

# Stolarsky Means in Many Variables

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**Abstract:** We give in this article two possible explicit extensions of Stolarsky means to the multi-variable case. They attain all main properties of Stolarsky means and coincide with them in the case of two variables.

**Keywords:** extended mean values; logarithmic convexity; multi-variable case

**MSC:** 26A51; 60E15

## 1. Introduction

There is a huge number of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean values, defined for positive variables  $x, y; x \neq y$ , as

$$E_{r,s}(x, y) := \left( \frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)}, \quad rs(r-s) \neq 0.$$

Those means can be continuously extended on the domain

$$\{(r, s; x, y) | r, s \in \mathbb{R}; x, y \in \mathbb{R}_+\}$$

by the following

$$E_{r,s}(x, y) = \begin{cases} \left( \frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)}, & rs(r-s) \neq 0; \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0; \\ \left( \frac{x^s - y^s}{s(\log x - \log y)} \right)^{1/s}, & s \neq 0, r = 0; \\ \sqrt{xy}, & r = s = 0; \\ x, & y = x > 0, \end{cases}$$

and in this form has been introduced by Keneth Stolarsky in [1].

Most of the classical two variable means are just special cases of the class  $E$ . For example,  $E_{1,2} = \frac{x+y}{2}$  is the arithmetic mean,  $E_{-r,r} = E_{0,0} = \sqrt{xy}$  is the geometric mean,  $E_{0,1} = \frac{x-y}{\log x - \log y}$  is the logarithmic mean,  $E_{1,1} = (x^x / y^y)^{\frac{1}{x-y}} / e$  is the identric mean, etc. More generally, the  $r$ -th power mean  $\left( \frac{x^r + y^r}{2} \right)^{1/r}$  is equal to  $E_{r,2r}$  ([2]).

Characteristic properties of Stolarsky means are:

1. Symmetry in variables,  $E_{r,s}(x, y) = E_{r,s}(y, x)$ ;
2. Symmetry in parameters,  $E_{r,s}(x, y) = E_{s,r}(x, y)$ ;

3. Means  $E_{r,s}(x, y)$  are homogeneous of order one i.e,  $E_{r,s}(tx, ty) = tE_{r,s}(x, y), t > 0$ .
4. Means  $E_{r,s}(x, y)$  are monotone increasing in both parameters  $r$  and  $s$ .

By two articles ([3,4]) published in Amer.Math. Monthly, this class of means attains popularity in a wide audience. As a result, great number of papers are produced investigating its most subtle properties. In this sense we quote here papers [5,6]. A comparison of Stolarsky and Gini means is given in [7–9], weighted variants in [10,11]. F. Qi in [12] find intervals of  $r, s$  where these means are logarithmically convex/concave, etc.

Furthermore, there are several papers attempting to define an extension of the class  $E$  to  $n, n > 2$  variables. Unfortunately, this is done in a highly implicit mode ([5,6,13–15]).

Here is an illustration of this point; J. Merikoski ([13]) has proposed the following generalization of the Stolarsky mean  $E_{r,s}$  to several variables

$$E_{r,s}(X) := \left[ \frac{L(X^s)}{L(X^r)} \right]^{\frac{1}{s-r}}, r \neq s,$$

where  $X = (x_1, \dots, x_n)$  is an  $n$ -tuple of positive numbers and

$$L(X^s) := (n-1)! \int_{I_{n-1}} \prod_{i=1}^n x_i^{su_i} du_1 \cdots du_{n-1}.$$

The symbol  $I_{n-1}$  stands for the Euclidean simplex which is defined by

$$I_{n-1} := \{(u_1, \dots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1; u_1 + \dots + u_{n-1} \leq 1\}.$$

In this article we shall expose two possible explicit formulae of Stolarsky means in  $n \geq 2$  variables which preserve its main properties and coincide for  $n = 2$ .

The first one is given by the following

Let  $X_n = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ . Then,

$$e_{r,s}(X_n) = e_{r,s}(x_1, x_2, \dots, x_n) := \left( \frac{r^2 x_1^{ns} + x_2^{ns} + \dots + x_n^{ns} - n(x_1 x_2 \dots x_n)^s}{s^2 x_1^{nr} + x_2^{nr} + \dots + x_n^{nr} - n(x_1 x_2 \dots x_n)^r} \right)^{\frac{1}{n(s-r)}}, rs(s-r) \neq 0,$$

represents an extension of Stolarsky means to the multi-variable case.

