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Article

C^*-Algebra Valued Partial b-Metric Spaces and Fixed Point Results with an Application

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Abstract: In this paper, we enlarge the class of C^*-algebra valued partial metric spaces as well as the class of C^*-algebra valued b-metric spaces by introducing the class of C^*-algebra valued partial b-metric spaces and utilize the same to prove our fixed point results. We furnish an example to highlight the utility of our main result. Finally, we apply our result in order to examine the existence and uniqueness of a solution for the system of Fredholm integral equations.

Keywords: C^*-algebra; C^*-algebra valued partial b-metric space; fixed point

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1. Introduction

The theory of fixed point is a very active area of research despite having a history of more than hundred years. The strength of fixed point theory lies in its application, which is spread throughout the existing literature fixed point theory. In the field of metric fixed point theory, the first important and significant result was proved by Banach [1] in 1922. The celebrated Banach contraction principle has been extended and generalized in numerous different directions (see [2–11]). To enhance the domain of applicability, I.A. Bakhtin [2], S. Czerwik [5], introduced the concept of b-metric space as a noted improvement of metric spaces and proved fixed point results as an analogue of Banach contraction principle. In the recent past, several research articles dealing with the fixed point theory for single-valued and multivalued mappings in b-metric spaces and by now there exists a considerable literature in such spaces (see [12–14]). On the other hand, with a similar quest, Matthews [3] employed another way to enlarge the class of metric spaces by introducing the notion of partial metric spaces and established an analogue of Banach contraction principle in such spaces. Thereafter, several metrical fixed point results were extended to partial metric spaces that were essentially inspired by Matthews (see [15–17]). Motivated by these two ideas of b-metric spaces and partial metric spaces, Shukla [18] introduced the notion of partial b-metric spaces that is a genuinely sharper version of both b-metric spaces and partial metric spaces and utilize the same to prove fixed point results in such spaces. Later on, many researchers proved some existence and uniqueness results on a fixed point in partial b-metric spaces (see [19–21]).

In 2014, Ma et al. [22] established the notion of C^*-algebra valued metric spaces (in short C^*-avMS) by replacing the range set $\mathbb{R}$ with an unital C^*-algebra, which is more general class than the class of metric spaces and utilized the same to prove some fixed point results is such spaces. In 2015, Ma et al. [23] introduce the notion of C^*-algebra valued b-metric spaces as a generalization of C^*-avMS and proved some fixed point results also used their results as an application for an integral type operator. Very soon, Chandok [24], generalized the class of C^*-avMS by introducing the class of C^*-algebra valued partial metric spaces and utilize the same to prove some fixed point theorems.

Inspired by foregoing observations, we enlarge the class of C^*-avbMS and C^*-avPMS by introducing the class of C^*-avPbMS and utilize the same to prove fixed point result. We also furnish some examples which demonstrate the utility of our results. Moreover, we apply our main result to examine the existence and uniqueness of a solution for the system of integral type operators.

This paper consists of five sections, wherein Section 1 begins with an introduction. In Section 2, we first recall some related definitions and remarks thereafter we introduce the notion of C^*-algebra valued partial b-metric space and discuss its related properties. In Section 3, we define the contraction condition in the setting of C^*-algebra valued partial b-metric space thereafter we prove fixed point result besides giving an example in support of our main result and give two corollaries. In Section 4, we apply our main result to examine the existence and uniqueness of a solution for the system of Fredholm integral equation and in the last section, we accomplish the conclusion part.

2. Preliminaries

Throughout the paper, we denote $\mathcal{A}$ by an unital (*i.e.*, unity element I) C^*-algebra with linear involution $*$, such that, for all $a, b \in \mathcal{A}$, $(ab)^* = b^*a^*$, and $a^{**} = a$. A positive element $a \in \mathcal{A}$ is denoted by $0_{\mathcal{A}} \preccurlyeq a$, where $0_{\mathcal{A}}$ is a zero element in $\mathcal{A}$. If $a = a^*$ and $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is non-invertible } \} \subseteq [0, \infty)$. The partial ordering on $\mathcal{A}$ can be defined as follows: $a \preccurlyeq b$ if and only if $0_{\mathcal{A}} \preccurlyeq b - a$. The pair $(\mathcal{A}, *)$ is said to be an unital $*$-algebra, if it contains the unity element I. A unital $*$-algebra $(\mathcal{A}, *)$ is called a Banach $*$-algebra, if it satisfies $\|a^*\| = \|a\|$ along with a complete sub-multiplicative norm. A Banach $*$-algebra satisfying $\|a^*a\| = \|a\|^2$, for all $a \in \mathcal{A}$ is called a C^*-algebra.

