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Multipolar Intuitionistic Fuzzy Hyper BCK-Ideals in Hyper BCK-Algebras

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Abstract: In 2020, Kang et al. introduced the concept of a multipolar intuitionistic fuzzy set of finite degree, which is a generalization of a k-polar fuzzy set, and applied it to a BCK/BCI-algebra. The specific purpose of this study was to apply the concept of a multipolar intuitionistic fuzzy set of finite degree to a hyper BCK-algebra. The notions of the k-polar intuitionistic fuzzy hyper BCK-ideal, the k-polar intuitionistic fuzzy weak hyper BCK-ideal, the k-polar intuitionistic fuzzy s-weak hyper BCK-ideal, the k-polar intuitionistic fuzzy strong hyper BCK-ideal and the k-polar intuitionistic fuzzy reflexive hyper BCK-ideal are introduced herein, and their relations and properties are investigated. These concepts are discussed in connection with the k-polar lower level set and the k-polar upper level set.

Keywords: k-polar intuitionistic fuzzy hyper BCK-ideal; k-polar intuitionistic fuzzy weak hyper BCK-ideal; k-polar intuitionistic fuzzy s-weak hyper BCK-ideal; k-polar intuitionistic fuzzy strong hyper BCK-ideal; k-polar intuitionistic fuzzy reflexive hyper BCK-ideal

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1. Introduction

As is well known, the fuzzy set, which was first introduced by Zadeh [1], dealt with the membership degree that is represented by only one function, the so-called truth function. As a generalization of the fuzzy set, the notion of the intuitionistic fuzzy set is introduced by Atanassove [2]. In 2014, Chen et al. [3] introduced an m-polar fuzzy set which is an extension of the bipolar fuzzy set, and then this notion was applied to graph theory, algebraic structure, the decision making problem, etc. For BCK/BCI-algebra, see [4–6]; for graph theory, see [7–10]; and see [11–14] for the decision making problem. In [15], Kang et al. introduced the concept of a multipolar intuitionistic fuzzy set of finite degree as a generalization of an intuitionistic fuzzy set, and they applied it to BCK/BCI-algebras. The hyperstructure theory was introduced by Marty [16] in 1934 at the 8th Congress of Scandinavian Mathematicians. Jun et al. [17,18] applied the hyperstructure theory to BCK-algebras, and they introduced a hyper BCK-algebra which is a generalization of a BCK-algebra. They studied hyper ideal theory in hyper BCK-algebras. Jun and Xin discussed the fuzzy set theory of hyper BCK-ideals in hyper BCK-algebras (see [19]), and Bakhshi et al. [20] studied fuzzy (positive, weak) implicative hyper BCK-ideals. In 2004, Borzooei and Jun [21] considered the intuitionistic fuzzy set theory of hyper BCK-ideals in hyper BCK-algebras.

In this paper, using the concept of a multipolar intuitionistic fuzzy set of finite degree, we introduce the notions of the k-polar intuitionistic fuzzy hyper BCK-ideal (briefly, k-pIF hBCK-ideal), the k-polar intuitionistic fuzzy weak hyper BCK-ideal (briefly, k-pIF weak hBCK-ideal), the k-polar intuitionistic fuzzy s-weak hyper BCK-ideal (briefly, k-pIF s-weak hBCK-ideal), the k-polar intuitionistic fuzzy strong hyper BCK-ideal (briefly, k-pIF strong hBCK-ideal) and the k-polar intuitionistic fuzzy reflexive hyper BCK-ideal (briefly, k-pIF reflexive hBCK-ideal). We investigate several properties and their relations. We discuss k-pIF (weak, s-weak, strong, reflexive) the hBCK-ideal in relation to k-polar upper and lower level sets.

2. Preliminaries

Let $\mathcal{H}$ be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets D and C of $\mathcal{H}$, denote by $D \circ C$ the set $\bigcup_{a \in D, b \in C} a \circ b$. We shall use $\zeta \circ \eta$ instead of $\zeta \circ \{\eta\}$, $\{\zeta\} \circ \eta$, or $\{\zeta\} \circ \{\eta\}$.

By a *hyper BCK-algebra* (briefly, hBCK-algebra), we mean a nonempty set $\mathcal{H}$ endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms (see [17]):

(HK1) $(\zeta \circ \varrho) \circ (\eta \circ \varrho) \ll \zeta \circ \eta$;
(HK2) $(\zeta \circ \eta) \circ \varrho = (\zeta \circ \varrho) \circ \eta$;
(HK3) $\zeta \circ \mathcal{H} \ll \{\zeta\}$;
(HK4) $\zeta \ll \eta$ and $\eta \ll \zeta$ imply $\zeta = \eta$,

for all $\zeta, \eta, \varrho \in \mathcal{H}$, where $\zeta \ll \eta$ is defined by $0 \in \zeta \circ \eta$ and for every $D, C \subseteq \mathcal{H}$, $D \ll C$ is defined by $\forall a \in D$, $\exists b \in C$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in $\mathcal{H}$.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall \zeta, \eta \in \mathcal{H})(\zeta \circ \eta \ll \{\zeta\}). \tag{1}$$

A subset D of a hBCK-algebra $\mathcal{H}$ is called

- A *hyper BCK-ideal* (briefly, hBCK-ideal) of $\mathcal{H}$ (see [17]) if

$$0 \in D, \tag{2}$$
$$(\forall \zeta, \eta \in \mathcal{H})(\zeta \circ \eta \ll D,\ \eta \in D \Rightarrow \zeta \in D). \tag{3}$$

- A *weak hyper BCK-ideal* (briefly, weak hBCK-ideal) of $\mathcal{H}$ (see [17]) if it satisfies (2) and

$$(\forall \zeta, \eta \in \mathcal{H})(\zeta \circ \eta \subseteq D,\ \eta \in D \Rightarrow \zeta \in D). \tag{4}$$

- A *strong hyper BCK-ideal* (briefly, strong hBCK-ideal) of $\mathcal{H}$ (see [18]) if it satisfies (2) and

$$(\forall \zeta, \eta \in \mathcal{H})((\zeta \circ \eta) \cap D \neq \emptyset,\ \eta \in D \Rightarrow \zeta \in D). \tag{5}$$

- A *reflexive hyper BCK-ideal* (briefly, reflexive hBCK-ideal) of $\mathcal{H}$ (see [18]) if it is a hBCK-ideal of $\mathcal{H}$ which satisfies:

$$(\forall \zeta \in \mathcal{H})(\zeta \circ \zeta \subseteq D). \tag{6}$$

Every hBCK-algebra $\mathcal{H}$ satisfies the following assertions.

$$(\forall \zeta \in \mathcal{H})(\zeta \circ 0 \ll \{\zeta\},\ 0 \circ \zeta = \{0\},\ 0 \circ 0 = \{0\}), \tag{7}$$
$$(\forall \zeta \in \mathcal{H})(0 \ll \zeta,\ \zeta \ll \zeta,\ \zeta \in \zeta \circ 0), \tag{8}$$
$$(\forall \zeta, \eta \in \mathcal{H})(\zeta \circ 0 \ll \{\eta\} \Rightarrow \zeta \ll \eta), \tag{9}$$
$$(\forall \zeta, \eta, \varrho \in \mathcal{H})(\eta \ll \varrho \Rightarrow \zeta \circ \varrho \ll \zeta \circ \eta), \tag{10}$$
$$(\forall \zeta, \eta, \varrho \in \mathcal{H})(\zeta \circ \eta = \{0\} \Rightarrow \zeta \circ \varrho \ll \eta \circ \varrho,\ (\zeta \circ \varrho) \circ (\eta \circ \varrho) = \{0\}), \tag{11}$$

For any subsets C, A and B of a hBCK-algebra $\mathcal{H}$, the following assertions are valid.

$$C \subseteq A \Rightarrow C \ll A, \tag{12}$$
$$C \ll \{0\} \Rightarrow C = \{0\}, \tag{13}$$
$$C \ll C,\ C \circ A \ll C,\ (C \circ A) \circ B = (C \circ B) \circ A, \tag{14}$$
$$C \circ \{0\} = \{0\} \Rightarrow C = \{0\}. \tag{15}$$

For any family $\{b_i \mid i \in \Gamma\}$ of real numbers, we define

$$\bigvee\{b_i \mid i \in \Gamma\} := \begin{cases} \max\{b_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ \sup\{b_i \mid i \in \Gamma\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{b_i \mid i \in \Gamma\} := \begin{cases} \min\{b_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ \inf\{b_i \mid i \in \Gamma\} & \text{otherwise.} \end{cases}$$

If $\Gamma = \{1, 2\}$, we will also use $b_1 \vee b_2$ and $b_1 \wedge b_2$ instead of $\bigvee\{b_i \mid i \in \Gamma\}$ and $\bigwedge\{b_i \mid i \in \Gamma\}$, respectively.