**Remark 1.** We assume that there exist  $j, k; 1 \leq j < k \leq n$ , such that  $x_j \neq x_k$ .

It is of interest to examine the inner structure of those means. For example, applying the formula

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)[(x-y)^2 + (y-z)^2 + (z-x)^2],$$

we obtain that

$$e_{r,s}(x_1, x_2, x_3) = (A_{r,s}(x_1, x_2, x_3))^{1/3} (B_{r,s}(x_1, x_2, x_3))^{2/3},$$

where

$$A_{r,s}(x_1, x_2, x_3) := \left( \frac{x_1^s + x_2^s + x_3^s}{x_1^r + x_2^r + x_3^r} \right)^{1/(s-r)}$$

is the well-known Gini mean, and

$$B_{r,s}(x_1, x_2, x_3) := \left( \frac{r^2 (x_1^s - x_2^s)^2 + (x_2^s - x_3^s)^2 + (x_3^s - x_1^s)^2}{s^2 (x_1^r - x_2^r)^2 + (x_2^r - x_3^r)^2 + (x_3^r - x_1^r)^2} \right)^{1/(2(s-r))}$$

is the new mean in 3 variables which coincides with the Stolarsky mean  $E_{r,s}(x_1, x_2)$  whenever  $x_3 = x_1$  or  $x_3 = x_2$ .

This notion leads to the second, more general representation of Stolarsky means in many variables.

Let  $A_n = (a_1, a_2, \dots, a_n)$ ,  $X_n = (x_1, x_2, \dots, x_n)$ ,  $Y_n = (y_1, y_2, \dots, y_n)$ ;  $A_n, X_n, Y_n \in \mathbb{R}_+^n$ .

Then

$$E_{r,s}(A_n; X_n, Y_n) := \left( \frac{r^2 a_1 (x_1^s - y_1^s)^2 + a_2 (x_2^s - y_2^s)^2 + \dots + a_n (x_n^s - y_n^s)^2}{s^2 a_1 (x_1^r - y_1^r)^2 + a_2 (x_2^r - y_2^r)^2 + \dots + a_n (x_n^r - y_n^r)^2} \right)^{\frac{1}{2(s-r)}},$$

represents another multi-variable variant of Stolarsky means.

It will be shown in the sequel that both means  $e_{r,s}(X_n)$  and  $E_{r,s}(A_n; X_n, Y_n)$  are monotone increasing in parameters  $r$  and  $s$ . An intriguing task is to determine some necessary and sufficient conditions for their monotonicity in  $n$ . Although the solution is relatively simple in the second case and reduces to the monotonicity of sequences  $X_n$  and  $Y_n$  (independently of  $A_n$ ), this question is much more complicated for the means  $e_{r,s}(X_n)$ .

For example, means  $e_{0,0}(X_n)$  are monotone increasing/decreasing in  $n$  if and only if  $x_n \geq g(z(X_{n-1}), e_{0,0}(X_{n-1}))$ ,  $n \geq 3$ , where  $z(X_n)$  is the geometric mean of numbers  $X_n$  and  $g(z_n, e_{0,0}(X_n)) := z_n(e_{0,0}(X_n)/z_n)^{3(n+1)/(n+2)}$ .

## 2. Results and Proofs

Recall that the Jensen functional  $J_n(p, x; f)$  is defined on an interval  $I \subseteq \mathbb{R}$  by

$$J_n(p, x; f) := \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right),$$

where  $f: I \rightarrow \mathbb{R}$ ,  $x = (x_1, x_2, \dots, x_n) \in I^n$  and  $p = \{p_i\}_1^n$  is a positive weight sequence.

Another well known assertion is the following

**Jensen's inequality:** If  $f$  is twice continuously differentiable and  $f'' \geq 0$  on an interval  $I$ , then  $f$  is convex on  $I$  and the inequality

$$J_n(p, x; f) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq 0$$

holds for each  $x := (x_1, \dots, x_n) \in I^n$  and any positive weight sequence  $p := \{p_i\}_1^n$  with  $\sum_{i=1}^n p_i = 1$ .

The next two properties of Jensen functionals will be of importance in the sequel.

**Theorem 1** ([16,17]). Let  $f, g: I \rightarrow \mathbb{R}$  be twice continuously differentiable functions. Assume that  $g$  is strictly convex and  $\phi$  is a continuous and strictly monotone function on  $I$ .

Then the expression

$$\phi^{-1}\left(\frac{J_n(p, x; f)}{J_n(p, x; g)}\right), \quad (n \geq 2),$$

represents a mean value of the numbers  $x_1, \dots, x_n \in I$ , that is

$$\min\{x_1, \dots, x_n\} \leq \phi^{-1}\left(\frac{J_n(p, x; f)}{J_n(p, x; g)}\right) \leq \max\{x_1, \dots, x_n\},$$

if and only if the relation

$$f''(t) = \phi(t)g''(t)$$

holds for each  $t \in I$ .

