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Stability and Bifurcation in a Predator–Prey Model with the Additive Allee Effect and the Fear Effect †

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Abstract: We proposed and analyzed a predator–prey model with both the additive Allee effect and the fear effect in the prey. Firstly, we studied the existence and local stability of equilibria. Some sufficient conditions on the global stability of the positive equilibrium were established by applying the Dulac theorem. Those results indicate that some bifurcations occur. We then confirmed the occurrence of saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. Those theoretical results were demonstrated with numerical simulations. In the bifurcation analysis, we only considered the effect of the strong Allee effect. Finally, we found that the stronger the fear effect, the smaller the density of predator species. However, the fear effect has no influence on the final density of the prey.

Keywords: fear effect; additive allee effect; saddle-node bifurcation; transcritical bifurcation; hopf bifucation

1. Introduction

In 1931, to study the relationship between the growth of a species and its density, Allee [1] proposed the effect later called the Allee effect, which means that the population size will decrease if it is too sparse. The Allee effect occurs due to lots of factors, including inbreeding, depression [2], difficulty in finding spouses [3], social dysfunction at low-densities [4], and so on. In the following, we mention two single species models with Allee effects.

The first one proposed by Bazykin [5] is described by the following equation.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)(x - m), \tag{1}$$

where r denotes the intrinsic per capita growth rate of the population and K is the carrying capacity of the environment. Model (1) is said to havecq strong Allee effect if $0 < m < K$ and to have a weak Alleee effect if $m \leq 0$. To study the dynamics, Bazykin introduced a population threshold, which is the minimum population size for the species to survive. It is shown that with a strong Allee effect, the population must surpass this threshold in order to grow. However, there is no threshold for a weak effect.

Further, in a study on how mating affects a population's reproductive rate, Dennis [6] found that not only can a lack of mates affect it, but also the mating function has a great influence on the birth rate in the population growth rate. To describe the Allee effect of prey, the isometric hyperbolic function is used. Under such circumstances, the Allee effect is called additive. The single species model with an additive Allee effect proposed in [6] is as follows.

$$\frac{dx}{dt} = \left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x+a} \right] x, \tag{2}$$

where m and a are constants, which reflect the degree of Allee effect. Biologically, m denotes the rate for level of Allee effect and a represents the population size of the prey specie whose fitness is half its maximum value. Note that if $0 < m < ar$ then (2) has the weak Allee effect and if $m > ar$ then it has the strong Allee effect. For sparse populations experiencing the Allee effect, Dennis demonstrated with numerical simulations that the critical density, the growth, and the extinction probability can be obtained. Until now, many researchers have paid a great deal of attention on the impact of Allee effect on predation (see [7–18]). For example, Liu et al. [9] showed that a system with gestation delay and an additive Allee effect is unstable if economic interest increases through zero, which may occur in the case of an Allee effect (strong or weak). In [10], they found the extinction of species due to the Allee effect.

Research has indicated that predators can not only kill prey directly but also affect the behavior of prey, and the latter is more lethal than the former. In fact, all animals show many kinds of anti-predator responses, such as changes of foraging behavior, habitat usage, physiology, and so on ([19–23]). To describe that, the concept of fear in the prey was introduced and studied ([24–30]). In particular, Wang et al. [29] for the first time proposed the following predator–prey model with the cost of fear:

$$\begin{aligned} \frac{du}{dt} &= r_0 u f(k,v) - du - au^2 - g(u)v, \\ \frac{dv}{dt} &= -mv + cg(u)v, \end{aligned} \tag{3}$$

where k is the level of fear, which is due to anti-predator behaviors of the prey; g is the functional response. Based on the biological background, the following reasonable assumptions are imposed,

$$\begin{aligned} &f(0,v) = 1, \qquad f(k,0) = 1, \quad \lim_{k \to +\infty} f(k,v) = 0, \\ &\lim_{v \to +\infty} f(k,v) = 0, \quad \frac{\partial f(k,v)}{\partial k} < 0, \quad \frac{\partial f(k,v)}{\partial v} < 0. \end{aligned}$$

Taking the linear functional response, i.e., $g(u) = pu$, Wang et al. found that if $d < r_0 < d + \frac{am}{cp}$ then $E_1(\frac{r_0-d}{a}, 0)$ is globally asymptotically stable and if $r_0 > d + \frac{am}{cp}$ then the unique positive equilibrium E_2 is globally asymptotically stable. Moreover, analysis reveals that the fear factor does not change the stability of the equilibrium when it exists. In (3), the fear factor affects the intrinsic growth rate. Then, inspired by [29] Sasmal [30] considered the case wherein the fear factor impacts the growth rate and the growth rate has the strong Allee effect. The model studied is given by

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right)(x - \theta)\frac{1}{1+fy} - axy, \\ \frac{dy}{dt} &= a\alpha xy - my, \end{aligned} \tag{4}$$

where f represents the effect of fear. It was found that (4) undergoes a subcritical Hopf-bifurcation at $m = \frac{1+\theta}{2}$. Moreover, changing the parameter values of θ and m can produce bi-stability or stable oscillatory coexistence of both prey and predator. It was further observed that the change of f can only change the density of predator at the positive equilibrium but not the stability of the equilibrium.

To the best of our knowledge, so far there is not much being done on predator–prey models with both the additive Allee effect and the fear effect. This motivated us to modify (4) by replacing the Allee effect with the additive Allee effect. Precisely, we studied the following model:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - x - \frac{m}{x+a}\right)\frac{1}{1+fy} - bxy, \\ \frac{dy}{dt} &= \alpha bxy - ny, \end{aligned} \tag{5}$$

where $\frac{1}{1+fy}$ and $\frac{m}{x+a}$ stand for the fear effect and additive Allee effect, respectively; r is the intrinsic growth rate of prey; b is the predation rate; α is the conversion coefficient; and n is the death rate of the predator. As we known, the relationship between prey and predator has always been the focus of scholars [31–37]; hence, this paper will enrich the literature in this field.

The remaining part of this paper is organized as follows. First, we study the existence and local stability of equilibria of (5) in Sections 2 and 3, respectively. Then we provide sufficient conditions ensuring the global stability of the positive equilibrium in Section 4. In Section 5 is the bifurcation analysis, which includes saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. These theoretical results are supported with numerical simulations in Section 6. The paper concludes with a discussion on the impact of the fear effect.

2. Existence of Equilibria

Obviously, system (5) always has the trivial equilibrium $E_0(0,0)$. In order to obtain the other equilibria, we consider the two nullclines:

$$\begin{array}{rcl} rx\left(1-x-\frac{m}{x+a}\right)\frac{1}{1+fy}-bxy & = & 0, \\ \alpha bxy-ny & = & 0. \end{array} \tag{6}$$

Note that $y=0$ if $x=0$ from the second line of (6). Additionally, from this equation, we get $y=0$ (which corresponding to the boundary equilibria) and $y\neq 0$ with $x=\frac{n}{\alpha b}$ (which corresponds to the positive or internal equilibrium).

We first study the existence of boundary equilibria. Substituting $x=0$ into 1st line of (6) gives

$$rx\left(1-x-\frac{m}{x+a}\right)=0,$$

or

$$x^2+(a-1)x+m-a=0. \tag{7}$$

Denote

$$\Delta(m)=(a+1)^2-4m.$$

Let $m^*=\frac{(a+1)^2}{4}$. Then $\Delta(m)=0$ when $m=m^*$ and hence (7) only has one root, denoted by $x_1=\frac{1-a}{2}$; $\Delta(m)>0$ when $m<m^*$ and hence it has two roots, denoted by $x_2=\frac{1-a-\sqrt{\Delta(m)}}{2}$ and $x_3=\frac{1-a+\sqrt{\Delta(m)}}{2}$; $\Delta(m)<0$ when $m>m^*$ and hence it has no real roots. Note that $a\leq m^*$ and $a=m^*$ if and only if $a=1$. Based on the above discussion, we can have the following result on the existence of boundary equilibria.

Lemma 1. *The following results on the existence of boundary equilibria of (5) are true.*

(i) *Suppose $a\in(0,1)$. Then the existence of boundary equilibria in addition to E_0 is summarized in Table 1.*
(ii) *Suppose $a=1$. Then besides E_0, there is also another boundary equilibrium $E_4=(x_4,0)=(\sqrt{1-m},0)$ only when $0<m<1$.*
(iii) *Suppose $a>1$. Then besides E_0, there is also another boundary equilibrium $E_5=(x_5,0)\left(\frac{1-a+\sqrt{\Delta(m)}}{2},0\right)$ only when $0<m<a<m^*$.*

Next, we consider the existence of positive equilibria. In this case, we have $x^*=\frac{n}{\alpha b}$. Substituting it into 1st line of (6) gives

$$bfy^2+by-r\left(1-x^*-\frac{m}{x^*+a}\right)=0.$$

The above equation has positive solutions only when $1 - x^* - \frac{m}{x^*+a} > 0$, and in this case it has only one positive solution $y^* = \frac{-b+\sqrt{\Delta}}{2bf}$, where $\Delta = b^2 + 4bfr\left(1 - x^* - \frac{m}{x^*+a}\right)$. Additionally, in other cases, there is no positive root. That is summarized in the following result.