A *multipolar intuitionistic fuzzy set of finite degree k* (briefly, *k-pIF set*) over a universe $\mathcal{H}$ is a mapping

$$(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}}) : \mathcal{H} \to [0,1]^k \times [0,1]^k, \zeta \mapsto (\widetilde{\mathcal{K}}(\zeta), \widetilde{\mathcal{M}}(\zeta)) \tag{16}$$

where $\widetilde{\mathcal{K}} : \mathcal{H} \to [0,1]^k$ and $\widetilde{\mathcal{M}} : \mathcal{H} \to [0,1]^k$ are k-pF sets over a universe $\mathcal{H}$ such that $\widetilde{\mathcal{K}}(\zeta) + \widetilde{\mathcal{M}}(\zeta) \leq \tilde{1}$ for all $\zeta \in \mathcal{H}$, that is,

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) + (\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq 1$$

for all $\zeta \in \mathcal{H}$ and $i = 1, 2, \ldots, k$.

Given a k-pIF set $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ over a universe $\mathcal{H}$, we consider the sets

$$U(\widetilde{\mathcal{K}}, \tilde{s}) := \{\zeta \in \mathcal{H} \mid \widetilde{\mathcal{K}}(\zeta) \geq \tilde{s}\} \text{ and } L(\widetilde{\mathcal{M}}, \tilde{t}) := \{\zeta \in \mathcal{H} \mid \widetilde{\mathcal{M}}(\zeta) \leq \tilde{t}\}, \tag{17}$$

where $\tilde{s} = (s_1, s_2, \ldots, s_k) \in [0,1]^k$ and $\tilde{t} = (t_1, t_2, \ldots, t_k) \in [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$, that is,

$$U(\widetilde{\mathcal{K}}, \tilde{s}) := \{\zeta \in \mathcal{H} \mid (\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq s_i \text{ for all } i = 1, 2, \ldots, k\}$$

and

$$L(\widetilde{\mathcal{M}}, \tilde{t}) := \{\zeta \in \mathcal{H} \mid (\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq t_i \text{ for all } i = 1, 2, \ldots, k\}$$

which is called a *k-polar upper* (resp., *lower*) level set of $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$. It is clear that $U(\widetilde{\mathcal{K}}, \tilde{s}) = \bigcap_{i=1}^k U(\widetilde{\mathcal{K}}, \tilde{s})^i$ and $L(\widetilde{\mathcal{M}}, \tilde{t}) = \bigcap_{i=1}^k L(\widetilde{\mathcal{M}}, \tilde{t})^i$ where

$$U(\widetilde{\mathcal{K}}, \tilde{s})^i = \{\zeta \in \mathcal{H} \mid (\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq s_i\} \text{ and } L(\widetilde{\mathcal{M}}, \tilde{t})^i = \{\zeta \in \mathcal{H} \mid (\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq t_i\}.$$

3. k-Polar Intuitionistic Fuzzy Hyper BCK-Ideals

Unless otherwise stated, $\mathcal{H}$ shall represent a hyper BCK-algebra.

Definition 1. *A k-pIF set $\widetilde{\mathcal{K}}$ on $\mathcal{H}$ is called a k-polar intuitionistic fuzzy hyper BCK-ideal (briefly, k-pIF hBCK-ideal) of $\mathcal{H}$ if it satisfies*

$$(\forall \zeta, \eta \in \mathcal{H}) \left(\zeta \ll \eta \Rightarrow \widetilde{\mathcal{K}}(\zeta) \geq \widetilde{\mathcal{K}}(\eta),\ \widetilde{\mathcal{M}}(\zeta) \leq \widetilde{\mathcal{M}}(\eta) \right), \tag{18}$$

$$(\forall \zeta, \eta \in \mathcal{H}) \left(\begin{array}{l} \widetilde{\mathcal{K}}(\zeta) \geq \min \left\{ \bigwedge \{ \widetilde{\mathcal{K}}(a) \mid a \in \zeta \circ \eta \}, \widetilde{\mathcal{K}}(\eta) \right\} \\ \widetilde{\mathcal{M}}(\zeta) \leq \max \left\{ \bigvee \{ \widetilde{\mathcal{M}}(a) \mid a \in \zeta \circ \eta \}, \widetilde{\mathcal{M}}(\eta) \right\} \end{array} \right), \tag{19}$$

that is, $(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\eta)$ *and* $(\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq (\pi_i \circ \widetilde{\mathcal{M}})(\eta)$ *for all* $\zeta, \eta \in \mathcal{H}$ *with* $\zeta \ll \eta$*, and*

$$\left\{ \begin{array}{l} (\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min \left\{ \bigwedge \{ (\pi_i \circ \widetilde{\mathcal{K}})(a) \mid a \in \zeta \circ \eta \}, (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} \\ (\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq \max \left\{ \bigvee \{ (\pi_i \circ \widetilde{\mathcal{M}})(a) \mid a \in \zeta \circ \eta \}, (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \right\} \end{array} \right. \tag{20}$$

for all $\zeta, \eta \in \mathcal{H}$ *and* $i = 1, 2, \ldots, k$*.*

Example 1. *Let* $\mathcal{H} = \{0, 1, 2\}$ *be a set with the hyperoperation "*$\circ$*", which is given by Table 1.*

Then $\mathcal{H}$ *is a hBCK-algebra (see [17]). Let* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *be a* 4*-polar intuitionistic fuzzy set over* $\mathcal{H}$ *given by*

$$\begin{array}{l} (\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}}) : \mathcal{H} \to [0,1]^4 \times [0,1]^4, \\ \zeta \mapsto \left\{ \begin{array}{ll} \left((\frac{1}{3}, 0.33), (0.71, \frac{2}{m+7}), (\frac{1}{n+3}, 0.21), (0.63, 0.25) \right) & \text{if } \zeta = 0, \\ \left((\frac{1}{6}, 0.43), (0.51, \frac{2}{m+5}), (\frac{1}{2n+3}, 0.32), (0.63, 0.25) \right) & \text{if } \zeta = 1, \\ \left((\frac{1}{9}, 0.53), (0.21, \frac{2}{m+3}), (\frac{1}{3n+3}, 0.39), (0.32, 0.43) \right) & \text{if } \zeta = 2, \end{array} \right. \end{array}$$

where m *and* n *are natural numbers. It is routine to verify that* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *is a* 4*-polar intuitionistic fuzzy hBCK-ideal of* $\mathcal{H}$*.*

Table 1. Cayley table for the hyperoperation "$\circ$".