**Theorem 2** ([18]). Let  $f_s$  be a twice continuously differentiable function on the interval  $J := (c, d)$  for each parameter  $s \in I := (a, b)$ . If  $s \rightarrow f_s''(x)$  is log-convex on  $I$  for each  $x \in J$ , then the expression

$$s \rightarrow \Phi_f(w, x; s) := \sum_{i=1}^n w_i f_s(x_i) - f_s\left(\sum_{i=1}^n w_i x_i\right),$$

is log-convex on  $I$  for each  $x := (x_1, \dots, x_n) \in J^n$ , where  $w = \{w_i\}_1^n$  is any positive weight sequence.

**Lemma 1.** A function  $F$  is convex on an interval  $I$  if and only if the ratio

$$\frac{F(s) - F(r)}{s - r}$$

is monotone increasing in both  $r$  and  $s$  for  $r, s \in I$ .

In the following two theorems we shall prove that our expressions  $e_{r,s}(X_n)$  and  $E_{r,s}(A_n; X_n, Y_n)$ , extended to the whole  $(r, s)$  plane, are actually means which preserve all main properties of the ordinary Stolarsky means and coincide with them for  $n = 2$ .

**Theorem 3.** Let,

$$e_{r,s}(x_1, x_2, \dots, x_n) = \begin{cases} \left( \frac{r^2(\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s)}{s^2(\sum_1^n x_i^{nr} - n(\prod_1^n x_i)^r)} \right)^{1/(n(s-r))} & , rs(s-r) \neq 0; \\ \left( \frac{2}{ns^2} \frac{\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s}{n \sum_1^n \log^2 x_i - (\sum_1^n \log x_i)^2} \right)^{1/(ns)} & , r = 0, s \neq 0; \\ \exp\left(\frac{-2}{ns} + \frac{\sum_1^n x_i^{ns} \log x_i - (\sum_1^n \log x_i)(\prod_1^n x_i)^s}{\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s}\right) & , r = s \neq 0; \\ \exp\left(\frac{n^2 \sum_1^n \log^3 x_i - (\sum_1^n \log x_i)^3}{3n(\sum_1^n \log^2 x_i - (\sum_1^n \log x_i)^2)}\right) & , r = s = 0. \end{cases}$$

Then

1. Expressions  $e_{r,s}(X_n)$  are means, that is,

$$\min\{x_1, x_2, \dots, x_n\} \leq e_{r,s}(x_1, x_2, \dots, x_n) \leq \max\{x_1, x_2, \dots, x_n\}.$$

2.  $e_{r,s}(X_n)$  are symmetric in parameters  $r$  and  $s$  i.e.,  $e_{r,s}(X_n) = e_{s,r}(X_n)$ .
3.  $e_{r,s}(X_n)$  are symmetric in all variables.
4.  $e_{r,s}(X_n)$  are homogeneous of order one.
5.  $e_{r,s}(X_n)$  are monotone increasing in both parameters  $r$  and  $s$ .
6.  $e_{r,s}(x_1, x_2) = E_{r,s}(x_1, x_2)$ .

**Proof.** Note that the Properties 2–4 are evident and can be proved directly.

We apply Theorem A for the proof of Property 1.

Namely, choose that  $g = f_r(y)$  and

$$f = f_s(y) := \begin{cases} (e^{sy} - sy - 1)/s^2 & , s \neq 0; \\ y^2/2 & , s = 0. \end{cases}$$

The conditions of Theorem A are fulfilled with

$$f''(y) = e^{sy}, \quad g''(y) = e^{ry}, \quad \phi(y) = e^{(s-r)y}, \quad \phi^{-1}(y) = \frac{1}{s-r} \log y,$$

for  $r \neq s$ .

Therefore, with  $p_i = 1/n$ , we obtain

$$\min\{y_i\}_1^n \leq \frac{1}{s-r} \log \left( \frac{r^2 \sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{s^2 \sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}} \right) \leq \max\{y_i\},$$

that is,

$$e^{\min\{y_i\}_1^n} \leq \left( \frac{r^2 \sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{s^2 \sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}} \right)^{1/(s-r)} \leq e^{\max\{y_i\}_1^n}.$$

In the case  $r = 0, s \neq 0$ , we have

$$f''(y) = e^{sy}, g''(y) = 1, \phi(y) = e^{sy}, \phi^{-1}(y) = \frac{1}{s} \log y.$$

Hence,

$$e^{\min\{y_i\}_1^n} \leq \left( \frac{2n \sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{s^2 \sum_1^n y_i^2 - (\sum_1^n y_i)^2} \right)^{1/s} \leq e^{\max\{y_i\}_1^n}.$$

Now, change of variables  $e^{y_i} = x_i$ ,  $s \rightarrow ns$ ,  $r \rightarrow nr$ , evidently leads to the desired results.