Table 1. Boundary equilibria besides E_0 of (5) with $a \in (0,1)$.

Condition	Boundary Equilibria
$a < m^* < m$	No
$a < m = m^*$	$E_1 = (x_1, 0) = (\frac{1-a}{2}, 0)$
$a < m < m^*$	$E_2 = (x_2, 0) = \left(\frac{1-a-\sqrt{\Delta(m)}}{2}, 0\right)$ and $E_3 = (x_3, 0) = \left(\frac{1-a+\sqrt{\Delta(m)}}{2}, 0\right)$
$0 < m = a < m^*$	E_3 (E_2 and E_0 coincide)
$0 < m < a < m^*$	E_2 ($x_2 < 0$) and E_3

Lemma 2. *Let $x^* = \frac{n}{\alpha b}$. Then (5) has positive equilibria only when $1 - x^* - \frac{m}{x^*+a} > 0$, and in this case, there is only one positive equilibrium $E^* = (x^*, y^*)$, where $y^* = \frac{-b+\sqrt{\Delta}}{2bf}$ with $\Delta = b^2 + 4bfr\left(1 - x^* - \frac{m}{x^*+a}\right)$. Additionally, in other cases, there is no positive equilibrium.*

3. Local Stability of Equilibria

The purpose of this section is to study the local stability of the equilibria obtained in Lemmas 1 and 2 one by one. Note that both E_4 and E_5 are in fact E_3.

Theorem 1. *The trivial equilibrium E_0 of (5) is a stable node if $a < m$ or $a = m = 1$, a saddle-node if $a = m \neq 1$, and a saddle if $a > m$.*

Proof. The Jacobian matrix of (5) at E_0 is

$$J(E_0) = \begin{pmatrix} r\left(1 - \frac{m}{a}\right) & 0 \\ 0 & -n \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = r(1 - \frac{m}{a})$ and $\lambda_2 = -n$. If $a < m$ then $\lambda_1 < 0$ and hence E_0 is a stable node while if $a > m$ then E_0 is a saddle as $\lambda_1 > 0$. What left is what happens when $a = m$, as in this case $\lambda_1 = 0$. To study the stability of E_0, we rescale t by $\tau = -nt$ and expand the resulting system from (5) in power series up to the third order around E_0 to get

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{b}{n}xy - \frac{r}{n}\left(\frac{1}{m} - 1\right)x^2 + \frac{rf}{n}\left(\frac{1}{m} - 1\right)x^2 y + \frac{r}{m^2 n}x^3 + P_1(x,y), \\ \frac{dy}{d\tau} &= y - \frac{\alpha b}{n}xy, \end{aligned}$$

where $P_1(x,y)$ is a power series in (x,y) with terms $x^i y^j$ satisfying $i + j \geq 4$. By applying Theorem 7.1 of Chapter 2 in [38], we see that E_0 is a saddle-node if $a = m \neq 1$ as the coefficient of x^2, $\frac{r}{n}(\frac{1}{m} - 1)$, is not 0; and E_0 is a stable node if $a = m = 1$ as in this case that coefficient of x^2 is 0 but $\frac{r}{m^2 n} \neq 0$. □

Next, we consider E_1.

Theorem 2. *The boundary equilibrium E_1 of (5) is a saddle-node if $\alpha b x_1 - n \neq 0$, but if $\alpha b x_1 - n = 0$ then E_1 is a saddle.*

Proof. The Jacobian matrix at E_1 is given by

$$\begin{pmatrix} A_1 & -bx_1 \\ 0 & \alpha b x_1 - n \end{pmatrix},$$

where $A_1 = r - 2rx_1 - \frac{amr}{(x_1+a)^2} = ar[1 - \frac{4m}{(a+1)^2}]$. Recall that when E_1 exists we have $m = m^* = \frac{(a+1)^2}{4}$ which implies that $A_1 = 0$. Thus the one eigenvalues of $J(E_1)$ is $\lambda_1 = 0$.

(i) $\lambda_2 = \alpha bx_1 - n \neq 0$, then the discussion on the stability of E_1 is similar to the last part of the proof of Theorem (8).

We first translate E_1 into the origin by the transformation $(X, Y) = (x - x_1, y)$ and expand the resulting system from (5) in power series up to the second order around the origin to get

$$\begin{aligned} \frac{dX}{dt} &= -bx_1Y - \frac{mrx_1}{(a+x_1)^3}X^2 - bXY + P_2(X,Y), \\ \frac{dY}{dt} &= (\alpha bx_1 - n)Y + \alpha bXY, \end{aligned} \tag{8}$$

where $P_2(X, Y)$ is a power series in (X, Y) with terms X^iY^j satisfying $i + j \geq 3$. Now we apply the transformation

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{bx_1}{\alpha bx_1 - n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and then the rescaling $\tau = (\alpha bx_1 - n)t$ to transform (8) into the following standard form:

$$\begin{aligned} \frac{dX_1}{d\tau} &= \frac{mrx_1}{(\alpha bx_1-n)(a+1)^3}X_1^2 + \left[\frac{2bmrx_1^2}{(a+1)^3(\alpha bx_1-n)^2} - \frac{b}{n} + \frac{\alpha b^2x_1}{(\alpha bx_1-n)^2}\right] X_1Y_1 \\ &\quad + \left[\frac{b^2x_1}{(\alpha bx_1-n)^2} - \frac{b^2mrx_1{}^3}{(a+x_1)^3(\alpha bx_1-n)^3} - \frac{\alpha b^3x_1{}^2}{(\alpha bx_1-n)^4}\right] Y_1^2 \\ &\quad + P_3(X_1,Y_1), \\ \frac{dY_1}{d\tau} &= Y_1 - \frac{\alpha b}{\alpha bx_1-n}X_1Y_1 + \frac{\alpha b^2x_1}{(\alpha bx_1-n)^2}Y_1^2, \end{aligned}$$

where $P_3(X_1, Y_1)$ is a power series in (X_1, Y_1) with terms $X_1^iY_1^j$ satisfying $i + j \geq 3$. Since the coefficient of X_1^2, $\frac{mrx_1}{n(a+1)^3}$ is not 0, we know that E_1 is a saddle-node by Theorem 7.1 of Chapter 2 in [38].

(ii) $\lambda_2 = \alpha bx_1 - n = 0$ and let $\tau_1 = -bx_1t$; then (8) change into the following form,

$$\begin{aligned} \frac{dX}{d\tau_1} &= Y + \frac{r}{(b(1-x_1)}X^2 + \frac{1}{x_1}XY + o(\mid X, Y\mid^2) = Y + P_4(X,Y), \\ \frac{dY}{d\tau_1} &= -\frac{\alpha}{x_1}XY = Q_4(X,Y). \end{aligned} \tag{9}$$

Let $Y + P_4(X, Y) = 0$; then we have the following implicit functions

$$\phi(X) = -\frac{r}{b(1-x_1)}X^2 - \frac{r}{bx_1(1-x_1)}X^3 - \frac{r}{bx_1{}^2(1-x_1)}X^4 + \cdots,$$

$$\psi(X) = \frac{\alpha r}{bx_1(1-x_1)}X^3 + \frac{\alpha r}{bx_1{}^3(1-x_1)}X^5 + \cdots,$$

and

$$\delta(X) = \frac{2rx_1 - \alpha b(1-x_1)}{bx_1(1-x_1)}X + [X]_2.$$

By Theorems 7.2 and 7.3 and the corollary (see page 120 to 121) of Chapter 2 in [38], we have $k = 2m + 1$, $m = 1$; $a_k = \frac{\alpha r}{bx_1(1-x_1)} > 0$, and thus E_1 is a saddle. The proof completes.

