Ricci Curvature Inequalities for Skew CR-Warped Product Submanifolds in Complex Space Forms

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Abstract: The fundamental goal of this study was to achieve the Ricci curvature inequalities for a skew CR-warped product (SCR W-P) submanifold isometrically immersed in a complex space form (CSF) in the expressions of the squared norm of mean curvature vector and warping functions (W-F). The equality cases were likewise examined. In particular, we also derived Ricci curvature inequalities for CR-warped product (CR W-P) submanifolds. To sustain this study, an example of these submanifolds is provided.

1. Introduction

There have been several studies in the past to demonstrate the geometries of submanifolds in the settings of almost Hermitian (A-H) and almost contact metric (A-C M) manifolds. By the operation of the almost complex structure J, the tangent space of a submanifold of an almost Hermitian manifold can be classified into holomorphic and totally real submanifolds. The notion of CR-submanifolds was introduced and studied by A. Bejancu [1] in 1981 as a generalization of holomorphic and totally real submanifolds. Thus, as to have a more profound knowledge of the geometry of CR-submanifolds of almost Hermitian "AH" manifolds, Chen [2] further explored these submanifolds and provided many fundamental results. In 1990 Chen [3] instigated a generalized class of submanifolds, namely, slant submanifolds. Moreover, advances in the geometry of CR-submanifolds and slant submanifolds stimulated various authors to search for the class of submanifolds which unifies the properties of all previously discussed submanifolds. In this context, N. Papaghuic [4] introduced the notion of semi-slant submanifolds in the framework of almost-Hermitian manifolds and showed that submanifolds belonging to this class enjoy many of the desired properties. Later, the contact variant of semi-slant submanifolds was studied by Cabrerizo et al. [5]. Recently, B. Sahin [6] investigated another class of submanifolds in the setting of almost Hermitian manifolds and he called these submanifolds Hemi-slant submanifolds. This class includes the CR-submanifolds and slant submanifolds.

In 1990, Ronsse [7] started the study of skew CR-submanifolds in the setting of almost Hermitian manifolds. Skew CR-submanifolds contain the classes of CR-submanifolds, semi-slant submanifolds and Hemi-slant submanifolds.

The acknowledgment of warped product manifolds appeared after the methodology of Bishop and O'Neill [8] on the manifolds of non positive curvature. By analyzing the way that a Riemannian product of manifolds cannot have non positive curvature, they represented warped product (W-P) manifolds for the class of manifolds of non-positive curvature which is characterized as follows:

Let $(S_1, \langle, \rangle_1)$ and $(S_2, \langle, \rangle_2)$ be two Riemannian manifolds with Riemannian metrics $\langle, \rangle_1$ and $\langle, \rangle_2$ respectively and g be a smooth positive function on S_1. If $\pi : S_1 \times S_2 \to S_1$ and $\eta : S_1 \times S_2 \to S_2$ are the projection maps given by $\pi(x, y) = x$ and $\eta(x, y) = y$ for every $(x, y) \in S_1 \times S_2$, then the *W-P manifold* is the product manifold $S_1 \times S_2$ holding the Riemannian structure such that

$$\langle U_1, U_2 \rangle = \langle \pi_* U_1, \pi_* U_2 \rangle_1 + (g \circ \pi)^2 \langle \eta_* U_1, \eta_* U_2 \rangle_2,$$

for all $U_1, U_2 \in TS$. The function g is called the *warping function* (W-F) of the warped product (W-P) manifold. If the W-F is constant, then the W-P is a trivial, i.e., simply Riemannian product. Further, if $U_1 \in TS_1$ and $U_2 \in TS_2$, then from Lemma 7.3 of [8], we have the following well-known result

$$D_{U_1} U_2 = D_{U_2} U_1 = (\frac{U_1 g}{g}) U_2, \tag{1}$$

where D is the Levi-Civita connection on S. In the light of the fact that W-P manifolds have various uses in physics and the theory of relativity [9], this has been a subject of broad interest. The idea of displaying the space-time close to black holes admits the W-P manifolds [10]. Schwartzschild space-time $T \times_k S^2$, is a model of W-P, wherein the base $T = R \times R^+$ is a half plane $k > 0$ and the fiber S^2 is the unit sphere. A cosmological model to show the universe as space-time, known as the Robertson–Walker model, is a W-P manifold [11].

Some common properties of W-P manifolds were concentrated on in [8]. B.-Y. Chen [12] played out an outward investigation of W-P submanifolds in a Kaehler manifold. From that point forward, numerous geometers have investigated W-P manifolds in various settings such as almost complex and almost contact manifolds, and different existence results have been researched (see the survey article [13–16]). Recently, B. Sahin [17] contemplated SCR W-P submanifolds in Kaehler manifolds and got some essential outcomes. Further, these submanifolds were explored by Haidar and Thakur in the context of cosymplectic manifolds [18].

In 1999, Chen [19] discovered a relationship between Ricci curvature and a squared mean curvature vector for a discretionary Riemannian manifold. More precisely, Chen proved the following theorem

Theorem 1. *Let $\phi : S^t \to \bar{S}^m(c)$ be an isometric immersion of a $t-$ dimensional Riemannian manifold into a Riemannian space form $\bar{S}^m(c)$.*

1. *For each unit tangent vector $\chi \in T_p S^t$, we have*

    $$\|\Pi\|^2(p) \geq \frac{4}{t^2} \{R^S(\chi) - (t-1)c\}$$

    where $\|\Pi\|^2(p)$ is the squared mean curvature and $R^S(\chi)$ the Ricci curvature of S^t at χ.
2. *If $\Pi(p) = 0$, then the unit tangent vector χ at p satisfies the equality case of (1) if and only if χ lies in the relative null space $\mathcal{N}_p$ at p.*
3. *The equality case holds identically for all unit tangent vectors at x if and only if either p is a totally geodesic point or $t = 2$ and p is a totally umbilical point.*

Theorem 1 was generalized for semi-slant submanifolds in Sasakian space form by Cioroboiu and Chen [20]. Further, D. W. Yoon [21] studied Chen Ricci inequality for slant submanifols in the framework of cosymplectic space forms. Motivated by Chen [19], Mihai and Ožgur [22] studied Chen Ricci inequality for real space forms with semi-symmetric connections. In [23] M. M. Tripathi formulated an improved relationship between Ricci curvature and squared mean curvature. More recently, Ali et al. [24] generalized Chen Ricci inequality for warped product submanifolds in spheres and provided some applications in mechanics and mathematical physics.

The class of SCR W-P submanifolds is rich in its geometric behavior; it contains classes of CR-warped product submanifolds, semi-slant warped product submanifolds and hemi-slant warped

product submanifolds. In the literature it was found that Ricci curvature for these warped product submanifolds in complex space forms has not been studied. In other words, we can say that Theorem 1 is an open problem for skew CR-warped product submanifolds in the setting of complex space forms.

In this study our point is to establish a connection between Ricci curvature and squared mean curvature for SCR W-P submanifolds in the setting of complex space forms.

2. Preliminaries

Let $\bar{S}$ be an A-H manifold with an almost complex structure J and a Hermitian metric $\langle,\rangle$, i.e., $J^2 = -I$ and $\langle JU_1, JU_2 \rangle = \langle U_1, U_2 \rangle$, for all vector fields U_1, U_2 on $\bar{S}$. If J is parallel with respect to the Levi-Civita connection $\bar{D}$ on $\bar{S}$, that is

$$(\bar{D}_{U_1} J) U_2 = 0, \tag{2}$$

for all $U_1, U_2 \in T\bar{S}$, then $(\bar{S}, J, \langle,\rangle, \bar{D})$ is called a *Kaehler manifold* (K-M).

A K-M $\bar{S}$ is called a *CSF* if it has constant holomorphic sectional curvature c denoted by $\bar{S}(c)$. The curvature tensor of the CSF $\bar{S}(c)$ is given by

$$\bar{R}(U_1, U_2, U_3, U_4) = \frac{c}{4}[\langle U_2, U_3 \rangle \langle U_1, U_4 \rangle - \langle U_1, U_3 \rangle \langle U_2, U_4 \rangle + \langle U_1, JU_3 \rangle \langle JU_2, U_4 \rangle$$
$$- \langle U_2, JU_3 \rangle \langle JU_1, U_4 \rangle + 2\langle U_1, JU_2 \rangle \langle JU_3, U_4 \rangle], \tag{3}$$

for any $U_1, U_2, U_3, U_4 \in T\bar{S}$.