The following definition was introduced by Ma et al. [22]:

Definition 1. *Let $A \neq \emptyset$. A mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av metric on A, if it satisfies the following for all $a, b, c \in A$:*

(i) *$d(a,b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a,b) = 0_{\mathcal{A}}$ iff $a = b$;*
(ii) *$d(a,b) = d(b,a)$;*
(iii) *$d(a,b) \preccurlyeq d(a,c) + d(c,b)$.*

The triplet $(A, \mathcal{A}, d)$ is called a C^*-avMS.

In 2015, again Ma et al. [23] introduced the notion of C^*-av b-metric space, as follows:

Definition 2. *Let $A \neq \emptyset$ and $s \in \mathcal{A}$ such that $s \succcurlyeq I$. A mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av b-metric on A, if it satisfies the following for all $a, b, c \in A$:*

(i) *$d(a,b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a,b) = 0_{\mathcal{A}}$ iff $a = b$;*
(ii) *$d(a,b) = d(b,a)$; and,*
(iii) *$d(a,b) \preccurlyeq s[d(a,c) + d(c,b)]$.*

The triplet $(A, \mathcal{A}, d)$ is called a C^*-avbMS.

Remark 1. *Clearly, if $s = 1$, then a C^*-avbMS reduces to a C^*-avMS.*

Now, we recall the definition of C^*-algebra valued partial metric space introduced by Chandok et al. [24].

Definition 3. *Let $A \neq \emptyset$. A mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av partial metric on A, if it satisfies the following for all $a, b, c \in A$:*

(*i*) $d(a,b) \succcurlyeq 0_{\mathcal{A}}$ *and* $a = b \Leftrightarrow d(a,a) = d(b,b) = d(a,b)$;
(*ii*) $d(a,a) \preccurlyeq d(b,a)$;
(*iii*) $d(a,b) = d(b,a)$; *and,*
(*iv*) $d(a,b) \preccurlyeq d(a,c) + d(c,b) - d(c,c)$.

The triplet $(A, \mathcal{A}, d)$ *is called a* C^**-avPMS.*

Remark 2. *Obviously, if* $d(a,a) = 0_{\mathcal{A}}$ *for all* $a \in A$, *then* $(A, \mathcal{A}, d)$ *is a* C^**-avMS.*

Now, we define C^*-algebra valued partial b-metric space (in short C^*-avPbMS), as follows:

Definition 4. *Let* $A \neq \emptyset$ *and* $s \in \mathcal{A}$ *such that* $s \succcurlyeq I$. *A mapping* $d : A \times A \to \mathcal{A}$ *is called a* C^**-av partial b-metric on* A, *if it satisfies the following for all* $a, b, c \in A$:

(*i*) $d(a,b) \succcurlyeq 0_{\mathcal{A}}$ *and* $a = b \Leftrightarrow d(a,a) = d(b,b) = d(a,b)$;
(*ii*) $d(a,a) \preccurlyeq d(b,a)$;
(*iii*) $d(a,b) = d(b,a)$;
(*iv*) $d(a,b) \preccurlyeq s[d(a,c) + d(c,b)] - d(c,c)$.

The triplet $(A, \mathcal{A}, d)$ *is called a* C^**-avPbMS.*

Observe that, a C^*-avPbMS $(A, \mathcal{P})$ is a generalization of both C^*-avbMS as well as C^*-avPMS. Obviously, every C^*-avbMS is a C^*-avPbMS with zero self distance and every C^*-avPMS is a C^*-avPbMS with $s = 1$, but converse is not true in general.

$$\begin{tikzcd}
 & MS \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & bMS \arrow[dl] \arrow[dd] \\
PMS \arrow[rr] \arrow[dd] & & PbMS \arrow[dd] & \\
 & C^*-avMS \arrow[rr, dotted] \arrow[dl, dotted] & & C^*-avbMS \arrow[dl] \\
C^*-avPMS \arrow[rr] & & C^*-avPbMS &
\end{tikzcd}$$