$\circ$	**0**	**1**	**2**
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Proposition 1. *Let* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *be a k-pIF hBCK-ideal of* $\mathcal{H}$*. Then*

(i) $(\forall \zeta \in \mathcal{H}) \left(\widetilde{\mathcal{K}}(0) \geq \widetilde{\mathcal{K}}(\zeta),\ \widetilde{\mathcal{M}}(0) \leq \widetilde{\mathcal{M}}(\zeta) \right)$*, that is,* $(\pi_i \circ \widetilde{\mathcal{K}})(0) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\zeta)$ *and* $(\pi_i \circ \widetilde{\mathcal{M}})(0) \leq (\pi_i \circ \widetilde{\mathcal{M}})(\zeta)$ *for all* $\zeta \in \mathcal{H}$ *and* $i = 1, 2, \ldots, k$*,*

(ii) *If* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *satisfies the condition*

$$(\forall T \subseteq \mathcal{H}) \left(\exists \zeta_0, \eta_0 \in T \text{ s.t. } \widetilde{\mathcal{K}}(\zeta_0) = \bigwedge_{\zeta \in T} \widetilde{\mathcal{K}}(\zeta), \widetilde{\mathcal{M}}(\eta_0) = \bigvee_{\eta \in T} \widetilde{\mathcal{M}}(\eta) \right), \tag{21}$$

then

$$(\forall \zeta, \eta \in \mathcal{H}) \left(\exists a, b \in \zeta \circ \eta \text{ s.t. } \left\{ \begin{array}{l} \widetilde{\mathcal{K}}(\zeta) \geq \min\{ \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta) \} \\ \widetilde{\mathcal{M}}(\zeta) \leq \max\{ \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(\eta) \} \end{array} \right. \right); \tag{22}$$

that is, for every $\zeta, \eta \in \mathcal{H}$ *there exist* $a, b \in \zeta \circ \eta$ *such that*

$$\begin{array}{l} (\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min\{ (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \} \\ (\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq \max\{ (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \} \end{array}$$

for $i = 1, 2, \ldots, k$*.*

Proof. (i) Since $0 \ll \zeta$ for all $\zeta \in \mathcal{H}$, it follows from (18) that $\widetilde{\mathcal{K}}(0) \geq \widetilde{\mathcal{K}}(\zeta)$ and $\widetilde{\mathcal{M}}(0) \leq \widetilde{\mathcal{M}}(\zeta)$ for all $\zeta \in \mathcal{H}$.

(ii) Assume that $\widetilde{\mathcal{K}}$ satisfies the condition (21). For any $\zeta, \eta \in \mathcal{H}$, there exists $a_0, b_0 \in \zeta \circ \eta$ such that $(\pi_i \circ \widetilde{\mathcal{K}})(a_0) = \bigwedge_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a)$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b_0) = \bigvee_{b \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{M}})(b)$. It follows from (20) that

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min\left\{\bigwedge\left\{(\pi_i \circ \widetilde{\mathcal{K}})(a) \mid a \in \zeta \circ \eta\right\}, (\pi_i \circ \widetilde{\mathcal{K}})(\eta)\right\} = \min\left\{(\pi_i \circ \widetilde{\mathcal{K}})(a_0), (\pi_i \circ \widetilde{\mathcal{K}})(\eta)\right\}$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq \max\left\{\bigvee\left\{(\pi_i \circ \widetilde{\mathcal{M}})(b) \mid b \in \zeta \circ \eta\right\}, (\pi_i \circ \widetilde{\mathcal{M}})(\eta)\right\} = \max\left\{(\pi_i \circ \widetilde{\mathcal{M}})(b_0), (\pi_i \circ \widetilde{\mathcal{M}})(\eta)\right\}$$

for $i = 1, 2, \ldots, k$. which proves (ii). □

Theorem 1. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF set over $\mathcal{H}$. If $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF hBCK-ideal of $\mathcal{H}$, then the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$.*

Proof. Let $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Assume that $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF hBCK-ideal of $\mathcal{H}$. It is clear that $0 \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $0 \in L(\widetilde{\mathcal{M}}, \tilde{t})$ by Proposition 1(i). Let $\zeta, \eta, u, v \in \mathcal{H}$ be such that $\zeta \circ \eta \ll U(\widetilde{\mathcal{K}}, \tilde{s})$, $\eta \in U(\widetilde{\mathcal{K}}, \tilde{s})$, $u \circ v \ll L(\widetilde{\mathcal{M}}, \tilde{t})$ and $v \in L(\widetilde{\mathcal{M}}, \tilde{t})$. Then $\zeta \circ \eta \ll U(\widetilde{\mathcal{K}}, \tilde{s})^i$, $\eta \in U(\widetilde{\mathcal{K}}, \tilde{s})^i$, $u \circ v \ll L(\widetilde{\mathcal{M}}, \tilde{t})^i$ and $v \in L(\widetilde{\mathcal{M}}, \tilde{t})^i$ for all $i = 1, 2, \ldots, k$. It follows that

$$(\forall a \in \zeta \circ \eta)\left(\exists a_0 \in U(\widetilde{\mathcal{K}}, \tilde{s})^i \text{ s.t. } a \ll a_0 \text{ and so } (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq (\pi_i \circ \widetilde{\mathcal{K}})(a_0)\right),$$

and

$$(\forall b \in u \circ v)\left(\exists b_0 \in L(\widetilde{\mathcal{M}}, \tilde{t})^i \text{ s.t. } b \ll b_0 \text{ and so } (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq (\pi_i \circ \widetilde{\mathcal{M}})(b_0)\right),$$

which imply that $(\pi_i \circ \widetilde{\mathcal{K}})(a) \geq s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b) \leq t_i$ for all $a \in \zeta \circ \eta$ and $b \in u \circ v$. Hence $\bigwedge_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq s_i$ and $\bigvee_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq t_i$, and so

$$\begin{aligned}
(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) &\geq \min\left\{\bigwedge\left\{(\pi_i \circ \widetilde{\mathcal{K}})(a) \mid a \in \zeta \circ \eta\right\}, (\pi_i \circ \widetilde{\mathcal{K}})(\eta)\right\} \geq s_i, \\
(\pi_i \circ \widetilde{\mathcal{M}})(u) &\leq \max\left\{\bigvee\left\{(\pi_i \circ \widetilde{\mathcal{M}})(b) \mid b \in u \circ v\right\}, (\pi_i \circ \widetilde{\mathcal{M}})(v)\right\} \leq t_i
\end{aligned}$$

for all $i = 1, 2, \ldots, k$. Thus $\zeta \in \bigcap_{i=1}^k U(\widetilde{\mathcal{K}}, \tilde{s})^i = U(\widetilde{\mathcal{K}}, \tilde{s})$ and $u \in \bigcap_{i=1}^k L(\widetilde{\mathcal{M}}, \tilde{t})^i = L(\widetilde{\mathcal{M}}, \tilde{t})$. Therefore $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are hBCK-ideals of $\mathcal{H}$. □

We need the following lemma for considering the converse of Theorem 1.