For the proof of Property 5. we shall use Theorem B.

By the function  $f_s(y)$  defined above, we have that  $f_s''(y) = e^{sy}$  is log-convex for  $s \in \mathbb{R}$ .

Hence, by Theorem B we obtain that the form

$$F(s) = \frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{ns^2},$$

is log-convex on  $\mathbb{R}$ .

Since a positive function is log-convex on  $I$  if its logarithm is convex on  $I$ , applying Lemma 1 we have that the form

$$\frac{\log F(s) - \log F(r)}{s-r} = \log \left( \frac{r^2 \sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{s^2 \sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}} \right)^{1/(s-r)},$$

is monotone increasing in both  $r$  and  $s$ .

The same change of variables  $e^{y_i} = x_i$ ,  $s \rightarrow ns$ ,  $r \rightarrow nr$ , proves the validity of Property 5.

Finally, for the Property 6. of Theorem 3, we have

$$\begin{aligned} e_{r,s}(x_1, x_2) &= \left( \frac{r^2 x_1^{2s} + x_2^{2s} - 2(x_1 x_2)^s}{s^2 x_1^{2r} + x_2^{2r} - 2(x_1 x_2)^r} \right)^{1/(2(s-r))} \\ &= \left( \frac{r^2 (x_1^s - x_2^s)^2}{s^2 (x_1^r - x_2^r)^2} \right)^{1/(2(s-r))} = \left| \frac{r (x_1^s - x_2^s)}{s (x_1^r - x_2^r)} \right|^{1/(s-r)} = E_{r,s}(x_1, x_2). \end{aligned}$$

□

**Theorem 4.** Let,

$$E_{r,s}(A_n; X_n, Y_n) = \begin{cases} \left( \frac{r^2 (\sum_1^n a_i (x_i^s - y_i^s)^2)}{s^2 (\sum_1^n a_i (x_i^r - y_i^r)^2)} \right)^{1/(2(s-r))} & , rs(s-r) \neq 0; \\ \left( \frac{\sum_1^n a_i (x_i^s - y_i^s)^2}{s^2 \sum_1^n a_i (\log x_i - \log y_i)^2} \right)^{1/(2s)} & , r = 0, s \neq 0; \\ \exp \left( \frac{-1}{s} + \frac{\sum_1^n a_i (x_i^s - y_i^s) (x_i^s \log x_i - y_i^s \log y_i)}{\sum_1^n a_i (x_i^s - y_i^s)^2} \right) & , r = s \neq 0; \\ \exp \left( \frac{\sum_1^n a_i (\log x_i - \log y_i) (\log^2 x_i - \log^2 y_i)}{2 \sum_1^n a_i (\log x_i - \log y_i)^2} \right) & , r = s = 0. \end{cases}$$

Then

1. Functions  $E_{r,s}(A_n; X_n, Y_n)$  are means.

2. Means  $E_{r,s}(A_n; X_n, Y_n)$  are symmetric in parameters  $r$  and  $s$ .
3. Means  $E_{r,s}(A_n; X_n, Y_n)$  are symmetric in variables, that is,  $E_{r,s}(A_n; X_n, Y_n) = E_{r,s}^n(A_n; Y_n, X_n)$ .
4. Means  $E_{r,s}(A_n; X_n, Y_n)$  are homogeneous of order one.
5.  $E_{r,s}(A_n; X_n, Y_n)$  are monotone increasing in both parameters  $r$  and  $s$ .
6.  $E_{r,s}(a_1; x_1, y_1) = E_{r,s}(x_1, y_1)$ .

**Remark 2.** We assume that there exists  $i$ ,  $1 \leq i \leq n$ , such that  $x_i \neq y_i$ .

**Proof.** Properties 2–6 are self-evident. For the rest of the proof we can assume that  $x_i > y_i$ ,  $i = 1, 2, \dots, n$ . Otherwise, we put  $x_i \in Y_n, y_i \in X_n$ .

Furthermore, because of symmetry, we take  $s \geq r$ .

