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Common Attractive Points of Generalized Hybrid Multi-Valued Mappings and Applications

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Abstract: In this paper, we first propose the concepts of (ζ,η,λ,π)-generalized hybrid multi-valued mappings, the set of all the common attractive points ($CA_{f,g}$) and the set of all the common strongly attractive points ($C_sA_{f,g}$), respectively for the multi-valued mappings f and g in a CAT(0) space. Moreover, we give some elementary properties in regard to the sets A_f, F_f and $CA_{f,g}$ for the multi-valued mappings f and g in a complete CAT(0) space. Furthermore, we present a weak convergence theorem of common attractive points for two (ζ,η,λ,π)-generalized hybrid multi-valued mappings in the above space by virtue of Banach limits technique and Ishikawa iteration respectively. Finally, we prove strong convergence of a new viscosity approximation method for two (ζ,η,λ,π)-generalized hybrid multi-valued mappings in CAT(0) spaces, which also solves a kind of variational inequality problem.

Keywords: generalized hybrid set-valued mapping; common attractive point; CAT(0) space

MSC: 47H09; 47H10

1. Introduction

In 1975, Baillon proved the first nonlinear ergodic theorem in a Hilbert space. In 1978, Reich obtained the almost convergence and nonlinear ergodic theorems. In 2010, Kocourek et al. [1] brought in (ζ,η)-generalized hybrid mappings in a Hilbert space for the first time. They also proved a mean convergence theorem for a generalized hybrid mapping that generalizes Baillon's nonlinear ergodic theorem. Let K be a nonempty subset of a real Hilbert space X. A mapping $f:K\to K$ is called (ζ,η)-generalized hybrid if there exist $\zeta,\eta\in\mathbb{R}$, such that

$$\zeta\|fa-fb\|^2+(1-\zeta)\|a-fb\|^2\le\eta\|fa-b\|^2+(1-\eta)\|a-b\|^2$$

for all $a,b\in K$. f is said to be nonexpansive if f is $(1,0)$-generalized hybrid; f is said to be nonspreading if f is $(2,1)$-generalized hybrid [2]; f is said to be hybrid if f is $(\frac{3}{2},\frac{1}{2})$-generalized hybrid [3]. It can be observed that the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings are all included in (ζ,η)-generalized hybrid mappings.

The set of attractive points was proposed by Takahashi et al. [4] in 2011. That is,

$$A_f=\{c\in E:\|fa-c\|\le\|a-c\|,\quad\forall a\in K\}.$$

They also obtained some fundamental properties for attractive points in a real Hilbert space. Using these properties, they proved a mean convergence theorem without convexity for finding an attractive point of a generalized hybrid mapping. Moreover, Takahashi et al. [5] gave the definition of a more general class of mappings, called $(\zeta, \eta, \lambda, \pi)$-normally generalized hybrid.

Definition 1 ([5])**.** *A mapping $f : K \to K$ is called $(\zeta, \eta, \lambda, \pi)$-normally generalized hybrid if there exist $\zeta, \eta, \lambda, \pi \in \mathbb{R}$, such that*

- $\zeta + \eta + \lambda + \pi \geq 0$;
- $\zeta + \eta \geq 0$ *or* $\zeta + \lambda \geq 0$;
- $\zeta \|fa - fb\|^2 + \eta \|a - fb\|^2 + \lambda \|fa - b\|^2 + \pi \|a - b\|^2 \leq 0, \quad \forall a, b \in K.$

The theory of multi-valued mappings is widely applied in many fields, such as control theory, convex optimization, differential equations, economics, and so on [6–13]. In recent years, there is a growing interest in developing an approximation method for fixed points and attractive points of multi-valued mappings. In 2017, Lili Chen et al. [14] raised the definitions of (ζ, η)-generalized hybrid multi-valued mappings in Banach spaces. By the way, they also gave the concepts of attractive points and strongly attractive points of (ζ, η)-generalized hybrid multi-valued mappings. In 2019, Lili Chen et al. [15] introduced the concepts of (ζ, η)-generalized hybrid multi-valued mappings and the corresponding definitions of common attractive points and common strongly attractive points in Hilbert spaces.

In this work, we firstly extend the definitions of $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings, the set of common attractive points ($CA_{f,g}$) and the set of common strongly attractive points ($C_sA_{f,g}$) of multi-valued mappings f and g to CAT(0) spaces. Furthermore, we present some essential properties in relation to the sets A_f, F_f and $CA_{f,g}$ for the multi-valued mappings f and g in a complete CAT(0) space. In addition, we obtain a weak convergence theorem of common attractive points for two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings in the above space by means of Banach limits technique and Ishikawa iteration respectively. Moreover, we give a strong convergence theorem of two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings by the use of a new viscosity approximation method in CAT(0) spaces, which also resolves a kind of variational inequality problem.

2. Preliminaries

Let (E, ρ) be a metric space. A geodesic path (or shortly a geodesic) joining a to b in E is a map $c : [0, l] \subseteq \mathbb{R} \to E$, such that $c(0) = x$, $c(l) = y$ and $\rho(c(s), c(t)) = \mid s - t \mid$ for all $s, t \in [0, l]$. The image of c is called a geodesic segment joining a and b when it is unique and denoted by $[a, b]$. We denote the unique point $w \in [a, b]$, such that $\rho(a, w) = \theta\rho(a, b)$ and $\rho(b, w) = (1 - \theta)\rho(a, b)$ by $(1 - \theta)a \oplus \theta b$, where $0 \leq \theta \leq 1$.

The metric space (E, ρ) is called a geodesic space if any two points of E are joined by a geodesic, and E is said to be uniquely geodesic if there is exactly one geodesic joining a and b for each $a, b \in E$.

A geodesic triangle $\Lambda(r_1, r_2, r_3)$ in a geodesic space (E, ρ) consists of three points in E (the vertices of Λ) and a geodesic segment between each pair of points (the edges of Λ). A comparison triangle for $\Lambda(r_1, r_2, r_3)$ in (E, ρ) is a triangle $\overline{\Lambda}(r_1, r_2, r_3) := (\bar{r}_1, \bar{r}_2, \bar{r}_3)$ in the Euclidean plane $\mathbb{R}^2$, such that $\rho_{\mathbb{R}^2}(\bar{r}_p, \bar{r}_q) = \rho(r_p, r_q)$ for all $p, q \in \{1, 2, 3\}$.