Lemma 1 ([22]). *Let D be a subset of $\mathcal{H}$. If K is a hBCK-ideal of $\mathcal{H}$ such that $D \ll K$, then D is contained in K.*

Theorem 2. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF set over $\mathcal{H}$ in which the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF hBCK-ideal of $\mathcal{H}$.*

Proof. Assume that the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Let $\zeta, \eta, u, v \in \mathcal{H}$ be such that $\zeta \ll \eta$, $u \ll v$, $\widetilde{\mathcal{K}}(\eta) = \tilde{s}$ and $\widetilde{\mathcal{M}}(v) = \tilde{t}$. Then $\eta \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $v \in L(\widetilde{\mathcal{M}}, \tilde{t})$, and so $\{\zeta\} \ll U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\{u\} \ll L(\widetilde{\mathcal{M}}, \tilde{t})$. It follows from Lemma 1 that $\{\zeta\} \subseteq U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\{u\} \subseteq L(\widetilde{\mathcal{M}}, \tilde{t})$, i.e., $\zeta \in U(\widetilde{\mathcal{K}}, \tilde{s})$

and $u \in L(\widetilde{\mathcal{M}},\tilde{t})$. Hence $\widetilde{\mathcal{K}}(\zeta) \geq \tilde{s} = \widetilde{\mathcal{K}}(\eta)$ and $\widetilde{\mathcal{M}}(u) \leq \tilde{t} = \widetilde{\mathcal{M}}(v)$. For any $\zeta,\eta,u,v \in \mathcal{H}$, let $\tilde{s} := \min\left\{\bigwedge_{a\in\zeta\circ\eta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta)\right\}$ and $\tilde{t} := \max\left\{\bigvee_{b\in u\circ v} \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v)\right\}$. Then $\eta \in U(\widetilde{\mathcal{K}},\tilde{s})$, $v \in L(\widetilde{\mathcal{M}},\tilde{t})$ and

$$
\begin{aligned}
\widetilde{\mathcal{K}}(c) &\geq \bigwedge_{a\in\zeta\circ\eta} \widetilde{\mathcal{K}}(a) \geq \min\left\{\bigwedge_{a\in\zeta\circ\eta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta)\right\} = \tilde{s}, \\
\widetilde{\mathcal{K}}(d) &\leq \bigvee_{b\in u\circ v} \widetilde{\mathcal{M}}(b) \leq \max\left\{\bigvee_{b\in u\circ v} \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v)\right\} = \tilde{t}
\end{aligned}
$$

for all $c \in \zeta\circ\eta$ and $d \in u\circ v$, i.e., $c \in U(\widetilde{\mathcal{K}},\tilde{s})$ and $d \in L(\widetilde{\mathcal{M}},\tilde{t})$. Thus $\zeta\circ\eta \subseteq U(\widetilde{\mathcal{K}},\tilde{s})$ and $u\circ v \subseteq L(\widetilde{\mathcal{M}},\tilde{t})$ which imply from (12) that $\zeta\circ\eta \ll U(\widetilde{\mathcal{K}},\tilde{s})$ and $u\circ v \ll L(\widetilde{\mathcal{M}},\tilde{t})$. Since $U(\widetilde{\mathcal{K}},\tilde{s})$ and $L(\widetilde{\mathcal{M}},\tilde{t})$ are hBCK-ideal of $\mathcal{H}$, we have $\zeta \in U(\widetilde{\mathcal{K}},\tilde{s})$ and $u \in L(\widetilde{\mathcal{M}},\tilde{t})$ which imply that $\widetilde{\mathcal{K}}(\zeta) \geq \tilde{s} = \min\left\{\bigwedge_{a\in\zeta\circ\eta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta)\right\}$ and $\widetilde{\mathcal{M}}(u) \leq \tilde{t} = \max\left\{\bigvee_{b\in u\circ v} \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v)\right\}$. Therefore $(\widetilde{\mathcal{K}},\widetilde{\mathcal{M}})$ is a k-pIF hBCK-ideal of $\mathcal{H}$. $\square$

Definition 2. *A k-pIF set $(\widetilde{\mathcal{K}},\widetilde{\mathcal{M}})$ over $\mathcal{H}$ is called a*

- *k-polar intuitionistic fuzzy weak hBCK-ideal (briefly, k-pIF weak hBCK-ideal) of $\mathcal{H}$ if it satisfies Proposition 1(i) and (19).*
- *k-polar intuitionistic fuzzy s-weak hBCK-ideal (briefly, k-pIF s-weak hBCK-ideal) of $\mathcal{H}$ if it satisfies Proposition 1(i) and (22).*
- *k-polar intuitionistic fuzzy strong hBCK-ideal (briefly, k-pIF strong hBCK-ideal) of $\mathcal{H}$ if it satisfies*

$$
\begin{aligned}
&(\forall \zeta,\eta \in \mathcal{H})\left(\bigwedge_{a\in\zeta\circ\zeta} \widetilde{\mathcal{K}}(a) \geq \widetilde{\mathcal{K}}(\zeta) \geq \min\left\{\bigvee_{b\in\zeta\circ\eta} \widetilde{\mathcal{K}}(b), \widetilde{\mathcal{K}}(\eta)\right\}\right), \\
&(\forall u,v \in \mathcal{H})\left(\bigvee_{b\in u\circ u} \widetilde{\mathcal{M}}(b) \leq \widetilde{\mathcal{M}}(u) \leq \max\left\{\bigwedge_{c\in u\circ v} \widetilde{\mathcal{M}}(c), \widetilde{\mathcal{M}}(v)\right\}\right),
\end{aligned}
\tag{23}
$$

that is,

$$
\bigwedge_{a\in\zeta\circ\zeta} (\pi_i\circ\widetilde{\mathcal{K}})(a) \geq (\pi_i\circ\widetilde{\mathcal{K}})(\zeta) \geq \min\left\{\bigvee_{b\in\zeta\circ\eta} (\pi_i\circ\widetilde{\mathcal{K}})(b), (\pi_i\circ\widetilde{\mathcal{K}})(\eta)\right\}
$$

and

$$
\bigvee_{b\in u\circ u} (\pi_i\circ\widetilde{\mathcal{M}})(b) \leq (\pi_i\circ\widetilde{\mathcal{M}})(u) \leq \max\left\{\bigwedge_{c\in u\circ v} (\pi_i\circ\widetilde{\mathcal{M}})(c), (\pi_i\circ\widetilde{\mathcal{M}})(v)\right\}
$$

for all $\zeta,\eta,u,v \in \mathcal{H}$ and $i = 1,2,\ldots,k$.

Example 2. *Let $\mathcal{H} = \{0,1,2\}$ be a set with the hyperoperation "$\circ$" which is given by Table 2.*

Then $\mathcal{H}$ is a hBCK-algebra (see [17]). Let $(\widetilde{\mathcal{K}},\widetilde{\mathcal{M}})$ be a 4-polar intuitionistic fuzzy set over $\mathcal{H}$ given by

$$
\begin{aligned}
&(\widetilde{\mathcal{K}},\widetilde{\mathcal{M}}) : \mathcal{H} \to [0,1]^4 \times [0,1]^4, \\
&\zeta \mapsto \begin{cases}
\Big((0.2\pi,0.2),(0.71,0.2),(\tfrac{1}{n+3},0.3),(0.63,\mu(3n))\Big) & \text{if } \zeta = 0, \\
\Big((0.1\pi,0.4),(0.51,0.3),(\tfrac{1}{n+5},0.5),(0.43,\mu(2n))\Big) & \text{if } \zeta = 1, \\
\Big((0.05\pi,0.6),(0.31,0.5),(\tfrac{1}{n+7},0.7),(0.32,\mu(n))\Big) & \text{if } \zeta = 2,
\end{cases}
\end{aligned}
$$

where n is a natural number and $\mu : \mathbb{N} \to [0,1], \zeta \mapsto \frac{0.5}{\zeta}$. It is routine to check that $(\widetilde{\mathcal{K}},\widetilde{\mathcal{M}})$ is a 4-polar intuitionistic fuzzy strong hBCK-ideal of $\mathcal{H}$.

Table 2. Cayley table for the binary operation "$*$".

$\circ$	**0**	**1**	**2**
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

We describe the relation between k-pIF weak hBCK-ideal and k-pIF s-weak hBCK-ideal in the following theorem.

