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Lucas Numbers Which Are Concatenations of Two Repdigits

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Abstract: In this paper, we find all Lucas numbers written in the form $\overline{c\cdots cd\cdots d}$, where $\overline{c\cdots cd\cdots d}$ is the concatenation of two repdigits in base 10 with $c,d\in\{0,1,\ldots,9\}$, $c\neq d$ and $c>0$.

Keywords: Lucas numbers; concatenations of two repdigits; logarithmic height; continued fraction

1. Introduction

Linear form in logarithms has many important applications in solving Diophantine equations [1–4]. In 2002, by applying linear form in logarithms, A. Dujella and B. Jadrijević [1] showed that the solutions to quartic Thue equations $x^4-4cx^3y+(6c+2)x^2y^2+4cxy^3+y^4=1$ are only $(x,y)=(\pm1,0)$ and $(0,\pm1)$ for an integer $c\geq 3$. Suppose that $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence given by $F_{n+2}=F_{n+1}+F_n$, with initial values $F_0=0$ and $F_1=1$ and let $\{L_n\}_{n\geq0}$ be the Lucas sequence defined by $L_{n+2}=L_{n+1}+L_n$, where $L_0=2$ and $L_1=1$. In 2011, F. Luca and R. Oyono [2] concluded that there is no solution (m,n,s) to the Diophantine equation $F_m^s+F_{m+1}^s=F_n$ for integers $m\geq 2, n\geq 1, s\geq 3$ by applying linear form in logarithms. There are many papers in the literature which solve Diophantine equations related to Fibonacci numbers and Lucas numbers [3–14]. In 2013, D. Marques and A. Togbé [3] found all solutions (n,a,b,c) to the Diophantine equation $F_n=2^a+3^b+5^c$ and $L_n=2^a+3^b+5^c$ for integers n,a,b,c with $0\leq\max\{a,b\}\leq c$. In 2019, B. D. Bitim [4] investigated the solutions (n,m,a) to the Diophantine equation $L_n-L_m=2\cdot3^a$ for nonnegative integers n,m,a with $n>m$. Let p be a prime number and $\max\{a,b\}\geq 2$, in 2009, F. Luca and P. Stănică [5] concluded that there are only finitely many positive integer solutions (n,p,a,b) to the Diophantine equation $F_n=p^a\pm p^b$.

Assume that $q\geq 2$ is an integer. A positive number $n\in\mathbb{N}$ is called a base $q-$repdigit if $n=c\frac{q^t-1}{q-1}$, for some $t\geq 1$ and $c\in\{1,2,\ldots,q-1\}$. When $q=10$, n is simply called a repdigit. We use $\overline{B_1\cdots B_t}_{(q)}$ to express an integer's base$-q$ representation which is the concatenation of the base$-q$ representations of positive integers $B_1,\ldots,B_t$. We ignore writing q if $q=10$. Then we can denote the repdigit n by $n=\underbrace{\overline{c\cdots c}}_{t\text{ times}}$ and the concatenation of two repdigits in base 10 is $\overline{\underbrace{c\cdots c}_{s\text{ times}}\underbrace{d\cdots d}_{t\text{ times}}}$, where $c,d\in\{0,1,\ldots,9\}$, $c\neq d, c>0, s\geq 1$ and $t\geq 1$. There are many papers in the literature on investigating Diophantine equations related to repdigits [8,9,11–21]. In 2000, Luca [15] proved that if $F_n=a\frac{10^m-1}{9}$ and $L_n=a\frac{10^m-1}{9}$ for some $a\in\{0,1,\ldots,9\}$ and $m\geq 1$, then $n=0,1,2,3,4,5,6,10$ and $n=0,1,2,3,4,5$ respectively. In 2012, all repdigits in base 10 expressible as sums of three Fibonacci numbers were found in [16]. In 2018, all repdigits in base 10 which are sums of four Fibonacci or Lucas numbers were determined in [17]. In 2019, all solutions to the Diophantine equation $F_n=\overline{\underbrace{a\cdots a}_{m\text{ times}}\underbrace{b\cdots b}_{l\text{ times}}}$ were found in [18], where $a,b\in\{0,1,\ldots,9\}$ and $a>0$. For the research of concatenations of two repdigits in balancing numbers, Padovan numbers and Tribonacci numbers, please refer to the literature [19–21] respectively.

