sigma(X \cup Y),$$

for each $X, Y \in P(E)$.

Theorem 1. *The rank function R of a matroid $M = (E, I)$ is submodular.*

Next, some concepts and notations concerning fuzzy sets or intuitionistic fuzzy sets are cited; see [9,18,19,26–38,40].

Definition 4. *Let X be a fixed set. Then*

$$A = \{(x, \mu_A(x))|x \in X\}$$

is called a fuzzy set, where $\mu_A(x)$ is the membership degree of x to A, $0 \leq \mu_A(x) \leq 1$. The collection of fuzzy sets on X is denoted by $FS(X)$.

Definition 5. *Let X be a fixed set. Then*

$$A = \{(x, \mu_A(x), \nu_A(x))|x \in X\}$$

is called an IFS (i.e., intuitionistic fuzzy set). For any $x \in X$, $\mu_A(x)$, $\nu_A(x)$ and $\pi_A(x)$ are called membership degree, non-membership degree and hesitancy degree, respectively, where $\mu_A(x), \nu_A(x), \pi_A(x) \geq 0$ and $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$. The collection of IFSs on X is denoted by $IFS(X)$. If for all $x \in X$, $\pi_A(x) = 0$, then $\mu_A(x) + \nu_A(x) = 1$ and IFS A is reduced to a fuzzy set. In this paper, we use $(\mu_\alpha, \nu_\alpha, \pi_\alpha)$ to denote intuitionistic fuzzy set and $(\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x))$ to denote intuitionistic fuzzy value.

For convenience and suitable for the study of $G - V$ intuitionistic fuzzy matroids later, an IFS $(\mu_\alpha, \nu_\alpha, \pi_\alpha)$ is abbreviated as (μ_α, π_α) and an IFV $(\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x))$ is denoted by $(\mu_\alpha(x), \pi_\alpha(x))$. Note that this notation is different from that in Definition 5.

Definition 6. *Let $(\mu_\alpha, \pi_\alpha) \in IFS(X)$ be an IFS. Then the accuracy function H of $(\mu_\alpha(x), \pi_\alpha(x)), (x \in X)$ is denoted by*

$$H(\mu_\alpha(x), \pi_\alpha(x)) = 1 - \pi_\alpha(x)$$

Definition 7. *Let $(\mu_\alpha, \pi_\alpha) \in IFS(X)$ be an IFS. Then the similarity function h of $(\mu_\alpha(x), \pi_\alpha(x))$ for any $x \in X$ is*

$$h(\mu_\alpha(x), \pi_\alpha(x)) = 1 - \tfrac{1-\mu_\alpha(x)}{1+\pi_\alpha(x).}$$

In the special case $\pi_\alpha(x) = 0$, we have $h(\mu_\alpha(x), \pi_\alpha(x)) = \mu_\alpha(x)$.

Let X be a finite set and $(\mu_\alpha, \pi_\alpha), (\mu_\beta, \pi_\beta) \in IFS(X)$ be IFSs and $x \in X$. We now introduce the following notation and results; see [40]:

1. $H_{(\mu_\alpha, \pi_\alpha)}(x) = H(\mu_\alpha(x), \pi_\alpha(x))$.
   $h_{(\mu_\alpha, \pi_\alpha)}(x) = h(\mu_\alpha(x), \pi_\alpha(x))$.
   $(\mu_\alpha, \pi_\alpha)(x) = (\mu_\alpha(x), \pi_\alpha(x))$.

2. $(\mu_\alpha, 0) = (\mu_\alpha, \pi_\alpha)$ if $\pi_\alpha(x) = 0$ for any $x \in X$.
3. $h(\mu_\alpha, \pi_\alpha) \leq h(\mu_\beta, \pi_\beta)$: for any $x \in X$, $h(\mu_\alpha(x), \pi_\alpha(x)) \leq h(\mu_\beta(x), \pi_\beta(x))$.
   $h(\mu_\alpha, \pi_\alpha) = h(\mu_\beta, \pi_\beta)$: for any $x \in X$, $h(\mu_\alpha(x), \pi_\alpha(x)) = h(\mu_\beta(x), \pi_\beta(x))$.
   $h(\mu_\alpha, \pi_\alpha) < h(\mu_\beta, \pi_\beta)$: $h(\mu_\alpha, \pi_\alpha) \leq h(\mu_\beta, \pi_\beta)$ and $h(\mu_\alpha(x), \pi_\alpha(x)) < h(\mu_\beta(x), \pi_\beta(x))$ for some $x \in X$.
4. $H(\mu_\alpha, \pi_\alpha) \leq H(\mu_\beta, \pi_\beta)$: for any $x \in X$, $H(\mu_\alpha(x), \pi_\alpha(x)) \leq H(\mu_\beta(x), \pi_\beta(x))$.
   $H(\mu_\alpha, \pi_\alpha) = H(\mu_\beta, \pi_\beta)$: for any $x \in X$, $H(\mu_\alpha(x), \pi_\alpha(x)) = H(\mu_\beta(x), \pi_\beta(x))$.
   $H(\mu_\alpha, \pi_\alpha) < H(\mu_\beta, \pi_\beta)$:$H(\mu_\alpha, \pi_\alpha) \leq H(\mu_\beta, \pi_\beta)$ and $H(\mu_\alpha(x), \pi_\alpha(x)) < H(\mu_\beta(x), \pi_\beta(x))$ for some $x \in X$.
5. $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$:$h(\mu_\alpha, \pi_\alpha) \leq h(\mu_\beta, \pi_\beta)$ and $H(\mu_\alpha, \pi_\alpha) \leq H(\mu_\beta, \pi_\beta)$.
   $(\mu_\alpha, \pi_\alpha) \prec (\mu_\beta, \pi_\beta)$:$h(\mu_\alpha, \pi_\alpha) < h(\mu_\beta, \pi_\beta)$ and $H(\mu_\alpha, \pi_\alpha) \leq H(\mu_\beta, \pi_\beta)$.
   $(\mu_\alpha, \pi_\alpha) = (\mu_\beta, \pi_\beta)$:$h(\mu_\alpha, \pi_\alpha) = h(\mu_\beta, \pi_\beta)$ and $H(\mu_\alpha, \pi_\alpha) = H(\mu_\beta, \pi_\beta)$.
6. $\mathrm{supp}(\mu_\alpha, \pi_\alpha) = \{x \in X | h(\mu_\alpha(x), \pi_\alpha(x)) > 0\}$.
7. $m(\mu_\alpha, \pi_\alpha) = inf\{h(\mu_\alpha(x), \pi_\alpha(x)) | x \in \mathrm{supp}(\mu_\alpha, \pi_\alpha)\}$.
8. $C_r(\mu_\alpha, \pi_\alpha) = \{x \in X | h(\mu_\alpha(x), \pi_\alpha(x)) \geq r\}$, where $0 \leq r \leq 1$.