Let S be a $n-$dimensional Riemannian manifold isometrically immersed in a $m-$ dimensional Riemannian manifold $\bar{S}$. Then, the Gauss and Weingarten formulas are $\bar{D}_{U_1} U_2 = D_{U_1} U_2 + \Gamma(U_1, U_2)$ and $\bar{D}_{U_1} \xi = -A_\xi U_1 + D^\perp_{U_1} \xi$ respectively, for all $U_1, U_2 \in TS$ and $\xi \in T^\perp S$, where D is the induced Levi-Civita connection on S, ξ is a vector field normal to S, Γ is the second fundamental form of S, $D^\perp$ is the normal connection in the normal bundle $T^\perp S$ and A_ξ is the shape operator of the second fundamental form. The second fundamental form Γ and the shape operator are related by the following formula

$$\langle \Gamma(U_1, U_2), \xi \rangle = \langle A_\xi U_1, U_2 \rangle. \tag{4}$$

The Gauss equation is given by

$$R(U_1, U_2, U_3, U_4) = \bar{R}(U_1, U_2, U_3, U_4) + \langle \Gamma(U_1, U_4), \Gamma(U_2, U_3) \rangle - \langle \Gamma(U_1, U_3), \Gamma(U_2, U_4) \rangle, \tag{5}$$

for all $U_1, U_2, U_3, U_4 \in TS$, where $\bar{R}$ and R are the curvature tensors of $\bar{S}$ and S respectively.

For any $U_1 \in TS$ and $\xi \in T^\perp S$, JU_1 and $J\xi$ can be decomposed as follows.

$$JU_1 = PU_1 + FU_1 \tag{6}$$

and

$$J\xi = t\xi + f\xi, \tag{7}$$

where PU_1 (resp. $t\xi$) is the tangential and FU_1 (resp. $f\xi$) is the normal component of JU_1 (resp. $J\xi$).

It is evident that $\langle JU_1, U_2 \rangle = \langle PU_1, U_2 \rangle$ for any $U_1, U_2 \in T_x S$; this implies that $\langle PU_1, Y_2 \rangle + \langle U_1, PU_2 \rangle = 0$. Thus, P^2 is a symmetric operator on the tangent space $T_x S$, for any $x \in S$. The eigenvalues of P^2 are real and diagonalizable. Moreover, for each $x \in S$, one can observe

$$L^\lambda_x = Ker\{P^2 + \lambda^2(x)I\}_x,$$

where I denotes the identity transformation on $T_x S$, and $\lambda(x) \in [0, 1]$ such that $-\lambda^2(x)$ is an eigenvalue of $P^2(x)$. Further, it is easy to observe that $KerF = L^1_x$ and $KerP = L^0_x$, where L^1_x is the maximal holomorphic sub space of $T_x S$ and L^0_x is the maximal totally real subspace of $T_x S$; these distributions

are denoted by L and $L^{\perp}$ respectively. If $-\lambda_1^2(x), \ldots, -\lambda_k^2(x)$ are the eigenvalues of P^2 at x, then $T_x S$ can be decomposed as

$$T_x S = L_x^{\lambda_1} \oplus L_x^{\lambda_2} \oplus \ldots L_x^{\lambda_k}.$$

Every $L_x^{\lambda_i}$, $1 \le i \le k$ is a $P-$invariant subspace of $T_x S$. Moreover, if $\lambda_i \ne 0$, then $L_x^{\lambda_i}$ is even dimensional the submanifold S of a Kaehler manifold $\bar{S}$ is a generic submanifold if there exists an integer k and functions λ_i $1 \le i \le k$ defined on S with $\lambda_i \in (0,1)$ such that

(i)   Each $-\lambda_i^2(x), 1 \le i \le k$, is a distinct eigenvalue of P^2 with

$$T_x S = L_x^T \oplus L_x^{\perp} \oplus L_x^{\lambda_1} \oplus \ldots, \oplus L_x^{\lambda_k}$$

for any $x \in S$.

(ii)   The distributions of $L_x^T, L_x^{\perp}$ and $L_x^{\lambda_i}$, $1 \le i \le k$ are independent of $x \in S$.

If in addition, each λ_i is constant on S, then S is called a skew CR-submanifold [7]. It is significant to recount that CR-submanifolds are a particular class of skew CR-submanifold for which $k = 1, L^T = \{0\}, L^{\perp} = \{0\}$ and λ_1 is constant. If $L^{\perp} = \{0\}, L^1 \ne \{0\}$ and $k = 1$, then S is a semi-slant submanifold, whereas if $L = \{0\}, L^{\perp} \ne \{0\}$ and $k = 1$, then S is a hemi-slant submanifold.

Definition 1. *A submanifold S of an A-H manifold $\bar{S}$ is said to be a "skew CR-submanifold of order 1" if S is a skew CR-submanifold with $k = 1$ and λ_1 is constant.*

We have the following characterization

Theorem 2. *Reference [3] let S be a submanifold of an A-H manifold $\bar{S}$. Then S is a slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda I.$$

Furthermore, if θ is a slant angle, then $\lambda = \cos^2 \theta$.

For any orthonormal basis $\{e_1, e_2, \ldots, e_t\}$ of the tangent space $T_x S$, the mean curvature vector $\Pi(x)$ and its squared norm are defined as follows.

$$\Pi(x) = \frac{1}{t} \sum_{i=1}^{t} \Gamma(e_i, e_i), \quad \|\Pi\|^2 = \frac{1}{t^2} \sum_{i,j=1}^{t} \langle \Gamma(e_i, e_i), \Gamma(e_j, e_j) \rangle, \tag{8}$$

where t is the dimension of S. If $\Gamma = 0$ then the submanifold is said to be totally geodesic and minimal if $\Pi = 0$. If $\Gamma(U_1, U_2) = \langle U_1, U_2 \rangle \Pi$ for all $U_1, U_2 \in TS$, then S is called totally umbilical (T-U).

The scalar curvature of $\bar{S}$ is denoted by $\bar{\tau}(\bar{S})$ and is defined as

$$\bar{\tau}(\bar{S}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}, \tag{9}$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{S}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\tau}(\bar{S}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{10}$$

In a similar way, the scalar curvature $\bar{\tau}(L_x)$ of a $L-$plane is given by

$$\bar{\tau}(L_x) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{11}$$

Let $\{e_1, \ldots, e_t\}$ be an orthonormal basis of the tangent space $T_x S$ and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^\perp S$, then we have

$$\Gamma_{pq}^r = \langle \Gamma(e_p, e_q), e_r \rangle \tag{12}$$

and

$$\|\Gamma\|^2 = \sum_{p,q=1}^{t} \langle \Gamma(e_p, e_q), \Gamma(e_p, e_q) \rangle. \tag{13}$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold S and in the Riemannian space form $\bar{S}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=t+1}^{m} (\Gamma_{pp}^r \Gamma_{qq}^r - (\Gamma_{pq}^r)^2). \tag{14}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_t\}$ on S is defined as

$$\bar{M}(U_1, U_2) = \sum_{i=1}^{t} \{ \langle \bar{R}(e_i, U_1)U_2, e_i \rangle \}, \tag{15}$$

for all $U_1, U_2 \in T_x S$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_n from $\{e_1, \ldots, e_t\}$ on S, which is governed by χ, then the Ricci curvature is defined by

$$R^S(\chi) = \sum_{\substack{p=1 \\ p \neq n}}^{t} \kappa(e_p \wedge e_n). \tag{16}$$

For a smooth function g on a Riemannian manifold S with Riemannian metric $\langle , \rangle$, the gradient of g is denoted by ∇g and is defined as

$$\langle \nabla g, U_1 \rangle = U_1 g, \tag{17}$$

for all $U_1 \in TS$.

Let the dimension of S be t and $\{e_1, e_2, \ldots, e_t\}$ be a basis of TS. Then as a result of (17), we get

$$\|\nabla g\|^2 = \sum_{i=1}^{t} (e_i(g))^2. \tag{18}$$

The Laplacian of g is defined by

$$\Delta g = \sum_{i=1}^{t} \{ (\nabla_{e_i} e_i)g - e_i e_i \psi \}. \tag{19}$$

For a W-P submanifold $S_1^{t_1} \times_g S_2^{t_2}$ isometrically immersed in a Riemannian manifold $\bar{S}$, we observe the well known result, which can be described as follows [25]:

$$\sum_{p=1}^{t_1} \sum_{q=1}^{t_2} \kappa(e_p \wedge e_q) = \frac{t_2 \Delta g}{g} = t_2 (\Delta \ln g - \|\nabla \ln g\|^2), \tag{20}$$

where t_1 and t_2 are the dimensions of the submanifolds $S_1^{t_1}$ and $S_2^{t_2}$ respectively.