A geodesic space E is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let Λ be a geodesic triangle in E and $\overline{\Lambda}$ be a comparison triangle in $\mathbb{R}^2$. Subsequently, the triangle is said to satisfy the CAT(0) inequality if

$$\rho(m, n) \leq \rho_{\mathbb{R}^2}(\overline{m}, \overline{n}),$$

for all $m, n \in \Lambda$ and all comparison points $\overline{m}, \overline{n} \in \overline{\Lambda}$.

If w, u, v are points in a CAT(0) space and if h is the midpoint of the segment $[u, v]$, then the CAT(0) inequality implies the so-called (CN) inequality, i.e.,

$$\rho^2(w, h) \leq \frac{1}{2}\rho^2(w, u) + \frac{1}{2}\rho^2(w, v) - \frac{1}{4}d^2(u, v).$$

Moreover, a uniquely geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [16].

Now, we collect some elementary facts about CAT(0) spaces.

Lemma 1 ([16])**.** *Let E be a CAT(0) space, $m, n, g, h \in E$ and $\theta \in [0, 1]$. Afterwards,*

$$\rho(\theta m \oplus (1-\theta)g, \theta n \oplus (1-\theta)h) \leq \theta\rho(m, n) + (1-\theta)\rho(g, h),$$

$$\rho(\theta m \oplus (1-\theta)g, n) \leq \theta\rho(m, n) + (1-\theta)\rho(g, n),$$

$$\rho^2(\theta m \oplus (1-\theta)g, n) \leq \theta\rho^2(m, n) + (1-\theta)\rho^2(g, n) - \theta(1-\theta)\rho^2(m, g).$$

Suppose that $\{a_k\}$ is a bounded sequence in a CAT(0) space E. For $a \in E$, put

$$r(a, \{a_k\}) = \limsup_{k\to\infty} \rho(a, a_k).$$

The asymptotic radius $r(\{a_k\})$ of $\{a_k\}$ is given by

$$r(\{a_k\}) = \inf\{r(a, \{a_k\}) : a \in E\},$$

and the asymptotic center $A(\{a_k\})$ of $\{a_k\}$ is the set

$$A(\{a_k\}) = \{a \in E : r(a, \{a_k\}) = r(\{a_k\})\}.$$

It follows from [17] that $A(\{a_k\})$ is made up of one point in a CAT(0) space. A sequence $\{a_k\} \subset E$ is said to be Δ-convergent to $a \in E$ if $A(\{a_{k_j}\}) = \{a\}$ for every subsequence $\{a_{k_j}\}$ of $\{a_k\}$.

Lemma 2 ([16])**.** *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 3 ([16])**.** *If K is a closed convex subset of a complete CAT(0) space and if $\{a_k\}$ is a bounded sequence in K, then the asymptotic center of $\{a_k\}$ is in K.*

Subsequently, the definition of $\Delta-$convergence and corresponding primary properties are presented below.

Let K be a nonempty closed convex subset of a complete CAT(0) space E. Afterwards, for any $e \in E$, we know that there exists a unique nearest point $\kappa \in K$, such that

$$\rho(e, \kappa) = \inf_{k\in K} \rho(e, k).$$

In this case, κ is the only nearest point of e in K.

Lemma 4 ([18])**.** *Assume $P_K : E \to K$ is a metric projection and $\{a_k\} \subseteq E$. If $\rho(a_{k+1}, a) \leq \rho(a_k, a)$ for any $a \in K$, then $\{P_K a_k\}$ converges strongly to some $a_0 \in K$.*

In 2008, Berg et al. [19] proposed the idea of quasilinearization in a metric space E. Each pair $(m, n) \in E \times E$ is denoted by $\overrightarrow{mn}$ and called a vector. Subsequently, quasilinearization is a map $\langle *, * \rangle : (E \times E) \times (E \times E) \to \mathbb{R}$ defined as

$$2\langle\overrightarrow{mn},\overrightarrow{rs}\rangle=\rho^2(m,s)+\rho^2(n,r)-\rho^2(m,r)-\rho^2(n,s),$$

for all $m,n,r,s\in E$. It can be observed easily that $\langle\overrightarrow{mn},\overrightarrow{rs}\rangle=\langle\overrightarrow{rs},\overrightarrow{mn}\rangle$, $\langle\overrightarrow{mn},\overrightarrow{rs}\rangle=-\langle\overrightarrow{nm},\overrightarrow{rs}\rangle$ and $\langle\overrightarrow{mc},\overrightarrow{rs}\rangle+\langle\overrightarrow{cn},\overrightarrow{rs}\rangle=\langle\overrightarrow{mn},\overrightarrow{rs}\rangle$, for all $m,n,c,r,s\in E$. We say that E satisfies Cauchy-Schwarz inequality if

$$\langle\overrightarrow{mn},\overrightarrow{rs}\rangle\leq\rho(m,n)\rho(r,s).$$

The necessary and sufficient condition for geodesic connected metric space to be CAT(0) space is that geodesic connected metric space satisfies Cauchy-Schwarz inequality [20].

In 2013, Dehghan and Rooin [21] presented a characterization of a metric projection in CAT(0) spaces by using the concept of quasilinearization.

Lemma 5 ([21])**.** *Let K be a nonempty convex subset of a complete CAT(0) space E, $p\in E$ and $q\in K$. Subsequently,*

$$q=P_Kp\quad if\ and\ only\ if\quad \langle\overrightarrow{qp},\overrightarrow{lq}\rangle\geq 0,$$

for all $l\in K$.

Lemma 6 ([22])**.** *Let E be a CAT(0) space and $a,b\in E$. For any $\xi\in[0,1]$, we set $a_\xi=\xi a+(1-\xi)b$. Afterwards, for each $g,h\in E$, we have*

(1) $\langle\overrightarrow{a_\xi g},\overrightarrow{a_\xi h}\rangle\leq\xi\langle\overrightarrow{ag},\overrightarrow{a_\xi h}\rangle+(1-\xi)\langle\overrightarrow{bg},\overrightarrow{a_\xi h}\rangle$;
(2) $\langle\overrightarrow{a_\xi g},\overrightarrow{ah}\rangle\leq\xi\langle\overrightarrow{ag},\overrightarrow{ah}\rangle+(1-\xi)\langle\overrightarrow{bg},\overrightarrow{ah}\rangle$ *and* $\langle\overrightarrow{a_\xi g},\overrightarrow{bh}\rangle\leq\xi\langle\overrightarrow{ag},\overrightarrow{bh}\rangle+(1-\xi)\langle\overrightarrow{bg},\overrightarrow{bh}\rangle$.