For the stability of E_2, we note that the Jacobian matrix at E_2 is

$$J(E_2) = \begin{pmatrix} A_2 & -bx_2 \\ 0 & \alpha bx_2 - n \end{pmatrix},$$

where $A_2 = r - 2rx_2 - \frac{amr}{(x_2+a)^2} = ar + r\sqrt{\Delta(m)} - \frac{4amr}{(a+1-\sqrt{\Delta(m)})^2}$. Recall that E_2 exists when $a \in (0,1)$ and $a < m < m^*$. It follows that

$$A_2 > ar + r\sqrt{\Delta(m)} - \frac{4amr}{(a+1)^2} = ar\Big(1 - \frac{m}{m^*}\Big) + r\sqrt{\Delta(m)} > 0.$$

As the two eigenvalues of $J(E_2)$ are $\lambda_1 = A_2$ and $\lambda_2 = \alpha b x_2 - n$, the following result follows immediately. □

Theorem 3. *The boundary equilibrium E_2 of (5) is always unstable. In particular, E_2 is an unstable node if $\alpha b x_2 - n > 0$, it is a saddle if $\alpha b x_2 - n < 0$, and it is a saddle-node if $\alpha b x_2 - n = 0$ (this proof is similar to Theorem 1 (i)).*

From the previous section, we can see that E_3 exists if $0 < a < 1$ and $a < m < m^*$; E_4 exists if $a = 1$ and $0 < m < 1$; E_5 exists if $a > 1$ and $0 < m < a < m^*$. Now, we study the stability of E_i ($i = 3$, 4, 5). The Jacobian matrix at E_i is

$$J(E_i) = \begin{pmatrix} A_i & -bx_i \\ 0 & \alpha b x_i - n \end{pmatrix},$$

where $A_i = r - 2rx_i - \frac{amr}{(x_i+a)^2} = rx_i[\frac{m}{(x_i+a)^2} - 1] = rx_i[\frac{4m}{(a+1+\sqrt{\Delta(m)})^2} - 1]$ for $i = 3$ and 5 and $A_4 = r - 2rx_4 - \frac{amr}{(x_4+a)^2}$. The eigenvalues of $J(E_i)$ are $\lambda_1 = A_i$ and $\lambda_2 = \alpha b x_i - n$. As in the discussion for E_i, one can easily show that $A_i < 0$ by using the conditions guaranteeing its existence. Therefore, the following theorem summarizes the results on stability of E_i.

Theorem 4. *For $i = 3$, 4, and 5, the boundary equilibrium E_i of (5) is a saddle if $\alpha b x_i - n > 0$; it is a stable node if $\alpha b x_i - n < 0$; and it is a saddle-node if $\alpha b x_i - n = 0$(this proof is similar to Theorem 1 (i)).*

Finally, we consider the stability of the positive equilibrium E^*.

Theorem 5. *The positive equilibrium E^* of (5) is locally asymptotically stable if $a > \sqrt{m} - x^*$ and unstable if $a < \sqrt{m} - x^*$.*

Proof. The Jacobian matrix of (5) at E^* is

$$J(E^*) = \begin{pmatrix} rx^*\left[\frac{m}{(1+fy^*)(x^*+a)^2} - \frac{1}{1+fy^*}\right] & -frx^*\left(1 - x^* - \frac{m}{x^*+a}\right) - bx^* \\ \alpha b y^* & 0 \end{pmatrix}.$$

Note that

$$\det(J(E^*)) = \alpha b y^*\left[frx^*(1 - x^* - \frac{m}{x^*+a}) + bx^*\right] > 0$$

from the condition on the existence of E^* and

$$\mathrm{tr}(J(E^*)) = rx^*\left[\frac{m}{(1+fy^*)(x^*+a)^2} - \frac{1}{1+fy^*}\right].$$

It is easy to see that $\mathrm{tr}(J(E^*)) < 0$ if $a > \sqrt{m} - x^*$, $\mathrm{tr}(J(E^*)) = 0$ if $a = \sqrt{m}x^*$, and $\mathrm{tr}((E^*)) > 0$ if $a < \sqrt{m} - x^*$. Therefore, both eigenvalues of $J(E^*)$ have negative real parts if $a > \sqrt{m} - x^*$, have positive real parts if $a < \sqrt{m} - x^*$, and have zero real parts if $a = \sqrt{m} - x^*$. Then the desired result follows. □

4. Global Asymptotical Stability of the Positive Equilibrium

In Theorem 5, we have shown that the positive equilibrium E^* of system (5) is locally asymptotically stable if $a > \sqrt{m} - x^*$. In this section, we provide some sufficient conditions on its global stability.

Theorem 6. *Suppose $a > \sqrt{m} - x^*$. Then the positive equilibrium E^* of system (5) is globally asymptotically stable in the interior of $\mathbb{R}^2_+$ if one of the following conditions holds.*

(i) $a < 1$, $m \leq a < m^*$, *and* $\alpha b x_3 - n > 0$;
(ii) $a = 1$, $m < a = m^*$, *and* $\alpha b x_4 - n > 0$;
(iii) $a > 1$, $m < a < m^*$, *and* $\alpha b x_5 - n > 0$.

Proof. Note that, in addition to E_0 and E^*, system (5) also has a boundary equilibrium E_3 when (i) holds, or E_4 when (ii) holds, or E_5 when (iii) holds. Under the conditions, both E_0 and E_i ($i = 3, 4, 5$) are saddles, which are unstable, but E^* is locally asymptotically stable. It is easy to see that all $\{(x, 0)|x \geq 0\}$, $\{(0, y)|y \geq 0\}$, and $\{(x, y)|x > 0, y > 0\}$ (the interior of $\mathbb{R}^2_+$) are positively invariant subsets of system (5). In order to show the global stability of E^* in the interior of $\mathbb{R}^2_+$, we only need to exclude the existence of closed orbits in it. For this purpose, we denote

$$
\begin{aligned}
F_1 &= rx\left(1 - x - \tfrac{m}{x+a}\right)\tfrac{1}{1+fy} - bxy, \\
F_2 &= \alpha bxy - ny.
\end{aligned}
$$

With the Dulac function $B(x, y) = \frac{1}{xy}$, we have

$$
D = \frac{\partial(BF_1)}{\partial x} + \frac{\partial(BF_2)}{\partial y} = -\frac{r[(x+a)^2 - m]}{y(fy+1)(x+a)^2} < 0
$$

in the interior of $\mathbb{R}^2_+$. By the Dulac Theorem, there is no closed orbit in the interior of $\mathbb{R}^2_+$. This completes the proof. □

5. Bifurcation Analysis

From the local stability analysis, we see that there are bifurcations occurring. In this section, we derive conditions on saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation.

Firstly, in order to prove the saddle-node bifurcation and transcritical bifurcation of system (5), we need the following Lemma (Sotomayor's Theorem in [39,40]).

Theorem 7 (Sotomayor's Theorem in [39,40])**.** *Consider the system as follows.*

$$
\dot{x} = f(x, \mu). \tag{10}
$$

Suppose $f(x_0, \mu_0) = 0$ at equilibrium x_0 holds. Additionally, assume that the matrix $A_{n\times n} = Df(x_0, \mu_0)$ has one characteristic root $\lambda = 0$, and both V and W are eigenvectors belonging to the eigenvalue $\lambda = 0$ of the matrix A and A^T, respectively. Then

(1) *Suppose*

$$
\begin{aligned}
W^T f_\mu(x_0, \mu_0) &\neq 0, \\
W^T[D^2 f_\mu(x_0, \mu_0)(V, V)] &\neq 0.
\end{aligned}
$$

Hence, when the bifurcation parameter μ has a critical value, that is, $\mu = \mu_0$, system (10) undergoes a saddle-node bifurcation at x_0.

(2) *Suppose*

$$\begin{aligned} W^T f_\mu(x_0, \mu_0) &= 0, \\ W^T [D f_\mu(x_0, \mu_0) V] &\neq 0, \\ W^T [D^2 f_\mu(x_0, \mu_0)(V, V)] &\neq 0. \end{aligned}$$

Hence, when μ is of a critical value, that is, $\mu = \mu_0$, system (10) undergoes a transcritical bifurcation at x_0.

By Table 1 of Lemma 2.1, when $a < 1$, system (5) has two boundary equilibria E_2 and E_3 if $a < m < m^*$, has one boundary equilibrium E_1 if $a < m = m*$, and has no boundary equilibrium if $a < m^* < m$. This suggests a bifurcation around E_1. The above analysis indicates that we can choose the parameter m in the additive Allee effect as the bifurcation parameter to obtain saddle-node bifurcation.