9. $R^{+}(\mu_\alpha, \pi_\alpha) = \{h(\mu_\alpha(x), \pi_\alpha(x)) | h(\mu_\alpha(x), \pi_\alpha(x)) > 0, x \in X\}$ is called the positive $h - range$ of (μ_α, π_α).
10. $|(\mu_\alpha, \pi_\alpha)| = \sum_{x\in X} h(\mu_\alpha(x), \pi_\alpha(x))$ is called the "cardinality" of an IFS.

Definition 8. *Let $(\mu_\alpha, \pi_\alpha), (\mu_\beta, \pi_\beta)$ be two intuitionistic fuzzy sets, $x \in X$. $(\mu_\gamma, \pi_\gamma) = (\mu_\alpha, \pi_\alpha) \vee (\mu_\beta, \pi_\beta)$ and $(\mu_\omega, \pi_\omega) = (\mu_\alpha, \pi_\alpha) \wedge (\mu_\beta, \pi_\beta)$ are called the union and intersection of (μ_α, π_α) and (μ_β, π_β), respectively, where (μ_γ, π_γ) is defined by*

$$(\mu_\gamma, \pi_\gamma)(x) = \begin{cases} (\mu_\alpha, \pi_\alpha)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) > h_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\beta, \pi_\beta)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) < h_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\alpha, \pi_\alpha)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) = h_{(\mu_\beta,\pi_\beta)}(x) \; and \quad H_{(\mu_\alpha,\pi_\alpha)}(x) \geq H_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\beta, \pi_\beta)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) = h_{(\mu_\beta,\pi_\beta)}(x) \; and \quad H_{(\mu_\alpha,\pi_\alpha)}(x) < H_{(\mu_\beta,\pi_\beta)}(x). \end{cases}$$

and (μ_ω, π_ω) is defined by

$$(\mu_\omega, \pi_\omega)(x) = \begin{cases} (\mu_\beta, \pi_\beta)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) > h_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\alpha, \pi_\alpha)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) < h_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\beta, \pi_\beta)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) = h_{(\mu_\beta,\pi_\beta)}(x) \; and \quad H_{(\mu_\alpha,\pi_\alpha)}(x) \geq H_{(\mu_\beta,\pi_\beta)}(x), \\ (\mu_\alpha, \pi_\alpha)(x), & if \quad h_{(\mu_\alpha,\pi_\alpha)}(x) = h_{(\mu_\beta,\pi_\beta)}(x) \; and \quad H_{(\mu_\alpha,\pi_\alpha)}(x) < H_{(\mu_\beta,\pi_\beta)}(x). \end{cases}$$

Definition 9. *Let E be a finite set and $\psi \subseteq IFS(E)$ be a nonempty family of fuzzy sets. The pair (E, ψ) is called a $G - V$ IFM on E if it satisfies the following conditions:*

1. *If $(\mu_\alpha, \pi_\alpha) \in \psi, (\mu_\beta, \pi_\beta) \in IFS(E)$, and $(\mu_\beta, \pi_\beta) \prec (\mu_\alpha, \pi_\alpha)$, then $(\mu_\beta, \pi_\beta) \in \psi$.*
2. *If $(\mu_\alpha, \pi_\alpha), (\mu_\beta, \pi_\beta) \in \psi$, and $|supp(\mu_\alpha, \pi_\alpha)| < |supp(\mu_\beta, \pi_\beta)|$, then there exists $(\mu_\omega, \pi_\omega) \in \psi$, such that:*
   (a) $(\mu_\alpha, \pi_\alpha) \prec (\mu_\omega, \pi_\omega) \preceq (\mu_\alpha, \pi_\alpha) \vee (\mu_\beta, \pi_\beta)$;
   (b) $m(\mu_\omega, \pi_\omega) \geq min\{m(\mu_\alpha, \pi_\alpha), m(\mu_\beta, \pi_\beta)\}$.

Suppose that (E, ψ) is a $G - V$ IFM. $(\mu_\alpha, \pi_\alpha) \in \psi$ is called an independent IFS and ψ is called the set of independent $IFSs$. $(\mu_\beta, \pi_\beta) \notin \psi$ is called a dependent IFS.

If for any $(\mu_\alpha, \pi_\alpha) \in IFS(E)$ and for any $x \in E$, $\pi_\alpha(x) = 0$, then $IFS(E)$ is actually $FS(E)$. Thus, (E, ψ) is reduced to $G - V$ FM.

Theorem 2. *Let (E, ψ) be a $G - V$ IFM. For each r, $0 \leq r \leq 1$, let*

$$I_r = \{C_r(\mu_\alpha, \pi_\alpha) | (\mu_\alpha, \pi_\alpha) \in \psi\}$$

Then for each r, $0 < r \leq 1$,

$$M_r = (E, I_r)$$

is a matroid.

Theorem 3. *Let (E, ψ) be a $G - V$ IFM. Let $M_r = (E, I_r)$ be a matroid on E defined in Theorem 2, where $0 < r \leq 1$. Then there is a finite sequence $r_0 < r_1 < \cdots < r_n$ such that:*

(i) $r_0 = 0, r_n = 1$.

(ii) $I_s \neq \{\phi\}$ *if* $0 < s \leq r_n$, $I_s = \{\phi\}$ *if* $s > r_n$.

(iii) *If* $r_i < s, t < r_{i+1}$*, then* $I_s = I_t$*, where* $0 \leq i \leq n-1$*.*

(iv) *If* $r_i < s < r_{i+1} < t < r_{i+2}$*, then* $I_s \subset I_t$*, where* $0 \leq i \leq n-2$*.*

Then the sequence $r_0, r_1, r_2, \cdots, r_n$ *is called the fundamental sequence of* (E, ψ)*. Moreover, if for any* i*,* $1 \leq i \leq n$*, let* $\bar{r}_i = \frac{r_{i-1}}{r_i}$*, then we can get a sequence of matroids* $M_{\bar{r}_n} \subset M_{\bar{r}_{n-1}} \subset \cdots \subset M_{\bar{r_2}} \subset M_{\bar{r_1}}$ *which is called the induced matroid sequence.*

Note that $C_r(\mu_\alpha, \pi_\alpha) = \{x \in E | h(\mu_\alpha(x), \pi_\alpha(x)) \geq r\}$*, where* $0 < r \leq 1$*, and* $I_r = \{C_r(\mu_\alpha, \pi_\alpha) | (\mu_\alpha, \pi_\alpha) \in \psi\}$*, so* $I_s = \{\phi\}$ *not but* $I_s = \{\phi\}$ *when* $s > r_n$*.*

A matroid sequence can be constructed from a $G-V\ IFM$ above. On the contrary, a $G-V\ IFM$ can be constructed from a matroid sequence.