In 2012, Kakavandi [23] proposed the following results in a complete CAT(0) space.

Lemma 7 ([23])**.** *A sequence $\{e_k\}$ in a complete CAT(0) space (E,ρ) Δ-converges to $e\in E$ if and only if* $\limsup_{k\to\infty}\langle\overrightarrow{ee_k},\overrightarrow{et}\rangle\leq 0$ *for all $t\in E$, and w-converges to $e\in E$ if $\lim\limits_{k\to\infty}\langle\overrightarrow{ee_k},\overrightarrow{et}\rangle=0$ for all $t\in E$.*

Definition 2 ([23])**.** *We say that a complete CAT(0) space (E,ρ) satisfies the (S) property if for any $(x,y)\in E\times E$ there exists a point $y_x\in E$, such that $[\overrightarrow{xy}]=[\overrightarrow{y_x x}]$.*

Obviously in metric spaces the strong convergence implies w-convergence, and w-convergence implies Δ-convergence, the Example 4.7 of [23] shows that the converse is not valid.

Let l^∞ be the Banach space of bounded sequences with supremum norm [14,24,25]. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Subsequently, we denote by $\mu(x)$ the value of μ at $x=(t_1,t_2,t_3,...)\in l^\infty$. Sometimes, we denote by $\mu_k(t_k)$ the value $\mu(x)$. A linear functional μ on l^∞ is said to be a mean if $\|\mu\|=\mu(\varepsilon)=1$, where $\varepsilon=(1,1,1,...)$. If $\mu_k(t_{k+1})=\mu_k(t_k)$, a mean μ is said to be a Banach limit on l^∞. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $x=(t_1,t_2,t_3,...)\in l^\infty$,

$$\liminf_{k\to\infty}t_k\leq\mu_k(t_k)\leq\limsup_{k\to\infty}t_k.$$

In particular, if $x=(t_1,t_2,t_3,...)\in l^\infty$ and $t_k\to t\in\mathbb{R}$, then we obtain $\mu(x)=\mu_k(t_k)=t$. A useful lemma would be given.

Lemma 8 ([24])**.** *Let F be a Hilbert space, let $\{a_k\}$ be a bounded sequence in F and let μ be a mean on l^∞. Afterwards, there exists a unique point $a_0\in\overline{co}\{a_k|k\in\mathcal{N}\}$, such that*

$$\mu_k\langle a_k,e\rangle=\langle a_0,e\rangle$$

for any $e\in F$.

3. Main Results

In this section, we shall prove a weak convergence theorem of common strongly attractive points for two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings in a complete CAT(0) space. Now, we present the following notions and lemmas in CAT(0) spaces which will be used in the sequel. Suppose E is a CAT(0) space and K is a nonempty subset of E, and let $f : K \to 2^K \backslash \{\emptyset\}$ be a multi-valued mapping. Let F_f be the set of all fixed points of the mapping f.

Let H be the Hausdorff distance, as defined by

$$H(P,Q) = \max\{\sup_{p\in P} \rho(p,Q), \sup_{q\in Q} \rho(q,P)\},$$

where $\rho(p,Q) = \inf\{\rho(p,\widehat{q}) : \widehat{q} \in Q\}$ and $\rho(q,P) = \inf\{\rho(q,\widehat{p}) : \widehat{p} \in P\}$.

Definition 3. *A mapping f defined on a CAT(0) space E is called $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued if there exist $\zeta, \eta, \lambda, \pi \in \mathbb{R}$, such that*

$$\zeta H^2(fm, fn) + \eta H^2(m, fn) + \lambda H^2(n, fm) + \pi\rho^2(m,n) \leq 0, \ \forall m,n \in E. \tag{1}$$

Example 1. *Let $E = \mathbb{R}$, and define $\rho(a,b) = |a-b|$ for all $a, b \in E$ and let H be the Hausdorff distance. It is not difficult to see that (E, ρ) is also a complete CAT(0) space. Let $K = [-1, 0]$, which is a closed convex subset of E, and let f be a multi-valued mapping on K defined by $f(x) = \{\frac{1-x}{2}, 0\}$ for each $x \in K$. Let $\zeta = 2$, $\eta = \lambda = -1$, $\pi = 0$, we will show that f is $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued, which is,*

$$2H^2(fa, fb) - H^2(a, fb) - H^2(b, fa) \leq 0, \ \forall a, b \in K.$$

Indeed, we have

$$\begin{aligned} 2H^2(fa,fb) &= 2H^2(\{\frac{1-a}{2}, 0\}, \{\frac{1-b}{2}, 0\}) = 2\left\{\max\left\{\inf\{\frac{|b-a|}{2}, \frac{|1-a|}{2}\}, \inf\{\frac{|b-a|}{2}, \frac{|1-b|}{2}\}\right\}\right\}^2 \\ &= \frac{(b-a)^2}{2}, \end{aligned}$$

and

$$\begin{aligned} H^2(a,fb) + H^2(b,fa) &= \left\{\max\left\{|a - \frac{1-b}{2}|, |a|\right\}\right\}^2 + \left\{\max\left\{|b - \frac{1-a}{2}|, |b|\right\}\right\}^2 \\ &= (a - \frac{1-b}{2})^2 + (b - \frac{1-a}{2})^2. \end{aligned}$$

Hence, we conclude

$$\begin{aligned} 2H^2(fa,fb) - H^2(a,fb) - H^2(b,fa) &= \frac{(b-a)^2}{2} - (a - \frac{1-b}{2})^2 - (b - \frac{1-a}{2})^2 \\ &= -\frac{1}{2} - \frac{3}{4}a^2 - \frac{3}{4}b^2 - 3ab + \frac{3}{2}(a+b) < 0. \end{aligned}$$

Therefore, f is $(2, -1, -1, 0)$-generalized hybrid multi-valued.

Definition 4. *The set of all attractive points of the mapping f is defined as*

$$A_f = \{e \in E : \ \rho(e, fu) \leq \rho(e, u), \ \forall u \in K\}.$$

Definition 5. *The set of all common attractive points of the multi-valued mappings f and g is defined as*

$$CA_{f,g} = \{e \in E : \max\{\rho(e, fu), \rho(e, gu)\} \leq \rho(e, u), \ \forall u \in K\}.$$

Definition 6. *The set of all strongly attractive points of the mapping f is denoted by*

$$SA_f = \{e \in E : H(e, fu) \leq \rho(e, u), \ \forall u \in K\}.$$

Definition 7. *The set of all common strongly attractive points of the multi-valued mappings f and g is defined as*

$$C_s A_{f,g} = \{e \in E : \max\{H(e, fu), H(e, gu)\} \leq \rho(e, u), \ \forall u \in K\}.$$

Ishikawa iterative process for two mappings f and g in CAT(0) spaces is as follows:

$$\begin{cases} b_k = \zeta_k a_k + (1 - \zeta_k) d_k, \\ a_{k+1} = \eta_k a_k + (1 - \eta_k) c_k, \end{cases} \tag{2}$$

where $c_k \in f b_k$ and $d_k \in g a_k$, $\zeta_k, \eta_k \in (0, 1)$.