Theorem 8. *Suppose $a < 1$ and $\alpha b(1-a) - 2n \neq 0$. Then (5) undergoes a saddle-node bifurcation from $E_1 = (x_1, 0) = \left(\frac{1-a}{2}, 0\right)$ at $m = m_{SN} = \frac{(a+1)^2}{4}$.*

Proof. When $a < 1$ and m_{SN}, (5) has the unique boundary equilibrium E_1. We apply Lemma 3 to study the bifurcation around E_1. Firstly, we easily see that the Jacobian matrix $J(E_1; m_{SN}) = \begin{pmatrix} 0 & -bx_1 \\ 0 & \alpha b x_1 - n \end{pmatrix}$ has the two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \alpha b x_1 - n = \frac{\alpha b(1-a)-2n}{2} \neq 0$. Choose the eigenvectors V and W associated with the eigenvalue λ_1 of $J(E_1; m_{SN})$ and $J(E_1; m_{SN})^T$ given respectively by

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\alpha b(1-a)}{\alpha b(1-a)-2n} \end{pmatrix}.$$

Define

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} rx\left(1 - x - \frac{m}{x+a}\right) \frac{1}{1+fy} - bxy \\ \alpha bxy - ny \end{pmatrix}.$$

Then

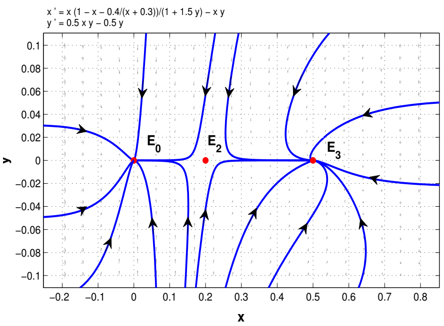
$$\begin{aligned} F_m(E_1; m_{SN}) &= \begin{pmatrix} -\frac{r(1-a)}{1+a} \\ 0 \end{pmatrix}, \\ D^2 F(E_1; m_{SN})(V, V) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_1; m_{SN})} \\ &= \begin{pmatrix} -\frac{2r}{a+1} \\ 0 \end{pmatrix}. \end{aligned}$$

It follows that

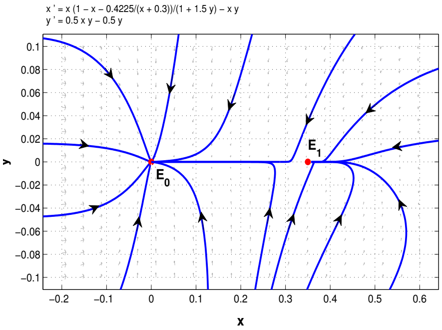
$$\begin{aligned} W^T F_m(E_1; m_{SN}) &= -\frac{r(1-a)}{1+a} \neq 0, \\ W^T [D^2 F(E_1; m_{SN})(V, V)] &= -\frac{2r}{1+a} \neq 0. \end{aligned}$$

Therefore, system (5) undergoes a saddle-node bifurcation at $m = m_{SN}$. □

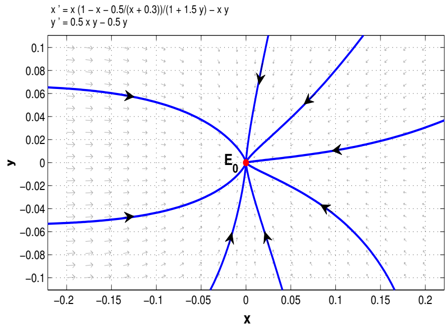
To illustrate the saddle-node bifurcation, we chose $r = 1$, $a = 0.3$, $f = 1.5$, $b = 1$, $\alpha = n = 0.5$. Then $m_{SN} = 0.425$. When $a < 0.4 = m < m_{SN}$, system (5) has two distinct boundary equilibria, E_2 and E_3; when $a < m = m_{SN}$, E_2 collapses to E_0 and only the boundary equilibrium E_3 remains. However, when $a < m_{SN} < m = 0.5$, the boundary equilibrium E_3 also disappears (see Figure 1).



(**a**) $m = 0.4 < m_{SN} = 0.4225$



(**b**) $m = m_{SN} = 0.4225$



(**c**) $m = 0.5 > 0.4225 = m_{SN}$

Figure 1. (**a**) Two distinct boundary equilibria and one trivial equilibrium when $m < m_{SN}$: there are two stable nodes E_0 and E_3, and a saddle E_2. (**b**) A boundary equilibrium and a trivial equilibrium when $m = m_{SN}$: E_3 is a saddle-node and E_0 is a stable node. (**c**) A trivial equilibrium when $m > m_{SN}$: E_0 is a stable node.

Next, by Table 1 of Lemma 2.1, system (5) has two boundary equilibria E_2 and E_3 if $a < m < m^*$, has one boundary equilibrium E_3 (E_2 and E_0 coincide)if $m = a < m*$, and has two boundary equilibria E_2 ($x_2 < 0$) and E_3 if $m < a < m^*$. This suggests a bifurcation around E_0. The above analysis indicates that we can choose the parameter a as the bifurcation produces transcritical bifurcation.

Theorem 9. *Suppose that $m < 1$. Then system (5) undergoes a transcritical bifurcation from E_0 at $a = a_{TC} = m$.*

Proof. The proof is similar to that of Theorem 7. We just verify the condition on transcritical bifurcation of Lemma 3. When $a = a_{TC} = m$, we have

$$J(E_0; a_{TC}) = \begin{pmatrix} 0 & 0 \\ 0 & -n \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -n \neq 0$. Choose the eigenvectors of $J(E_0; a_{TC}))$ and $J(E_0; a_{TC})^T$ associated with the eigenvalue λ_1 given respectively by

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad W = V.$$

Let F be defined as in the proof of Theorem 7. Then

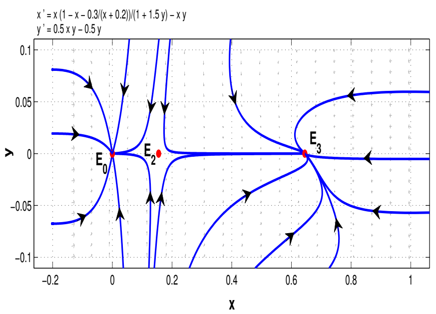
$$
\begin{aligned}
F_a(E_0; a_{TC}) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
DF_a(E_0; a_{TC})V &= \begin{pmatrix} \frac{mr}{(1+fy)(x+a)^2} & 0 \\ 0 & 0 \end{pmatrix}_{(E_0; a_{TC})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{r}{m} \\ 0 \end{pmatrix}, \\
D^2F(E_0; a_{TC})(V, V) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_0; a_{TC})} \\
&= \begin{pmatrix} -2r + \frac{2r}{m} \\ 0 \end{pmatrix}.
\end{aligned}
$$

Then we easily see that V and W satisfy

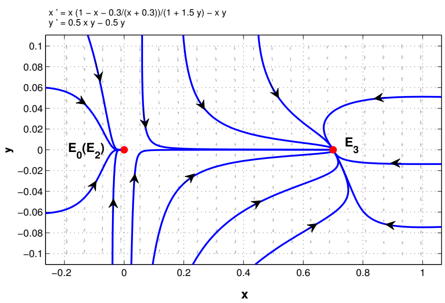
$$
\begin{aligned}
W^T F_a(E_0; a_{TC}) &= 0, \\
W^T[DF_a(E_0; a_{TC})V] &= \tfrac{r}{m} \neq 0, \\
W^T[D^2F(E_0; a_{TC})(V, V)] &= -2r + \tfrac{2r}{m} \neq 0.
\end{aligned}
$$

Therefore, system (5) undergoes transcritical bifurcation from E_0 at $a = a_{TC} = m$.

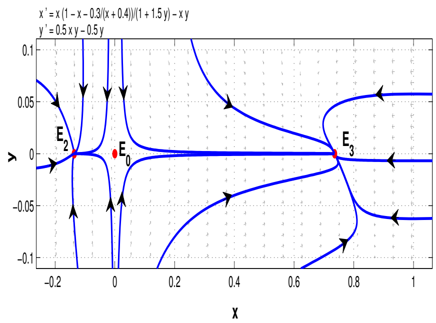
With $r = 1$, $m = 0.3$, $f = 1.5$, $b = 1$, $\alpha = 0.5$, Figure 2 shows the transcritical bifurcation with $a = 0.2$, $a = 0.3$, and $a = 0.4$. $\square$



(a) $a = 0.2 < 0.3 = a_{TC}$



(b) $a = a_{TC} = 0.3$



(c) $a = 0.4 > 0.3 = a_{TC}$

Figure 2. (**a**) Two distinct boundary equilibria and one trivial equilibrium when $a < a_{TC}$: two stable nodes E_2 and E_3, and a saddle E_0. (**b**) A boundary equilibrium and a trivial equilibrium when $a = a_{TC}$: E_3 is a stable node and E_0 is a saddle-node. (**c**) Two boundary equilibria and a trivial equilibrium when $a > a_{TC}$: E_2 is a saddle, both E_3 and E_0 are stable nodes.

In the remainder of this section, we consider Hopf bifurcation. From Theorem 5 and its proof, it is easily concluded that the positive equilibrium of system (5) is locally asymptotically stable if $a > \sqrt{m} - x^*$, is a center if $a = \sqrt{m} - x^*$, and through Hopf bifurcation loses its stability under appropriate parameters. In the following we choose a as the bifurcation parameter to show that.