In this paper, we find all Lucas numbers which are concatenations of two repdigits. More precisely, we have the following result.

Theorem 1. *If*

$$L_n = \overline{\underbrace{c\cdots c}_{s\ times}\underbrace{d\cdots d}_{t\ times}}, \tag{1}$$

with $c, d \in \{0, 1, \ldots, 9\}$, $c \neq d$, $c > 0$, $s \geq 1$ *and* $t \geq 1$, *then*

$$(n, L_n) \in \{(6, 18), (7, 29), (8, 47), (9, 76), (11, 199), (12, 322)\}.$$

2. Preliminaries

Firstly, the Binet's formula for Lucas sequence is

$$L_n = \alpha^n + \beta^n, n \geq 0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. For all positive integers n, we have the following inequality

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+1}. \tag{2}$$

Secondly, we recall the definition and properties for logarithmic height of an algebraic number. Let η be an algebraic number of degree m and suppose that the minimal primitive polynomial of η is $f(X) := a_0 \prod_{i=1}^{m}(X - \eta^{(i)}) \in \mathbb{Z}[X]$ with $a_0 > 0$. We give the logarithmic height of η by

$$h(\eta) := \frac{1}{m}\left(\log a_0 + \sum_{i=1}^{m} \log \max\{|\eta^{(i)}|, 1\}\right).$$

In this paper, for any two integers a and b, we denote the greatest common divisor of a and b by $\gcd(a, b)$. Specifically, $h(\eta) = \log \max\{|p|, q\}$ when $\eta = \frac{p}{q} \in \mathbb{Q}$ with $\gcd(p, q) = 1$ and $q > 0$. We have the following properties of the logarithmic height $h(\cdot)$:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$

$$h(\eta^k) = |k| h(\eta) \ \ (k \in \mathbb{Z}).$$

We need the following lemma to prove our theorem.

Lemma 1. *(see [22]) Let* $d_{\mathbb{L}}$ *be the degree of an algebraic number field* $\mathbb{L}$ *over* $\mathbb{Q}$ *and* $\mathbb{L} \subseteq \mathbb{R}$. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_l \in \mathbb{L}$ *be non-zero elements and let* $b_1, \ldots, b_l$ *be rational integers. If* $\Gamma := \gamma_1^{b_1} \cdots \gamma_l^{b_l} - 1 \neq 0$, *then*

$$|\Gamma| \geq \exp(-1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log B) A_1 \cdots A_l),$$

where A_j *are real numbers such that*

$$A_j \geq \max\{d_{\mathbb{L}} h(\gamma_j), |\log \gamma_j|, 0.16\}$$

for $j = 1, \ldots, l$ *and* $B \geq \max\{|b_1|, \ldots, |b_l|, 3\}$.

Thirdly, we need the following Lemma 2 and Lemma 3 to reduce some large upper bounds on the variables in the course of our calculations.

Lemma 2. *(see [23]) Let M be a positive integer and let $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number α such that $q > 6M$, and let A, B, τ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\tau q\| - M\|\alpha q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\alpha - v + \tau| < AB^{-\omega}$$

in positive integers u, v, and ω with $u \leq M$ and $w \geq \frac{\log(Aq/\epsilon)}{\log B}$.

Lemma 3. *(see [24]) Let τ be an irrational number, M be a positive integer and $\frac{p_k}{q_k}(k = 0, 1, 2, \dots)$ be all the convergents of the continued fraction $[a_0, a_1, \dots]$ of τ. Let N be such that $q_N > M$. Then putting $a_M := \max\{a_i : i = 0, 1, \dots, N\}$, the inequality*

$$|m\tau - n| > \frac{1}{(a_M + 2)m}$$

holds for all pairs (n, m) of integers with $0 < m < M$.