3. Skew CR-Warped Product Submanifolds

Recently, B. Sahin [17] demonstrated the existence of SCR W-P of the type $S = S_1 \times_f S_\perp$, where S_1 is a semi-slant submanifold as defined by N. Papaghuic [4] and $S_\perp$ is a totally real

submanifold. Throughout this section we consider the SCR W-P $S = S_1 \times_f S_\perp$ in a Kaehler manifold $\bar{S}$. Then it is evident that S is a proper SCR W-P of order 1. Moreover, the tangent space TS of S can be decomposed as follows.

$$TS = L^\theta \oplus L^T \oplus L^\perp, \tag{21}$$

where $L_x^\theta = L_x^{\lambda_1}$. If $L^\theta = \{0\}$, then S becomes a CR-warped product submanifold defined in [26]. If $L^T = \{0\}$, then S is reduced to a warped product hemi-slant submanifold [6]. Thus, skew CR-warped product submanifold presents a single platform to study the CR W-P submanifolds and W-P hemi-slant submanifolds.

Now, we have an example of SCR W-P submanifold in an A-H manifold

Example 1. *Let S be a submanifold in R^{12} defined by $x_1 = u, x_2 = v\,sech\alpha, x_3 = k\,tanh\beta, x_4 = k\,sech\beta, x_5 = u\,sech\beta, x_6 = u\,tanh\beta, y_1 = -v, y_2 = v\,tanh\alpha, y_3 = -r\,tanh\beta, y_4 = -r\,sech\beta, y_5 = 0, y_6 = 0.*

Then, we have the following basis of TS

$$U_1 = sech\beta\frac{\partial}{\partial x_5} + tanh\beta\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_1}, \quad U_2 = sech\alpha\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1} + tanh\alpha\frac{\partial}{\partial y_2},$$

$$U_3 = tanh\beta\frac{\partial}{\partial x_3} + sech\beta\frac{\partial}{\partial x_4}, \quad U_4 = -tanh\beta\frac{\partial}{\partial y_3} - sech\beta\frac{\partial}{\partial y_4},$$

$$U_5 = -k\,sech\beta\frac{\partial}{\partial x_3} + k\,tanh\beta\frac{\partial}{\partial x_4} + u\,tanh\beta\frac{\partial}{\partial x_5} - u\,sech\beta\frac{\partial}{\partial x_6} + r\,sech\beta\frac{\partial}{\partial y_3} - r\,tanh\beta\frac{\partial}{\partial y_4}.$$

It is straightforward to identify that $L^\theta = span\{U_1, U_2\}$ is a slant distribution with slant angle $60°$, $L = span\{U_3, U_4\}$ is a holomorphic distribution and JU_5 is orthogonal to S. Thus $L^\perp = span\{U_5\}$ is a totally real distribution. Moreover, it is easy to observe that L^θ, L and $L^\perp$ are integrable. If S_θ, S_T and $S_\perp$ are the integral manifolds of the distributions L^θ, L and $L^\perp$ respectively. Then the induced metric tensor of S is given by

$$ds^2 = \langle,\rangle_{S_\theta} + \langle,\rangle_{S_T} + (k^2 + u^2 + r^2)\langle,\rangle_{S_\perp}$$

or

$$ds^2 = \langle,\rangle_{S_1} + (k^2 + u^2 + r^2)\langle,\rangle_{S_\perp}.$$

Definition 2. *The warped product $S_1 \times_f S_2$ isometrically immersed in a Riemannian manifold $\bar{S}$ is called S_i totally geodesic if the partial second fundamental form Γ_i is zero identically. It is called S_i-minimal if the partial mean curvature vector Π^i becomes zero for $i = 1, 2$.*

Let $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{t_1=2p} = Je_p, e^1, \ldots, e^q, e^{q+1} = \sec\theta Pe^1, \ldots, e^{(t_2=2q)} = \sec\theta Pe^q, e^{t_2+1}, \ldots, e^{t_3}\}$ be a local orthonormal frame of vector fields such that $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{t_1=2p} = Je_p\}$ is an orthonormal basis of L, $\{e^1, \ldots, e^q, e^{q+1} = \sec\theta Pe^1, \ldots, e^{(t_2=2q)} = \sec\theta Pe^q\}$ is an orthonormal basis of L_θ and $\{e^{t_2+1}, \ldots, e^{t_3}\}$ is an orthonormal basis of $L^\perp$.

Throughout this paper we consider that the SCR W-P submanifold $S_1 \times_f S_\perp$ is $L-$minimal. Presently we have the following outcome for further applications

Lemma 1. *Let $S^t = S_1^{t_1+t_2} \times_f S_\perp^{t_3}$ be a $L-$minimal SCR W-P submanifold isometrically immersed in a Kaehler manifold; then*

$$\|\Pi\|^2 = \frac{1}{t^2}\sum_{r=t+1}^{m}(\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2, \tag{22}$$

where $\|\Pi\|^2$ represents squared mean curvature.

4. Ricci Curvature for Skew CR-Warped Product Submanifold

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping functions as follows:

Theorem 3. *Let* $S^t = S_1^{t_1+t_2} \times_f S_\perp^{s_3}$ *be a* $L-$*minimal SCR W-P submanifold isometrically immersed in a Complex space form* $\bar{S}^m(c)$. *If the holomorphic and slant distributions* L *and* L_θ *are integrable with integral submanifolds* $S_T^{t_1}$ *and* $S_\theta^{t_2}$ *respectively, then for each orthogonal unit vector field* $\chi \in T_xS$, *the tangent to* $S_T^{t_1}$, $S_\theta^{t_2}$ *or* $S_\perp^{t_3}$, *we have that*

(1) *The Ricci curvature satisfies the following expressions:*

    (i)   *If* $\chi \in TS_T^{t_1}$, *then*

$$\frac{1}{4}t^2\|\Pi\|^2 \geq R^S(\chi) + \frac{t_3\Delta f}{f} + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_1t_3 - \frac{1}{2}). \tag{23}$$

    (ii) $\chi \in TS_\theta^{t_2}$, *then*

$$\frac{1}{4}t^2\|\Pi\|^2 \geq R^S(\chi) + \frac{t_3\Delta f}{f} + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_1t_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{24}$$

    (iii)   *If* $\chi \in TS_\perp^{t_2}$, *then*

$$\frac{1}{4}t^2\|\Pi\|^2 \geq R^S(\chi) + \frac{t_3\Delta f}{f} + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_1t_3 + 1). \tag{25}$$

(2) *If* $\Gamma(x) = 0$ *for each point* $x \in S^t$, *then there is a unit vector field* χ *which satisfies the equality of (1) iff* S^t *is mixed totally geodesic and* $\chi \in \mathcal{N}_x$ *at* x.

(3) *For the equality case we have*

    (a)   *The equality of (23) holds identically for all unit vector fields tangential to* $S_T^{t_1}$ *at each* $x \in S^t$ *iff* S^t *is mixed TG and* $L-$*totally geodesic SCR W-P submanifold in* $\bar{S}^m(c)$.

    (b)   *The equality of (24) holds identically for all unit vector fields tangential to* S_θ *at each* $x \in S^t$ *iff* S *is mixed totally geodesic and either* S^t *is* L_θ- *totally geodesic SCR W-P submanifold or* S^t *is a* L_θ *totally umbilical in* $\bar{S}^m(c)$ *with dim* $L_\theta = 2$.

    (c)   *The equality of (25) holds identically for all unit vector fields tangential to* $S_\perp^{t_2}$ *at each* $x \in S^t$ *iff* S *is mixed totally geodesic and either* S^t *is* $L^\perp$- *totally geodesic SCR W-P or* S^t *is a* $L^\perp$ *totally umbilical in* $\bar{S}^m(c)$ *with dim* $L^\perp = 2$.