To prove Property 1, note that from the definition of Stolarsky means, for  $s > r \neq 0$  and each  $i = 1, 2, \dots, n$ , the bounds

$$y_i \leq \left( \frac{r(x_i^s - y_i^s)}{s(x_i^r - y_i^r)} \right)^{1/(s-r)} \leq x_i.$$

are known.

Hence,

$$(s(x_i^r - y_i^r))^2 y_i^{2(s-r)} \leq (r(x_i^s - y_i^s))^2 \leq (s(x_i^r - y_i^r))^2 x_i^{2(s-r)},$$

and

$$s^2 \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 y_i^{2(s-r)} \leq r^2 \sum_{i=1}^n a_i (x_i^s - y_i^s)^2 \leq s^2 \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 x_i^{2(s-r)},$$

wherefrom one easily obtains that

$$s^2 (\min\{y_i\})^{2(s-r)} \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 \leq r^2 \sum_{i=1}^n a_i (x_i^s - y_i^s)^2 \leq s^2 (\max\{x_i\})^{2(s-r)} \sum_{i=1}^n a_i (x_i^r - y_i^r)^2,$$

i.e.,

$$\min\{y_i\} \leq \left( \frac{r^2 (\sum_{i=1}^n a_i (x_i^s - y_i^s)^2)}{s^2 (\sum_{i=1}^n a_i (x_i^r - y_i^r)^2)} \right)^{1/2(s-r)} \leq \max\{x_i\}, \quad i = 1, 2, \dots, n.$$

The other cases follow simultaneously as a results of limit processes inside the definite fixed bounds.

For example, for  $r = s = 0$ , we have

$$\begin{aligned} E_{0,0}(A_n; X_n, Y_n) &= \exp \left( \frac{\sum_{i=1}^n a_i (\log x_i - \log y_i) (\log^2 x_i - \log^2 y_i)}{2 \sum_{i=1}^n a_i (\log x_i - \log y_i)^2} \right) \\ &= \exp \left( \frac{\sum_{i=1}^n a_i (\log x_i - \log y_i)^2 \log(\sqrt{x_i y_i})}{\sum_{i=1}^n a_i (\log x_i - \log y_i)^2} \right), \end{aligned}$$

and applying the inequality

$$\min\{y_i\} \leq y_i \leq \sqrt{x_i y_i} \leq x_i \leq \max\{x_i\},$$

we obtain

$$\min\{y_i\} \leq E_{0,0}(A_n; X_n, Y_n) \leq \max\{x_i\}, \quad i = 1, 2, \dots, n.$$

Therefore, the fact that expressions  $E_{r,s}(A_n; X_n, Y_n)$  are means is proved.  $\square$

For the proof of Property 5., let us recall some basic facts from Convexity Theory.

A function  $f$  is convex on an interval  $I$  if it is continuous on  $I$  and it is Jensen convex on  $I$ , that is for all  $x, y \in I$ ,

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right).$$

**Lemma 2.** A positive function  $g$  is log-convex on an interval  $I$  if it is continuous on  $I$  and the inequality

$$\alpha^2 g(s) + 2\alpha\beta g\left(\frac{s+t}{2}\right) + \beta^2 g(t) \geq 0,$$

holds for all  $\alpha, \beta \in \mathbb{R}$  and  $s, t \in I$ .

**Proof.** The above inequality holds for all  $\alpha, \beta \in \mathbb{R}$  if and only if

$$g(s)g(t) \geq g^2\left(\frac{s+t}{2}\right),$$

that is

$$\frac{\log g(s) + \log g(t)}{2} \geq \log g\left(\frac{s+t}{2}\right).$$

This means that  $\log \circ g$  is convex in the Jensen sense, and hence the continuity of  $g$  implies that it is log-convex.  $\square$

**Lemma 3.** Let the function  $h(x, y; s)$ ,  $x > y > 0$ , be defined as

$$h(s) = h(x, y; s) := \begin{cases} \frac{x^s - y^s}{s} & , s \neq 0; \\ \log(x/y) & , s = 0. \end{cases}$$

Then  $h(s)$  is log-convex on  $s \in \mathbb{R}$ .