3. Proof of Theorem 1

3.1. Bounding n

According to (1), we get

$$\begin{aligned} L_n &= \overline{\underbrace{c\cdots c}_{s\ times}\underbrace{d\cdots d}_{t\ times}} \\ &= \overline{\underbrace{c\cdots c}_{s\ times}} \cdot 10^t + \overline{\underbrace{d\cdots d}_{t\ times}} \\ &= \frac{1}{9}(c10^{s+t} - (c-d)10^t - d). \end{aligned} \tag{3}$$

Suppose that $n > 1000$. From inequality (2), we can get $\alpha^{n-1} \leq L_n < 10^{s+t}$ and $10^{s+t-1} \leq L_n \leq \alpha^{n+1}$, which implies that

$$(s+t)\log 10 - \log 10 - \log\alpha \leq n\log\alpha < (s+t)\log 10 + \log\alpha. \tag{4}$$

Thus, we can get

$$4.78(s+t) - 5.8 < n < 4.79(s+t) + 1. \tag{5}$$

From (5), we get $s + t > \frac{n-1}{4.79} > 208$ and $n > s + t$. According to (3) and Binet's formulae for Lucas sequences, we get

$$|9\alpha^n - c10^{s+t}| = |-9\beta^n - ((c-d)10^t + d)| \leq 9\alpha^{-n} + 9\cdot 10^t + 9 < 27\cdot 10^t, \tag{6}$$

which implies that

$$\left|\frac{9}{c}\alpha^n 10^{-s-t} - 1\right| < \frac{27}{10^s}. \tag{7}$$

Let $\Gamma_1 := \frac{9}{c}\alpha^n 10^{-s-t} - 1$, then $\Gamma_1 \neq 0$. If $\Gamma_1 = 0$, then $\alpha^n = \frac{10^{s+t}c}{9} \in \mathbb{Q}$, thus we have $\frac{10^{s+t}c}{9} = \frac{(1+\sqrt{5})^n}{2^n} = \frac{f+g\sqrt{5}}{2^n}$, where $f, g \in \mathbb{Z}, f > 0, g > 0$, this implies that $\sqrt{5} = \frac{\frac{10^{s+t}c2^n}{9} - f}{g} \in \mathbb{Q}$, which is impossible. According to Lemma 1, we take $l = 3$, $\gamma_1 = \frac{9}{c}, \gamma_2 = \alpha, \gamma_3 = 10$ and $b_1 = 1, b_2 = n, b_3 = -s-t$. Thus, we have $\mathbb{L} = \mathbb{Q}(\alpha), d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. Note that $h(\gamma_1) = h(\frac{9}{c}) \leq \log 9, h(\gamma_2) = \frac{1}{2}\log\alpha, h(\gamma_3) = \log 10$. Thus, we can take $A_1 = 2\log 9$, $A_2 = 0.5$, $A_3 = 4.8$. Note that $B = \max\{|b_1|, |b_2|, |b_3|, 3\} = \max\{1, n, s+t, 3\} = n$. Hence, we get

$$|\Gamma_1| > \exp(-C_1(1 + \log n)), \tag{8}$$

where $C_1 = 1.025 \times 10^{13}$. Thus from (7) and (8) , we can get

$$s\log 10 < C_1(1+\log n) + \log 27. \tag{9}$$

We rewrite Equation (3), then we get

$$\left|\alpha^n - \frac{c10^s - (c-d)}{9} \cdot 10^t\right| = \left|\beta^n + \frac{d}{9}\right| \leq \alpha^{-n} + 1 < 2. \tag{10}$$

It follows that

$$\left|\frac{c10^s - (c-d)}{9} \cdot \alpha^{-n} \cdot 10^t - 1\right| < \frac{2}{\alpha^n}. \tag{11}$$