Theorem 3. *In a hBCK-algebra, every k-pIF s-weak hBCK-ideal is a k-pIF weak hBCK-ideal.*

Proof. Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF s-weak hBCK-ideal of $\mathcal{H}$ and let $\zeta, \eta, u, v \in \mathcal{H}$. Then there exist $a \in \zeta \circ \eta$ and $b \in u \circ v$ such that $\widetilde{\mathcal{K}}(\zeta) \geq \min\{\widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta)\}$ and $\widetilde{\mathcal{M}}(u) \leq \max\{\widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v)\}$ by (22). Since $\widetilde{\mathcal{K}}(a) \geq \bigwedge_{b \in \zeta \circ \eta} \widetilde{\mathcal{K}}(b)$ and $\widetilde{\mathcal{M}}(b) \leq \bigvee_{c \in u \circ v} \widetilde{\mathcal{M}}(c)$, it follows that

$$\widetilde{\mathcal{K}}(\zeta) \geq \min\left\{\bigwedge\{\widetilde{\mathcal{K}}(b) \mid b \in \zeta \circ \eta\}, \widetilde{\mathcal{K}}(\eta)\right\},$$
$$\widetilde{\mathcal{M}}(u) \leq \max\left\{\bigvee\{\widetilde{\mathcal{M}}(c) \mid c \in u \circ v\}, \widetilde{\mathcal{M}}(v)\right\}.$$

Therefore $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF weak hBCK-ideal of $\mathcal{H}$. □

We consider a condition for a k-pIF weak hBCK-ideal to be a k-pIF s-weak hBCK-ideal.

Theorem 4. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF weak hBCK-ideal of $\mathcal{H}$ which satisfies the condition (21). Then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF s-weak hBCK-ideal of $\mathcal{H}$.*

Proof. For any $\zeta, \eta, u, v \in \mathcal{H}$, there exist $a_0 \in \zeta \circ \eta$ and $b_0 \in u \circ v$ such that $\widetilde{\mathcal{K}}(a_0) = \bigwedge_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a)$ and $\widetilde{\mathcal{M}}(b_0) = \bigvee_{b \in u \circ v} \widetilde{\mathcal{M}}(b)$; that is, $(\pi_i \circ \widetilde{\mathcal{K}})(a_0) = \bigwedge_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a)$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b_0) = \bigvee_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b)$ by (21). It follows that

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min\left\{\bigwedge\{(\pi_i \circ \widetilde{\mathcal{K}})(a) \mid a \in \zeta \circ \eta\}, (\pi_i \circ \widetilde{\mathcal{K}})(\eta)\right\} = \min\{(\pi_i \circ \widetilde{\mathcal{K}})(a_0), (\pi_i \circ \widetilde{\mathcal{K}})(\eta)\},$$
$$(\pi_i \circ \widetilde{\mathcal{M}})(u) \leq \max\left\{\bigvee\{(\pi_i \circ \widetilde{\mathcal{M}})(b) \mid b \in u \circ v\}, (\pi_i \circ \widetilde{\mathcal{M}})(v)\right\} = \max\{(\pi_i \circ \widetilde{\mathcal{M}})(b_0), (\pi_i \circ \widetilde{\mathcal{M}})(v)\}.$$

Therefore $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF s-weak hBCK-ideal of $\mathcal{H}$. □

Proposition 2. *Every k-pIF strong hBCK-ideal $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ of $\mathcal{H}$ satisfies the following assertions.*

(i) $(\forall \zeta \in \mathcal{H})(\widetilde{\mathcal{K}}(0) \geq \widetilde{\mathcal{K}}(\zeta), \widetilde{\mathcal{M}}(0) \leq \widetilde{\mathcal{M}}(\zeta))$; *that is,* $(\pi_i \circ \widetilde{\mathcal{K}})(0) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\zeta)$ *and* $(\pi_i \circ \widetilde{\mathcal{M}})(0) \leq (\pi_i \circ \widetilde{\mathcal{M}})(\zeta)$ *for all* $\zeta, u \in \mathcal{H}$ *and* $i = 1, 2, \ldots, k$,

(ii) $(\forall \zeta, \eta \in \mathcal{H})(\zeta \ll \eta \Rightarrow \widetilde{\mathcal{K}}(\zeta) \geq \widetilde{\mathcal{K}}(\eta), \widetilde{\mathcal{M}}(\zeta) \leq \widetilde{\mathcal{M}}(\eta))$; *that is,* $(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\eta)$ *and* $(\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq (\pi_i \circ \widetilde{\mathcal{M}})(\eta)$ *for all* $\zeta, \eta \in \mathcal{H}$ *with* $\zeta \ll \eta$ *and* $i = 1, 2, \ldots, k$.

(iii) $(\forall a, \zeta, \eta \in \mathcal{H})\left(a \in \zeta \circ \eta \Rightarrow \widetilde{\mathcal{K}}(\zeta) \geq \min\{\widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta)\}, \widetilde{\mathcal{M}}(\zeta) \leq \max\{\widetilde{\mathcal{M}}(a), \widetilde{\mathcal{M}}(\eta)\}\right)$.

Proof. (i) Since $0 \in \zeta \circ \zeta$ for all $\zeta \in \mathcal{H}$, we get

$$\widetilde{\mathcal{K}}(0) \geq \bigwedge_{a \in \zeta \circ \zeta} \widetilde{\mathcal{K}}(a) \geq \widetilde{\mathcal{K}}(\zeta) \text{ and } \widetilde{\mathcal{M}}(0) \leq \bigvee_{b \in \zeta \circ \zeta} \widetilde{\mathcal{M}}(b) \leq \widetilde{\mathcal{M}}(\zeta)$$

for all $\zeta \in \mathcal{H}$.

(ii) Let $\zeta, \eta \in \mathcal{H}$ be such that $\zeta \ll \eta$. Then $0 \in \zeta \circ \eta$ and thus $\bigvee_{b \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(b) \geq (\pi_i \circ \widetilde{\mathcal{K}})(0)$ and $\bigwedge_{c \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{M}})(c) \leq (\pi_i \circ \widetilde{\mathcal{M}})(0)$ for $i = 1, 2, \ldots, k$. It follows from (23) and (i) that

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min \left\{ \bigvee_{b \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(b), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} \geq \min \left\{ (\pi_i \circ \widetilde{\mathcal{K}})(0), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} = (\pi_i \circ \widetilde{\mathcal{K}})(\eta)$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq \max \left\{ \bigwedge_{c \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{M}})(c), (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \right\} \leq \max \left\{ (\pi_i \circ \widetilde{\mathcal{M}})(0), (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \right\} = (\pi_i \circ \widetilde{\mathcal{M}})(\eta)$$

for $i = 1, 2, \ldots, k$, that is, $\widetilde{\mathcal{K}}(\zeta) \geq \widetilde{\mathcal{K}}(\eta)$ and $\widetilde{\mathcal{M}}(\zeta) \leq \widetilde{\mathcal{M}}(\eta)$ for all $\zeta, \eta \in \mathcal{H}$ with $\zeta \ll \eta$.

(iii) Let $a, \zeta, \eta \in \mathcal{H}$ be such that $a \in \zeta \circ \eta$. Then

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min \left\{ \bigvee_{b \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(b), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} \geq \min \left\{ (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\}$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(\zeta) \leq \max \left\{ \bigwedge_{c \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{M}})(c), (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \right\} \leq \max \left\{ (\pi_i \circ \widetilde{\mathcal{M}})(a), (\pi_i \circ \widetilde{\mathcal{M}})(\eta) \right\}$$

for $i = 1, 2, \ldots, k$. Hence $\widetilde{\mathcal{K}}(\zeta) \geq \min \left\{ \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta) \right\}$ and $\widetilde{\mathcal{M}}(\zeta) \leq \max \left\{ \widetilde{\mathcal{M}}(a), \widetilde{\mathcal{K}}(\eta) \right\}$ for all $a, \zeta, \eta \in \mathcal{H}$ with $a \in \zeta \circ \eta$. $\square$

Corollary 1. *If $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$, then*

$$\widetilde{\mathcal{K}}(\zeta) \geq \min \left\{ \widetilde{\mathcal{K}}(\eta), \bigwedge_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a) \right\} \text{ and } \widetilde{\mathcal{M}}(\zeta) \leq \max \left\{ \widetilde{\mathcal{M}}(\eta), \bigvee_{b \in \zeta \circ \eta} \widetilde{\mathcal{K}}(b) \right\}$$

for all $\zeta, \eta \in \mathcal{H}$.