Example 1. *Let* $A = [0,1]$ *and* $\mathcal{A} = M_2(\mathbb{C})$, *the class of bounded and linear operators on a Hilbert space* $\mathbb{C}^2$. *Define* $d : A \times A \to \mathcal{A}$ *by (for all* $a, b \in A$*):*

$$d(a,b) = \begin{bmatrix} |a-b|^p & 0 \\ 0 & k|a-b|^p \end{bmatrix} + \begin{bmatrix} \max\{a,b\}^p & 0 \\ 0 & k\max\{a,b\}^p \end{bmatrix}$$

where $k \geq 0$ *and* $p > 1$. *Then,* $(A, \mathcal{A}, d)$ *is a* C^**-avPbMS with coefficient* $s = 2^{p-1}I$. *However, it is easy to see that* $(A, \mathcal{A}, d)$ *is neither a* C^**-avbMS nor* C^**-avPMS. To substantiate the claim, for any non-zero element* $a \in A$, *we have*

$$d(a,a) = \begin{bmatrix} a^p & 0 \\ 0 & ka^p \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{\mathcal{A}}.$$

Therefore, $(A, \mathcal{A}, d)$ *is not a* C^**-avbMS. Furthermore, for* $a = 0,\ b = 1$ *and* $c = 0.5$, *we obtain*

$$d(a,b) = \begin{bmatrix} |0-1|^p & 0 \\ 0 & k|0-1|^p \end{bmatrix} + \begin{bmatrix} \max\{0,1\}^p & 0 \\ 0 & k\max\{0,1\}^p \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2k \end{bmatrix}$$

and

$$d(a,c)+d(c,b)-d(c,c)=\begin{bmatrix}\frac{2}{2^p}+1 & 0\\ 0 & k(\frac{2}{2^p}+1)\end{bmatrix}.$$

Thus,

$$d(a,b)\succ d(a,c)+d(c,b)-d(c,c), \text{ for all } p>1.$$

Therefore, d is not C^*-avPMS on A.

Example 2. *Let $A=\mathbb{R}$ and $\mathcal{A}=M_3(\mathbb{C})$. Define $d: A\times A\to\mathcal{A}$ by (for all $a,b\in A$ and $p\geq 1$):*

$$d(a,b)=\begin{bmatrix}|a-b|^p & 0 & 0\\ 0 & |a-b|^p & 0\\ 0 & 0 & k|a-b|^p\end{bmatrix}+\alpha\begin{bmatrix}1&0&0\\0&1&0\\0&0&k\end{bmatrix}$$

where, $k\geq 0$ and $\alpha>0$. Observe that, d is C^*-avPbM and $(A,\mathcal{A},d)$ is a C^*-avPbMS with coefficient $s=2^{p-1}I$.

Example 3. *Let $(A,\mathcal{A},d_p)$ be a C^*-avPMS and $(A,\mathcal{A},d_b)$ a C^*-avbMS with coefficient $s\geq 1$ on A. Define a mapping $d: A\times A\to\mathcal{A}$ by (for all $a,b\in A$):*

$$d(a,b)=d_p(a,b)+d_b(a,b).$$

Subsequently, d is a C^*-avPbM and $(A,\mathcal{A},d)$ is a C^*-avPbMS.

Proof. It is easy to verify that the conditions $(i)-(iii)$ of Definition 4 are satisfied. To verify condition (iv) of Definition 4, we have (for all $a,b,c\in A$)

$$\begin{aligned}d(a,b) &= d_p(a,b)+d_b(a,b)\\ &\preceq d_p(a,c)+d_p(c,b)-d_p(c,c)+s[d_b(a,c)+d_b(c,b)]\\ &\preceq s[d_p(a,c)+d_p(c,b)]-d(c,c)+s[d_b(a,c)+d_b(c,b)]\\ &= s[d_p(a,c)+d_b(a,c)+d_p(c,b)+d_b(c,b)]-d(c,c)\\ &= s[d(a,c)+d(c,b)]-d(c,c).\end{aligned}$$

Therefore, d satisfies all the conditions of Definition 4. Hence, $(A,\mathcal{A},d)$ is a C^*-avPbMS. □