**Proof.** Indeed,  $h(s)$  is continuous on  $s \in \mathbb{R}$  and the inequality

$$\alpha^2 h(s) + 2\alpha\beta h\left(\frac{s+t}{2}\right) + \beta^2 h(t) \geq 0$$

holds, because

$$\begin{aligned} & \alpha^2 h(s) + 2\alpha\beta h\left(\frac{s+t}{2}\right) + \beta^2 h(t) \\ &= \alpha^2 \int_y^x u^{s-1} du + 2\alpha\beta \int_y^x u^{\frac{s+t}{2}-1} du + \beta^2 \int_y^x u^{t-1} du \\ &= \int_y^x (\alpha u^{s/2} + \beta u^{t/2})^2 u^{-1} du. \end{aligned}$$

Therefore Lemma 2 can be applied.  $\square$

**Lemma 4.** If, for positive  $u, v, w$ , the inequality

$$\alpha^2 u + 2\alpha\beta v + \beta^2 w \geq 0,$$

holds for each  $\alpha, \beta \in \mathbb{R}$ , then also

$$\alpha^2 u^p + 2\alpha\beta v^p + \beta^2 w^p \geq 0,$$

holds for each  $p > 0$ .

**Proof.** Obvious. Now we are enabled to prove Property 5. of Theorem 4. For this cause, denote

$$g_n(s) = \sum_1^n a_i h_i^2(s),$$

where  $h_i(s) := h(x_i, y_i; s)$  and  $a_i, i = 1, 2, \dots, n$  are positive numbers.

By Lemmas 2–4, we see that  $g_n(s)$  is log-convex in  $s, s \in \mathbb{R}$ , since

$$\alpha^2 g_n(s) + 2\alpha\beta g_n\left(\frac{s+t}{2}\right) + \beta^2 g_n(t) = \sum_1^n a_i (\alpha^2 h_i^2(s) + 2\alpha\beta h_i^2\left(\frac{s+t}{2}\right) + \beta^2 h_i^2(t)) \geq 0.$$

Therefore the function  $F(s) = \log g_n(s)$  is convex and, applying Lemma 1, we obtain that

$$\frac{\log g_n(s) - \log g_n(r)}{s - r} = \log \left( \frac{g_n(s)}{g_n(r)} \right)^{\frac{1}{s-r}} = 2 \log E_{r,s}(A_n; X_n, Y_n),$$

is monotone increasing in both  $r$  and  $s$ , which is equivalent with the Property 5 in the case  $s > r \neq 0$ .

By continuity, the proof of other cases follows immediately. For example, since for any  $\epsilon > 0$  we have

$$E_{r+\epsilon, s+\epsilon}(A_n; X_n, Y_n) \geq E_{r,s}(A_n; X_n, Y_n),$$

letting  $r \rightarrow s$ , we obtain

$$E_{s+\epsilon, s+\epsilon}(A_n; X_n, Y_n) \geq E_{s,s}(A_n; X_n, Y_n),$$

that is,  $E_{s,s}(A_n; X_n, Y_n)$  is monotone increasing in  $s$ .

□

Our task in the sequel is to investigate under what conditions the means  $e_{r,s}(X_n)$  and  $E_{r,s}(A_n; X_n, Y_n)$  are monotone increasing/decreasing in  $n$ .

For this cause we need the following two lemmas.

**Lemma 5.** Stolarsky means  $E_{r,s}(x, y)$  are monotone increasing in both variables  $x$  and  $y$ .

This is the well-known assertion ([1]).

**Lemma 6.** For two given sequences  $\{u_n\}$  and  $\{v_n\}$  of positive numbers, denote

$$w_n := \frac{u_n}{v_n}; \quad W_n := \frac{u_1 + u_2 + \dots + u_n}{v_1 + v_2 + \dots + v_n}.$$

If the sequence  $w_n$  is monotone decreasing/increasing, then the sequence  $W_n$  is also monotone decreasing/increasing.

**Proof.** Let  $w_n$  be a decreasing sequence. The other case can be treated similarly.

We prove firstly that  $v_{n+1} \sum_1^n u_i \geq u_{n+1} \sum_1^n v_i$ .

Indeed,

$$v_{n+1} \sum_1^n u_i = v_{n+1} \sum_1^n v_i w_i \geq v_{n+1} w_{n+1} \sum_1^n v_i = u_{n+1} \sum_1^n v_i.$$

Hence,



$$\sum_1^n u_i \sum_1^{n+1} v_i = \sum_1^n u_i \sum_1^n v_i + v_{n+1} \sum_1^n u_i \geq \sum_1^n u_i \sum_1^n v_i + u_{n+1} \sum_1^n v_i = \sum_1^n v_i \sum_1^{n+1} u_i,$$

i.e.,  $W_n \geq W_{n+1}$ .

□

**Theorem 5.** If both sequences  $\{X_n\}$  and  $\{Y_n\}$  are monotone decreasing (increasing), then means  $E_{r,s}(A_n; X_n, Y_n)$  are monotone decreasing (increasing) in  $n$ .