We use $\mathcal{F}(E)$ to denote the family of all the closed convex subsets of E. We can observe the following results.

Proposition 1. *Let E be a complete CAT(0) space and K be a nonempty closed convex subset of E. Let $f, g : K \to \mathcal{F}(K)$ be two mappings. If $CA_{f,g} \neq \emptyset$, then $F_f \cap F_g \neq \emptyset$. In particular, if $w \in CA_{f,g}$, then $P_K w \in F_f \cap F_g$.*

Proof. Let $w \in CA_{f,g}$, then $w \in A_f$ and $w \in A_g$. Thus, by the definition of metric projection there exists a unique element $P_K w \in K$, such that

$$\rho(w, P_K w) = \rho(w, K) \leq \rho(w, g(P_K w)). \tag{3}$$

Similarly, since $g(P_K w)$ is a closed and convex subset of K, there exists $P_{g(P_K w)} w \in g(P_K w)$ such that

$$\rho(w, P_{g(P_K w)} w) = \rho(w, g(P_K w)). \tag{4}$$

On the other hand, $w \in A_g$ implies that $\rho(w, gu) \leq \rho(w, u)$ for all $u \in K$, especially,

$$\rho(w, g(P_K w)) \leq \rho(w, P_K w). \tag{5}$$

Combining with Equations (3)–(5), we deduce that

$$\rho(w, P_K w) = \rho(w, P_{g(P_K w)} w) = \rho(w, g(P_K w)) = \rho(w, K).$$

Because of the uniqueness, we get $P_K w = P_{g(P_K w)} w \in g(P_K w)$, which implies that $P_K w \in F_g$. Similarly, we can claim $P_K w \in F_f$. Hence, $P_K w \in F_f \cap F_g$ and $F_f \cap F_g \neq \emptyset$. $\square$

Proposition 2. *Let E be a complete CAT(0) space and let K be a nonempty subset of E. Let $f : K \to \mathcal{F}(K)$ be a quasi-nonexpansive mapping(i.e. for each $p \in F_f$, $H(p, fu) \leq d(p, u)$ holds for all $u \in K$). Subsequently, $A_f = F_f$.*

Proof. First of all, it is not hard to see that $F_f \subseteq A_f$. Now, we will show that $A_f \subseteq F_f$. Let $w \in A_f$, then, for any $v \in E$, we have

$$\rho(w, fv) \leq H(w, fv) \leq \rho(w, v),$$

which implies that

$$\rho(w, fw) \leq \rho(w, w) = 0.$$

Because fw is closed, we obtain $w \in F_f$. $\square$

Proposition 3. *Let E be a complete CAT(0) space and let K be a nonempty subset of E. Suppose $\{a_k\} \subseteq K$ is a bounded sequence and $f : K \to 2^E \backslash \{\emptyset\}$ is a multi-valued mapping, such that $\rho(a_k, b_k) \to 0$, $b_k \in fa_k$. Then*

(1) the sequences $\{\rho(a_k, c)\}$, $\{\rho(b_k, c)\}$ and $\{\rho(c, fa_k)\}$ are bounded for all $c \in K$;
(2) $\mu_k \rho(a_k, c) = \mu_k \rho(b_k, c)$ for any Banach limit μ on l^∞.

Proof. Suppose $k \in \mathcal{N}$, $c \in K$. We deduce that the sequence $\{\rho(a_k, c)\}$ is bounded, since $\{a_k\}$ is bounded. Combined with $\rho(a_k, b_k) \to 0$ and $b_k \in fa_k$, it follows that $\{\rho(b_k, c)\}$ is bounded. Moreover, the sequence $\{\rho(c, fa_k)\}$ is bounded, since $\rho(c, fa_k) = \inf\limits_{d \in fa_k} \rho(c, d) \leq \rho(b_k, c)$. Subsequently, we have

$$\rho(a_k, c) \leq \rho(a_k, b_k) + \rho(b_k, c), \tag{6}$$

and

$$\rho(b_k, c) \leq \rho(b_k, a_k) + \rho(a_k, c). \tag{7}$$

Both sides of formulas (6) and (7) are applied to the Banach limit μ, combined with $\rho(a_k, b_k) \to 0$, we can get

$$\mu_k \rho(a_k, c) = \mu_k \rho(b_k, c).$$

$\square$

Theorem 1. *Let E be a complete CAT(0) space and K be a nonempty subset of E. Let $f, g : K \to 2^E \backslash \{\emptyset\}$ be two multi-valued mappings. Suppose that $C_s A_{f,g} \neq \emptyset$. If the sequence $\{a_k\}$ is defined by (2), where $\{\zeta_k\}$, $\{\eta_k\}$ are sequences in (0,1) with $\varliminf\limits_{k \to \infty} (1 - \eta_k)(1 - \zeta_k)\eta_k > 0$, then the following conclusions hold:*

(1) the sequence $\{a_k\}$ is bounded;
(2) $\lim_{k \to \infty} \rho(a_k, w)$ exists for each $w \in C_s A_{f,g}$;
(3) $\lim_{k \to \infty} \rho(a_k, ga_k) = 0$.