Let $\Gamma_2 := \frac{c10^s-(c-d)}{9} \cdot \alpha^{-n} \cdot 10^t - 1$, then $\Gamma_2 \neq 0$. If $\Gamma_2 = 0$, then $\alpha^n = \frac{c10^s-(c-d)}{9} \cdot 10^t \in \mathbb{Q}$, which is false. According to Lemma 1, we take $l = 3, \gamma_1 = \frac{c10^s-(c-d)}{9}, \gamma_2 = \alpha, \gamma_3 = 10$ and $b_1 = 1, b_2 = -n, b_3 = t$. Thus, we have $\mathbb{L} = \mathbb{Q}(\alpha), d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. From (9), we can get

$$\begin{aligned} h(\gamma_1) &\leq h(c10^s - (c-d)) + h(9) \\ &\leq 3\log 9 + s\log 10 + \log 2 \\ &\leq C_1(1+\log n) + \log 27 + 3\log 9 + \log 2 \\ &\leq 1.03 \cdot 10^{13} \cdot (1+\log n), \end{aligned} \tag{12}$$

and we have $h(\gamma_2) = \frac{1}{2}\log\alpha, h(\gamma_3) = \log 10$. Thus, we can take $A_1 = 2.06 \cdot 10^{13} \cdot (1+\log n)$, $A_2 = 0.5$, $A_3 = 4.8$. Note that $B = \max\{|b_1|, |b_2|, |b_3|, 3\} = \max\{1, n, t, 3\} = n$. Hence, we get

$$|\ \Gamma_2\ | > \exp(-C_2(1+\log n)^2), \tag{13}$$

where $C_2 = 4.8 \times 10^{25}$. Thus from (11) and (13) , we can get

$$n\log\alpha < C_2(1+\log n)^2 + \log 2, \tag{14}$$

this implies that $n < 4.8 \times 10^{29}$. Hence we can conclude that

$$s + t < \frac{n+5.8}{4.78} < 1.01 \cdot 10^{29}.$$

To sum up, we have the lemma as follows.

Lemma 4. *If (n, c, d, s, t) is a solution in non-negative integers of Equation (1), with $c, d \in \{0, 1, \ldots, 9\}$, $c \neq d$ and $c > 0$, then*

$$s + t < n < 4.8 \cdot 10^{29}, s + t < 1.01 \cdot 10^{29}.$$

3.2. Reducing the Bound on n

We use the Lemmas 2 and 3 to reduce the bound for n. Let

$$\Lambda_1 := (s+t)\log 10 - n\log\alpha - \log\frac{9}{c}.$$

From (7), we conclude that

$$\left|e^{-\Lambda_1} - 1\right| < \frac{27}{10^s}. \tag{15}$$

If $s \geq 2$, then $|e^{-\Lambda_1} - 1| < \frac{27}{10^s} < \frac{1}{2}$, which implies that $\frac{1}{2} < e^{-\Lambda_1} < \frac{3}{2}$. If $\Lambda_1 > 0$, then $0 < \Lambda_1 < e^{\Lambda_1} - 1 = e^{\Lambda_1}(1 - e^{-\Lambda_1}) < \frac{54}{10^s}$. If $\Lambda_1 < 0$, then $0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = e^{-\Lambda_1} - 1 < \frac{27}{10^s}$. In any case, it is always holds true $0 < |\Lambda_1| < \frac{54}{10^s}$, which implies

$$0 < \left|(s+t)\frac{\log 10}{\log\alpha} - n - \frac{\log\frac{9}{c}}{\log\alpha}\right| < \frac{\frac{54}{\log\alpha}}{10^s}. \tag{16}$$