    (d)   *The equality case of (1) holds identically for all unit tangent vectors to* S^t *at each* $x \in S^t$ *iff either* S^t *is totally geodesic submanifold or* M^t *is a mixed totally geodesic totally umbilical and* L *totally geodesic submanifold with dim* $S_\theta^{t_2} = 2$ *and dim* $S_\perp^{t_3} = 2$.

where t_1, t_2 *and* t_3 *are the dimensions of* $S_T^{t_1}$, $S_\theta^{t_2}$ *and* $S_\perp^{t_3}$ *respectively.*

Proof. Suppose that $S^t = S_1^{t_1+t_2} \times_f S_\perp^{t_3}$ be a SCR W-P submanifold of a CSF. From Gauss equation, we have

$$t^2\|\Pi\|^2 = 2\tau(S^t) + \|\Gamma\|^2 - 2\bar{\tau}(S^t). \tag{26}$$

Let $\{e_1, \ldots, e_{t_1}, e_{t_1+1}, \ldots, e_{t_2}, \ldots e_t\}$ be a local orthonormal frame of vector fields on S^t such that $\{e_1, \ldots, e_{t_1}\}$ is tangential to $S_T^{t_1}$, $\{e_{t_1+1}, \ldots, e_{t_2}\}$ is tangential to $S_\theta^{t_2}$ and $\{e_{t_2+1}, \ldots, e_t\}$ is the tangent to $S_\perp^{t_3}$. Thus, the unit tangent vector $\chi = e_A \in \{e_1, \ldots, e_t\}$ can be expanded (26) as follows.

$$t^2\|\Pi\|^2 = 2\tau(S^t) + \frac{1}{2}\sum_{r=t+1}^{m}\{(\Gamma_{11}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r - \Gamma_{AA}^r)^2 + (\Gamma_{AA}^r)^2\}$$
$$- \sum_{r=t+1}^{m}\sum_{1\leq i\neq j\leq t}\Gamma_{ii}^r\Gamma_{jj}^r - 2\bar{\tau}(S^t). \tag{27}$$

The above expression can be represented as

$$t^2\|\Pi\|^2 = 2\tau(S^t) + \frac{1}{2}\sum_{r=t+1}^{m}\{(\Gamma_{11}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2$$
$$+ (2\Gamma_{AA}^r - (\Gamma_{11}^r + \cdots + \Gamma_{tt}^r))^2\} + 2\sum_{r=t+1}^{m}\sum_{1\leq i<j\leq t}(\Gamma_{ij}^r)^2$$
$$- 2\sum_{r=t+1}^{m}\sum_{1\leq i<j\leq t}\Gamma_{ii}^r\Gamma_{jj}^r - 2\bar{\tau}(S^t).$$

In view of the assumption that SCR W-P submanifold $S_1 \times_f S_\perp$ is $L-$minimal submanifold, the preceding expression takes the form

$$t^2\|\Pi\|^2 = 2\tau(S^t) + \frac{1}{2}\sum_{r=t+1}^{m}\{(\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2$$
$$+ \frac{1}{2}\sum_{r=t+1}^{m}(2\Gamma_{AA}^r - (\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r))^2$$
$$+ \sum_{r=t+1}^{m}\sum_{1\leq i<j\leq t}(\Gamma_{ij}^r)^2 - \sum_{r=t+1}^{m}\sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}\Gamma_{ii}^r\Gamma_{jj}^r - 2\bar{\tau}(S^t) \tag{28}$$
$$+ \sum_{r=t+1}^{m}\sum_{\substack{a=1 \\ a\neq A}}^{t}(\Gamma_{aA}^r)^2 + \sum_{r=t+1}^{m}\sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}(\Gamma_{ij}^r)^2 - \sum_{r=t+1}^{m}\sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}\Gamma_{ii}^r\Gamma_{jj}^r.$$

Equation (14) can be written as

$$\sum_{\substack{1\leq p<q\leq t \\ p,q\neq A}}\bar{\kappa}_{pq} - \sum_{\substack{1\leq p<q\leq t \\ p,q\neq A}}\kappa_{pq} = \sum_{r=t+1}^{m}\sum_{\substack{1\leq p<q\leq t \\ p,q\neq A}}(\Gamma_r^{pq})^2 - \sum_{r=t+1}^{m}\sum_{\substack{1\leq p<q\leq t \\ p,q\neq A}}\Gamma_{pp}^r\Gamma_{qq}^r.$$

Substituting this value in (28), we derive

$$t^2\|\Pi\|^2 = 2\tau(S^t) + \frac{1}{2}\sum_{r=t+1}^{m}\{(\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2$$
$$+ \frac{1}{2}\sum_{r=t+1}^{m}(2\Gamma_{AA}^r - (\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r))^2$$
$$+ \sum_{r=t+1}^{m}\sum_{1\leq i<j\leq t}(\Gamma_{ij}^r)^2 - \sum_{r=t+1}^{m}\sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}\Gamma_{ii}^r\Gamma_{jj}^r - 2\bar{\tau}(S^t) \tag{29}$$
$$+ \sum_{r=t+1}^{m}\sum_{\substack{a=1 \\ a\neq A}}^{t}(\Gamma_{aA}^r)^2 + \sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}\bar{\kappa}_{ij} - \sum_{\substack{1\leq i<j\leq t \\ i,j\neq A}}\kappa_{ij}.$$

On the other hand, from (9) we have

$$
\tau(S^t) = \sum_{1 \leq i < j \leq t} \kappa(e_i \wedge e_j) = \sum_{\alpha=1}^{t_1+t_2} \sum_{\beta=t_1+t_2+1}^{t} \kappa(e_\alpha \wedge e_\beta) + \sum_{1 \leq \alpha < \gamma \leq t_1} \kappa(e_\alpha \wedge e_\gamma)
$$
$$
+ \sum_{t_1+1 \leq l < 0 \leq t_2} \kappa(e_l \wedge e_0) + \sum_{t_2+1 \leq u < v \leq t} \kappa(e_u \wedge e_v).
\tag{30}
$$

Using (9) and (20), we derive

$$
\tau(S^t) = \frac{t_3 \Delta f}{f} + \bar{\tau}(S_T^{t_1}) + \bar{\tau}(S_\theta^{t_2}) + \bar{\tau}(S_\perp^{t_3}).
$$

Using this in (29), we get

$$
\begin{aligned}
t^2 \|\Pi\|^2 = &\frac{t_3 \Delta f}{f} + \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1 t_1+1}^r + \dots \Gamma_{t_2 t_2}^r + \dots + \Gamma_{tt}^r)^2 \\
&+ \frac{1}{2} \sum_{r=t+1}^{m} (2\Gamma_{AA}^r - (\Gamma_{t_1+1 t_1+1}^r + \dots \Gamma_{t_2 t_2}^r + \dots + \Gamma_{tt}^r))^2 \\
&+ \sum_{r=t+1}^{m} \sum_{1 \leq \alpha < \beta \leq t_1} (\Gamma_{\alpha\alpha}^r \Gamma_{\beta\beta}^r - (\Gamma_{\alpha\beta}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{t_1+1 \leq p < q \leq t_2} (\Gamma_{pp}^r \Gamma_{qq}^r - (\Gamma_{pq}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{t_2+1 \leq s < n \leq t} (\Gamma_{ss}^r \Gamma_{nn}^r - (\Gamma_{sn}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{1 \leq i < j \leq t} (\Gamma_{ij}^r)^2 - \sum_{r=t+1}^{m} \sum_{\substack{1 \leq i < j \leq t \\ i,j \neq A}} (\Gamma_{ii}^r \Gamma_{jj}^r) \\
&- 2\bar{\tau}(S^t) + \sum_{\substack{1 \leq i < j \leq t \\ i,j \neq A}} \bar{\kappa}_{ij} + \bar{\tau}(S_T^{t_1}) + \bar{\tau}(S_\theta^{t_2}) + \bar{\tau}(S_\perp^{t_3}).
\end{aligned}
\tag{31}
$$

Considering unit tangent vector $\chi = e_A$, we have three choices: χ is the tangent to the base manifold $S_T^{t_1}$ or $S_\theta^{t_2}$, or to the fiber $S_\perp^{t_3}$.