**Proof.** We shall prove the “decreasing” part of Theorem 5. The proof of the other part is analogous.

Hence, we assume that both sequences  $\{X_n\}$  and  $\{Y_n\}$  are monotone decreasing. In the case  $s > r \neq 0$ , denote

$$u_n := a_n r^2 (x_n^s - y_n^s)^2; \quad v_n := a_n s^2 (x_n^r - y_n^r)^2.$$

By Lemma 5, we have

$$w_n = \frac{u_n}{v_n} = \frac{r^2 (x_n^s - y_n^s)^2}{s^2 (x_n^r - y_n^r)^2} \geq \frac{r^2 (x_{n+1}^s - y_{n+1}^s)^2}{s^2 (x_{n+1}^r - y_{n+1}^r)^2} = w_{n+1}.$$

Therefore the sequence  $w_n$  is monotone decreasing and, by Lemma 6, this implies  $W_n \geq W_{n+1}$ , that is,

$$\begin{aligned} & \frac{r^2 a_1 (x_1^s - y_1^s)^2 + a_2 (x_2^s - y_2^s)^2 + \cdots + a_n (x_n^s - y_n^s)^2}{s^2 a_1 (x_1^r - y_1^r)^2 + a_2 (x_2^r - y_2^r)^2 + \cdots + a_n (x_n^r - y_n^r)^2} \\ & \geq \frac{r^2 a_1 (x_1^s - y_1^s)^2 + a_2 (x_2^s - y_2^s)^2 + \cdots + a_n (x_n^s - y_n^s)^2 + a_{n+1} (x_{n+1}^s - y_{n+1}^s)^2}{s^2 a_1 (x_1^r - y_1^r)^2 + a_2 (x_2^r - y_2^r)^2 + \cdots + a_n (x_n^r - y_n^r)^2 + a_{n+1} (x_{n+1}^r - y_{n+1}^r)^2}. \end{aligned}$$

Since  $s > r$ , this is equivalent to  $E_{r,s}(A_n; X_n, Y_n) \geq E_{r,s}(A_{n+1}; X_{n+1}, Y_{n+1})$ .

In the cases  $r = s \neq 0, s > 0 = r$  and  $r = s = 0$  one should take

$$u_n = a_n (x_n^s - y_n^s) (x_n^s \log x_n - y_n^s \log y_n), \quad v_n = a_n (x_n^s - y_n^s)^2;$$

$$u_n = a_n (x_n^s - y_n^s)^2, \quad v_n = s^2 a_n (\log x_n - \log y_n)^2;$$

$$u_n = a_n (\log x_n - \log y_n) (\log^2 x_n - \log^2 y_n), \quad v_n = 2a_n (\log x_n - \log y_n)^2,$$

respectively, and proceed as above.

□

On the other hand, the problem of monotonicity in  $n$  for means  $e_{r,s}(X_n)$  seems significantly harder. We are able to solve it only in the simplest case  $r = s = 0$ .

**Theorem 6.** The means  $e_{0,0}(X_n)$  are monotone increasing/decreasing in  $n$  if and only if

$$x_n \geq z(X_{n-1}) (e_{0,0}(X_{n-1}) / z(X_{n-1}))^{3n/(n+1)}, \quad n \geq 3,$$

where  $z(X_n)$  denotes the geometric mean of numbers  $X_n$ .

**Proof.** We have

$$\log e_{0,0}(X_n) = \frac{b_n(X_n)}{3a_n(X_n)},$$

with

$$a_n(X_n) = \frac{1}{n} \sum_1^n \log^2 x_i - \left( \frac{1}{n} \sum_1^n \log x_i \right)^2 = \frac{1}{n} \sum_1^n \log^2(x_i/z(X_n)) \geq 0,$$

and

$$b_n(X_n) = \frac{1}{n} \sum_1^n \log^3 x_i - \left( \frac{1}{n} \sum_1^n \log x_i \right)^3.$$

Note that for  $x_n = z(X_{n-1})$  we have  $e_{0,0}(X_n) = e_{0,0}(X_{n-1})$ . Therefore by Taylor expansion around this point, we obtain

$$a_n(X_n) = \frac{n-1}{n} [a_{n-1}(X_{n-1}) + \frac{1}{n} \log^2(x_n/z(X_{n-1}))],$$

and

$$b_n(X_n) = \frac{n-1}{n} [b_{n-1}(X_{n-1}) + \frac{3 \log z(X_{n-1})}{n} \log^2(x_n/z(X_{n-1})) + \frac{n+1}{n^2} \log^3(x_n/z(X_{n-1}))].$$