Proof. Let $w \in C_s A_{f,g}$. Then by (2), we get

$$\begin{aligned} \rho^2(b_k, w) &= \rho^2(\zeta_k a_k + (1 - \zeta_k) d_k, w) \\ &\leq \zeta_k \rho^2(a_k, w) + (1 - \zeta_k) \rho^2(d_k, w) \\ &\leq \zeta_k \rho^2(a_k, w) + (1 - \zeta_k) H^2(ga_k, w) \\ &\leq \zeta_k \rho^2(a_k, w) + (1 - \zeta_k) \rho^2(a_k, w) \\ &= \rho^2(a_k, w), \end{aligned}$$

and

$$\begin{aligned} \rho^2(a_{k+1}, w) &= \rho^2(\eta_k a_k + (1 - \eta_k) c_k, w) \\ &\leq \eta_k \rho^2(a_k, w) + (1 - \eta_k) \rho^2(c_k, w) \\ &\leq \eta_k \rho^2(a_k, w) + (1 - \eta_k) H^2(fb_k, w) \\ &\leq \eta_k \rho^2(a_k, w) + (1 - \eta_k) \rho^2(b_k, w) \\ &\leq \rho^2(a_k, w). \end{aligned}$$

It follows that the limit $\lim_{k\to\infty} \rho(a_k, w)$ exists and the sequence $\{a_k\}$ is bounded. Now we show the last conclusion holds. Because E is a complete CAT(0) space, then

$$\begin{aligned}\rho^2(a_{k+1}, w) &= \rho^2(\eta_k a_k + (1-\eta_k)c_k, w)\\ &\leq \eta_k \rho^2(a_k, w) + (1-\eta_k)\rho^2(b_k, w),\end{aligned}$$

among

$$\rho^2(b_k, w) \leq \zeta_k \rho^2(a_k, w) + (1-\zeta_k)\rho^2(d_k, w) - (1-\zeta_k)\zeta_k \rho^2(a_k, d_k).$$

Thus we have

$$\begin{aligned}\rho^2(a_{k+1}, w) &= \rho^2(\eta_k a_k + (1-\eta_k)c_k, w)\\ &\leq \eta_k \rho^2(a_k, w) + (1-\eta_k)\zeta_k \rho^2(a_k, w) + (1-\eta_k)(1-\zeta_k)\rho^2(d_k, w)\\ &\quad - (1-\eta_k)(1-\zeta_k)\zeta_k \rho^2(a_k, d_k)\\ &\leq \rho^2(a_k, w) - (1-\eta_k)(1-\zeta_k)\zeta_k \rho^2(a_k, d_k).\end{aligned}$$

Therefore, we get

$$(1-\eta_k)(1-\zeta_k)\zeta_k \rho^2(a_k, d_k) \leq \rho^2(a_k, w) - \rho^2(a_{k+1}, w).$$

Since $\underline{\lim}_{k\to\infty} (1-\eta_k)(1-\zeta_k)\zeta_k > 0$, we obtain that $\lim_{k\to\infty} \rho(a_k, d_k) = 0$. Noticing that $d_k \in ga_k$, we get

$$\rho(a_k, d_k) \geq \rho(a_k, ga_k) \to 0,\ k \to \infty \tag{8}$$

which completes the proof. □

Now, we show the existence of common attractive points for two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings by Ishikawa iterative process in a CAT(0) space.

Theorem 2. *Let E be a complete CAT(0) space satisfying the (S) property and K be a nonempty closed convex subset of E. Let $f, g : K \to \mathcal{F}(K)$ be two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings satisfying $\zeta + \eta + \lambda + \pi \geq 0$, either $\zeta + \eta > 0$ or $\zeta + \lambda > 0$. Suppose $C_s A_{f,g} \neq \emptyset$. If $\{a_k\}$ is a sequence generated by (2) satisfying $\rho(a_k, fa_k) \to 0$, where $\{\zeta_k\}$, $\{\eta_k\}$ are sequences in (0,1) with $\underline{\lim}_{k\to\infty} (1-\eta_k)(1-\zeta_k)\eta_k > 0$, then there is a subsequence $\{a_{n_k}\}$ of $\{a_k\}$, such that $\{a_{n_k}\}$ $w-$converges to $q \in CA_{f,g}$.*

Proof. Because g is a $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mapping, for any $a, b \in K$, we have

$$\begin{aligned}\zeta H^2(ga, gb) + \eta\rho^2(a, gb) + \lambda\rho^2(b, ga) + \pi\rho^2(a, b) &\leq \zeta H^2(ga, gb) + \eta H^2(a, gb) + \lambda H^2(b, ga) + \pi\rho^2(a, b)\\ &\leq 0.\end{aligned}$$

Now, we consider the following two cases.

Case I. If $\zeta + \eta > 0$, then

$$\zeta H^2(ga_k, gb) + \eta\rho^2(a_k, gb) + \lambda\rho^2(b, ga_k) + \pi\rho^2(a_k, b) \leq 0,$$

where

$$H(ga_k, gb) = \max\{\sup_{x\in ga_k} \rho(x, gb), \sup_{y\in gb} \rho(y, ga_k)\}.$$

Subsequently, for any $x \in ga_k$, we get

$$\zeta\rho^2(x, gb) + \eta\rho^2(a_k, gb) + \lambda\rho^2(b, ga_k) + \pi\rho^2(a_k, b) \leq 0.$$

By the conclusion (3) of Theorem 1, there exists $z_k \in ga_k$, such that

$$\lim_{k\to\infty} \rho(a_k, z_k) = 0.$$

Meanwhile, we notice that

$$\zeta\rho^2(z_k, gb) + \eta\rho^2(a_k, gb) + \lambda\rho^2(b, ga_k) + \pi\rho^2(a_k, b) \leq 0.$$

On the other hand, we choose $y_k \in ga_k$, such that $\rho(b, y_k) = \rho(b, ga_k)$. We can obtain that

$$\rho(b, ga_k) = \rho(b, y_k) \geq \rho(b, a_k) - \rho(a_k, y_k).$$

Making use of Banach limit μ_k and due to Proposition 3, we observe that

$$\zeta\mu_k\rho^2(a_k, gb) + \eta\mu_k\rho^2(a_k, gb) + \lambda\mu_k\rho^2(a_k, b) + \pi\mu_k\rho^2(a_k, b) \leq 0,$$

which implies

$$(\zeta + \eta)\mu_k\rho^2(a_k, gb) \leq -(\lambda + \pi)\mu_k\rho^2(a_k, b),$$

for all $b \in K$. Since $\zeta + \eta + \lambda + \pi \geq 0$ and $\zeta + \eta > 0$, we obtain

$$\begin{aligned}\mu_k\rho^2(a_k, gb) &\leq -\frac{\lambda + \pi}{\zeta + \eta}\mu_k\rho^2(a_k, b). \\ &\leq \mu_k\rho^2(a_k, b).\end{aligned}$$

Case II. If $\zeta + \lambda > 0$, then

$$\zeta H^2(gb, ga_k) + \eta\rho^2(b, ga_k) + \lambda\rho^2(a_k, gb) + \pi\rho^2(b, a_k) \leq 0.$$