Theorem 4. *Let* $0 = s_0 < s_1 < s_2 < \cdots < s_n \leq 1$ *be a finite sequence. Suppose that* $M_{s_1}, M_{s_2}, \cdots, M_{s_{n-1}}, M_{s_n}$ *(*$M_{s_i} = (E, I_{s_i}), 1 \leq i \leq n$*) is a matroid sequence on a finite set E and satisfies* $I_{s_{i+1}} \subset I_{s_i}$ *(*$0 \leq i \leq n-1$*). For any* $0 \leq s \leq 1$*, let*

$$I_s = \begin{cases} I_{s_i},\ if & s_{i-1} < s \leq s_i,\ 0 \leq i \leq n, \\ \{\phi\},\ if & s_n < s \leq 1. \end{cases}$$

and let

$$\psi^* = \{(\mu_\alpha, \pi_\alpha) \in IFS(E) | C_s(\mu_\alpha, \pi_\alpha) \in I_s, 0 < s \leq 1\}.$$

Then (E, ψ^*) *is a* $G-V\ IFM$ *and its induced matroid sequence is* $M_{s_n} \subset M_{s_{n-1}} \subset \cdots \subset M_{s_2} \subset M_{s_1}$*, where for* $1 \leq i \leq n$*,* $M_{s_i} = (E, I_{s_i})$*.*

Theorem 5. *Let* (E, ψ) *be a* $G-V\ IFM$*, and for each* r*, let* $0 < r \leq 1$*,* $M_r = (E, I_r)$ *be a matroid defined by Theorem 2. Let* $\psi^* = \{(\mu_\alpha, \pi_\alpha) \in IFS(E) | C_r(\mu_\alpha, \pi_\alpha) \in I_r, 0 < r \leq 1\}$*. Then* $\psi = \psi^*$*.*

Theorem 6. *Let* (E, ψ) *be a* $G-V\ IFM$ *and* $(\mu_\alpha, \pi_\alpha) \in IFS(E)$*. Then* $(\mu_\alpha, \pi_\alpha) \in \psi$ *if and only if* $C_\lambda(\mu_\alpha, \pi_\alpha) \in I_\lambda$ *for each* $\lambda \in R^+(\mu_\alpha, \pi_\alpha)$*.*

Theorem 7. *Suppose that* (E, ψ) *is a* $G-V\ IFM$ *with the fundamental sequence* $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$*. If* $I_r = I_{r_i}$ *for any* $r_{i-1} < r \leq r_i$ *(*$0 \leq i \leq n$*), then* (E, ψ) *is called a closed* $G-V\ IFM$*.*

3. Bases of $G-V\ IFMs$

Based on $G-V\ IFMs$ and bases of matroids or fuzzy matroids, we propose the concept of the intuitionistic fuzzy basis of a $G-V\ IFM$.

Definition 10. *Let* (E, ψ) *be a* $G-V\ IFM$*.* $(\mu_\alpha, \pi_\alpha) \in \psi$ *is said to be maximal in* ψ*. If for any* $(\mu_\beta, \pi_\beta) \in \psi$ *and* $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$*, then* $(\mu_\alpha, \pi_\alpha) = (\mu_\beta, \pi_\beta)$*. I.e., there does not exist* $(\mu_\beta, \pi_\beta) \in \psi$ *such that* $(\mu_\alpha, \pi_\alpha) \prec (\mu_\beta, \pi_\beta)$*).*

An intuitionistic fuzzy basis (IFB for short) of a $G-V\ IFM$ (E, ψ) is a maximal member $(\mu_\alpha, \pi_\alpha) \in \psi$.

Suppose that (μ_α, π_α) is an IFB and $(\mu_\beta, \pi_\beta) \in IFS(E)$. Let $h(\mu_\beta, \pi_\beta) = h(\mu_\alpha, \pi_\alpha)$ and $H(\mu_\beta, \pi_\beta) < H(\mu_\alpha, \pi_\alpha)$; then $(\mu_\beta, \pi_\beta) \preceq (\mu_\alpha, \pi_\alpha)$ and $|(\mu_\beta, \pi_\beta)| = |(\mu_\alpha, \pi_\alpha)|$. Obviously, $(\mu_\beta, \pi_\beta) \in \psi$. Therefore, (μ_β, π_β) here is called an intuitionistic fuzzy sub-basis ($IFSB$ for short) with respect to IFB (μ_α, π_α) for a $G-V\ IFM$ (E, ψ). Generally, there are infinite $IFSBs$ for a $G-V\ IFM$ and their cardinality is the same as that of the corresponding IFB.

Definition 11. *An IFS (μ_α, π_α) is an elementary IFS if $R^+(\mu_\alpha, \pi_\alpha) = 1$. If (μ_α, π_α) is an elementary IFS with A =supp(μ_α, π_α) and $R^+(\mu_\alpha, \pi_\alpha) = \{r\}$, then (μ_α, π_α) is denoted by $\omega(A, r)$ with support A and height r.*

Theorem 8. *Suppose that $(\mu_\alpha, \pi_\alpha) \in \psi$ is an IFB of a $G - V$ IFM (E, ψ), then $\pi_\alpha(x) = 0$ for each $x \in E$.*

Proof. Assume that there exists an $x_0 \in E$ such that $\pi_\alpha(x_0) = \eta > 0$. Let $h(\mu_\beta, \pi_\beta) = h(\mu_\alpha, \pi_\alpha)$ for each $x \in E$ and

$$\pi_\beta(x) = \begin{cases} \pi_\alpha(x), \ if & x \in E \ and \ x \neq x_0, \\ \dfrac{\eta}{2} \quad , \ if & x = x_0. \end{cases}$$

Then $H(\mu_\alpha, \pi_\alpha) < H(\mu_\beta, \pi_\beta)$. It follows that $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$. However, for each $\lambda \in R^+(\mu_\alpha, \pi_\alpha)$, we have $C_\lambda(\mu_\beta, \pi_\beta) = C_\lambda(\mu_\alpha, \pi_\alpha) \in I_\lambda$. Then $(\mu_\beta, \pi_\beta) \in \psi$ from Theorem 6. This contradicts the hypothesis.

Here, we will use Theorem 6 to prove the next theorem. □

Theorem 9. *Let (E, ψ) be a $G - V$ IFM with the fundamental sequence $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ and suppose (μ_α, π_α) is an IFB of (E, ψ); then*

$$R^+(\mu_\alpha, \pi_\alpha) \subseteq \{r_1, r_2, \cdots, r_n\}.$$

Proof. Let (μ_α, π_α) be an *IFB* of (E, ψ). Then $(\mu_\alpha, \pi_\alpha) \in \psi$. It follows that $C_\lambda(\mu_\alpha, \pi_\alpha) \in I_\lambda$ for each $\lambda \in R^+(\mu_\alpha, \pi_\alpha)$.

Assume that there is an $s \in R^+(\mu_\alpha, \pi_\alpha)$ such that $r_i < s < r_{i+1}$. We take $\varepsilon = (r_{i+1} - s)/2$ and let (μ_β, π_β) be the elementary *IFS* which is defined by supp$(\mu_\beta, \pi_\beta) = C_s(\mu_\alpha, \pi_\alpha)$ and $R^+(\mu_\beta, \pi_\beta) = s + \varepsilon$.