Since  $b_{n-1}(X_{n-1})/3a_{n-1}(X_{n-1}) = \log e_{0,0}(X_{n-1})$ , we finally get

$$\begin{aligned} \log e_{0,0}(X_n) - \log e_{0,0}(X_{n-1}) &= \frac{b_n(X_n)}{3a_n(X_n)} - \frac{b_{n-1}(X_{n-1})}{3a_{n-1}(X_{n-1})} \\ &= \frac{\log^2(x_n/z(X_{n-1}))}{3na_n(X_n)} [3 \log z(X_{n-1}) + \frac{n+1}{n} \log(x_n/z(X_{n-1})) - b_{n-1}(X_{n-1})/a_{n-1}(X_{n-1})], \end{aligned}$$

and the proof follows.  $\square$

### 3. Conclusions

In this article we give two explicit generalizations of Stolarsky means to the multi-variable case and proved that they preserve all main properties of the original means. Let us note that other subtle properties are not equally transposed. For example, log-convexity of  $E_{r,s}(x_1, x_2)$  entirely depends on parameters  $r, s$  ([12]), but in the case of means  $B_{r,s}(x_1, x_2, x_3)$ , mentioned in the Introduction, it also depends on  $x_3 \leq \sqrt{x_1 x_2}$ .

Furthermore, many open questions can be proposed. For example, is monotone increase of the sequences  $\{X_n\}$  and  $\{Y_n\}$  necessary for  $E_{r,s}(A_n; X_n, Y_n)$  to be increasing in  $n$ ?

Or, is the monotonicity in variables possible for the means  $e_{r,s}(X_n)$  only if  $n = 2$  or  $n = 3$ ?

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### References

1. Stolarsky, K.B. Generalizations of the logarithmic mean. *Math. Mag.* **1995**, *48*, 87–92. [CrossRef]
2. Simic, S. Another converse of Jensen's Inequality. Available online: <http://vuir.vu.edu.au/17742/1/simic.pdf> (accessed on 20 March 2020).
3. Stolarsky, K.B. The power and generalized logarithmic means. *Amer. Math. Mon.* **1980**, *87*, 545–548. [CrossRef]
4. Leach E.B.; Sholander, M.C. Extended mean values. *Amer. Math. Mon.* **1978**, *85*, 84–90. [CrossRef]

5. Leach, E.B.; Sholander, M.C. Multi-variable extended mean values. *J. Math. Anal. Appl.* **1984**, *104*, 390–407. [[CrossRef](#)]
6. Páles, Z. Inequalities for differences of powers. *J. Math. Anal. Appl.* **1988**, *131*, 271–281. [[CrossRef](#)]
7. Neuman E.; Páles, Z. On comparison of Stolarsky and Gini means. *J. Math. Anal. Appl.* **2003**, *278*, 274–284. [[CrossRef](#)]
8. Neuman E.; Sándor, J. Inequalities involving Stolarsky and Gini means. *Math. Pannonica* **2003**, *14*, 29–44.
9. Czinder, P.; Páles, Z. An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means. *J. Ineq. Pure Appl. Math.* **2004**, *5*, 42.
10. Witkowski, A. Weighted extended mean values. *Colloq. Math.* **2004**, *100*, 111–117. [[CrossRef](#)]
11. Witkowski, A. Convexity of weighted Stolarsky means. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 73.
12. Qi, F. Logarithmic convexity of extended mean values. *Proc. Amer. Math. Soc.* **2001**, *130*, 1787–1796. [[CrossRef](#)]
13. Merikoski, J. K. Extending means of two variables to several variables. *J. Ineq. Pure Appl. Math.* **2004**, *5*, 65.
14. Pečarić, J.; Šimić, V. Stolarsky-Tobey mean in  $n$  variables. *Math. Ineq. Appl.* **1999**, *2*, 325–334. [[CrossRef](#)]
15. Horvath, I.; Pečarić, J. New versions of weighted multidimensional functional and Stolarsky means. *Acta Math. Hung.* **2015**, *147*, 81–96. [[CrossRef](#)]
16. Simic, S. On a class of mean values including Jensen functionals. *Publ. Math. Debrecen* **2008**, *73*, 461–469.
17. Simic, S. An extension of Stolarsky means to the multivariable case. *Int. J. Math. Math. Sci.* **2009**, 32857. [[CrossRef](#)]
18. Simic, S. On certain new inequalities in Information Theory. *Acta Math. Hungar.* **2009**, *124*, 353–361. [[CrossRef](#)]



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