By a similar argument, for $z_k \in ga_k$, we have

$$\zeta\rho^2(z_k, gb) + \eta\rho^2(b, ga_k) + \lambda\rho^2(a_k, gb) + \pi\rho^2(b, a_k) \leq 0.$$

Making use of Banach limit μ_k, we can get

$$\zeta\mu_k\rho^2(a_k, gb) + \eta\mu_k\rho^2(b, a_k) + \lambda\mu_k\rho^2(a_k, gb) + \pi\mu_k\rho^2(b, a_k) \leq 0,$$

which implies that

$$(\zeta + \lambda)\mu_k\rho^2(a_k, gb) \leq -(\eta + \pi)\mu_k\rho^2(a_k, b)$$

holds for all $b \in K$. Since $\zeta + \eta + \lambda + \pi \geq 0$ and $\zeta + \lambda > 0$, we get

$$\begin{aligned}\mu_k\rho^2(a_k, gb) &\leq -\frac{\eta + \pi}{\zeta + \lambda}\mu_k\rho^2(a_k, b) \\ &\leq \mu_k\rho^2(a_k, b).\end{aligned}$$

Therefore, we deduce

$$\mu_k\rho^2(a_k, gb) \leq \mu_k\rho^2(a_k, b) \tag{9}$$

for any $b \in K$.

From Theorem 1, it follows that the sequence $\{a_k\}$ is bounded. Subsequently, there exists a subsequence $\{a_{n_k}\}$ of $\{a_k\}$, such that $\{a_{n_k}\}$ $\Delta-$converges to $q \in K$. Because E satisfies the (S) property, we deduce that $\{a_{n_k}\}$ $w-$converges to q.

By Lemma 7, for any $b \in K$, we have $\lim_{k\to\infty} \langle \overrightarrow{qa_{n_k}}, \overrightarrow{qb} \rangle = 0$, that is

$$2 \lim_{k\to\infty} \langle \overrightarrow{qa_{n_k}}, \overrightarrow{qb} \rangle = \lim_{k\to\infty} [\rho^2(q,b) + \rho^2(a_{n_k}, q) - \rho^2(a_{n_k}, b)] = 0. \tag{10}$$

From (9), it follows that

$$-\mu_{n_k}\rho^2(a_{n_k}, b) \le -\mu_{n_k}\rho^2(a_{n_k}, gb).$$

By adding $\mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, b) + \rho^2(q, gb))$ to both sides of the above inequality, we can conclude that

$$\begin{aligned} &-\mu_{n_k}\rho^2(a_{n_k}, b) + \mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, b) + \rho^2(q, gb)) \\ &\le -\mu_{n_k}\rho^2(a_{n_k}, gb) + \mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, b) + \rho^2(q, gb)), \end{aligned}$$

which yields

$$\begin{aligned} &\rho^2(q, gb) + \mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, b) - \rho^2(a_{n_k}, b)) \\ &\le \rho^2(q, b) + \mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, gb) - \rho^2(a_{n_k}, gb)). \end{aligned} \tag{11}$$

Noticing that gb is closed and convex, we can take $m_{n_k} \in gb$, such that

$$\rho(a_{n_k}, m_{n_k}) = \rho(a_{n_k}, gb).$$

From (10), it follows that

$$\mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, gb) - \rho^2(a_{n_k}, gb)) \le \mu_{n_k}(\rho^2(a_{n_k}, q) + \rho^2(q, m_{n_k}) - \rho^2(a_{n_k}, m_{n_k})) = 0. \tag{12}$$

From (10), (11) and (12), we get $\rho^2(q, gb) \le \rho^2(q, b)$. Similarly, we can deduce that

$$\rho^2(q, fb) \le \rho^2(q, b),$$

which yields $q \in CA_{f,g}$. $\square$

By a similar method, we can obtain the following result on account of Theorem 2.

Corollary 1. *Let E be a complete CAT(0) space and K be a nonempty closed convex subset of E. Let $f, g : K \to \mathcal{F}(K)$ be two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings satisfying $\zeta + \eta + \lambda + \pi \ge 0$, either $\zeta + \eta > 0$ or $\zeta + \lambda > 0$. Suppose that $C_sA_{f,g} \ne \emptyset$. If $\{a_k\}$ is a sequence generated by (2) such that $\{a_k\}$ $w-$converges to q, $\rho(a_k, fa_k) \to 0$ and $\rho(a_k, d_k) \to 0$ in which $d_k \in ga_k$, then $q \in CA_{f,g}$.*

Here, we omit the proof of Corollary 1, since it is essentially similar to the proof of Theorem 2.

4. Application

In 2000, Moudafi [26] gave a viscosity approximation method for finding fixed points of nonexpansive mappings. Exactly, suppose that X is a Hilbert space and C is a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set F_T. Staring with an arbitrary initial point $a_0 \in H$, define a sequence $\{a_k\}$, by

$$a_{k+1} = \zeta_k f(a_k) + (1 - \zeta_k)Ta_k,$$

where $f : C \to C$ is a contraction and $\{\zeta_k\}$ is a sequence in (0,1). In [26], under certain appropriate conditions imposed on $\{\zeta_k\}$, the author proved that $\{a_k\}$ converges strongly to a fixed point a^* of T, which satisfies the following variational inequality:

$$\langle (I - f)a^*, a - a^* \rangle \geq 0, \quad a \in F_T.$$

In 2012, Shi and Chen [27] used the property $\mathcal{P}$ to generalize the result of Moudafi to CAT(0) space and Wangkeeree and Preechasilp [22] omitted the property $\mathcal{P}$ from Shi and Chen's result by the concept of quasi-linearization introduced by Berg and Nikolaev [19]. Immediately after, Panyanak and Suantai [28] extended the results of [22] to multivalued nonexpansive mappings with the endpoint condition. Next, we prove strong convergence of a new viscosity approximation method for a finite family of $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings in CAT(0) spaces.