If we let $(\mu_\omega, \pi_\omega) = (\mu_\alpha, \pi_\alpha) \vee (\mu_\beta, \pi_\beta)$, then for each $r \in (0, 1]$, we have

$$C_r(\mu_\omega, \pi_\omega) = \begin{cases} C_s(\mu_\alpha, \pi_\alpha) \in I_s, \ if \ r \in (s, s+\varepsilon], \\ C_r(\mu_\alpha, \pi_\alpha) \in I_r, \ if \ r \ \notin (s, s+\varepsilon]. \end{cases}$$

By Theorem 6, we have $(\mu_\omega, \pi_\omega) \in \psi$.

By the hypothesis, for (μ_α, π_α), we have that there exists $x_0 \in supp(\mu_\alpha, \pi_\alpha)$ such that $h(\mu_\alpha(x_0), \pi_\alpha(x_0)) = s$. Thus $h(\mu_\omega(x_0), \pi_\omega(x_0)) = s + \varepsilon$. Since $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\omega, \pi_\omega)$, $(\mu_\alpha, \pi_\alpha) \prec (\mu_\omega, \pi_\omega)$. This contradicts that (μ_α, π_α) is an *IFB*. □

Theorem 10. *Suppose that (E, ψ) is a $G - V$ IFM and $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ is the fundamental sequence of (E, ψ). Then (E, ψ) is closed if and only if for any $(\mu_\alpha, \pi_\alpha) \in \psi$, there exists an IFB $(\mu_\beta, \pi_\beta) \in \psi$ such that $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$.*

Proof. Assume that for any $(\mu_\alpha, \pi_\alpha) \in \psi$, there exists an *IFB* $(\mu_\beta, \pi_\beta) \in \psi$ such that $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$. If (E, ψ) is not closed.

Let i_0 be a positive integer such that if $r_{i_0-1} < t < r_{i_0}$, then $I_{r_{i_0}} \subset I_t$, where

$$I_r = \{C_r(\mu_\alpha, \pi_\alpha) | (\mu_\alpha, \pi_\alpha) \in \psi\}.$$

Let A be a basis of I_t but not a basis of $I_{r_{i_0}}$. Let $\omega(A, t)$ be the elementary *IFS*. Obviously, $C_r(\omega(A, t)) = A \in I_r$ for any $r \in (0, t]$, and $C_r(\omega(A, t)) = \emptyset \in I_r$ for any $r \in (t, 1]$. It follows that $\omega(A, t) \in \psi$.

Suppose that $(\mu_\beta, \pi_\beta) \in \psi$ is an *IFB* and $\omega(A, t) \preceq (\mu_\beta, \pi_\beta)$. Then $C_t(\mu_\beta, \pi_\beta) \in I_t$ and $A \subseteq C_t(\mu_\beta, \pi_\beta)$. Since A is a crisp basis of I_t, $A = C_t(\mu_\beta, \pi_\beta)$.

Since $I_{r_{i_0}} \subseteq I_t$ and by Theorem 5 and Theorem 9, we have

$$A = C_t(\mu_\beta, \pi_\beta) = C_{r_{i_0}}(\mu_\beta, \pi_\beta) \in I_{r_{i_0}}.$$

Conversely, suppose that (E, ψ) is closed. Let $(\mu_\alpha, \pi_\alpha) \in \psi$ and $R^+(\mu_\alpha, \pi_\alpha) = \{t_1, t_2, \cdots, t_k\}$, where $t_1 < t_2 < \cdots < t_k$. Let $\{t_{p_1}, t_{p_2}, \cdots, t_{p_s}\}$ be a subsequence of $\{t_1, t_2, \cdots, t_k\}$ and $\{r_{q_1}, r_{q_2}, \cdots, r_{q_s}\}$ be a subsequence of $\{r_0, r_1, \cdots, r_n\}$ such that $t_{p_1} = t_1$, and for a given t_{p_j}, there is $r_{q_j} = \min\{r_i | r_i \geq t_{p_j}\}$, and for a given r_{q_j} there is $t_{p_{j+1}} = \min\{t_i | t_i > r_{q_j}\}$. It follows that

(1) $t_{p_1} = t_1$;
(2) $r_{q_{j-1}} < t_{p_j} < r_{q_j}, j = 1, 2, \cdots, s$;
(3) If $t_{p_j} < t_i < t_{p_{j+1}}$, then $r_{q_{j-1}} < t_i \leq r_{q_j}$;
(4) If $t_{p_s} > t_i$, then $r_{q_{s-1}} < t_i \leq r_{q_s}$.

Let $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1$ be a nested sequence such that

(a) $\text{supp}(\mu_\alpha, \pi_\alpha) \subseteq A_1$, where A_1 is a basis of (E, I_{r_1});
(b) For $i \geq 2$ (where i is an integer), we have $q_{j-1} < i < q_j (q_0 = 0)$ and A_i is a maximal subset of A_{i-1} in I_{r_i} that contains $C_{t_{p_j}}(\mu_\alpha, \pi_\alpha)$;
(c) For $q_t < i \leq n$ (where i is an integer), we have A is a maximal subset of A_{i-1} in I_{r_i}.

Let $(\mu_{\beta_i}, \pi_{\beta_i})$ be the elementary IFS $\omega(A_i, r_i)$, where $i \in [1, n]$. If $(\mu_\beta, \pi_\beta) = \bigvee_{i=1}^{n}(\mu_{\beta_i}, \pi_{\beta_i})$, then we can easily get $C_r(\mu_\beta, \pi_\beta) \in I_r, r \in (0, 1]$. By Theorem 6, $(\mu_\beta, \pi_\beta) \in \psi$. From the construction of (μ_β, π_β), $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta) \in \psi$ and (μ_β, π_β) is an IFB for (E, ψ), the conclusion is established. □

4. The Judgement of an IFB for a $G - V$ IFM

From the proof of Theorem 10, we can get the following result.

Theorem 11. *Suppose that (E, ψ) is a closed $G - V$ IFM with the fundamental sequence $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ and the induced matroid sequence $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$, where $M_{r_i} = (E, I_{r_i})(1 \leq i \leq n)$. Let $(\mu_\alpha, \pi_\alpha) \in IFS(E)$. If (μ_α, π_α) is an IFB of (E, ψ), then $supp(\mu_\alpha, \pi_\alpha) = C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha)$ is a basis of matroid (E, I_{r_1}).*

Proof. Suppose that $m(\mu_\alpha, \pi_\alpha)$ is an IFB of IFM. Then $R^+(\mu_\alpha, \pi_\alpha) \subseteq \{r_1, r_2, \cdots, r_n\}$ and $C_r(\mu_\alpha, \pi_\alpha) \in I_r$ for any $r \in R^+(\mu_\alpha, \pi_\alpha)$.