The continued fraction of $\frac{\log 10}{\log\alpha}$ is $[a_0, a_1, a_2, a_3, a_4, \dots] = [4, 1, 3, 1, 1, 1, 6, \dots]$, and let $\frac{p_k}{q_k}$ be its kth convergent. Note that $s + t < 1.01 \cdot 10^{29}$ by Lemma 4. It is easy to see that $\frac{\log 10}{\log\alpha}$ is irrational. In fact, if $\frac{\log 10}{\log\alpha} = \frac{p}{q}(p, q \in \mathbb{Z}$ and $p > 0, q > 0, \gcd(p, q) = 1)$, then $\alpha^p = 10^q \in \mathbb{Q}$, which is an absurdity. For all $c \in \{1, \dots, 8\}$, according to (16) and Lemma 2, we take $M = 1.01 \cdot 10^{29}$ and $q_{60} > 6M$, hence we get the minimum value of ϵ is $0.061483\dots$ and $s < 34$. If $c = 9$, from (16), we get

$$0 < \left|(s+t)\frac{\log 10}{\log\alpha} - n\right| < \frac{\frac{54}{\log\alpha}}{10^s}. \tag{17}$$

According to Lemma 3, we take $M = 1.01 \cdot 10^{29}$ and $q_{60} > M$, hence we get $a_M := \max\{a_i : i = 0, 1, \dots, 60\} = 106$ and we have

$$\left|(s+t)\frac{\log 10}{\log\alpha} - n\right| > \frac{1}{(a_M + 2)(s+t)} > \frac{1}{108 \cdot 1.01 \cdot 10^{29}}. \tag{18}$$

Thus, from (17) and (18), we get

$$\frac{1}{108 \cdot 1.01 \cdot 10^{29}} < \frac{\frac{54}{\log\alpha}}{10^s},$$

this leads to $s < 34$. So we always have $s < 34$.

Let

$$\Lambda_2 := t\log 10 - n\log\alpha + \log\frac{c10^s - (c-d)}{9}.$$

From (11) and $n > 1000$, we conclude that

$$\left|e^{\Lambda_2} - 1\right| < \frac{2}{\alpha^n} < \frac{1}{2}, \tag{19}$$

which implies that $\frac{1}{2} < e^{\Lambda_2} < \frac{3}{2}$. If $\Lambda_2 > 0$, then $0 < \Lambda_2 < e^{\Lambda_2} - 1 < \frac{2}{\alpha^n}$. If $\Lambda_2 < 0$, then $0 < |\Lambda_2| < e^{-\Lambda_2} - 1 = e^{-\Lambda_2}(1 - e^{\Lambda_2}) < \frac{4}{\alpha^n}$. In any case, since $0 < |\Lambda_2| < \frac{4}{\alpha^n}$, thus we have

$$0 < \left|t\frac{\log 10}{\log\alpha} - n + \frac{\log\frac{c10^s - (c-d)}{9}}{\log\alpha}\right| < \frac{\frac{4}{\log\alpha}}{\alpha^n}, \tag{20}$$

where $s \leq 33, c \in \{1, \dots, 9\}$ and $d \in \{0, 1, \dots, 9\}$. For inequality (20), we consider the following two cases: if $(s, c, d) \neq (1, 1, 0)$, according to (20) and Lemma 2, we take $M = 1.01 \times 10^{29}$ and $q_{60} > 6M$, hence we obtain 25 negative values of ϵ , the minimum value in the values of positive ϵ is $0.00004477\dots$ and $n < 171$. For the values of (s, c, d) corresponding to the 25 negative values of ϵ , we take $q_{63} > 6M$, according to (20) and Lemma 2, we get the minimum value in the values of ϵ is $0.005613\dots$ and $n < 168$. If $(s, c, d) = (1, 1, 0)$, from (20), we get

$$0 < \left|t\frac{\log 10}{\log\alpha} - n\right| < \frac{\frac{4}{\log\alpha}}{\alpha^n}. \tag{21}$$

According to Lemma 3, we take $M = 1.01 \cdot 10^{29}$ and $q_{60} > M$, hence we get $a_M := \max\{a_i : i = 0, 1, \ldots, 60\} = 106$ and we have

$$\left| t\frac{\log 10}{\log\alpha} - n \right| > \frac{1}{(a_M + 2)t} > \frac{1}{108 \cdot 1.01 \cdot 10^{29}}. \tag{22}$$