Theorem 10. *Under the assumptions on the existence of the positive equilibrium E^* of system (5), that is, $1 - x^* - \frac{m}{x^*+a} > 0$, then there is a supercritical Hopf bifurcation from E^* at $a = a_H = \sqrt{m} - x^*$, where $x^* = \frac{n}{\alpha b}$.*

Proof. Recall that the characteristic equation of the Jacobian matrix $J(E^*)$ is

$$\lambda^2 - \mathrm{tr}(J(E^*))\lambda + \det(J(E^*)) = 0,$$

where

$$\begin{aligned} \det(J(E^*)) &= \alpha b y^* \left[f r x^* (1 - x^* - \tfrac{m}{x^*+a}) + b x^* \right] > 0 \\ \mathrm{tr}(J(E^*)) &= r x^* \left[\tfrac{m}{(1+fy^*)(x^*+a)^2} - \tfrac{1}{1+fy^*} \right]. \end{aligned}$$

Clearly, E^* is a center when $a = a_H$ and

$$\frac{d}{da}[\mathrm{tr}(J(E^*))]\Big|_{a=a_H} = \frac{-2rx^*}{\sqrt{m}(1+fy^*)} \neq 0.$$

Thus a Hopf bifurcation from E^* occurs at $a = a_H$. To discuss the stability (direction) of bifurcated periodic orbits, we compute the first Lyapunov number l_1 at E^* as follows.

Firstly, we translate E^* to the origin by the transformation $x = X_2 + x^*$ and $y = Y_2 + y^*$ and rewrite the resultant system as

$$\begin{aligned} \tfrac{dX_2}{dt} &= \alpha_{10} X_2 + \alpha_{01} Y_2 + \alpha_{20} {X_2}^2 + \alpha_{11} X_2 Y_2 + \alpha_{02} {Y_2}^2 + \alpha_{30} {X_2}^3 \\ &\quad + \alpha_{21} {X_2}^2 Y_2 + \alpha_{12} X_2 {Y_2}^2 + \alpha_{03} {Y_2}^3 + P_5(X_2, Y_2), \\ \tfrac{dY_2}{dt} &= \beta_{10} X_2 + \beta_{01} Y_2 + \beta_{11} X_2 Y_2, \end{aligned}$$

where

$$\begin{aligned} \alpha_{10} &= r x^* \left[\tfrac{m}{(1+fy^*)(x^*+a)^2} - \tfrac{1}{1+fy^*} \right], \\ \alpha_{01} &= -f r x^* (1 - x^* - \tfrac{m}{x^*+a}) - b x^*, \\ \alpha_{20} &= -\tfrac{r x^* f}{\sqrt{m}(1+fy^*)}, \\ \alpha_{11} &= -\left[\tfrac{r(1-x^*-\sqrt{m})f}{(1+fy^*)^2} + b \right], \\ \alpha_{02} &= \tfrac{r x^* (1-x^*-\sqrt{m}) f^2}{(1+fy^*)^3}, \\ \alpha_{30} &= \tfrac{r(x^*-\sqrt{m})}{m(1+fy^*)}, \\ \alpha_{21} &= -\tfrac{r x^* f}{\sqrt{m}(1+fy^*)^2}, \\ \alpha_{12} &= -\tfrac{r(1-x^*-\sqrt{m}) f^2}{(1+fy^*)^3}, \\ \alpha_{03} &= -\tfrac{r x^* (1-x^*-\sqrt{m}) f^3}{(1+fy^*)^4}, \\ \beta_{10} &= b \alpha y^*, \\ \beta_{01} &= 0, \\ \beta_{11} &= b \alpha, \end{aligned}$$

and $P_5(X_2, Y_2)$ is a power series in (X_2, Y_2) with terms ${X_2}^i {Y_2}^j$ satisfying $i + j \geq 4$. Let $\Delta = \alpha_{10}\beta_{01} - \alpha_{01}\beta_{10}$. Then

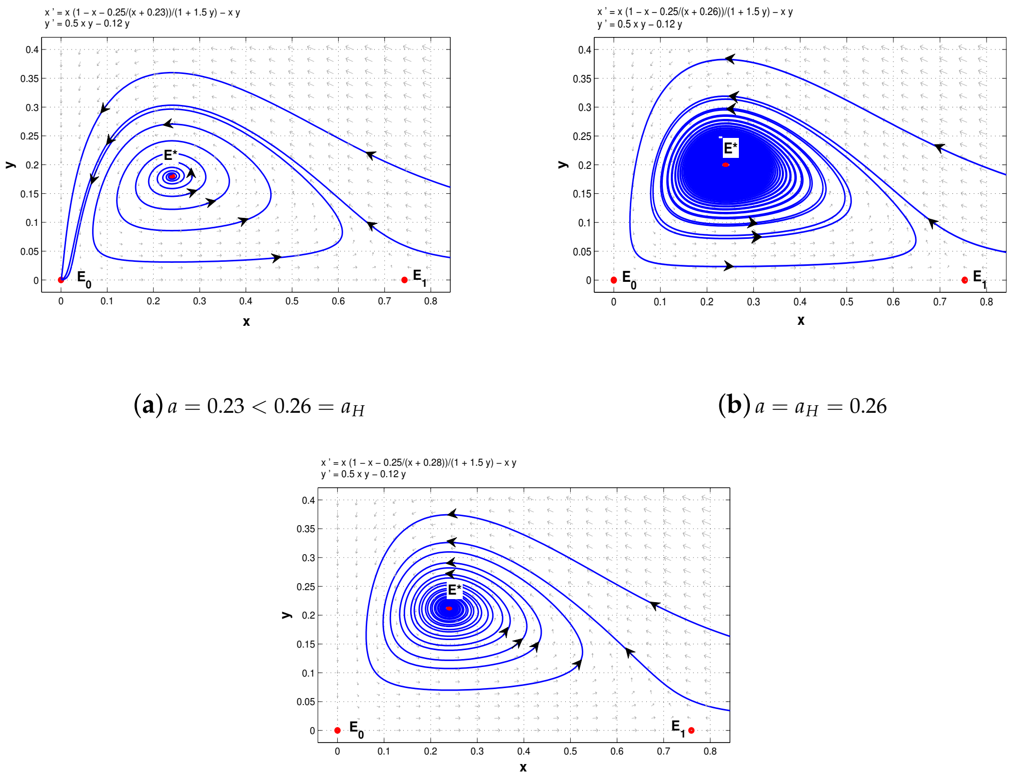
$$\begin{aligned}
l_1 &= -\tfrac{3\pi}{2\alpha_{01}\Delta^{3/2}}\{[\alpha_{10}\beta_{01}(\alpha_{11}^2 + \alpha_{11}\beta_{02} + \alpha_{02}\beta_{11}) + \alpha_{10}\alpha_{01}(\beta_{11}^2 + \alpha_{20}\beta_{11} + \alpha_{11}\beta_{02}) \\
&\quad + \beta_{10}^2(\alpha_{11}\alpha_{02} + 2\alpha_{02}\beta_{02}) - 2\alpha_{10}\beta_{01}(\beta_{02}^2 - \alpha_{20}\alpha_{02}) - 2\alpha_{10}\alpha_{01}(\alpha_{20}^2 - \beta_{20}\beta_{02}) \\
&\quad - \alpha_{01}^2(2\alpha_{20}\beta_{20} + \beta_{11}\beta_{20}) + (\alpha_{01}\beta_{10} - 2\alpha_{10}^2)(\beta_{11}\beta_{02} - \alpha_{11}\alpha_{20})] \\
&\quad - (\alpha_{10}^2 + \alpha_{01}\beta_{01})[3(\beta_{10}\beta_{03} - \alpha_{01}\alpha_{30}) + 2\alpha_{01}(\alpha_{21} + \beta_{12}) + (\beta_{01}\alpha_{12} - \alpha_{01}\beta_{21})]\}\Big|_{a=a_H} \\
&= -\tfrac{3\pi}{2\alpha_{01}\Delta^{3/2}}\{\alpha_{11}\alpha_{02}\beta_{10}^2 - \alpha_{01}\alpha_{11}\alpha_{20}\beta_{10} - \alpha_{01}\alpha_{12}\beta_{10}^2\} \\
&= -\frac{3\pi r}{2\sqrt{b\alpha m x^* y^*}\sqrt{r(1-x^*-\sqrt{m})f + by^{*2}f^2 + 2by^* f + b}}.
\end{aligned}$$

As $a = a_H = \sqrt{m} - x^*$, it follows from

$$0 = r\left(1 - x^* - \frac{m}{x^* + a}\right)\frac{1}{1 + fy^*} - by^* = r(1 - x^* - \sqrt{m})\frac{1}{1 + fy^*} - by^*$$

that $r(1 - x^* - \sqrt{m}) = (1 + fy^*)by^* > 0$. Then $l_1 < 0$ his meas that E^* is destabilized through a supercritical Hopf bifurcation at $a = a_H$. $\square$

Figure 3 shows the Hopf bifurcation while Figure 4 further indicates that E^* is locally stable when $a = a_H$ (which can be confirmed with center manifold theory). Here $r = 1$, $m = 0.25$, $f = 1.5$, $b = 1$, $\alpha = 0.5$, and $n = 0.12$.