Assume that $\text{supp}(\mu_\alpha, \pi_\alpha) = C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha)$ is not a basis of matroid (E, I_{r_1}); then there exists a basis A of (E, I_{r_1}) such that $\text{supp}(\mu_\alpha, \pi_\alpha) = C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha) \subset A$. Let

$$h(\mu_\omega(x), \pi_\omega(x)) = \begin{cases} r_1 & , if \quad x \in A \setminus C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha), \\ h(\mu_\alpha(x), \pi_\alpha(x)), & if \quad x \in C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha), \\ 0 & , \quad otherwise. \end{cases}$$

Then $(\mu_\alpha, \pi_\alpha) \prec (\mu_\omega, \pi_\omega)$, $R^+(\mu_\omega, \pi_\omega) \subseteq \{r_1, r_2, \cdots, r_n\}$ and $C_{r_1}(\mu_\omega, \pi_\omega) = A \in I_{r_1}$. Thus, for any $r_1 < r < m(\mu_\alpha, \pi_\alpha)$,

$$C_r(\mu_\omega, \pi_\omega) = C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha) \in I_{m(\mu_\alpha, \pi_\alpha)} \subseteq I_r,$$

and for any $m(\mu_\alpha, \pi_\alpha) \leq r \leq 1$,

$$C_r(\mu_\omega, \pi_\omega) = C_r(\mu_\alpha, \pi_\alpha) \in I_r.$$

From Theorem 6, it follows that $(\mu_\omega, \pi_\omega) \in \psi$. Since $(\mu_\alpha, \pi_\alpha) \prec (\mu_\omega, \pi_\omega)$, it contradicts that (μ_α, π_α) is an IFB of IFM. □

The following necessary and sufficient condition can be used to judge whether a fuzzy set is a fuzzy basis.

Theorem 12 ([25])**.** *Let (E, ψ) be a closed $G - V$ fuzzy matroid on E and $0 = r_0 < r_1 < \cdots < r_n \leq 1$ be the fundamental sequence. Let $\mu \in FS(E)$. Suppose $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$ is the induced matroid sequence (where $M_{r_i} = (E, I_{r_i})$,$i = 1, 2, \cdots, n$). Then μ is a fuzzy basis of (E, ψ) if and only if μ satisfies:*

(i) A_1=suppμ *is a basis of matroid* (E, I_{r_1}).

(ii) *There exists a sequence $A_2, \cdots, A_{n-1}, A_n$ ($A_i \in I_{r_i}$) which satisfies A_i is a maximal subset of A_{i-1} in I_{r_i}($i = 2, 3, 4, \cdots, n$) and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n$ such that for any $x \in A_n$, $\mu(x) = r_n$ and for any $x \in A_i \setminus A_{i+1}$, $\mu(x) = r_i$, where $i = 1, 2, 3, \cdots, n-1$.*

This result can be extended to intuitionistic fuzzy sets.

Theorem 13. *Suppose (E, ψ) is a closed $G - V$ IFM with the fundamental sequence $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ and the induced matroid sequence $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$, where $M_{r_i} = (E, I_{r_i})$ ($1 \leq i \leq n$). Let $(\mu_\alpha, \pi_\alpha) \in IFS(E)$; then (μ_α, π_α) is an IFB of (E, ψ) if and only if (μ_α, π_α) satisfies:*

(I) $\pi_\alpha(x) = 0$ *for each* $x \in E$;

(II) *The set $A_1 = supp(\mu_\alpha, \pi_\alpha)$ is a crisp basis of matroid (E, I_{r_1});*

(III) *There exists a sequence $A_2, \cdots, A_{n-1}, A_n$ ($A_i \in I_{r_i}$) which satisfies A_i is a maximal subset of A_{i-1} in I_{r_i} ($i = 2, 3, \cdots, n$) and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n$ such that for any $x \in A_n$, $h(\mu_\alpha(x), \pi_\alpha(x)) = r_n$, and for any $x \in A_i \setminus A_{i+1}$ ($i = 1, 2, \cdots, n-1$), $h(\mu_\alpha(x), \pi_\alpha(x)) = r_i$.*

Proof. By Theorem 8 and Theorem 11, we have

(I) $\pi_\alpha(x) = 0$ for each $x \in E$;

(II) The set A_1=supp(μ_α, π_α) is a basis of matroid (E, I_{r_1}).

Now we just prove that (III) holds.

Let $A_i = C_{r_i}(\mu_\alpha, \pi_\alpha)$ ($2 \leq i \leq n$). By the hypothesis, we have $C_{r_n}(\mu_\alpha, \pi_\alpha) \subseteq C_{r_{n-1}}(\mu_\alpha, \pi_\alpha) \subseteq \cdots \subseteq C_{r_2}(\mu_\alpha, \pi_\alpha) \subseteq C_{r_1}(\mu_\alpha, \pi_\alpha)$, That is $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_2 \subseteq A_1$.

Next, we will prove A_i is a maximal subset of A_{i-1} in I_{r_i}, where $k + 1 \leq i \leq n$.

Note that A_1=supp(μ_α, π_α) is the basis of (E, I_{r_1}).

Assume that there exists $A_i \in I_{r_i}$ ($2 \leq i \leq n$) such that A_i is not a maximal subset of A_{i-1} in $I_{r_{i-1}}$. Then there is $B \in I_{r_i}$ such that $A_i \subset B$ and B is a maximal subset of A_{i-1}.

Let $(\mu_\beta, \pi_\beta) \in IFS(E)$ and $\pi_\beta(x) = 0$ for each $x \in E$, and if $i = 2$, let

$$h(\mu_\beta(x), \pi_\beta(x)) = \begin{cases} r_1 \quad , & x \in A_1 \setminus B, \\ r_2 \quad , & x \in B \setminus A_2, \\ h(\mu_\alpha(x), \pi_\alpha(x)), & x \in A_2. \end{cases}$$

If $3 \leq i \leq n$, let

$$h(\mu_\beta(x), \pi_\beta(x)) = \begin{cases} r_j \quad , & x \in A_j \setminus A_{j+1}, \\ r_{i-1} \quad , & x \in A_{i-1} \setminus B, \\ r_i \quad , & x \in B \setminus A_i, \\ h(\mu_\alpha(x), \pi_\alpha(x)), & x \in A_i. \end{cases}$$

where $j = 1, 2, \cdots, i-2$. Then $(\mu_\alpha, \pi_\alpha) \preceq (\mu_\beta, \pi_\beta)$. Since $C_{r_i}(\mu_\beta, \pi_\beta) = B \in I_{r_i}$, it follows that $C_{r_j}(\mu_\beta, \pi_\beta) = A_j \in I_{r_j}$, for any $1 \leq j \leq i-1$, and $C_{r_j}(\mu_\beta, \pi_\beta) = C_{r_j}(\mu_\alpha, \pi_\alpha) \in I_{r_j}$ for any $i + 1 \leq j \leq n$. Then, by Theorem 6, $(\mu_\beta, \pi_\beta) \in \psi$, which contradicts that (μ_α, π_α) is an IFB of (E, ψ).