Thus, from (21) and (22), we get

$$\frac{1}{108 \cdot 1.01 \cdot 10^{29}} < \frac{\frac{4}{\log\alpha}}{\alpha^n},$$

which leads to $n < 153$. In summary, we have $n < 171$. This contradicts the assumption $n > 1000$. Finally, we search for the solutions to (1) in the range $n \le 1000$ by applying a program written in Mathematica and we obtain the solutions $(n, L_n) \in \{(6, 18), (7, 29), (8, 47), (9, 76), (11, 199), (12, 322)\}$. We complete the proof.

4. Conclusions and Future Research

For a fixed integer $k \ge 2$, let $\{F_n^{(k)}\}_{n \ge 2-k}$ be the $k-$generalized Fibonacci sequence defined by $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$ with the initial values $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$, $F_1^{(k)} = 1$ and $\{L_n^{(k)}\}_{n \ge 2-k}$ be the $k-$generalized Lucas sequence given by $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)}$ with the initial values $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \cdots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2, L_1^{(k)} = 1$. Suppose that $c, d \in \{0, 1, \ldots, 9\}$, $c \ne d, c > 0, s \ge 1$ and $t \ge 1$, our aim is to solve the two Diophantine equations

$$F_n^{(k)} = \overline{\underbrace{c \cdots c}_{s\ times}\underbrace{d \cdots d}_{t\ times}} \tag{23}$$

and

$$L_n^{(k)} = \overline{\underbrace{c \cdots c}_{s\ times}\underbrace{d \cdots d}_{t\ times}}. \tag{24}$$

For $k = 2$ and $k = 3$, the Diophantine Equation (23) has been solved in [18] and [21], respectively. In this paper, we solve the Diophantine Equation (24) for the case of $k = 2$. Our future research work is to solve the Diophantine Equations (23) and (24) completely for the case of $k \ge 3$. In addition, for the main Mathematica programs used in this paper, readers can refer to Appendix A.

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Appendix A. Mathematica Programs

We give the main Mathematica programs used in this paper as follows :

- $\alpha = \frac{1+\sqrt{5}}{2}; \gamma = \frac{\log[10]}{\log[\alpha]};$
- Generates a list of the first n terms in $\gamma's$ continued fraction representation:

$$ContinuedFraction[\gamma, n]$$

- The denominator of the nth$(n = 0, 1, 2, \ldots)$ convergent of $\gamma's$ continued fraction:

$$q[n\_] := Module[\{\gamma = \frac{\log[10]}{\log[\frac{1+\sqrt{5}}{2}]}\}, Last[Denominator[Convergents[\gamma, n+1]]]];$$

- The function $\|x\|$ which denotes the distance from x to the nearest integer:

$$cldist[x\_, jd\_] := Module[\{\}, Abs[N[Round[x] - x, jd]]];$$

- The number $\epsilon := \|\tau q\| - M\|\alpha q\|$ in Lemma 2:

$$epsilon[\tau\_, q\_, M\_, \alpha\_, jd\_] := Module[\{\}, cldist[\tau * q, jd] - M * cldist[\alpha * q, jd]];$$

- The number $\tau := -\frac{\log\frac{9}{c}}{\log\alpha}$ in (16): $\tau[c\_] := -\frac{\log[\frac{9}{c}]}{\log[\alpha]}$;
- The number $\tau := \frac{\log\frac{c10^s-(c-d)}{9}}{\log\alpha}$ in (20): $\tau[s\_, c\_, d\_] := \frac{\log[\frac{c10^s-(c-d)}{9}]}{\log[\alpha]}$;
- The nth term of Lucas sequence L_n:

$$Lucas[n\_] := Module[\{\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}\}, Simplify[Expand[\alpha^n + \beta^n]]];$$

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