Proposition 4. *Let K be a nonempty convex subset of a CAT(0) space E, and f be a $((\zeta, \eta, \lambda, \pi))$-generalized hybrid multi-valued mapping defined on K with $F_f \neq \emptyset$, which satisfies $\zeta + \eta + \lambda + \pi \geq 0$ and, either $\zeta + \eta > 0$ or $\zeta + \lambda > 0$. Subsequently, f is quasi-nonexpansive.*

Proof. Because of Definition 3, for any $a, b \in K$, we get

$$\zeta H^2(fa, fb) + \eta H^2(a, fb) + \lambda H^2(b, fa) + \pi \rho^2(a, b) \leq 0.$$

Let $b \in K$ be a fixed point of f, then we have

$$\zeta H^2(fa, b) + \eta H^2(a, b) + \lambda H^2(b, fa) + \pi \rho^2(a, b) \leq 0,$$

and, hence,

$$(\zeta + \lambda) H^2(fa, b) \leq -(\eta + \pi) \rho^2(a, b).$$

Since $\zeta + \eta + \lambda + \pi \geq 0$ and $\zeta + \lambda > 0$, we deduce that

$$H^2(fa, b) \leq -\frac{\eta + \pi}{\zeta + \lambda} \rho^2(a, b),$$

which implies that f is quasi-nonexpansive. Similarly, we get the desired result in the case of $\zeta + \eta > 0$. $\square$

Theorem 3. *Let K be a closed convex subset of a complete CAT(0) space E, which satisfies the (S) property, and let $f, g : K \to \mathcal{F}(K)$ be $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings satisfying $\zeta + \eta + \lambda + \pi \geq 0$ and, either $\zeta + \eta > 0$ or $\zeta + \lambda > 0$. Let $CA_{f,g} \neq \emptyset$ and f' be a contraction on K with coefficient $0 < \theta < 1$. Let $\{a_k\}$ be a sequence that is generated by*

$$\begin{cases} b_k = \zeta_k a_k + (1 - \zeta_k) d_k, \\ a_k = \eta_k f'(a_k) + (1 - \eta_k) c_k, \end{cases}$$

where $\{\zeta_k\}, \{\eta_k\} \subset (0,1)$ satisfy $\underline{\lim}_{k \to \infty} \zeta_k (1 - \zeta_k) > 0$ and $\lim_{k \to \infty} \eta_k = 0$, $c_k \in f b_k$, and $d_k \in g a_k$. If $\rho(a_k, f a_k) \to 0$ as $k \to \infty$, then there is a subsequence of $\{a_k\}$ converges strongly to $\tilde{a}$, such that $\tilde{a} = P_{\mathcal{F}} f'(\tilde{a})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{a} f'(\tilde{a})}, \overrightarrow{a \tilde{a}} \rangle \geq 0, \quad a \in \mathcal{F}. \tag{13}$$

Proof. Because $CA_{f,g} \neq \emptyset$, from Proposition 1, then $\mathcal{F} = F_f \bigcap F_g \neq \emptyset$. Moreover, from Proposition 4, it follows that f and g are quasi-nonexpansive. It follows from Lemma 2 that $F_f = A_f$ and $F_g = A_g$. Now, we show that $\{a_k\}$ is bounded. Indeed, for any $p \in \mathcal{F}$, we have

$$\begin{aligned}\rho(a_k, p) &= \rho(\eta_k f'(a_k) + (1-\eta_k)c_k, p)\\ &\leq \eta_k \rho(f'(a_k), p) + (1-\eta_k)\rho(c_k, p)\\ &\leq \eta_k \rho(f'(a_k), p) + (1-\eta_k)H(fb_k, p)\\ &\leq \eta_k \rho(f'(a_k), p) + (1-\eta_k)\rho(b_k, p),\end{aligned}$$

among which

$$\begin{aligned}\rho(b_k, p) &= \rho(\zeta_k a_k + (1-\zeta_k)d_k, p)\\ &\leq \zeta_k \rho(a_k, p) + (1-\zeta_k)\rho(d_k, p)\\ &\leq \zeta_k \rho(a_k, p) + (1-\zeta_k)H(ga_k, p)\\ &\leq \zeta_k \rho(a_k, p) + (1-\zeta_k)\rho(a_k, p) = \rho(a_k, p).\end{aligned}$$

Subsequently,

$$\begin{aligned}\rho(a_k, p) &\leq \eta_k \rho(f'(a_k), p) + (1-\eta_k)\rho(b_k, p)\\ &\leq \eta_k \rho(f'(a_k), p) + (1-\eta_k)\rho(a_k, p)\\ &\leq \eta_k(\rho(f'(a_k), f'(p)) + \rho(f'(p), p)) + (1-\eta_k)\rho(a_k, p)\\ &\leq (\theta\eta_k + 1 - \eta_k)\rho(a_k, p) + \eta_k \rho(f'(p), p),\end{aligned}$$

which implies

$$\rho(a_k, p) \leq \frac{1}{1-\theta}\rho(f'(p), p).$$

We can obtain that $\{a_k\}$ is bounded, which implies that $\{b_k\}$, $\{c_k\}$, $\{d_k\}$ and $\{f'(a_k)\}$ are bounded. Next, we show that $\lim_{k\to\infty} \rho(a_k, ga_k) = 0$ and $\lim_{k\to\infty} \rho(a_k, c_k) = 0$. Indeed, for any $p \in \mathcal{F}$, we have

$$\begin{aligned}\rho^2(a_k, p) &= \rho^2(\eta_k f'(a_k) + (1-\eta_k)c_k, p)\\ &\leq \eta_k \rho^2(f'(a_k), p) + (1-\eta_k)\rho^2(c_k, p)\\ &\leq \eta_k \rho^2(f'(a_k), p) + (1-\eta_k)\rho^2(b_k, p),\end{aligned}$$

among which

$$\begin{aligned}\rho^2(b_k, p) &= \rho^2(\zeta_k a_k + (1-\zeta_k)d_k, p)\\ &\leq \zeta_k \rho^2(a_k, p) + (1-\zeta_k)\rho^2(d_k, p) - \zeta_k(1-\zeta_k)\rho^2(a_k, d_k).\end{aligned}$$

Afterwards,

$$\begin{aligned}\rho^2(a_k, p) &\leq \eta_k \rho^2(f'(a_k), p) + (1-\eta_k)[\zeta_k \rho^2(a_k, p) + (1-\zeta_k)\rho^2(d_k, p) - \zeta_k(1-\zeta_k)\rho^2(a_k, d_k)]\\ &\leq \eta_k[\rho(f'(a_k), f'(p)) + \rho(f'(p), p)]^2 + (1-\eta_k)[\zeta_k \rho^2(a_k, p) + (1-\zeta_k)\rho^2(a_k, p)\\ &\quad - \zeta_k(1-\zeta_k)\rho^2(a_k, d_k)]\\ &\leq \eta_k[\theta\rho(a_k, p) + \rho(f'(p), p)]^2 + (1-\eta_k)[\rho^2(a_k, p) - \zeta_k(1-\zeta_k)\rho^2(a_k, d_k)],\end{aligned}$$

which implies

$$(1-\eta_k)\zeta_k(1-\zeta_k)\rho^2(a_k,d_k) \leq \eta_k[(\theta\rho^2(a_k,p)+\rho(f'(p),p))^2-\rho^2(a_k,p)].$$