Case 1: If $\chi \in S_T^{t_1}$, then we need to choose a unit vector field from $\{e_1, \dots, e_{t_1}\}$. Let $\chi = e_1$; then by (15) and the assumption that the submanifolds is $L-$minimal, we have

$$
\begin{aligned}
t^2 \|\Pi\|^2 \geq &R^S(\chi) + \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1 t_1+1}^r + \dots \Gamma_{t_2 t_2}^r + \dots + \Gamma_{tt}^r)^2 \\
&+ \frac{t_3 \Delta f}{f} + \frac{1}{2} \sum_{r=t+1}^{m} (2\Gamma_{11}^r - (\Gamma_{t_1+1 t_1+1}^r + \dots \Gamma_{t_2 t_2}^r + \dots + \Gamma_{tt}^r))^2 \\
&+ \sum_{r=t+1}^{m} \sum_{1 \leq \alpha < \beta \leq t_1} (\Gamma_{\alpha\alpha}^r \Gamma_{\beta\beta}^r - (\Gamma_{\alpha\beta}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{t_1+1 \leq p < q \leq t_2} (\Gamma_{pp}^r \Gamma_{qq}^r - (\Gamma_{pq}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{t_2+1 \leq s < n \leq t} (\Gamma_{ss}^r \Gamma_{nn}^r - (\Gamma_{sn}^r)^2) \\
&+ \sum_{r=t+1}^{m} \sum_{1 \leq i < j \leq t} (\Gamma_{ij}^r)^2 - \sum_{r=t+1}^{m} \sum_{2 \leq i < j \leq t} (\Gamma_{ii}^r \Gamma_{jj}^r) \\
&- 2\bar{\tau}(S^t) + \sum_{2 \leq i < j \leq t} \bar{\kappa}(e_i, e_j) + \bar{\tau}(S_T^{t_1}) + \bar{\tau}(S_\theta^{t_2}) + \bar{\tau}(S_\perp^{t_3}).
\end{aligned}
\tag{32}
$$

Putting $U_1, U_3 = e_i, \ U_2, U_4 = e_j$ in the formula (3), we have

$$2\bar{\tau}(S) = \frac{c}{4}[t(t-1) + 3t_1 + 3t_2\cos^2\theta] \tag{33}$$

$$\sum_{2\leq i<j\leq t} \bar{\kappa}(e_i, e_j) = \frac{c}{8}[(t-1)(t-2) + 3(t_1-1) + 3t_2\cos^2\theta]$$

$$\bar{\tau}(S_T^{t_1}) = \frac{c}{8}[t_1(t_1-1) + 3t_1]$$

$$\bar{\tau}(S_\theta^{t_2}) = \frac{c}{8}[t_2(t_2-1) + 3t_2\cos^2\theta]$$

$$\bar{\tau}(S_\perp^{t_3}) = \frac{c}{8}[t_3(t_3-1)].$$

Using these values in (32), we get

$$\begin{aligned}
t^2\|\Pi\|^2 \geq & R^S(\chi) + \frac{1}{2}t^2\|\Pi\|^2 + \frac{1}{2}\sum_{r=t+1}^{m}(2\Gamma_{11}^r - (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{tt}^r))^2 \\
& + \frac{t_3\Delta f}{f} + \sum_{r=t+1}^{m}\sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t_2}(\Gamma_{ij}^r)^2 \\
& + \sum_{r=t+1}^{m}\sum_{i=1}^{t_1}\sum_{k=t_2+1}^{t}(\Gamma_{ik}^r)^2 + \sum_{r=t+1}^{m}\sum_{\beta=2}^{t_1}\Gamma_{11}^r\Gamma_{\beta\beta}^r \\
& - \sum_{r=t+1}^{m}\sum_{i=2}^{t_1}\sum_{j=t_1+1}^{t_2}\Gamma_{ii}^r\Gamma_{jj}^r - \sum_{r=t+1}^{m}\sum_{i=2}^{t_1}\sum_{k=t_2+1}^{t}\Gamma_{ii}^r\Gamma_{kk}^r \\
& + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_3t_1 - \frac{1}{2}).
\end{aligned} \tag{34}$$

In view of the assumption that the submanifold is $L-$minimal, then

$$\sum_{r=t+1}^{m}\sum_{\beta=2}^{t_1}\Gamma_{11}^r\Gamma_{\beta\beta}^r = \sum_{r=t+1}^{m}(\Gamma_{11}^r)^2$$

$$-\sum_{r=t+1}^{m}\sum_{i=2}^{t_1}\Big[\sum_{j=t_1+1}^{t_2}\Gamma_{ii}^r\Gamma_{jj}^r + \sum_{k=t_2+1}^{t}\Gamma_{ii}^r\Gamma_{kk}^r\Big] = \sum_{r=t+1}^{m}\sum_{j=t_1+1}^{t}\Gamma_{11}^r\Gamma_{jj}^r.$$

Utilizing that in (34), we have

$$\begin{aligned}
t^2\|\Pi\|^2 \geq & R^S(\chi) + \frac{1}{2}t^2\|\Pi\|^2 + \frac{1}{2}\sum_{r=t+1}^{m}(2\Gamma_{11}^r - (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{nn}^r))^2 \\
& + \frac{t_3\Delta f}{f} + \sum_{r=t+1}^{m}\sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t_2}(\Gamma_{ij}^r)^2 \\
& + \sum_{r=t+1}^{m}\sum_{i=1}^{t_1}\sum_{k=t_2+1}^{t}(\Gamma_{ik}^r)^2 - \sum_{r=t+1}^{m}(\Gamma_{11}^r)^2 + \sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t}\Gamma_{ii}^r\Gamma_{jj}^r \\
& + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_3t_1 - \frac{1}{2}).
\end{aligned} \tag{35}$$

The third term on the right hand side can be written as

$$\frac{1}{2} \sum_{r=t+1}^{m} (2\Gamma_{11}^r - (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{t_2t_2}^r + \cdots + \Gamma_{nn}^r))^2$$

$$= 2 \sum_{r=t+1}^{m} (\Gamma_{11}^r)^2 + \frac{1}{2} t^2 \|\Pi\|^2 - 2 \sum_{r=t+1}^{m} \Big[\sum_{j=t_1+1}^{t_2} \Gamma_{11}^r \Gamma_{jj}^r \qquad (36)$$

$$+ \sum_{k=t_2+1}^{t} \Gamma_{11}^r \Gamma_{kk}^r \Big].$$

Combining above two expressions, we have

$$\frac{1}{2} t^2 \|\Pi\|^2 \geq R^S(\chi) + \sum_{r=t+1}^{m} (\Gamma_{11}^r)^2 - \sum_{r=t+1}^{m} \sum_{j=t_1+1}^{t} \Gamma_{11}^r \Gamma_{jj}^r$$

$$+ \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{t_2t_2}^r + \cdots + \Gamma_{nn}^r)^2$$

$$+ \sum_{r=t+1}^{m} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t} (\Gamma_{ij}^r)^2 + \frac{t_3 \Delta f}{f} \qquad (37)$$

$$+ \frac{c}{4} (t - t_1 t_2 - t_2 t_3 - t_3 t_1 - \frac{1}{2}),$$

or equivalently

$$\frac{1}{4} t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{1}{4} \sum_{r=t+1}^{m} (2\Gamma_{11}^r - (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{t_2t_2}^r + \cdots + \Gamma_{nn}^r))^2$$

$$+ \sum_{r=t+1}^{m} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t} (\Gamma_{ij}^r)^2 + \frac{t_3 \Delta f}{f} \qquad (38)$$

$$+ \frac{c}{4} (t - t_1 t_2 - t_2 t_3 - t_3 t_1 - \frac{1}{2}),$$

which proves the inequality (*i*) of (1).