Let $(A,\mathcal{A},d)$ be a C^*-avPbMS. Afterwards, open ball of center $a\in A$ and radius $0_{\mathcal{A}}\prec\epsilon\in\mathcal{A}$ is defined by:

$$B_d(a,\epsilon)=\{b\in A: d(a,b)\prec d(a,a)+\epsilon\}.$$

Similarly, the closed ball with center $a\in A$ and radius $\epsilon>0$ is defined by:

$$B_d[a,\epsilon]=\{b\in A: d(a,b)\preceq d(a,a)+\epsilon\}.$$

The family of open balls (for all $a\in A$ and $\epsilon\succ 0_{\mathcal{A}}$)

$$\mathcal{U}_d=\{B_d(a,\epsilon): a\in A, \epsilon\succ 0_{\mathcal{A}}\},$$

forms a basis of some topology τ_d on A.

Lemma 1. *Let (A,τ_d) be a topological space and $f: A\to A$. If f is continuous, then every sequence $\{a_n\}\subseteq A$, such that $a_n\to a$ implies $fa_n\to fa$. The converse holds if A is metrizable.*

Definition 5. *A sequence $\{a_n\}$ in $(A, \mathcal{A}, d)$ is called convergent (with respect to $\mathcal{A}$) to a point $a \in A$, if for given $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|d(a_n, a) - d(a, a)\| < \epsilon$, for all $n > k$. We denote it by*

$$\lim_{n\to\infty} d(a_n, a) = d(a, a).$$

Definition 6. *A sequence $\{a_n\}$ in $(A, \mathcal{A}, d)$ is called Cauchy (with respect to $\mathcal{A}$), if $\lim\limits_{n\to\infty} d(a_n, a_m)$ exists and it is finite.*

Definition 7. *The triplet $(A, \mathcal{A}, d)$ is called complete C^*-avPbMS if every Cauchy sequence in A is convergent to some point a in A such that*

$$\lim_{n\to\infty} d(a_n, a_m) = \lim_{n\to\infty} d(a_n, a) = d(a, a).$$

The following example shows that the limit of convergence in C^*-avPbMS may or may not be unique.

Example 4. *Let $A = \mathbb{R}_+$ and $\mathcal{A} = M_3(\mathbb{R})$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$ and $p \geq 1$):*

$$d(a,b) = \begin{bmatrix} \max\{a,b\}^p & 0 \\ 0 & k\max\{a,b\}^p \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

where, $k \geq 0$ and $\alpha > 0$. Then d is C^*-avPbM and $(A, \mathcal{A}, d)$ is a C^*-avPbMS with coefficient $s = 2^{p-1}I$. Now, we construct a constant sequence $\{a_n\}$ in A by $a_n = k$. Choose, $b \in A$, such that $b \geq k$. Subsequently, we have

$$\begin{aligned} d(a_n, b) &= \begin{bmatrix} \max\{a_n,b\}^p & 0 \\ 0 & k\max\{a_n,b\}^p \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} b^p & 0 \\ 0 & kb^p \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \\ &= d(b,b). \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} d(a_n, b) = d(b, b)$, for all $b \geq k$. Hence, the limit of convergence in C^*-avPbMS may not be unique.

3. Fixed Point Results

The following definition is utilized in our results:

Definition 8. *Let $(A, \mathcal{A}, d)$ be a C^*-avPbMS. A mapping $f : A \to A$ is said to be C_b^*-contraction if there exists $\rho \in \mathcal{A}$ with $\|s\rho\| < 1$ such that*

$$d(fa, fb) \preccurlyeq \rho^* d(a,b)\rho, \ \ \forall\, a, b \in A. \tag{1}$$

Our main result runs, as follows:

Theorem 1. *Let $(A, \mathcal{A}, d)$ be a complete C^*-avPbMS and $f : A \to A$ be a C_b^*-contraction. Then f has a unique fixed point $a \in A$ such that $d(a, a) = 0_{\mathcal{A}}$.*

Proof. Choose $a_0 \in A$ for constructing an iterative sequence $\{a_n\}$ by:

$$a_1 = fa_0,\ a_2 = fa_1 = f^2a_0,\ a_3 = fa_2 = f^3a_0, ..., a_n = fa_{n-1} = f^na_0, \cdots .$$