Conversely, from condition (II) (III), A_1 =supp(μ_α, π_α) is a crisp basis of matroid (E, I_{r_1}), $R^+(\mu_\alpha, \pi_\alpha) \subseteq \{r_1, r_2, \cdots, r_n\}$ and $C_{r_i}(\mu_\alpha, \pi_\alpha) = A_i \in I_{r_i}$ for any $r_i \in R^+(\mu_\alpha, \pi_\alpha)$ ($i = 1, 2, \cdots, n$). It follows that $(\mu_\alpha, \pi_\alpha) \in \psi$ from Theorem 6. □

(μ_α, π_α) is not an IFB of (E, ψ). Since $(\mu_\alpha, \pi_\alpha) \in \psi$ and (E, ψ) is a closed IFM, there exists an IFB (μ_β, π_β) of (E, ψ) such that $(\mu_\alpha, \pi_\alpha) \prec (\mu_\beta, \pi_\beta)$, so $m(\mu_\alpha, \pi_\alpha) \leq m(\mu_\beta, \pi_\beta)$ and $\text{supp}(\mu_\alpha, \pi_\alpha) \subseteq \text{supp}(\mu_\beta, \pi_\beta)$.

Case 1. $\text{supp}(\mu_\alpha, \pi_\alpha) = \text{supp}(\mu_\beta, \pi_\beta)$. Since (μ_β, π_β) is an IFB of (E, ψ), then $\pi_\beta(x) = 0$ for each $x \in E$ and $A_1 = \text{supp}(\mu_\alpha, \pi_\alpha) = \text{supp}(\mu_\beta, \pi_\beta)$ is a basis of matroid (E, I_{r_1}). As $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n$ and A_i is a maximal subset of A_{i-1}, where $A_i \in I_{r_i}$ $(i = 2, 3, \cdots, n)$, for any $x \in A_n$, $h(\mu_\beta(x), \pi_\beta(x)) = r_n$ and for any $x \in A_i \setminus A_{i+1}$ $(i = 1, 2, \cdots, n-1)$, $h(\mu_\beta(x), \pi_\beta(x)) = r_i$, for any $x \in \text{supp}(\mu_\alpha, \pi_\alpha) = \text{supp}(\mu_\beta, \pi_\beta)$, we have $h(\mu_\alpha(x), \pi_\alpha(x)) = h(\mu_\beta(x), \pi_\beta(x))$. Since $\pi_\alpha(x) = \pi_\beta(x) = 0$ for each $x \in E$, $H(\mu_\alpha(x), \pi_\alpha(x)) = H(\mu_\beta(x), \pi_\beta(x))$. It follows that $(\mu_\alpha, \pi_\alpha) = (\mu_\beta, \pi_\beta)$, which contradicts that $(\mu_\alpha, \pi_\alpha) \prec (\mu_\beta, \pi_\beta)$, $m(\mu_\alpha, \pi_\alpha) \leq m(\mu_\beta, \pi_\beta)$.

Case 2. $\text{supp}(\mu_\alpha, \pi_\alpha) \subset \text{supp}(\mu_\beta, \pi_\beta)$. Since (μ_β, π_β) is an IFB of (E, ψ), $C_{m(\mu_\beta, \pi_\beta)}(\mu_\beta, \pi_\beta) = \text{supp}(\mu_\beta, \pi_\beta)$ is a basis of matroid (E, I_{r_1}). From condition (II), $C_{m(\mu_\alpha, \pi_\alpha)}(\mu_\alpha, \pi_\alpha) = \text{supp}(\mu_\alpha, \pi_\alpha)$ is also a basis of matroid (E, I_{r_1}). Then $\text{supp}(\mu_\alpha, \pi_\alpha) = \text{supp}(\mu_\beta, \pi_\beta)$, which is in contradiction with $\text{supp}(\mu_\alpha, \pi_\alpha) \subset \text{supp}(\mu_\beta, \pi_\beta)$.

Therefore, (μ_α, π_α) is an IFB of (E, ψ).

The following corollary is obvious.

Corollary 1. *Suppose (E, ψ) is a closed $G - V$ IFM with the fundamental sequence $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ and the induced matroid sequence $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$, where $M_{r_i} = (E, I_{r_i})$ $(1 \leq i \leq n)$. Let $(\mu_\alpha, 0) \in IFS(E)$. Then $(\mu_\alpha, 0)$ is an IFB of (E, ψ) if and only if the IFS $(\mu_\alpha, 0)$ satisfies:*

(1) *A_1 is a crisp basis of (E, I_{r_1}), where $A_1 = supp(\mu_\alpha, 0)$.*

(2) *There exist $A_2, \cdots, A_{n-1}, A_n$ $(A_i \in I_{r_i})$ which satisfy $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n$ and A_i is a maximal subset of A_{i-1} $(i = 2, 3, \cdots, n)$ such that $h(\mu_\alpha(x), 0) = \mu_\alpha(x) = r_n$ for any $x \in A_n$, and $h(\mu_\alpha(x), 0) = \mu_\alpha(x) = r_i$ for any $x \in A_i \setminus A_{i+1}$, $i = 1, 2, \cdots, n-1$.*

Theorem 14. *Let E be a finite set. Suppose that there is the same fundamental sequence $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ and the same induced matroid sequence $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$ for $G - V$ fuzzy matroid $(E, \overline{\psi})$ and $G - V$ IFM (E, ψ), where $M_{r_i} = (E, I_{r_i})$ $(i = 1, 2, \cdots, n-1)$. Then $\mu_\alpha \in FS(E)$ is a fuzzy basis of $FM = (E, \overline{\psi})$ if and only if $(\mu_\alpha, 0) \in IFS(E)$ is an IFB of (E, ψ).*

Proof. By the hypothesis and Theorem 12, we have μ_α is a fuzzy basis of $(E, \overline{\psi})$ if and only if the fuzzy set μ_α satisfies:

(1) A_1 is a basis of (E, I_{r_1}), where $A_1 = \text{supp}\mu_\alpha$.

(2) There exist $A_2, \cdots, A_{n-1}, A_n$ which satisfy A_i is a maximal subset of A_{i-1} $(i = 2, 3, \cdots, n)$ and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n$ such that for any $x \in A_n$, $\mu_\alpha(x) = r_n$, and for any $x \in A_i \setminus A_{i+1}$ $(i = 1, 2, \cdots, n-1)$, $\mu_\alpha(x) = r_i$.

These two conditions hold if and only if $(\mu_\alpha, 0)$ satisfies:

(1) $A_1 = supp(\mu_\alpha, 0)$ is a crisp basis of matroid (E, I_{r_1}).