Case 2. If χ is tangential to $S_\theta^{t_2}$, we choose the unit vector from $\{e_{t_1+1}, \ldots, e_{t_2}\}$. Suppose $\chi = e_{t_2}$; then from (28), we deduce

$$t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2$$

$$+ \frac{t_3 \Delta f}{f} + \frac{1}{2} \sum_{r=t+1}^{m} ((\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r) - 2\Gamma_{t_2t_2}^r)^2$$

$$+ \sum_{r=t+1}^{m} \sum_{1 \leq \alpha < \beta \leq t_1} (\Gamma_{\alpha\alpha}^r \Gamma_{\beta\beta}^r - (\Gamma_{\alpha\beta}^r)^2) + \sum_{r=t+1}^{m} \sum_{t_1+1 \leq s < n \leq t_2} (\Gamma_{ss}^r \Gamma_{nn}^r - (\Gamma_{sn}^r)^2)$$

$$+ \sum_{r=t+1}^{m} \sum_{t_2+1 \leq p < q \leq t} (\Gamma_{pp}^r \Gamma_{qq}^r - (\Gamma_{pq}^r)^2) + \sum_{r=t+1}^{m} \sum_{1 \leq i < j \leq t} (\Gamma_{ij}^r)^2 \qquad (39)$$

$$- \sum_{r=t+1}^{m} \sum_{\substack{1 \leq i < j \leq n \\ i,j \neq t_2}} (\Gamma_{ii}^r \Gamma_{jj}^r) - 2\bar\tau(S^t) + \sum_{\substack{1 \leq i < j \leq t \\ i,j \neq t_2}} \bar\kappa(e_i, e_j)$$

$$+ \bar\tau(S_T^{t_1}) + \bar\tau(S_\theta^{t_2} + \bar\tau(S_\perp^{t_3})).$$

From (3) by putting $U_1, U_3 = e_i$, $U_2, U_3 = e_j$, one can compute

$$\sum_{\substack{1 \le i < j \le t \\ i,j \ne t_2}} \bar{\kappa}(e_i, e_j) = \frac{c}{8}[(t-1)(t-2) + 3t_1 + 3t_2 \cos^2 \theta]$$

$$\bar{\tau}(S_T^{t_1}) = \frac{c}{8}[t_1(t_1 - 1) + 3t_1]$$

$$\bar{\tau}(S_\theta^{t_3}) = \frac{c}{8}[t_2(t_2 - 1) + 3t_2 \cos^2 \theta]$$

$$\bar{\tau}(S_\perp^{t_3}) = \frac{c}{8}[t_3(t_3 - 1)].$$

Using these values together with (33) in (39) and applying similar techniques as in Case 1, we obtain

$$
\begin{aligned}
t^2 \|\Pi\|^2 \ge R^S(\chi) &+ \frac{1}{2} \sum_{r=t+1}^{m} ((\Gamma_{t_1+1 t_1+1}^r + \dots \Gamma_{t_2 t_2}^r + \dots + \Gamma_{tt}^r) - 2\Gamma_{t_2 t_2}^r)^2 \\
&+ \frac{1}{2} t^2 \|\Pi\|^2 + \frac{t_3 \Delta f}{f} + \sum_{r=t+1}^{m} \sum_{1 \le i < j \le n} (\Gamma_{ij}^r)^2 \\
&+ \sum_{r=t+1}^{m} \left[\sum_{n=t_1+1}^{t_2-1} \Gamma_{t_2 t_2}^r \Gamma_{nn}^r + \sum_{l=t_2+1}^{t} \Gamma_{t_2 t_2}^r \Gamma_{ll}^r \right] \\
&\sum_{r=1}^{m} \sum_{i=1}^{t_1} \left[\sum_{j=t_1+1}^{t_2-1} \Gamma_{ii}^r \Gamma_{jj}^r + \sum_{k=t_2+1}^{t} \Gamma_{ii}^r \Gamma_{kk}^r \right] \\
&+ \frac{c}{4}(t - t_1 t_2 - t_2 t_3 - t_3 t_1 + 1).
\end{aligned}
\tag{40}
$$

By the assumption that the submanifold S^t is $L-$minimal, one can conclude

$$\sum_{r=1}^{m} \sum_{i=1}^{t_1} \left[\sum_{j=t_1+1}^{t_2-1} \Gamma_{ii}^r \Gamma_{jj}^r + \sum_{k=t_2+1}^{t} \Gamma_{ii}^r \Gamma_{kk}^r \right] = 0.$$

The second and seventh terms on right hand side of (40) can be solved as follows:

$$
\begin{aligned}
\frac{1}{2} \sum_{r=t+1}^{m} &((\Gamma_{t_1+1 t_1+1}^r + \dots + \Gamma_{tt}^r) - 2\Gamma_{t_2 t_2}^r)^2 + \sum_{r=t+1}^{m} \left[\sum_{n=t_1+1}^{t_2-1} \Gamma_{t_2 t_2}^r \Gamma_{nn}^r + \sum_{l=t_2+1}^{t} \Gamma_{t_2 t_2}^r \Gamma_{ll}^r \right] \\
&= \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1 t_1+1}^r + \dots + \Gamma_{nn}^r)^2 + 2 \sum_{r=t+1}^{m} (\Gamma_{t_2 t_2}^r)^2 \\
&- 2 \sum_{r=t+1}^{m} \sum_{j=t_1+1}^{t} \Gamma_{t_2 t_2}^r \Gamma_{jj}^r + \sum_{r=t+1}^{m} \sum_{n=t_1+1}^{t} \Gamma_{t_2 t_2}^r \Gamma_{nn}^r - \sum_{r=t+1}^{m} (\Gamma_{t_2 t_2}^r)^2 \\
&= \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma_{t_1+1 t_1+1}^r + \dots + \Gamma_{nn}^r)^2 + \sum_{r=t+1}^{m} (\Gamma_{t_2 t_2}^r)^2 \\
&- \sum_{r=t+1}^{m} \sum_{j=t_1+1}^{t} \Gamma_{nn}^r \Gamma_{jj}^r.
\end{aligned}
\tag{41}
$$

By utilizing those two values in (40), we arrive at

$$
\begin{aligned}
\frac{1}{2}t^2\|\Pi\|^2 \geq{}& R^S(\chi) + \sum_{r=t+1}^{m}(\Gamma_{t_2t_2}^r)^2 - \sum_{r=t+1}^{m}\sum_{i=t_1+1}^{t}\Gamma_{nn}^r\Gamma_{jj}^r \\
&+ \frac{1}{2}\sum_{r=t+1}^{m}(\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_n^rn)^2 + \frac{1}{2}t^2\|\Pi\|^2 + \frac{t_3\Delta f}{f} \\
&+ \sum_{r=t+1}^{m}\sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t}(\Gamma_{ij}^r)^2 + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_3t_1 + 1).
\end{aligned}
\tag{42}
$$

By using similar steps as in Case 1, the above inequality can be written as

$$
\begin{aligned}
\frac{1}{4}t^2\|\Pi\|^2 \geq{}& R^S(\chi) + \frac{1}{4}\sum_{r=t+1}^{m}(2\Gamma_{t_2t_2}^r - (\Gamma_{t_1+1t_1+1}^r + \cdots + \Gamma_{nn}^r))^2 \\
&+ \frac{t_3\Delta f}{f} + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_1t_3 + 1).
\end{aligned}
\tag{43}
$$

The last inequality leads to inequality (ii) of (1).

Case 3. If χ is tangential to $S_\perp^{t_3}$, then we choose the unit vector field from $\{e_{t_2+1},\ldots,e_n\}$. Suppose the vector χ is e_n. Then from (28)

$$
\begin{aligned}
t^2\|\Pi\|^2 \geq{}& R^S(\chi) + \frac{1}{2}\sum_{r=t+1}^{m}(\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r)^2 \\
&+ \frac{t_3\Delta f}{f} + \frac{1}{2}\sum_{r=t+1}^{m}((\Gamma_{t_1+1t_1+1}^r + \ldots \Gamma_{t_2t_2}^r + \cdots + \Gamma_{tt}^r) - 2\Gamma_{tt}^r)^2 \\
&+ \sum_{r=t+1}^{m}\sum_{1\leq\alpha<\beta\leq t_1}(\Gamma_{\alpha\alpha}^r\Gamma_{\beta\beta}^r - (\Gamma_{\alpha\beta}^r)^2) + \sum_{r=t+1}^{m}\sum_{t_1+1\leq s<n\leq t_2}(\Gamma_{ss}^r\Gamma_{nn}^r - (\Gamma_{sn}^r)^2) \\
&+ \sum_{r=t+1}^{m}\sum_{t_2+1\leq p<q\leq t}(\Gamma_{pp}^r\Gamma_{qq}^r - (\Gamma_{pq}^r)^2) + \sum_{r=t+1}^{m}\sum_{1\leq i<j\leq n}(\Gamma_{ij}^r)^2 \\
&- \sum_{r=t+1}^{m}\sum_{1\leq i<j\leq t-1}\Gamma_{ii}^r\Gamma_{jj}^r - 2\bar\tau(S^t) + \sum_{1\leq i<j\leq t-1}\bar\kappa(e_i,e_j) \\
&+ \bar\tau(S_T^{t_1}) + \bar\tau(S_\theta^{t_2}) + \bar\tau(S_\perp^{t_3}).
\end{aligned}
\tag{44}
$$