Corollary 2. *Every k-pIF strong hBCK-ideal is a k-pIF hBCK-ideal and a k-pIF s-weak hBCK-ideal (and hence a k-pIF weak hBCK-ideal).*

In general, a k-pIF (weak) hBCK-ideal may not be a k-pIF strong hBCK-ideal. In fact, the 4-polar intuitionistic fuzzy hBCK-ideal $\widetilde{\mathcal{K}}$ of $\mathcal{H}$ in Example 1 is not a 4-polar intuitionistic fuzzy strong hBCK-ideal of $\mathcal{H}$ since

$$(\pi_3 \circ \widetilde{\mathcal{K}})(2) = \tfrac{1}{3n+3} < \tfrac{1}{2m+3} = (\pi_3 \circ \widetilde{\mathcal{K}})(1) = \min \left\{ (\pi_3 \circ \widetilde{\mathcal{K}})(1), \bigvee_{\zeta \in 2 \circ 1} (\pi_3 \circ \widetilde{\mathcal{K}})(\zeta) \right\}$$

and/or

$$(\pi_3 \circ \widetilde{\mathcal{M}})(2) = 0.39 > 0.32 = (\pi_3 \circ \widetilde{\mathcal{M}})(1) = \max \left\{ (\pi_3 \circ \widetilde{\mathcal{M}})(1), \bigwedge_{\zeta \in 2 \circ 1} (\pi_3 \circ \widetilde{\mathcal{M}})(\zeta) \right\}.$$

It is clear that every k-pIF hBCK-ideal of $\mathcal{H}$ is a k-pIF weak hBCK-ideal of $\mathcal{H}$. However, the converse is not true in general, as seen in the following example.

Example 3. *Let $\mathcal{H} = \{0,1,2\}$ be a hBCK-algebra as in Example 1. Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a 3-polar intuitionistic fuzzy set over $\mathcal{H}$ given by*

$$(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}}) : \mathcal{H} \to [0,1]^3 \times [0,1]^3,$$
$$\zeta \mapsto \begin{cases} \Big((0.4n, 0.3), (\frac{1}{n+2}, 0.2), (0.6, 0.3n)\Big) & \text{if } \zeta = 0, \\ \Big((0.1n, 0.6), (\frac{1}{3n+2}, 0.5), (0.2, 0.6n)\Big) & \text{if } \zeta = 1, \\ \Big((0.3n, 0.5), (\frac{1}{2n+2}, 0.3), (0.5, 0.4n)\Big) & \text{if } \zeta = 2, \end{cases}$$

where n is a natural number. Then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a 3-polar, intuitionistic, fuzzy weak hBCK-ideal of $\mathcal{H}$. Note that $1 \ll 2$, $(\pi_2 \circ \widetilde{\mathcal{K}})(1) = \frac{1}{3n+2} < \frac{1}{2n+2} = (\pi_2 \circ \widetilde{\mathcal{K}})(2)$ and/or $(\pi_3 \circ \widetilde{\mathcal{M}})(1) = 0.6n > 0.4n = (\pi_3 \circ \widetilde{\mathcal{M}})(2)$; that is, $\widetilde{\mathcal{K}}(1) \not\geq \widetilde{\mathcal{K}}(2)$ and/or $\widetilde{\mathcal{M}}(1) \not\leq \widetilde{\mathcal{M}}(2)$. Hence $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is not a 3-polar intuitionistic fuzzy hBCK-ideal of $\mathcal{H}$.

We have a characterization of a k-pIF weak hBCK-ideal in the similar way to the proofs of Theorems 1 and 2.

Theorem 5. *Given a k-pIF set $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ over $\mathcal{H}$, the following are equivalent.*

(i) *$(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF weak hBCK-ideal of $\mathcal{H}$.*
(ii) *The k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are weak hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$.*

Theorem 6. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF set over $\mathcal{H}$. If $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$, then the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$.*

Proof. Assume that $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$ and let $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$ be such that $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are nonempty. Then there exists $a \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $b \in L(\widetilde{\mathcal{M}}, \tilde{t})$, and so $\widetilde{\mathcal{K}}(a) \geq \tilde{s}$ and $\widetilde{\mathcal{M}}(b) \leq \tilde{t}$; that is, $(\pi_i \circ \widetilde{\mathcal{K}})(a) \geq s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b) \leq t_i$ for all $i = 1, 2, \ldots, k$. It is clear that $0 \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $0 \in L(\widetilde{\mathcal{M}}, \tilde{t})$ by Proposition 2(1). Let $\zeta, \eta, u, v \in \mathcal{H}$ be such that $\eta \in U(\widetilde{\mathcal{K}}, \tilde{s})$, $(\zeta \circ \eta) \cap U(\widetilde{\mathcal{K}}, \tilde{s}) \neq \emptyset$, $v \in L(\widetilde{\mathcal{M}}, \tilde{t})$ and $(u \circ v) \cap L(\widetilde{\mathcal{M}}, \tilde{t}) \neq \emptyset$. Then there exist $a_0 \in (\zeta \circ \eta) \cap U(\widetilde{\mathcal{K}}, \tilde{s})$ and and $b_0 \in (u \circ v) \cap L(\widetilde{\mathcal{M}}, \tilde{t})$. Hence $\widetilde{\mathcal{K}}(a_0) \geq \tilde{s}$ and $\widetilde{\mathcal{M}}(b_0) \leq \tilde{t}$, i.e., $(\pi_i \circ \widetilde{\mathcal{K}})(a_0) \geq s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b_0) \leq t_i$, for $i = 1, 2, \ldots, k$. It follows that

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min\left\{ \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} \geq \min\left\{ \pi_i \circ \widetilde{\mathcal{K}})(a_0), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} \geq s_i$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(u) \leq \max\left\{ \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\} \leq \max\left\{ \pi_i \circ \widetilde{\mathcal{M}})(b_0), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\} \leq t_i$$

for all $i = 1, 2, \ldots, k$. Hence $\zeta \in \bigcap_{i=1}^k U(\widetilde{\mathcal{K}}, \tilde{s})^i = U(\widetilde{\mathcal{K}}, \tilde{s})$ and $u \in \bigcap_{i=1}^k L(\widetilde{\mathcal{M}}, \tilde{t})^i = L(\widetilde{\mathcal{M}}, \tilde{t})$. Therefore $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$. $\square$

Theorem 7. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF set over $\mathcal{H}$ which satisfies the condition*

$$(\forall T \subseteq \mathcal{H}) \left(\exists \zeta_0, \eta_0 \in T \text{ s.t. } \widetilde{\mathcal{K}}(\zeta_0) = \bigvee_{\zeta \in T} \widetilde{\mathcal{K}}(\zeta),\ \widetilde{\mathcal{M}}(\eta_0) = \bigwedge_{\eta \in T} \widetilde{\mathcal{M}}(\eta) \right). \tag{24}$$

If the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$, then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$.