(a) $a = 0.23 < 0.26 = a_H$

(b) $a = a_H = 0.26$

(c) $a = 0.28 > 0.26 = a_H$

Figure 3. (**a**) When $a < a_H$, E^* is unstable. (**b**) When $a = a_H$, a stable periodic orbit bifurcated from E^*. (**c**) When $a > a_H$, E^* is stable.

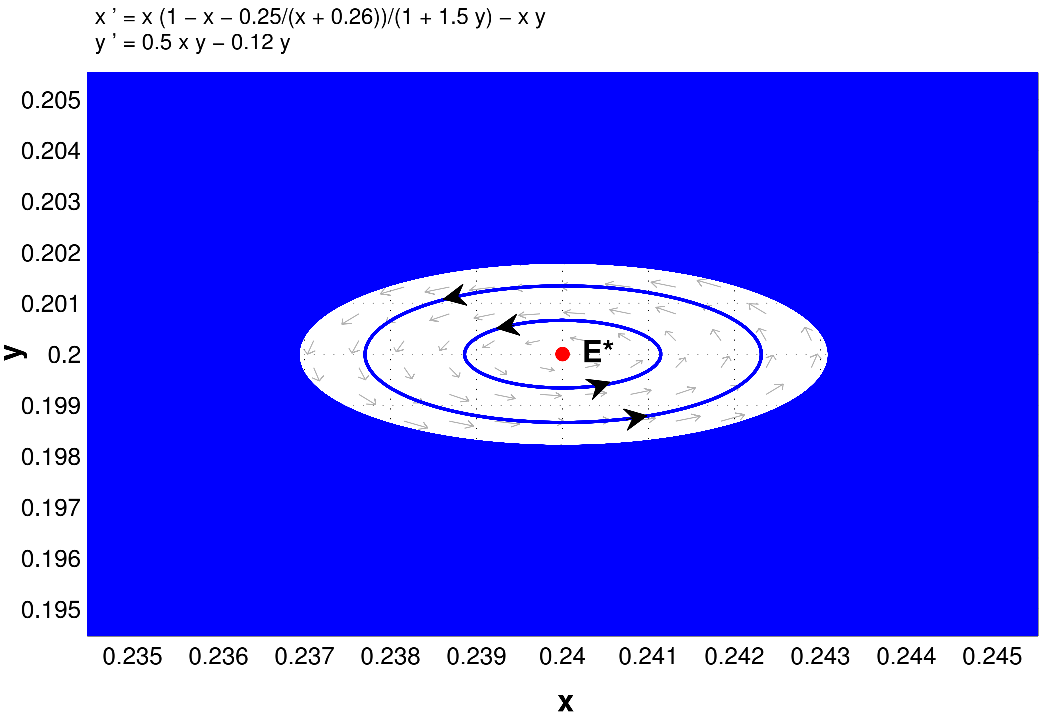


Figure 4. When $a = a_H = 0.26$, a stable periodic orbit bifurcated form E^* which is locally stable.

6. Numerical Simulations

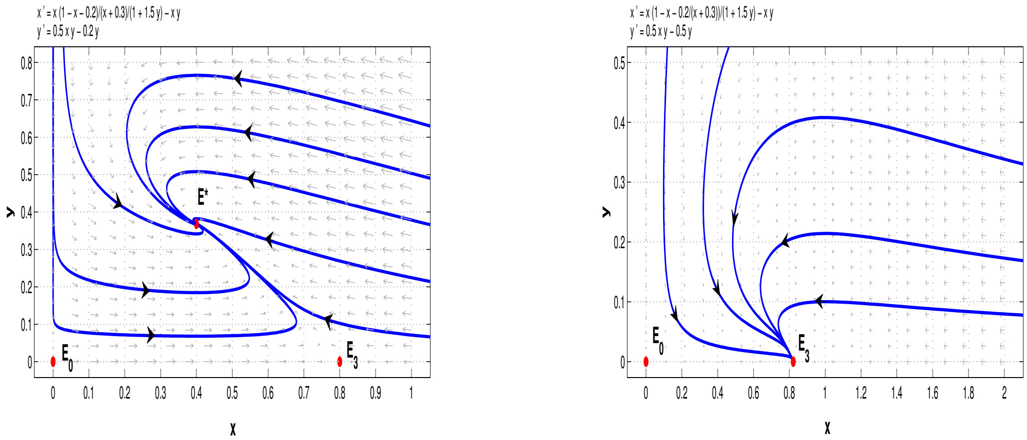
In Sections 3 and 4, we studied the stability of the equilibria of system (5). In this section, we use numerical simulations to demonstrate different scenarios of the dynamics according to whether $a = 1$ or not.

Example 1. *Firstly, we consider the following special case of system (5) with* $a = 0.3$*:*

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{m}{x+0.3}\right)\frac{1}{1+1.5y} - xy, \\ \frac{dy}{dt} &= 0.5xy - ny. \end{aligned} \tag{11}$$

We distinguish four cases to illustrate the complicated dynamics of system (11).

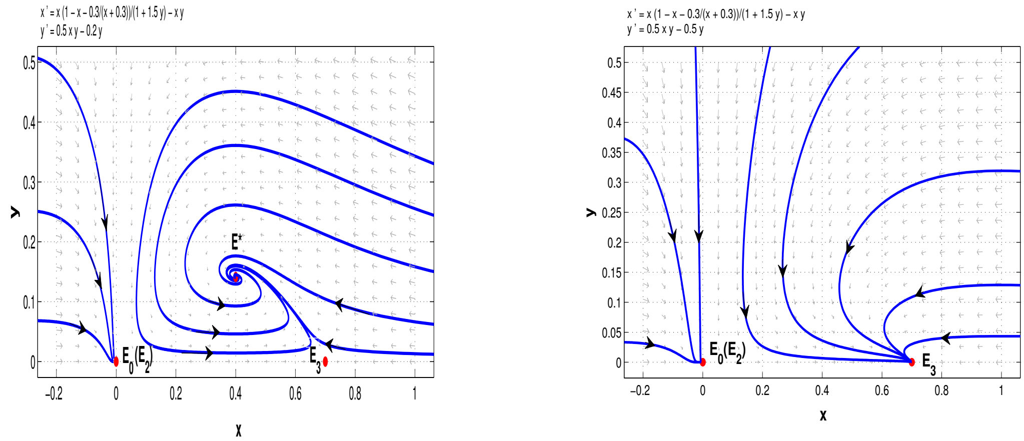
First, we choose $0.2 = m < a = 0.3$. If $n = 0.2$, then the conditions of Theorems 1 and 4–6 are satisfied, and hence, for system (11), E_0 and E_3 are saddle points and E^* is a stable node (see Figure 5a); but if $n = 0.5$, then E_0 is a saddle point and E_3 is a stable node (see Figure 5b).



(**a**) $m = 0.2 < 0.3 = a$ and $n = 0.2$ (**b**) $m = 0.2 < 0.3 = a$ and $n = 0.5$

Figure 5. (**a**) With $m = 0.2 < 0.3 = a$ and $n = 0.2$, E_0 and E_3 are saddle points. (**b**) With $m = 0.2 < 0.3 = a$ and $n = 0.5$, E^* is a stable node.

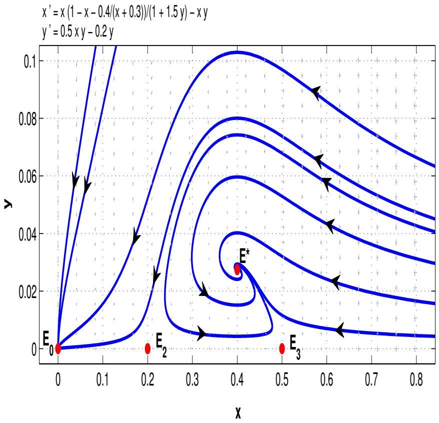
Next, let $m = a = 0.3$. When $n = 0.2$, system (11) has a saddle-node E_0 (which coincides with E_2), a saddle point E_3, and a stable node E^* (see Figure 6a), while when $n = 0.5$, it has a saddle-node E_0 (coincides with E_2) and a stable node E_3 (see Figure 6b).