From (3), one can compute

$$
\sum_{1\leq i<j\leq t-1}\bar\kappa(e_i,e_j) = \frac{c}{8}[(t-1)(t-2) + 3t_1 + 3(t_2 - 1)\cos^2\theta]
$$

$$
\bar\tau(S_T^{t_1}) = \frac{c}{8}[t_1(t_1 - 1) + 3t_1]
$$

$$
\bar\tau(S_\theta^{t_2}) = \frac{c}{8}[t_2(t_2 - 1) + 3t_2\cos^2\theta]
$$

$$
\bar\tau(t_\perp^{t_3}) = \frac{c}{8}[t_3(t_3 - 1)].
$$

By usage of those values together with (33) in (44), and analogously to Case 1 and Case 2, we obtain

$$t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{1}{2} t^2 \|\Pi\|^2 + \frac{1}{2} \sum_{r=t+1}^{m} ((\Gamma^r_{t_1+1t_1+1} + \dots \Gamma^r_{t_2t_2} + \dots + \Gamma^r_{tt}) - 2\Gamma^r_{tt})^2$$
$$+ \frac{t_3 \Delta f}{f} + \sum_{r=t+1}^{m} \sum_{1 \leq i < j \leq t} (\Gamma^r_{ij})^2$$
$$+ \sum_{r=t+1}^{m} \sum_{q=t_1+1}^{t-1} \Gamma^r_{tt} \Gamma^r_{qq} - \sum_{r=t+1}^{m} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t-1} \Gamma^r_{ii} \Gamma^r_{jj}$$
$$+ \frac{c}{4} (t - t_1 t_2 - t_2 t_3 - t_1 t_3 + 1 - \frac{3}{2} \cos^2 \theta). \tag{45}$$

Again, using the assumption that S^t is $L - minimal$, it is easy to verify

$$\sum_{r=t+1}^{m} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t-1} \Gamma^r_{ii} \Gamma^r_{jj} = 0. \tag{46}$$

Using in (45), we obtain

$$t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{1}{2} t^2 \|\Pi\|^2 + \frac{1}{2} \sum_{r=t+1}^{m} ((\Gamma^r_{t_1+1t_1+1} + \dots \Gamma^r_{t_2t_2} + \dots + \Gamma^r_{tt}) - 2\Gamma^r_{tt})^2$$
$$+ \frac{t_3 \Delta f}{f} + \sum_{r=t+1}^{m} \sum_{1 \leq i < j \leq n} (\Gamma^r_{ij})^2 + \sum_{r=t+1}^{m} \sum_{q=t_1+1}^{t-1} \Gamma^r_{tt} \Gamma^r_{qq}$$
$$+ \frac{c}{4} (t - t_1 t_2 - t_2 t_3 - t_1 t_3 + 1 - \frac{3}{2} \cos^2 \theta). \tag{47}$$

The third and sixth terms on the right hand side of (47) in a similar way as in Case 1 and Case 2 can be simplified as

$$\frac{1}{2} \sum_{r=t+1}^{m} ((\Gamma^r_{t_1+1t_1+1} + \dots \Gamma^r_{t_2t_2} + \dots + \Gamma^r_{tt}) - 2\Gamma^r_{tt})^2 + \sum_{r=t+1}^{m} \sum_{q=t_1+1}^{t-1} \Gamma^r_{tt} \Gamma^r_{qq}$$
$$= \frac{1}{2} \sum_{r=t+1}^{m} (\Gamma^r_{t_1+1t_1+1} + \dots \Gamma^r_{t_2t_2} + \dots + \Gamma^r_{tt})^2 + \sum_{r=t+1}^{m} (\Gamma^r_{tt})^2$$
$$- \sum_{r=t+1}^{m} \sum_{j=t_1+1}^{t} \Gamma^r_{tt} \Gamma^r_{jj}. \tag{48}$$

By combining (47) and (48) and using similar techniques as used in Case 1 and Case 2, we can derive

$$\frac{1}{4} t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{1}{4} \sum_{r=t+1}^{m} (2\Gamma^r_{tt} - (\Gamma^r_{t_1+1t_1+1} + \dots + \Gamma^r_{tt}))^2$$
$$+ \frac{t_3 \Delta f}{f} + \frac{c}{4} (t - t_1 t_2 - t_2 t_3 - t_1 t_3 + 1 - \frac{3}{2} \cos^2 \theta). \tag{49}$$

The last inequality leads to inequality (*iii*) in (1).

Next, we explore the equality cases of (1). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold S^t in the CSF $\bar{S}^m(c)$ at any point $x \in S^t$; the relative null space was defined by B.-Y. Chen [19], as follows:

$$\mathcal{N}_x = \{U_1 \in T_x S^t : \Gamma(U_1, U_2) = 0, \forall U_2 \in T_x S^t\}.$$

For $A \in \{1, \dots, t\}$ a unit vector field e_A tangential to S^t at x satisfies the equality sign of (23) identically iff

$$(i) \sum_{p=1}^{t_1} \sum_{q=t_1+1}^{t} \Gamma_{pq}^r = 0 \ (ii) \sum_{b=1}^{t} \sum_{\substack{A=1 \\ b \neq A}}^{t} \Gamma_{bA}^r = 0 \ (iii) \ 2\Gamma_{AA}^r = \sum_{q=t_1+1}^{t} \Gamma_{qq}^r, \tag{50}$$

such that $r \in \{t+1, \ldots m\}$ the condition (i) implies that S^t is mixed totally geodesic SCR W-P submanifold. Combining statements (ii) and (iii) with the fact that S^t is $L-$minimal, we get that the unit vector field $\chi = e_A \in \mathcal{N}_x$. The converse is trivial; this proves statement (2).

For a SCR W-P submanifold, the equality sign of (23) holds identically for all unit tangent vector belong to $S_T^{t_1}$ at x iff

$$(i) \sum_{p=1}^{t_1} \sum_{q=t_1+1}^{t} \Gamma_{pq}^r = 0 \ (ii) \sum_{b=1}^{t} \sum_{\substack{A=1 \\ b \neq A}}^{t_1} \Gamma_{bA}^r = 0 \ (iii) \ 2\Gamma_{pp}^r = \sum_{q=t_1+1}^{t} \Gamma_{qq}^r, \tag{51}$$

where $p \in \{1, \ldots, t_1\}$ and $r \in \{t+1, \ldots, m\}$. Since S^t is $L-$minimal SCR W-P submanifold, the third condition implies that $\Gamma_{pp}^r = 0$, $p \in \{1, \ldots, t_1\}$. Using this in the condition (ii), we conclude that S^t is $L-$totally geodesic SCR W-P submanifold in $\bar{S}^m(c)$ and totally mixed geodesicness follows from the condition (i), which proves (a) in the statement (3).

For a SCR W-P submanifold, the equality sign of (24) holds identically for all unit tangent vector fields tangential to $S_\theta^{t_2}$ at x if and only if

$$(i) \sum_{p=1}^{t_1} \sum_{q=t_1+1}^{t} \Gamma_{pq}^r = 0 \ (ii) \sum_{b=1}^{t} \sum_{\substack{A=t_1+1 \\ b \neq A}}^{t_2} \Gamma_{bA}^r = 0 \ (iii) \ 2\Gamma_{KK}^r = \sum_{q=t_1+1}^{t} \Gamma_{qq}^r, \tag{52}$$

such that $K \in \{t_1+1, \ldots, t_2\}$ and $r \in \{t+1, \ldots, m\}$. From the condition (iii) two cases emerge; that is,

$$\Gamma_{KK}^r = 0, \ \forall K \in \{t_1+1, \ldots, t_2\} \ \text{and} \ r \in \{t+1, \ldots, m\} \ \text{or} \ dim \, S_\theta^{t_2} = 2. \tag{53}$$

If the first case of (52) is satisfied, then by virtue of condition (ii), it is easy to conclude that S^t is a $D_\theta-$ totally geodesic SCR W-P submanifold in $\bar{S}^m(c)$. This is the first case of part (b) of statement (3).

For a SCR W-P submanifold, the equality sign of (25) holds identically for all unit tangent vector fields tangent to $S_\perp^{t_3}$ at x if and only if

$$(i) \sum_{p=1}^{t_1} \sum_{q=t_1+1}^{t} \Gamma_{pq}^r = 0 \ (ii) \sum_{b=1}^{t} \sum_{\substack{A=t_2+1 \\ b \neq A}}^{t_3} \Gamma_{bA}^r = 0 \ (iii) \ 2\Gamma_{LL}^r = \sum_{q=t_1+1}^{t} \Gamma_{qq}^r, \tag{54}$$

such that $L \in \{t_2+1, \ldots, t\}$ and $r \in \{t+1, \ldots, m\}$. From the condition (iii) two cases arise; that is,

$$\Gamma_{LL}^r = 0, \ \forall L \in \{t_2+1, \ldots, t\} \ \text{and} \ r \in \{t+1, \ldots, m\} \ \text{or} \ dim \, S_\perp^{t_3} = 2. \tag{55}$$

If the first case of (54) is satisfied, then by virtue of condition (ii), it is easy to conclude that S^t is a $L_\perp-$ totally geodesic SCR W-P submanifold in $\bar{S}^m(c)$. This is the first case of part (c) of statement (3).