Noticing that $\varliminf\limits_{k\to\infty}\zeta_k(1-\zeta_k)>0$, $\lim\limits_{k\to\infty}\eta_k=0$ and $\{a_k\}$ is bounded, we get

$$\lim_{k\to\infty}\rho(a_k,d_k)=0.$$

Let $k\to\infty$, then

$$\rho(a_k,ga_k)\leq\rho(a_k,d_k)\to 0$$

yields $\rho(a_k,ga_k)\to 0$. On the other hand, we notice that

$$\rho(a_k,c_k)=\rho(\eta_k f'(a_k)+(1-\eta_k)c_k,c_k)\leq\eta_k\rho(f'(a_k),c_k),$$

which implies $\rho(a_k,c_k)\to 0$ as $k\to\infty$.

Now, we show that $\{a_k\}$ contains a subsequence converging strongly to $\tilde{a}$, such that $\tilde{a}=P_{\mathcal{F}}(f'\tilde{a})$, which is equivalent to the following variational inequality:

$$\langle\overrightarrow{\tilde{a}f'(\tilde{a})},\overrightarrow{a\tilde{a}}\rangle\geq 0,\quad a\in\mathcal{F}.$$

Because $\{a_k\}$ is bounded, there exists a subsequence $\{a_{n_k}\}$ of $\{a_k\}$ such that $\{a_{n_k}\}$ $\Delta-$converges to some $\tilde{a}\in K$. Since E satisfies the (S) property, we deduce that $\{a_{n_k}\}$ $w-$converges to $\tilde{a}$. Because $\lim\limits_{k\to\infty}\rho(a_k,ga_k)=0$ and $\lim\limits_{k\to\infty}\rho(a_k,fa_k)=0$, similar to the proof of Theorem 2, we can get $\tilde{a}\in CA_{f,g}=\mathcal{F}$.

It follows from (1) of Lemma 6 that

$$\begin{aligned}
\rho^2(a_{n_k},\tilde{a}) &= \langle\overrightarrow{a_{n_k},\tilde{a}},\overrightarrow{a_{n_k},\tilde{a}}\rangle \\
&\leq \eta_{n_k}\langle\overrightarrow{f'(a_{n_k})\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle+(1-\eta_{n_k})\langle\overrightarrow{c_{n_k}\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle \\
&\leq \eta_{n_k}\langle\overrightarrow{f'(a_{n_k})\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle+(1-\eta_{n_k})\rho(c_{n_k},\tilde{a})\rho(a_{n_k},\tilde{a}) \\
&\leq \eta_{n_k}\langle\overrightarrow{f'(a_{n_k})\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle+(1-\eta_{n_k})H(fb_{n_k},\tilde{a})\rho(a_{n_k},\tilde{a}) \\
&\leq \eta_{n_k}\langle\overrightarrow{f'(a_{n_k})\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle+(1-\eta_{n_k})\rho(a_{n_k},\tilde{a})\rho(a_{n_k},\tilde{a}),
\end{aligned}$$

which implies $\rho^2(a_{n_k},\tilde{a})\leq\langle\overrightarrow{f'(a_{n_k})\tilde{a}},\overrightarrow{a_{n_k}\tilde{a}}\rangle$. Because $\{a_{n_k}\}$ converges weakly to $\tilde{a}$, it follows from Lemma 7 that $\rho(a_{n_k},\tilde{a})\to 0$ as $k\to\infty$.

Finally, we show that $\tilde{a}=P_F(f'(\tilde{a}))$ which solves the variational inequality (13). For any $q\in\mathcal{F}$, we deduce

$$\begin{aligned}
\rho^2(a_{n_k},q) &= \rho^2(\eta_{n_k}f'(a_{n_k})+(1-\eta_{n_k})c_{n_k},q) \\
&\leq \eta_{n_k}\rho^2(f'(a_{n_k}),q)+(1-\eta_{n_k})\rho^2(c_{n_k},q)-\eta_{n_k}(1-\eta_{n_k})\rho^2(f'(a_{n_k}),c_{n_k}) \\
&\leq \eta_{n_k}\rho^2(f'(a_{n_k}),q)+(1-\eta_{n_k})H^2(fb_{n_k},q)-\eta_{n_k}(1-\eta_{n_k})\rho^2(f'(a_{n_k}),c_{n_k}) \\
&\leq \eta_{n_k}\rho^2(f'(a_{n_k}),q)+(1-\eta_{n_k})\rho^2(a_{n_k},q)-\eta_{n_k}(1-\eta_{n_k})\rho^2(f'(a_{n_k}),c_{n_k}),
\end{aligned}$$

which implies that

$$\rho^2(a_{n_k},q)\leq\rho^2(f'(a_{n_k}),q)-(1-\eta_{n_k})\rho^2(f'(a_{n_k}),c_{n_k}).$$

Noticing that $\lim\limits_{k\to\infty}\rho(a_k,c_k)=0$ and $\lim\limits_{k\to\infty}\eta_k=0$, taking $k\to\infty$, we obtain

$$\rho^2(\tilde{a}, q) \leq \rho^2(f'(\tilde{a}), q) - \rho^2(f'(\tilde{a}), \tilde{a}).$$

Therefore, we claim

$$2\langle \overrightarrow{\tilde{a}f'(\tilde{a})}, \overrightarrow{q\tilde{a}} \rangle = \rho^2(\tilde{a}, \tilde{a}) + \rho^2(f'(\tilde{a}), q) - \rho^2(\tilde{a}, q) - \rho^2(f'(\tilde{a}), \tilde{a}) \geq 0.$$

It means that $\tilde{a}$ solves the variational inequality (13) and $\tilde{a} = P_F(f(\tilde{a}))$ by Lemma 5. $\square$

5. Conclusions

In this paper, we mainly obtain a weak convergence theorem of common attractive points for two $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings in the complete CAT(0) space by virtue of Banach limits technique and Ishikawa iteration, respectively. However, we do not know whether the common attractive points must be the common strongly attractive points. We will continue to study this problem. Furthermore, we prove the strong convergence of a new viscosity approximation method for a finite family of $(\zeta, \eta, \lambda, \pi)$-generalized hybrid multi-valued mappings in CAT(0) spaces, which also solves a kind of variational inequality problem.

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