(2) For the above $A_i, i = 1, 2, \cdots, n$, we have for any $x \in A_n$, $h(\mu_\alpha(x), 0) = \mu_\alpha(x) = r_n$, and for any $x \in A_i \setminus A_{i+1}$ $(i = 1, 2, \cdots, n-1)$, $h(\mu_\alpha(x), 0) = \mu_\alpha(x) = r_i$.

□

5. A Tree Structure of a Closed $G - V$ IFM

From Theorem 13, a tree structure of a closed $G - V$ IFM is proposed below, which is similar to the tree structure introduced in [25].

Let (E, ψ) be a closed $G - V$ IFM on E, $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ be the fundamental sequence and $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$ be the IFM-induced matroid sequence (where $M_{r_i} = (E, I_{r_i})$

$(1 \leq i \leq n)$). Suppose that (μ_α, π_α) is an IFB of (E, ψ) and $B_1 = $supp$(\mu_\alpha, \pi_\alpha)$ is a crisp basis of matroid (E, I_{r_1}). Then, from Theorem 13, there exists a sequence $B_{2,1}, \cdots, B_{n-1,1}, B_{n,1}$ $(B_{i,1} \in I_{r_i}, i = 2, 3, \cdots, n)$ such that $B_{i,1}$ is a maximal subset of $B_{i-1,1}$ $(i = 2, 3, \cdots, n)$ in I_{r_i} and $B_1 \supseteq B_{2,1} \supseteq \cdots \supseteq B_{n-1,1} \supseteq B_{n,1}$. Obviously, $C_{r_i}(\mu_\alpha, \pi_\alpha) = B_{i,1}, i = 1, 2, \cdots, n$. The number of the sequence $B_1, B_{2,1}, \cdots, B_{n-1,1}, B_{n,1}$ is determined by the number of the maximal subsets of the previous maximal subset in the next level based on the same IFB (μ_α, π_α). Obviously, each of the sequence can be constructed a brunch of a tree. All the sequences of the same IFB (μ_α, π_α) can be constructed a tree. Since there are many $IFBs$, there are many trees which become a forest. The forest is called a tree structure of the closed $G - V$ IFM (E, ψ) (Figure 1).

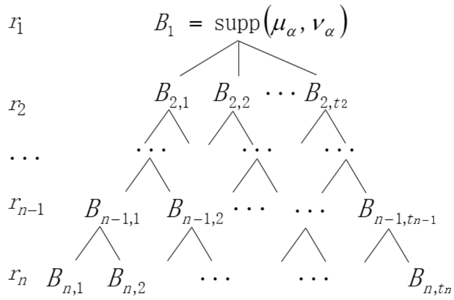

Figure 1. The tree structure of a closed G-V IFM.

Definition 12. *The set of trees constructed by the sequences in Theorem 13 is the tree structure of a closed* $G - V$ IFM (E, ψ), *denoted by* $T(E, \psi)$ *(T for short) (Figure 1), which is defined below.*

Remark 1. *There is one branch corresponding to a leaf in T and vice versa. From Theorem 13 and the construction of T, a branch of T and an IFB of* (E, ψ) *are one-to-one corresponding. Thus, for* (E, ψ)*, the number of the IFB is equal to the number of leaves* $(B_{n,j})$ *of T.*

Example 1. *Let* $E = \{a, b, c\}$, $I_1 = \{\emptyset, \{a\}, \{b\}\}$, $I_{1/3} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$, $I_{1/5} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Then* (E, I_1), $(E, I_{1/3})$ *and* $(E, I_{1/5})$ *are all matroids, and* $I_{1/5}$, $I_{1/3}$, I_1. *Let*

$$I_r = \begin{cases} I_{1/5}, \; 0 < r \leq \dfrac{1}{5}, \\ I_{1/3}, \; \dfrac{1}{5} < r \leq \dfrac{1}{3}, \\ I_1 \quad, \; \dfrac{1}{3} < r \leq 1. \end{cases}$$

and let $\psi = \{(\mu_\alpha, \pi_\alpha) \in IFS(E) | C_r(\mu_\alpha, \pi_\alpha) \in I_r\}$, *where* $r \in (0, 1]$. *From Definition 2.16,* (E, ψ) *is a closed* $G - V$ IFM. *The tree structure T is shown in Figure 2.*

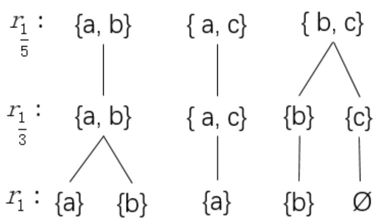

Figure 2. The tree structure of Example 5.3.

From Figure 2, there are three trees and five leaves in T. By Remark 1, there are five $IFBs$ of (E, ψ), which are as follows:

$$(\mu_{\alpha_1}(x), \pi_{\alpha_1}(x)) = \begin{cases} (1,0), \ x = a, \\ (\frac{1}{3},0), \ x = b, \\ (0,0), \ x = c. \end{cases}$$

$$(\mu_{\alpha_2}(x), \pi_{\alpha_2}(x)) = \begin{cases} (\frac{1}{3},0), \ x = a, \\ (1,0), \ x = b, \\ (0,0), \ x = c. \end{cases}$$

$$(\mu_{\alpha_3}(x), \pi_{\alpha_3}(x)) = \begin{cases} (1,0), \ x = a, \\ (0,0), \ x = b, \\ (\frac{1}{3},0), \ x = c. \end{cases}$$

$$(\mu_{\alpha_4}(x), \pi_{\alpha_4}(x)) = \begin{cases} (0,0), \ x = a, \\ (1,0), \ x = b, \\ (\frac{1}{5},0), \ x = c. \end{cases}$$

$$(\mu_{\alpha_5}(x), \pi_{\alpha_5}(x)) = \begin{cases} (0,0), \ x = a, \\ (\frac{1}{5},0), \ x = b, \\ (\frac{1}{3},0), \ x = c. \end{cases}$$

Then the values of the similarity function h for the five $IFBs$ are below:

$$h(\mu_{\alpha_1}(x), \pi_{\alpha_1}(x)) = \begin{cases} 1, \ x = a, \\ \frac{1}{3}, \ x = b, \\ 0, \ x = c. \end{cases}$$

$$h(\mu_{\alpha_2}(x), \pi_{\alpha_2}(x)) = \begin{cases} \frac{1}{3}, \ x = a, \\ 1, \ x = b, \\ 0, \ x = c. \end{cases}$$

$$h(\mu_{\alpha_3}(x), \pi_{\alpha_3}(x)) = \begin{cases} 1, \ x = a, \\ 0, \ x = b, \\ \frac{1}{3}, \ x = c. \end{cases}$$

$$h(\mu_{\alpha_4}(x), \pi_{\alpha_4}(x)) = \begin{cases} 0, \ x = a, \\ 1, \ x = b, \\ \frac{1}{5}, \ x = c. \end{cases}$$

$$h(\mu_{\alpha_5}(x), \pi_{\alpha_5}(x)) = \begin{cases} 0, \ x = a, \\ \frac{1}{5}, \ x = b, \\ \frac{1}{3}, \ x = c. \end{cases}$$

Next, we discuss the properties of T for (E, ψ).