For the other case, assume that S^t is not $L_\perp-$totally geodesic SCR W-P submanifold and dim $S_\perp^{t_3} = 2$. Then condition (ii) of (54) implies that S^t is $L_\perp-$ totally umbilical SCR W-P submanifold in $\bar{S}(c)$, which is second case of this part. This verifies part (c) of (3).

To prove (d) using parts (a), (b) and (c) of (3), we combine (51), (52) and (54). For the first case of this part, assume that $dim S_\theta^{t_2} \neq 2$ and $dim S_\perp^{t_3} \neq 2$. From parts (a), (b) and (c) of statement (3) we concluded that M^t is $L-$totally geodesic, $L_\theta-$ is totally geodesic and $D_\perp-$ is a totally geodesic submanifold in $\bar{S}^m(c)$. Hence S^t is a totally geodesic submanifold in $\bar{S}^m(c)$.

For another case, suppose that first case is not satisfied. Then parts (a), (b) and (c) provide that S^t is mixed totally geodesic and $L-$ totally geodesic submanifold of $\bar{S}^m(c)$ with $dim S_\theta^{t_2} = 2$ and $dim S_\perp^{t_3} = 2$. From the conditions (b) and (c) it follows that S^t is $L_\theta-$ and $L_\perp-$totally umbilical SCR W-P submanifolds and from (a) it is $L-$totally geodesic, which is part (d). This proves the theorem. $\square$

If, $S_\theta^{t_2} = \{0\}$ then the SCR W-P submanifold becomes the CR W-P submanifold. In this case we have the following corollary

Corollary 1. *Let $S^t = S_T^{t_1} \times_f S_\perp^{t_3}$ be a CR W-P submanifold isometrically immersed in a CSF $\bar{S}^m(c)$. Then for each orthogonal unit vector field $\chi \in T_x S^t$, either tangent to $S_T^{t_1}$ or $S_\perp^{t_3}$, we have*

(1)   The Ricci curvature satisfy the following inequalities

   (i)   If $\chi \in S_T^{t_1}$, then

$$\frac{1}{4} t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{t_3 \Delta f}{f} + \frac{c}{4}(t - t_1 t_3 - \frac{1}{2}). \tag{56}$$

   (ii)   If $\chi \in S_\perp^{t_3}$, then

$$\frac{1}{4} t^2 \|\Pi\|^2 \geq R^S(\chi) + \frac{t_3 \Delta f}{f} + \frac{c}{4}(t - t_1 t_3 + 1). \tag{57}$$

(2)   If $H(x) = 0$, then each point $x \in S^t$ there is a unit vector field χ which satisfies the equality case of (1) if and only if S^t is mixed totally geodesic and χ lies in the relative null space $\mathcal{N}_x$ at x.

(3)   For the equality case we have

   (a)   The equality case of (56) holds identically for all unit vector fields tangent to $S_T^{t_1}$ at each $x \in S^t$ iff S^t is mixed totally geodesic and $L-$totally geodesic CR W-P submanifold in $\bar{S}^m(c)$.

   (b)   The equality case of (57) holds identically for all unit vector fields tangent to $S_\perp^{t_3}$ at each $x \in M^t$ iff S^t is mixed totally geodesic and either S^t is $L_\perp$- totally geodesic CR-warped product or S^t is a $L_\perp$ totally umbilical in $\bar{S}^m(c)$ with $\dim L_\perp = 2$.

   (c)   The equality case of (1) holds identically for all unit tangent vectors to S^t at each $x \in S^t$ if and only if either S^t is totally geodesic submanifold or S^t is a mixed totally geodesic totally umbilical and $L-$ totally geodesic submanifold with $\dim S_\perp^{t_3} = 2$

   where t_1 and t_3 are the dimensions of $S_T^{t_1}$ and $S_\perp^{t_3}$ respectively.

In view of (20) we have the another version of the Theorem 2 as follows:

Theorem 4. *Let $S^t = S_1^{t_1+t_2} \times_f S_\perp^{t_3}$ be a $L-$minimal SCR W-P submanifold isometrically immersed in a CSF $\bar{M}(c)$. If the holomorphic and slant distributions L and L_θ are integrable with integral submanifolds $S_T^{t_1}$ and $S_\theta^{t_2}$ respectively. Then for each orthogonal unit vector field $\chi \in T_x S$, either tangent to $S_T^{t_1}$, $S_\theta^{t_2}$ or $S_\perp^{t_3}$, we have*

(1)   The Ricci curvature satisfy the following inequalities

   (i)   If $\chi \in T S_T^{t_1}$, then

$$\frac{1}{4} t^2 \|\Pi\|^2 \geq R^S(\chi) + t_3(\Delta \ln f - \|\nabla \ln f\|^2) + \frac{c}{4}(t - t_1 t_2 - t_2 t_3 \\ - t_1 t_3 - \frac{1}{2}). \tag{58}$$

(ii) $\chi \in TS_\theta^{t_2}$, then

$$\frac{1}{4}t^2\|\Pi\|^2 \geq R^S(\chi) + t_3(\Delta lnf - \|\nabla lnf\|^2) + \frac{c}{4}(t - t_1t_2 - t_2t_3$$
$$- t_1t_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{59}$$

(iii)   If $\chi \in TS_\perp^{t_2}$, then

$$\frac{1}{4}t^2\|\Pi\|^2 \geq R^S(\chi) + t_3(\Delta lnf - \|\nabla lnf\|^2) + \frac{c}{4}(t - t_1t_2 - t_2t_3 - t_1t_3 + 1). \tag{60}$$

(2)   If $\Gamma(x) = 0$ for each point $x \in S^t$, then there is a unit vector field χ which satisfies the equality of (1) iff S^t is mixed totally geodesic and $\chi \in \mathcal{N}_x$ at x.

(3)   For the equality case we have

    (a)   The equality of (58) holds identically for all unit vector fields tangent to $S_T^{t_1}$ at each $x \in S^t$ iff S^t is mixed TG and $L-$totally geodesic SCR W-P submanifold in $\bar{S}^m(c)$.

    (b)   The equality of (59) holds identically for all unit vector fields tangent to S_θ at each $x \in S^t$ iff S is mixed totally geodesic and either S^t is L_θ- totally geodesic SCR W-P submanifold or S^t is a L_θ totally umbilical in $\bar{S}^m(c)$ with dim $L_\theta = 2$.

    (c)   The equality of (60) holds identically for all unit vector fields tangent to $S_\perp^{t_2}$ at each $x \in S^t$ iff S is mixed totally geodesic and either S^t is $L^\perp$- totally geodesic SCR W-P or S^t is a $L^\perp$ totally umbilical in $\bar{S}^m(c)$ with dim $L^\perp = 2$.

    (d)   The equality case of (1) holds identically for all unit tangent vectors to S^t at each $x \in S^t$ iff either S^t is totally geodesic submanifold or M^t is a mixed totally geodesic totally umbilical and L totally geodesic submanifold with dim $S_\theta^{t_2} = 2$ and dim $S_\perp^{t_3} = 2$.

Where t_1, t_2 and t_3 are the dimensions of $S_T^{t_1}$, $S_\theta^{t_2}$ and $S_\perp^{t_3}$ respectively.

5. Conclusions

In the present study we obtained some fundamental results for skew CR-warped product submanifolds in the frame of complex space forms. Further, some inequalities in terms of Ricci curvature and squared norm of mean curvature vector were derived. In particular, a Ricci curvature for CR-warped product submanifolds was also discussed. Recently, we also studied warped product submanifolds in complex space forms (see [15,16]) and obtained some inequalities in terms of squared norm of second fundamental form, slant function and the warping functions, but the results obtained in the present study are dissimilar from the previous works of the authors and were proved by using different techniques.

Abbreviations

W-P	Warped product
W-F	Warping function
CSF	Complex Space form

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