Proof. Assume that the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Then $\zeta \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\eta \in L(\widetilde{\mathcal{M}}, \tilde{t})$ for some $\zeta, \eta \in \mathcal{H}$, and so $\zeta \circ \zeta \ll \{\zeta\} \subseteq U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\eta \circ \eta \ll \{\eta\} \subseteq L(\widetilde{\mathcal{M}}, \tilde{t})$. Using Lemma 1, we get $\zeta \circ \zeta \subseteq U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\eta \circ \eta \subseteq L(\widetilde{\mathcal{M}}, \tilde{t})$. Hence for every $a \in \zeta \circ \zeta$ and $b \in \eta \circ \eta$, we get $a \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $b \in L(\widetilde{\mathcal{M}}, \tilde{t})$. Thus $(\pi_i \circ \widetilde{\mathcal{K}})(a) \geq s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b) \leq t_i$ for all $i = 1, 2, \ldots, k$. It follows that

$$\bigwedge_{a \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq s_i = (\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \text{ and } \bigvee_{b \in \eta \circ \eta} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq t_i = (\pi_i \circ \widetilde{\mathcal{M}})(\eta)$$

for $i = 1, 2, \ldots, k$. For any $\zeta, \eta, u, v \in \mathcal{H}$, put $\tilde{d} := \min \left\{ \bigvee_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta) \right\}$ and $\tilde{e} := \max \left\{ \bigwedge_{b \in u \circ v} \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v) \right\}$, that is, $d_i := \min \left\{ \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\}$ and $e_i := \max \left\{ \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\}$ for $i = 1, 2, \ldots, k$. Then $U(\widetilde{\mathcal{K}}, \tilde{d})$ and $L(\widetilde{\mathcal{M}}, \tilde{e})$ are strong hBCK-ideals of $\mathcal{H}$ by hypothesis. The condition (24) implies that there exists $a_0 \in \zeta \circ \eta$ and $b_0 \in u \circ v$ such that $\widetilde{\mathcal{K}}(a_0) = \bigvee_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a)$ and $\widetilde{\mathcal{M}}(b_0) = \bigwedge_{b \in u \circ v} \widetilde{\mathcal{M}}(b)$, i.e., $(\pi_i \circ \widetilde{\mathcal{K}})(a_0) = \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a)$ and $(\pi_i \circ \widetilde{\mathcal{M}})(b_0) = \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b)$ for $i = 1, 2, \ldots, k$. Hence

$$(\pi_i \circ \widetilde{\mathcal{K}})(a_0) = \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq \min \left\{ \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\} = d_i$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(b_0) = \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq \max \left\{ \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\} = e_i$$

for $i = 1, 2, \ldots, k$, which imply that $a_0 \in \bigcap_{i=1}^{k} U(\widetilde{\mathcal{K}}, \tilde{d})^i = U(\widetilde{\mathcal{K}}, \tilde{d})$ and $b_0 \in \bigcap_{i=1}^{k} L(\widetilde{\mathcal{M}}, \tilde{e})^i = L(\widetilde{\mathcal{M}}, \tilde{e})$. Hence $(\zeta \circ \eta) \cap U(\widetilde{\mathcal{K}}, \tilde{d}) \neq \emptyset$ and $(u \circ v) \cap L(\widetilde{\mathcal{M}}, \tilde{e}) \neq \emptyset$, and thus $\zeta \in U(\widetilde{\mathcal{K}}, \tilde{d})$ and $u \in L(\widetilde{\mathcal{M}}, \tilde{e})$. It follows that

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq d_i = \min \left\{ \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\}$$

and

$$(\pi_i \circ \widetilde{\mathcal{M}})(u) \leq e_i = \max \left\{ \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\}$$

for $i = 1, 2, \ldots, k$. Therefore $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$. $\square$

Definition 3. *A k-pIF set* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *over* $\mathcal{H}$ *is called a k-pIF reflexive hBCK-ideal of* $\mathcal{H}$ *if it satisfies:*

$$(\forall \zeta, \eta, u, v \in \mathcal{H}) \left(\widetilde{\mathcal{K}}(\eta) \leq \bigwedge_{a \in \zeta \circ \zeta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{M}}(v) \geq \bigvee_{b \in u \circ u} \widetilde{\mathcal{M}}(b) \right), \tag{25}$$

$$(\forall \zeta, \eta, u, v \in \mathcal{H}) \left(\begin{array}{l} \widetilde{\mathcal{K}}(\zeta) \geq \min \left\{ \bigvee_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a), \widetilde{\mathcal{K}}(\eta) \right\} \\ \widetilde{\mathcal{M}}(u) \leq \max \left\{ \bigwedge_{b \in u \circ v} \widetilde{\mathcal{M}}(b), \widetilde{\mathcal{M}}(v) \right\} \end{array} \right), \tag{26}$$

that is, $(\pi_i \circ \widetilde{\mathcal{K}})(\eta) \leq \bigwedge_{a \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(a)$, $(\pi_i \circ \widetilde{\mathcal{M}})(v) \geq \bigvee_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b)$, *and*

$$(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq \min \left\{ \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a), (\pi_i \circ \widetilde{\mathcal{K}})(\eta) \right\},$$

$$(\pi_i \circ \widetilde{\mathcal{M}})(u) \leq \max \left\{ \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b), (\pi_i \circ \widetilde{\mathcal{M}})(v) \right\}$$

for all $\zeta, \eta, u, v \in \mathcal{H}$ *and* $i = 1, 2, \ldots, k$.

Theorem 8. *Every k-pIF reflexive hBCK-ideal is a k-pIF strong hBCK-ideal.*

Proof. Straightforward. □

Theorem 9. *If* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *is a k-pIF reflexive hBCK-ideal of* $\mathcal{H}$, *then the k-polar upper level set* $U(\widetilde{\mathcal{K}}, \tilde{s})$ *and the k-polar lower level set* $L(\widetilde{\mathcal{M}}, \tilde{t})$ *are reflexive hBCK-ideals of* $\mathcal{H}$ *for all* $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ *with* $\tilde{s} + \tilde{t} \leq \tilde{1}$.

Proof. Assume that $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF reflexive hBCK-ideal of $\mathcal{H}$. Then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$ by Theorem 8, and so $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF hBCK-ideal of $\mathcal{H}$. It follows from Theorem 1 that the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Let $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$ be such that $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are nonempty. Then $\widetilde{\mathcal{K}}(c) \geq \tilde{s}$ and $\widetilde{\mathcal{M}}(c') \leq \tilde{t}$ for some $c, c' \in \mathcal{H}$. For any $\zeta, \eta \in \mathcal{H}$, let $b \in \zeta \circ \zeta$ and $b' \in \eta \circ \eta$. The condition (25) implies that $\widetilde{\mathcal{K}}(b) \geq \bigwedge_{a \in \zeta \circ \zeta} \widetilde{\mathcal{K}}(a) \geq \widetilde{\mathcal{K}}(c) \geq \tilde{s}$ and $\widetilde{\mathcal{M}}(b') \leq \bigvee_{a' \in \eta \circ \eta} \widetilde{\mathcal{M}}(a') \leq \widetilde{\mathcal{M}}(c') \leq \tilde{t}$, that is, $b \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $b' \in L(\widetilde{\mathcal{M}}, \tilde{t})$. Thus $\zeta \circ \zeta \subseteq U(\widetilde{\mathcal{K}}, \tilde{s})$ and $\eta \circ \eta \subseteq L(\widetilde{\mathcal{M}}, \tilde{t})$ for all $\zeta, \eta \in \mathcal{H}$, and therefore $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are reflexive hBCK-ideals of $\mathcal{H}$. □

Lemma 2 ([18])**.** *Every reflexive hBCK-ideal is a strong hBCK-ideal.*

We need additional conditions to induce the converse of Theorem 9.