We denote $\Delta_0 = d(a_0, a_1)$. Now, we assert that $\lim_{n,m\to\infty} d(a_n, a_{n+1}) = 0_{\mathcal{A}}$. On setting $a = a_n$ and $b = a_{n+1}$ in (1), we get

$$\begin{aligned} d(a_n, a_{n+1}) &= d(fa_{n-1}, fa_n) = \rho^* d(a_{n-1}, a_n)\rho \\ &\preccurlyeq (\rho^*)^2 d(a_{n-2}, a_{n-1})\rho^2 \\ &\preccurlyeq \dots \\ &\preccurlyeq (\rho^*)^n d(a_0, a_1)\rho^n \\ &\preccurlyeq (\rho^*)^n \Delta_0 \rho^n. \end{aligned}$$

Because $d(a_n, a_n) \preccurlyeq d(a_n, a_{n+1})$, we have

$$\lim_{n\to\infty} d(a_n, a_n) = 0_{\mathcal{A}}. \tag{2}$$

For any $n, p \in \mathbb{N}$, we have

$$\begin{aligned} d(a_n, a_{n+p}) &\preccurlyeq s[d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+p})] - d(a_{n+1}, a_{n+1}) \\ &\preccurlyeq sd(a_n, a_{n+1}) + s^2[d(a_{n+1}, a_{n+2}) + d(a_{n+2}, a_{n+p})] \\ &\quad -d(a_{n+1}, a_{n+1}) - d(a_{n+2}, a_{n+2}) \\ &\preccurlyeq sd(a_n, a_{n+1}) + s^2 d(a_{n+1}, a_{n+2}) + ... + \\ &\quad s^{n+p-1}[d(a_{n+p-2}, a_{n+p-1}) + d(a_{n+p-1}, a_{n+p})] \\ &\quad -d(a_{n+1}, a_{n+1}) - ... - d(a_{n+p-1}, a_{n+p-1}) \\ &\preccurlyeq s(\rho^*)^n \Delta_0 \rho^n + s^2(\rho^*)^{n+1}\Delta_0\rho^{n+1} + ... + \\ &\quad s^{n+p-1}(\rho^*)^{n+p-2}\Delta_0\rho^{n+p-2} + s^{n+p-1}(\rho^*)^{n+p-1}\Delta_0\rho^{n+p-1} \\ &\quad -(\rho^*)^{n+1} d(a_0, a_0)\rho^{n+1} - ... - (\rho^*)^{n+p-1} d(a_0, a_0)\rho^{n+p-1} \\ &= \sum_{k=1}^{p-1} s^k (\rho^*)^{n+k-1}\Delta_0\rho^{n+k-1} + s^{n+p-1}(\rho^*)^{n+p-1}\Delta_0\rho^{n+p-1} \\ &\quad -(\rho^*)^{n+1} d(a_0, a_0)\rho^{n+1} - ... - (\rho^*)^{n+p-1} d(a_0, a_0)\rho^{n+p-1} \\ &= \sum_{k=1}^{p-1} \left((\rho^*)^{n+k-1} s^{\frac{k}{2}} \Delta_0^{\frac{1}{2}}\right)\left(\Delta_0^{\frac{1}{2}} s^{\frac{k}{2}} \rho^{n+k-1}\right) \\ &\quad +\left((\rho^*)^{n+p-1} s^{\frac{n+p-1}{2}} \Delta_0^{\frac{1}{2}}\right)\left(\Delta_0^{\frac{1}{2}} s^{\frac{n+p-1}{2}} \rho^{n+p-1}\right) \\ &\quad -\left((\rho^*)^{n+1} d(a_0, a_0)^{\frac{1}{2}}\right)\left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+1}\right) - ... - \\ &\quad \left((\rho^*)^{n+p-1} d(a_0, a_0)^{\frac{1}{2}}\right)\left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+p-1}\right) \\ &= \sum_{k=1}^{p-1} \left(\Delta_0^{\frac{1}{2}} s^{\frac{k}{2}} \rho^{n+k-1}\right)^*\left(\Delta_0^{\frac{1}{2}} s^{\frac{k}{2}} \rho^{n+k-1}\right) \\ &\quad +\left(\Delta_0^{\frac{1}{2}} s^{\frac{n+p-1}{2}} \rho^{n+p-1}\right)^*\left(\Delta_0^{\frac{1}{2}} s^{\frac{n+p-1}{2}} \rho^{n+p-1}\right) \\ &\quad -\left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+1}\right)^*\left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+1}\right) - ... - \\ &\quad \left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+p-1}\right)^*\left(d(a_0, a_0)^{\frac{1}{2}} \rho^{n+p-1}\right) \\ &= \sum_{k=1}^{p-1} |\Delta_0^{\frac{1}{2}} s^{\frac{k}{2}} \rho^{n+k-1}|^2 + |\Delta_0^{\frac{1}{2}} s^{\frac{n+p-1}{2}} \rho^{n+p-1}|^2 \\ &\quad -|d(a_0, a_0)^{\frac{1}{2}} \rho^{n+1}|^2 - ... - |d(a_0, a_0)^{\frac{1}{2}} \rho^{n+p-1}|^2 \end{aligned}$$