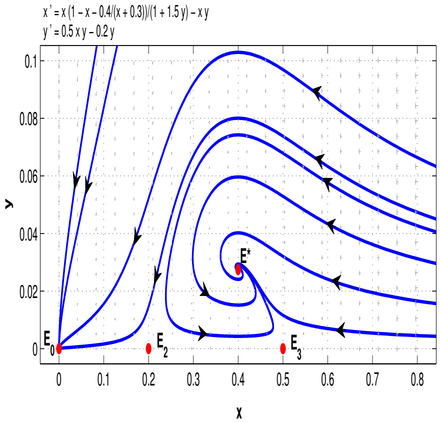
(**a**) $m = a = 0.3$ and $n = 0.2$ (**b**) $m = a = 0.3$ and $n = 0.5$

Figure 6. (**a**) With $m = a = 0.3$ and $n = 0.2$, E_0 is a saddle-node, E_3 is a saddle point, and E^* is a stable node. (**b**) With $m = a = 0.3$ and $n = 0.5$, E_0 is a saddle node and E_3 is a stable node.

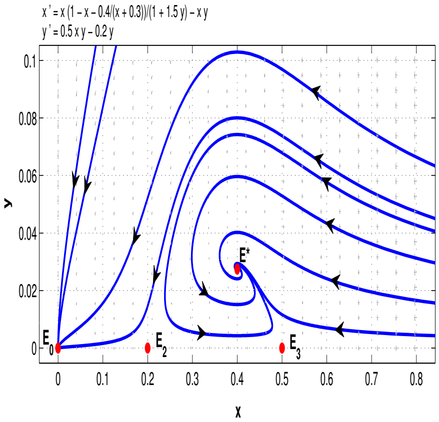
Now, take $m = 0.4 > 0.3 = a$. When $n = 0.1$, system (11) has a stable node E_0, a saddle-node E_2, and a saddle E_3(see Figure 7a); when $n = 0.2$, there is a stable node E_0, two saddles E_2 and E_3, and a stable node E^* (see Figure 7b); when $n = 0.25$, E_0 is a stable node, E_2 is a saddle, and E_3 is a saddle-node (see Figure 7c); when $n = 0.5$, E_0 is a stable node, E_2 is a saddle, and E_3 is a stable node (see Figure 7d).



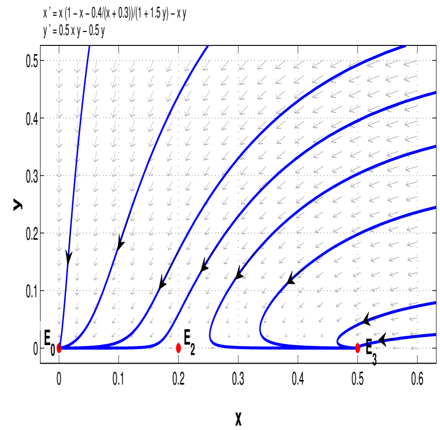
(**a**) $m = 0.4 > 0.3 = a$ and $n = 0.1$



(**b**) $m = 0.4 > 0.3 = a$ and $n = 0.2$



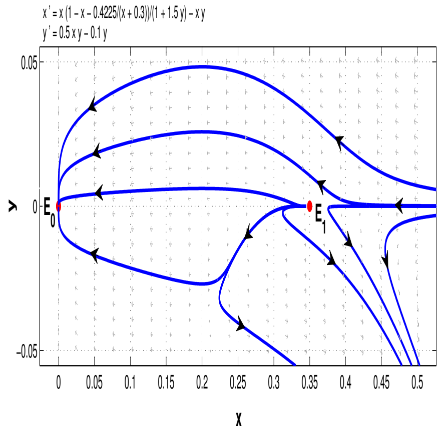
(**c**) $m = 0.4 > 0.3 = a$ and $n = 0.25$



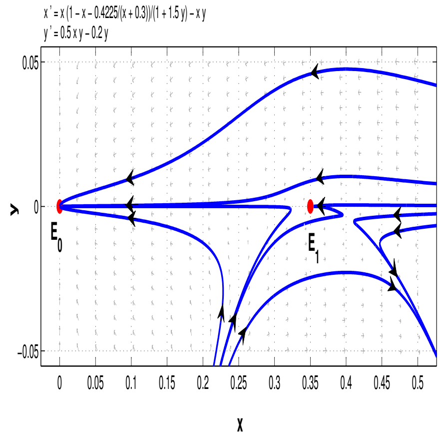
(**d**) $m = 0.4 > 0.3 = a$ and $n = 0.5$

Figure 7. (**a**) With $m = 0.4 > 0.3 = a$ and $n = 0.1$, there is a stable node E_0, a saddle-node E_2, and a saddle E_3. (**b**) With $m = 0.4 > 0.3 = a$ and $n = 0.2$, there is a stable node E_0, two saddles E_2 and E_3, and a stable node E^*. (**c**) With $m = 0.4 > 0.3 = a$ and $n = 0.25$, there is a stable node E_0, a saddle E_2, and a saddle-node E_3. (**d**) With $m = 0.4 > 0.3 = a$ and $n = 0.5$, there is a stable node E_0, a saddle E_2, and a stable node E_3.

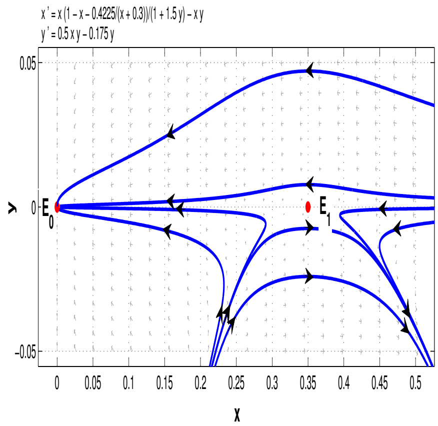
Finally, pick $m = m^* = 0.4225 > 0.3 = a$. When $n = 0.1$ and $n = 0.2$, we see that the equilibrium E_0 of system (11) is a stable node and E_1 is a saddle-node (see Figure 8a,b); when $n = 0.175$, E_0 is also a stable node, however, E_1 is a saddle (see Figure 8c).



(**a**) $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.1$



(**b**) $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.2$



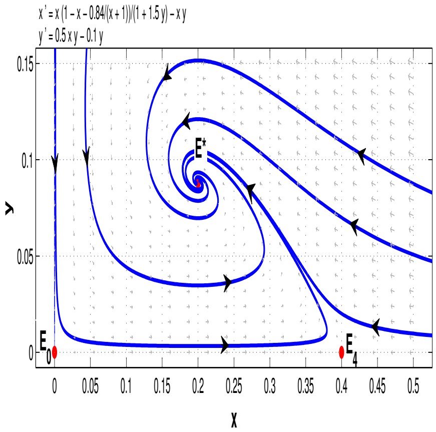
(**c**) $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.175$

Figure 8. (**a**) When $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.1$, E_0 is a stable node and E_1 is a saddle-node. (**b**) When $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.2$, E_0 is a stable node and E_1 is a saddle-node. (**c**) When $m = m^* = 0.4225 > 0.3 = a$ and $n = 0.175$, E_0 is a stable node and E_1 is a saddle.

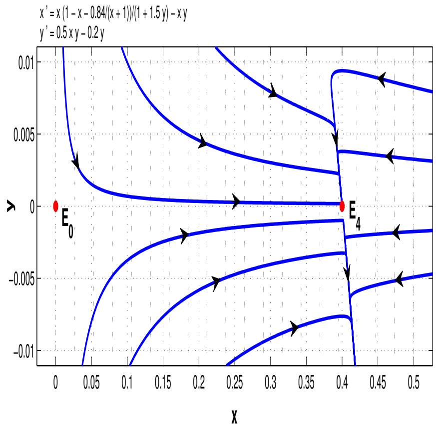
Example 2. *This time we let $a = 1$ and consider the following system:*

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{m}{x+1}\right)\frac{1}{1+1.5y} - xy, \\ \frac{dy}{dt} &= 0.5xy - ny. \end{aligned} \tag{12}$$

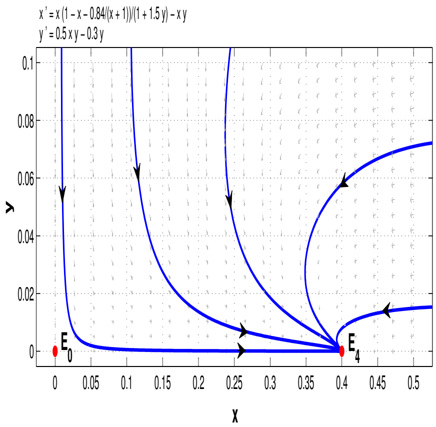
For system (12), we have $m^* = 1$. Choose $m = 0.84 < a$. From Theorems 1, 4, and 5, we can see that when $n = 0.1$, it has two saddle points E_0 and E_4, and a stable node E^* (see Figure 9a); when $n = 0.2$, E_0 is a saddle and E_4 is a saddle-node (see Figure 9b); when $n = 0.3$, E_0 is a saddle and E_4 is a saddle-node (see Figure 9c).