Theorem 10. *Let* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *be a k-pIF set over* $\mathcal{H}$ *which satisfies the condition* (24)*. If the k-polar upper level set* $U(\widetilde{\mathcal{K}}, \tilde{s})$ *and the k-polar lower level set* $L(\widetilde{\mathcal{M}}, \tilde{t})$ *are reflexive hBCK-ideals of* $\mathcal{H}$ *for all* $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ *with* $\tilde{s} + \tilde{t} \leq \tilde{1}$, *then* $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ *is a k-pIF reflexive hBCK-ideal of* $\mathcal{H}$.

Proof. Assume that the k-polar upper level set $U(\widetilde{\mathcal{K}}, \tilde{s})$ and the k-polar lower level set $L(\widetilde{\mathcal{M}}, \tilde{t})$ are reflexive hBCK-ideals of $\mathcal{H}$ for all $(\tilde{s}, \tilde{t}) \in [0,1]^k \times [0,1]^k$ with $\tilde{s} + \tilde{t} \leq \tilde{1}$. Then $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$ by Lemma 2. Using Theorem 7, we know that $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$ and so (26) is valid. For any $\zeta, \eta, u, v \in \mathcal{H}$, let $(\pi_i \circ \widetilde{\mathcal{K}})(\eta) = s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(v) = t_i$ for $i = 1, 2, \ldots, k$. Since $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are reflexive hBCK-ideals of $\mathcal{H}$, we get $\zeta \circ \zeta \subseteq U(\widetilde{\mathcal{K}}, \tilde{s})$

and $u \circ u \subseteq L(\widetilde{\mathcal{M}}, \tilde{t})$. Hence $c \in U(\widetilde{\mathcal{K}}, \tilde{s})$ for all $c \in \zeta \circ \zeta$ and $d \in L(\widetilde{\mathcal{M}}, \tilde{t})$ for all $d \in u \circ u$. Thus $(\pi_i \circ \widetilde{\mathcal{K}})(c) \geq s_i$ and $(\pi_i \circ \widetilde{\mathcal{M}})(d) \leq t_i$ which imply that

$$\bigwedge_{c \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(c) \geq s_i = (\pi_i \circ \widetilde{\mathcal{K}})(\eta), \ \bigvee_{d \in u \circ u} (\pi_i \circ \widetilde{\mathcal{M}})(d) \leq t_i = (\pi_i \circ \widetilde{\mathcal{M}})(v)$$

for all $i = 1, 2, \ldots, k$. Therefore $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF reflexive hBCK-ideal of $\mathcal{H}$. $\square$

Theorem 11. *Let $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ be a k-pIF strong hBCK-ideal of $\mathcal{H}$ which satisfies the condition (24). Then $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pF reflexive hBCK-ideal of $\mathcal{H}$ if and only if $\bigwedge_{a \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq (\pi_i \circ \widetilde{\mathcal{K}})(0)$ and $\bigvee_{b \in u \circ u} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq (\pi_i \circ \widetilde{\mathcal{M}})(0)$ for all $\zeta, u \in \mathcal{H}$ and $i = 1, 2, \ldots, k$.*

Proof. Assume that $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF strong hBCK-ideal of $\mathcal{H}$ which satisfies the condition (24). The necessity is clear. Assume that $\bigwedge_{a \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq (\pi_i \circ \widetilde{\mathcal{K}})(0)$ and $\bigvee_{b \in u \circ u} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq (\pi_i \circ \widetilde{\mathcal{M}})(0)$ for all $\zeta, u \in \mathcal{H}$ and $i = 1, 2, \ldots, k$. Since $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pF hBCK-ideal of $\mathcal{H}$ by Corollary 2, we have $(\pi_i \circ \widetilde{\mathcal{K}})(0) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\eta)$ and $(\pi_i \circ \widetilde{\mathcal{M}})(0) \leq (\pi_i \circ \widetilde{\mathcal{M}})(v)$ for all $\eta, v \in \mathcal{H}$ and $i = 1, 2, \ldots, k$. It follows that $\bigwedge_{a \in \zeta \circ \zeta} (\pi_i \circ \widetilde{\mathcal{K}})(a) \geq (\pi_i \circ \widetilde{\mathcal{K}})(\eta)$ and $\bigvee_{b \in u \circ u} (\pi_i \circ \widetilde{\mathcal{M}})(b) \leq (\pi_i \circ \widetilde{\mathcal{M}})(v)$ for all $\zeta, \eta, u, v \in \mathcal{H}$ and $i = 1, 2, \ldots, k$. For any $\zeta, \eta, u, v \in \mathcal{H}$ and $i = 1, 2, \ldots, k$, let $s_i := \min\left\{(\pi_i \circ \widetilde{\mathcal{K}})(\eta), \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a)\right\}$ and $t_i := \max\left\{(\pi_i \circ \widetilde{\mathcal{M}})(v), \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b)\right\}$. The condition (24) implies that there exists $a_0 \in \zeta \circ \eta$ such that $\widetilde{\mathcal{K}}(a_0) = \bigvee_{a \in \zeta \circ \eta} \widetilde{\mathcal{K}}(a)$ and so $\widetilde{\mathcal{K}}(a_0) \geq \tilde{s}$, i.e., $a_0 \in U(\widetilde{\mathcal{K}}, \tilde{s})$, and there exists $b_0 \in u \circ v$ such that $\widetilde{\mathcal{M}}(b_0) = \bigwedge_{b \in u \circ v} \widetilde{\mathcal{M}}(b)$ and so $\widetilde{\mathcal{M}}(b_0) \leq \tilde{t}$, i.e., $b_0 \in L(\widetilde{\mathcal{M}}, \tilde{t})$. Hence $(\zeta \circ \eta) \cap U(\widetilde{\mathcal{K}}, \tilde{s}) \neq \emptyset$ and $(u \circ v) \cap L(\widetilde{\mathcal{M}}, \tilde{t}) \neq \emptyset$. Since $U(\widetilde{\mathcal{K}}, \tilde{s})$ and $L(\widetilde{\mathcal{M}}, \tilde{t})$ are strong hBCK-ideals of $\mathcal{H}$ by Theorem 6, it follows that $\zeta \in U(\widetilde{\mathcal{K}}, \tilde{s})$ and $u \in L(\widetilde{\mathcal{M}}, \tilde{t})$. Hence $(\pi_i \circ \widetilde{\mathcal{K}})(\zeta) \geq s_i = \min\left\{(\pi_i \circ \widetilde{\mathcal{K}})(\eta), \bigvee_{a \in \zeta \circ \eta} (\pi_i \circ \widetilde{\mathcal{K}})(a)\right\}$ and $(\pi_i \circ \widetilde{\mathcal{M}})(u) \leq t_i = \max\left\{(\pi_i \circ \widetilde{\mathcal{M}})(v), \bigwedge_{b \in u \circ v} (\pi_i \circ \widetilde{\mathcal{M}})(b)\right\}$. Therefore $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{M}})$ is a k-pIF reflexive hBCK-ideal of $\mathcal{H}$. $\square$

4. Conclusions

In 2020, Kang (together with Song and Jun) introduced the concept of a k-polar intuitionistic fuzzy set in a universe, and applied it to BCK/BCI-algebras. In this article, we have applied the k-polar intuitionistic fuzzy set to hyper BCK-algebra. We have introduced the notions of the k-pIF hBCK-ideal, the k-pIF weak hBCK-ideal, the k-pIF s-weak hBCK-ideal, the k-pIF strong hBCK-ideal and the k-pIF reflexive hBCK-ideal, and have investigated related properties and their relations. We have discussed k-pIF (weak, s-weak, strong, reflexive) hBCK-ideals in relation to k-polar upper and lower level sets. In the future work, we will use the idea and results in this paper to study other hyper algebraic structures, for example, hyper hoop, hyper BCI-algebra, hyper equality algebra and hyper MV-algebra.

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