$$
\begin{aligned}
&\preccurlyeq \sum_{k=1}^{p-1} \|\Delta_0^{\frac{1}{2}} s^{\frac{k}{2}} \rho^{n+k-1}\|^2 I + \|\Delta_0^{\frac{1}{2}} s^{\frac{n+p-1}{2}} \rho^{n+p-1}\|^2 I \\
&\quad -\|d(a_0,a_0)^{\frac{1}{2}} \rho^{n+1}\|^2 I - \ldots - \|d(a_0,a_0)^{\frac{1}{2}} \rho^{n+p-1}\|^2 I \\
&\preccurlyeq \|\Delta_0\| \sum_{k=1}^{p-1} \|s^k\| \|\rho^{2(n+k-1)}\| I + \|\Delta_0\| \|s^{n+p-1}\| \|\rho^{n+p-1}\|^2 I \\
&\quad -\|d(a_0,a_0)\| \|\rho^{2(n+1)}\| I - \ldots - \|d(a_0,a_0)\| \rho^{2(n+p-1)}\| I \\
&\preccurlyeq \|\Delta_0\| \|s\| \|\rho^{2n}\| \frac{(1-(\|s\|\|\rho\|^2)^{p-1})}{1-\|s\|\|\rho\|^2} I + \|\Delta_0\| \|s^{n+p-1}\| \|\rho^{n+p-1}\|^2 I \\
&\quad -\|d(a_0,a_0)\| \|\rho^{2(n+1)}\| I - \ldots - \|d(a_0,a_0)\| \rho^{2(n+p-1)}\| I \\
&\preccurlyeq \|\Delta_0\| \|s\| \|\rho^{2n}\| I + \|\Delta_0\| \|s^{n+p-1}\| \|\rho^{n+p-1}\|^2 I \\
&\quad -\|d(a_0,a_0)\| \|\rho^{2(n+1)}\| I - \ldots - \|d(a_0,a_0)\| \rho^{2(n+p-1)}\| I \\
&\to 0_{\mathcal{A}} \quad (\text{as } \; n \to \infty).
\end{aligned} \tag{3}
$$

Thus, $\{a_n\}$ is a Cauchy sequence in A. Now, by the completeness of A, there exists $a \in A$ such that

$$\lim_{n\to\infty} d(a_n, a_m) = \lim_{n\to\infty} d(a_n, a) = d(a, a).$$

By employing (3), we have

$$\lim_{n\to\infty} d(a_n, a) = d(a, a) = 0_{\mathcal{A}}.$$

Now, we will show that a is a fixed point f. For any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
d(fa, a) &\preccurlyeq s[d(fa, a_{n+1}) + d(a_{n+1}, a)] - d(a_{n+1}, a_{n+1}) \\
&= s[d(fa, fa_n) + d(a_{n+1}, a)] - d(a_{n+1}, a_{n+1}) \\
&\preccurlyeq s[\rho^* d(a, a_n)\rho + d(a_{n+1}, a)] - d(a_{n+1}, a_{n+1}) \\
&\to 0_{\mathcal{A}} \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, a is a fixed point of f. To show the uniqueness of the fixed point, suppose $a, b \in A$, such that $fa = a$ & $fb = b$. Then, by the definition of C_b^*-contraction, we have