(**a**) $m = 0.5 < 1 = a = m^*$ and $n = 0.1$



(**b**) $m = 0.5 < 1 = a = m^*$ and $n = 0.2$



(**c**) $m = 0.5 < 1 = a = m^*$ and $n = 0.3$

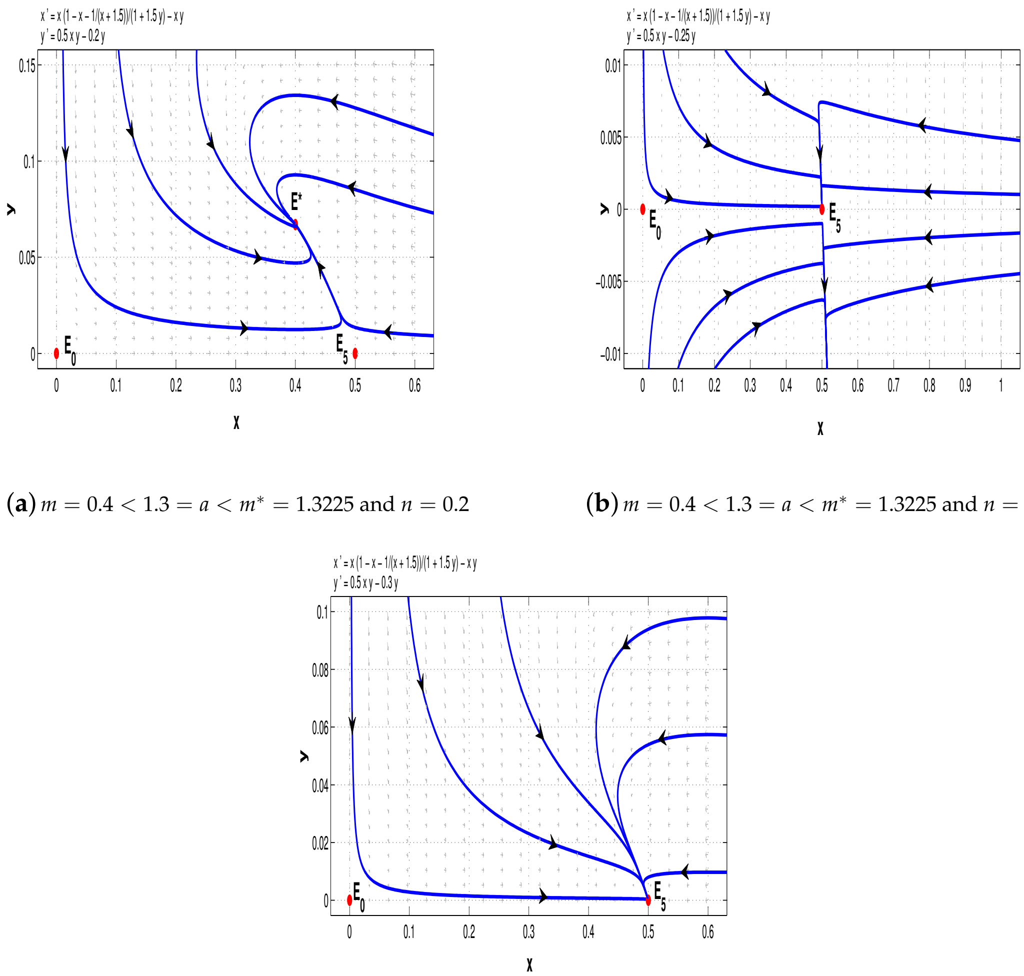
Figure 9. (**a**) When $m = 0.5 < 1 = a = m^*$ and $n = 0.1$, there are two saddle points E_0 and E_4, and a stable node E^*. (**b**) When $m = 0.5 < 1 = a = m^*$ and $n = 0.2$, E_0 is a saddle and E_4 is a saddle-node. (**c**) When $m = 0.5 < 1 = a = m^*$ and $n = 0.3$, E_0 is a saddle and E_4 is a stable node.

Example 3. *Now we consider*

$$
\begin{aligned}
\frac{dx}{dt} &= x\left(1 - x - \frac{m}{x+1.3}\right)\frac{1}{1+1.5y} - xy, \\
\frac{dy}{dt} &= 0.5xy - ny,
\end{aligned}
\tag{13}
$$

where $a = 1.5$.

In this case, $m^* = 1.3225$. Take $m = 1$. From Theorems 1, 4, and 5, we can know that we should let $n = 0.2$. Then system (13) has two saddle points E_0 and E_5, and a stable node E^* (see Figure 10a); when we let $n = 0.25$, it has a saddle point E_0 and saddle-node E_5 (see Figure 10b); but when we let $n = 0.3$, it has a saddle point E_0 and stable node E_5 (see Figure 10c).



(**a**) $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.2$

(**b**) $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.25$

(**c**) $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.3$

Figure 10. (**a**) When $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.2$, both E_0 and E_5 are saddle points and E^* is a stable node. (**b**) When $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.25$, E_0 is a saddle point and E_5 is a saddle-node. (**c**) When $m = 0.4 < 1.3 = a < m^* = 1.3225$ and $n = 0.3$, E_0 is a saddle point and E_5 is a stable node.

7. Discussion and Conclusions

In this paper, we mainly focused on the impact of the additive Allee effect. In this section, we first discuss the influence of the fear effect on the coexistence of the two species. For this purpose, we regard x^* and y^* functions of f. Differentiating both sides of

$$\begin{array}{rcl} r\left(1 - x^* - \frac{m}{x^*+a}\right)\frac{1}{1+fy^*} - by^* & = & 0, \\ \alpha b x^* - n & = & 0, \end{array}$$

with respect to f gives

$$\begin{pmatrix} \frac{r - \frac{mr}{(x^*+a)^2}}{1+fy^*} & -\frac{rf\left(1-x^*-\frac{m}{x^*+a}\right)}{(1+fy^*)^2} - b \\ \alpha b & 0 \end{pmatrix} \begin{pmatrix} \frac{dx^*}{df} \\ \frac{dy^*}{dy} \end{pmatrix} = \begin{pmatrix} -\frac{ry^*(1-x^*-\frac{m}{x^*+a})}{(1+fy^*)^2} \\ 0 \end{pmatrix}.$$

It follows that

$$\frac{dx^*}{df} = 0 \qquad \text{and} \qquad \frac{dy^*}{df} = \frac{dy^*}{df} = -\frac{ry^*\left(1 - x^* - \frac{m}{x^*+a}\right)}{rf\left(1 - x^* - \frac{m}{x^*+a}\right) + b(1+fy^*)^2} < 0.$$

Thus the fear effect has no influence at the size of the prey at the coexistence equilibrium (final size of prey) but enhancing it will make the final size of the predator decease. This is the same as in [29,30]. Figure 11 shows the relationship between the intensity of fear effect and the final size of the predator.

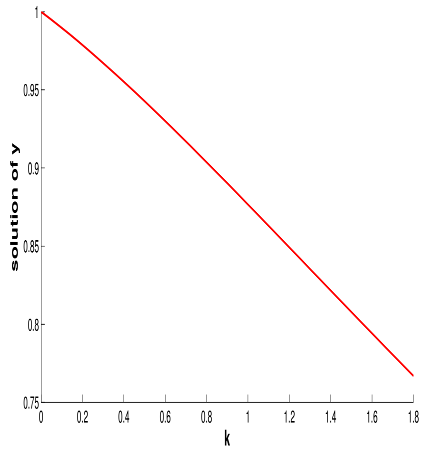


Figure 11. Relationship between the fear effect intensity and the final size of the predator.

We briefly summarize our findings to conclude this paper below.

In this paper, we proposed and studied a predator–prey model with the additive Allee effect and the fear effect. Though much has been done for predator–prey model with the Allee effect and the fear effect, to the best of our knowledge, the combined impact of these two factors has not been investigated. The findings here have some similarities and differences from those for system (4) with the strong Allee effect. For our model, both additive the Allee effect and the fear effect can affect the number and stability of equilibria. For example, the trivial equilibrium can be a stable node, or a saddle-node, or a saddle point. These results suggest possible bifurcations. By applying Sotomayor's theorem, we established conditions for the occurrence of saddle-node bifurcation and transcritical bifurcation from boundary equilibria. We also studied Hopf bifurcation from the positive (or coexistence) equilibrium. By calculating the first Lyapunov number, we know that the Hopf bifurcation is supercritical. Finally, the fear effect only affects the final size of the predator. These results indicate that the additive Allee effect can produce much more complex dynamics that the multiplicative Allee effect can.

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