Theorem 15. *Let (E, ψ) be a closed $G - V$ IFM on E, $0 = r_0 < r_1 < \cdots < r_n \leq 1$ be the fundamental sequence and $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$ (where $M_{r_i} = (E, I_{r_i})$ $(1 \leq i \leq n)$) be the induced matroid sequence. Let T be the tree structure of (E, ψ). Then each basis $B_i^{k_i}$ of the induced matroid $(E, I_{r_i})(i = 1, 2, \cdots, n$. k_i is a positive integer) is in r_i level of T.*

Proof. For any $i(i = 1, 2, \cdots, n)$, if $i = 1$, since each basis $B_1^{k_1}$ of matriod $M_{r_1} = (E, I_{r_1})$ is the root of each tree in T, $B_1^{k_1}$ is in r_1 level.

If $i \neq 1(i = 2, 3, \cdots, n)$, for any basis $B_i^{k_i}$ of matroid $M_{r_i} = (E, I_{r_i})$—since $(E, I_{r_i}) \subset (E, I_{r_{i-1}})$, it follows that $B_i^{k_i} \in I_{r_{i-1}}$—then there exists a basis $B_{i-1}^{k_{i-1}}$ of $(E, I_{r_{i-1}})$ such that $B_i^{k_i} \subseteq B_{i-1}^{k_{i-1}}$. Obviously, $B_i^{k_i}$ is a maximal subset of $B_{i-1}^{k_{i-1}}$ in I_{r_i}. It implies that $B_i^{k_i}$ is in r_i level of T.

Note that the converse of Theorem 15 does not hold. In Example 1, $\{a, b\}$,$\{a, c\}$ are both the bases of matroid $(E, I_{1/3})$ in the second level, but $\{b\}$,$\{c\}$ are not the bases. $\square$

Theorem 16. *Let (E, ψ) be a closed $G - V$ IFM on E, $0 = r_0 < r_1 < \cdots < r_n \leq 1$ be the fundamental sequence and $M_{r_1} \supset M_{r_2} \supset \cdots \supset M_{r_n}$ (where $M_{r_i} = (E, I_{r_i})$ $(1 \leq i \leq n)$) be the induced matroid sequence. Let T be the tree structure of (E, ψ). Suppose that $\mathbf{B}_i$ is the collection of the sets in r_i level of T, where $i = 1, 2, \cdots, n$. Let $J_{r_i} = \{X \mid X \subseteq B, B \in \mathbf{B}_i\}$. Then $J_{r_i} = I_{r_i}$.*

Proof. For any $Y \in I_{r_i}$, by the hypothesis, there is a basis B of matroid (E, I_{r_i}) such that $Y \subseteq B$. By Theorem 15, all bases of (E, I_{r_i}) are in r_i level T, where $i = 1, 2, \cdots, n$. Then $B \in \mathbf{B}_i$. It implies that $Y \in \{X|X \subseteq B, B \in \mathbf{B}_i\} = J_{r_i}$. Thus, $I_{r_i} \subseteq J_{r_i}$.

On the other hand, for any $Y \in J_{r_i}$, there exists a set $B \in \mathbf{B}_i$ in r_i $(i = 1, 2, \cdots, n)$ level of T such that $Y \subseteq B$. By Theorem 13, $B \in I_{r_i}$, $Y \in I_{r_i}$. That implies that $J_{r_i} \subseteq I_{r_i}$.

Therefore, $J_{r_i} = I_{r_i}$. $\square$

Remark 2. *Let (E, ψ) be a closed $G - V$ IFM on E and T be its tree structure. Suppose that $\mathbf{B}_i$ is the collection of the maximal subsets in r_i level of T. Then the bases of $M_{r_i} = (E, I_{r_i})$ $(i = 1, 2, \cdots, n)$ belong to $\mathbf{B}_i$.*

Theorem 17. *Let (E, ψ) be a closed $G - V$ IFM on E and T be its tree structure. Suppose that the sequence $B_1, B_2, \cdots, B_n$ (B_i is in $i - th$ level) of T satisfying $B_n \neq \emptyset$ and $B_1 \supset B_2 \supset \cdots \supset B_n$. For any $x \in B_n$, let $(\mu_\alpha, \pi_\alpha) \in IFS(E)$ and $k_n = h(\mu_\alpha(x), \pi_\alpha(x))$ and for any $x \in B_i \setminus B_{i+1}$ $(i = 1, 2, \cdots, n-1)$, let $k_i = h(\mu_\alpha(x), \pi_\alpha(x))$. Then $0 = k_0, k_1, k_2, \cdots, k_n$ is the fundamental sequence of (E, ψ).*

Proof. Let $0 = r_0 < r_1 < r_2 < \cdots < r_n \leq 1$ be the fundamental sequence of (E, ψ). By the hypothesis and Theorem 13, (μ_α, π_α) is a fuzzy basis of (E, ψ). Thus $R^+(\mu_\alpha, \pi_\alpha) \subseteq \{r_1, r_2, \cdots, r_n\}$. Suppose that a sequence $B_1, B_2, \cdots, B_n$ satisfies $B_n \neq \emptyset$ and $B_1 \supset B_2 \supset \cdots \supset B_n$. It follows that $B_i \backslash B_{i+1} \neq \emptyset$ $(i = 1, 2, \cdots, n-1)$. Then $k_i = h(\mu_\alpha, \pi_\alpha) \neq 0$ for any i $(i = 1, 2, \cdots, n-1)$ and $R^+(\mu_\alpha, \pi_\alpha) = \{k_1, k_2, \cdots, k_n\}$. Thus $\{k_1, k_2, \cdots, k_n\} \subseteq \{r_1, r_2, \cdots, r_n\}$. That implies that $\{k_1, k_2, \cdots, k_n\} = \{r_1, r_2, \cdots, r_n\}$.

Therefore, $k_0, k_1, k_2, \cdots, k_n$ is the fundamental sequence of (E, ψ). $\square$

6. Conclusions

In this paper, the IFB of $G - V$ $IFMs$ was defined by using the related concept of $G - V$ fuzzy matroids. Some conclusions of $G - V$ fuzzy matroids have been extended to $G - V$ $IFMs$. Especially, the judgement of an IFB was presented and proven, and the tree structure of closed $G - V$ $IFMs$ and its properties were discussed. We will discuss another important concept and its properties of $G - V$ $IFMs$–intuitionistic fuzzy circuits in a subsequent article.

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Abbreviations

The following abbreviations are used in this manuscript:

$G-V$ fuzzy matroid or $G-V\ FM$	Fuzzy matroid proposed by Goetschel and Voxman
IFM	Intuitionistic fuzzy matroid
IFB	Intuitionistic fuzzy basis
FS	Fuzzy set
IFS	Intuitionistic fuzzy set

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