$$d(a, b) = d(fa, fb) \preccurlyeq \rho^* d(a, b)\rho,$$

so that

$$\|d(a,b)\| = \|d(fa, fb)\| \leq \|\rho^* d(a,b)\rho\| \leq \|\rho^*\| \|d(a,b)\| \|\rho\| = \|\rho\|^2 \|d(a,b)\|$$

a contradiction. Hence, $a = b$, that is, f has a unique fixed point. Now, to show that $d(a,a) = 0_{\mathcal{A}}$. Suppose on contrary that $d(a,a) \neq 0_{\mathcal{A}}$. Subsequently, we have

$$\|d(a,a)\| = \|d(fa, fa)\| \leq \|\rho^* d(a,a)\rho\| \leq \|\rho^*\| \|d(a,a)\| \|\rho\| = \|\rho\|^2 \|d(a,a)\|$$

a contradiction. Therefore, $d(a,a) = 0_{\mathcal{A}}$. This completes the proof. □

To exhibit the utility of Theorem 1, we give the following example.

Example 5. *Let $A = [0,1]$, and $\mathcal{A} = M_2(\mathbb{C})$. Define $d : A \times A \to \mathcal{A}$ by:*

$$d(a,b) = \begin{bmatrix} |a-b|^2 & 0 \\ 0 & k|a-b|^2 \end{bmatrix} + \begin{bmatrix} \max\{a,b\}^2 & 0 \\ 0 & k\max\{a,b\}^2 \end{bmatrix}$$

where $k \geq 0$. Then, $(A, \mathcal{A}, d)$ is a complete C^*-avPbMS.

Define a map $f : A \to A$ by:

$$fa = \frac{a}{3}, \text{ for all } a \in A.$$

Observe that, $d(fa, fb) \preccurlyeq \rho^* d(a, b)\rho$, (for all $a, b \in A$) satisfies

$$\rho = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \in \mathbb{A} \text{ and } \|\rho\| = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} < 1.$$

Thus, all of the hypothesis of Theorem 1 are satisfied and $a = 0$ is unique fixed point of f.

In Theorem 1, by setting $s = I$ with zero self distance, which is, $d(a, a) = 0_{\mathcal{A}}$ for all $a \in A$, we obtain the result due to Ma et al. [22].

Corollary 1. *Let $(A, \mathcal{A}, d)$ be a complete C^*-avMS and $f : A \to A$ be a C_b^*-contraction. Afterwards, f has a unique fixed point $a \in A$.*

In Theorem 1, by setting $d(a, a) = 0_{\mathcal{A}}$ for all $a \in A$, we obtain the result due to Ma et al. [23].

Corollary 2. *Let $(A, \mathcal{A}, d)$ be a complete C^*-avPbMS and $f : A \to A$ be a C_b^*-contraction. Afterwards, f has a unique fixed point $a \in A$, such that $d(a, a) = 0_{\mathcal{A}}$.*

4. Application

As an application of Theorem 1, we find an existence and uniqueness result for a type of following integral equation:

$$a(\mu) = \int_E G(\mu, \nu, a(\nu))d\nu + h(\mu), \ \mu, \nu \in E, \tag{4}$$

where E is a measurable set, $G : E \times E \times \mathbb{R} \to \mathbb{R}$ and $h \in L^\infty(E)$.

Let $A = L^\infty(E)$, $H = L^2(E)$ and $L(H) = \mathcal{A}$. Define $d : A \times A \to \mathcal{A}$ by (for all $h, k, I \in A$, $p \geq 1$ and $\|\rho\| = k < 1$):

$$d(h, k) = \pi_{|h-k|^p + I},$$

where $\pi_u : H \to H$ is the multiplicative operator, which is defined by:

$$\pi_u(\phi) = u.\phi.$$

Now, we state and prove our result, as follows:

Theorem 2. *Suppose that, (for all $a, b \in A$)*

(1) there exist a continuous function $\psi : E \times E \to \mathbb{R}$ and $k \in (0, 1)$, such that

$$| G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu)) | \leq k \mid \psi(\mu, \nu) \mid (\mid a(\nu) - b(\nu) \mid + I - k^{-1} I),$$

for all $\mu, \nu \in E$.

(2) $\sup\limits_{\mu \in E} \int_E \mid \psi(\mu, \nu) \mid d\nu \leq 1$.

Subsequently, the integral Equation (4) has a unique solution in A.

Proof. Define $f : A \to A$ by:

$$fa(\mu) = \int_E G(\mu, \nu, a(\nu))d\nu + h(\mu), \ \forall \ \mu, \nu \in E.$$

Set $\rho = kI$, then $\rho \in \mathcal{A}$. For any $u \in H$ and $p \geq 1$, we have

$$
\begin{aligned}
\|d(fa, fb)\| &= \sup_{\|u\|=1} (\pi_{|fa-fb|^p+I}u, u) \\
&= \sup_{\|u\|=1} \int_E \left[\left| \int_E G(\mu,\nu,a(\nu)) - G(\mu,\nu,b(\nu)) d\nu \right|^p \right] u(\mu) u(\bar{\mu}) d\mu \\
&\quad + \sup_{\|u\|=1} \int_E u(\mu) u(\bar{\mu}) d\mu I \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E |G(\mu,\nu,a(\nu)) - G(\mu,\nu,b(\nu))| d\nu \right]^p |u(\mu)|^2 d\mu \\
&\quad + \sup_{\|u\|=1} \int_E |u(\mu)|^2 d\mu I \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E |k\psi(\mu,\nu)(a(\nu) - b(\nu) + I - k^{-1}I)| d\nu \right]^p |u(\mu)|^2 d\mu + I \\
&\leq k^p \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\mu,\nu)| d\nu \right]^p |u(\mu)|^2 d\mu \|a-b\|_\infty^p \\
&\leq k \sup_{\mu \in E} \int_E |\psi(\mu,\nu)| d\nu \sup_{\|u\|=1} \int_E |u(\mu)|^2 d\mu \|a-b\|_\infty^p \\
&\leq k\|a-b\|_\infty^p \\
&= \|\rho\| \, \|d(a,b)\|.
\end{aligned}
$$

Hence, the mapping f is a C_b^*-contraction with $\|\rho\| < 1$, so one can verify that all of the requirements of Theorem 1 are satisfied. Thus, the Fredholm integral Equation (4) has a unique solution, which is, f has a unique fixed point. □

Now, we give the following example in support of Theorem 2:

Example 6. *Let $E = [0,1]$, $A = L^\infty(E)$, and $H = L^2(E)$. Define $d : A \times A \to L(H)$ by:*

$$d(h,k) = \pi_{|h-k|^2+I},$$

where $\pi_u : H \to H$ is the multiplicative operator, which is defined by:

$$\pi_u(\phi) = u.\phi.$$

Subsequently, $(A, \mathcal{A}, d)$ is a complete C^*-avPbMS. Consider a function $\psi : E \times E \to \mathbb{R}$ defined by $\psi(\mu,\nu) = 1$ for all $\mu,\nu \in E$. Hence, we obtain

$$\sup_{\mu \in E} \int_E | \psi(\mu,\nu) | \, d\nu \leq 1.$$

Now, we define $G : E \times E \times \mathbb{R} \to \mathbb{R}$ by $G(\mu,\nu,a(\nu)) = (\mu-\nu)a(\nu)$. Let f be a self-mapping on A by:

$$fa(\mu) = \int_E G(\mu,\nu,a(\nu)) d\nu, \ \forall \, \mu,\nu \in E.$$

Observe that, $\|d(fa, fb)\| \leq \|\rho\| \|d(a,b)\|$, (for all $a,b \in A$) satisfies with $\rho = kI$ for any $k \in \left[0, \frac{1}{2}\right]$. Thus, all of the hypothesis of Theorem 2 are satisfied and we have a unique $a(\mu)$ with $fa = a$, which is required unique solution of Equation (4).

5. Conclusions

As the C^*-algebra valued metric space is a relatively new addition to the existing literature; therefore, in this note, we endeavor to further enrich this notion by introducing the idea of C^*-algebra valued partial b-metric space, wherein we generalized the notion of C^*-algebra valued partial metric space as well as the notion of C^*-algebra valued b-metric space. Our main result (i.e., Theorem 1) is an analogue of Banach contraction principle. An example is also included in order to highlight the realized improvements in our newly proved result. Finally, we apply Theorem 1 to examine the existence and uniqueness of a solution for the system of Fredholm integral equation. On the other hand, our main result remains possible for many generalized contractions, namely, weak contraction, Geraghty contraction, Suzuki contraction, and F-contraction etc.

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