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Article

Five-Dimensional Contact CR-Submanifolds in $\mathbb{S}^7(1)$ †

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† Dedicated to the memory of Professor Aurel Bejancu.

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Abstract: Due to the remarkable property of the seven-dimensional unit sphere to be a Sasakian manifold with the almost contact structure (φ, ξ, η), we study its five-dimensional contact CR-submanifolds, which are the analogue of CR-submanifolds in (almost) Kählerian manifolds. In the case when the structure vector field ξ is tangent to M, the tangent bundle of contact CR-submanifold M can be decomposed as $T(M) = H(M) \oplus E(M) \oplus \mathbb{R}\xi$, where $H(M)$ is invariant and $E(M)$ is anti-invariant with respect to φ. On this occasion we obtain a complete classification of five-dimensional proper contact CR-submanifolds in $\mathbb{S}^7(1)$ whose second fundamental form restricted to $H(M)$ and $E(M)$ vanishes identically and we prove that they can be decomposed as (multiply) warped products of spheres.

Keywords: contact CR-submanifold; nearly totally geodesic submanifold; warped product; seven-dimensional unit sphere

MSC: 53C15; 53C25; 53B25

1. Introduction

Let M be a Riemannian submanifold of the seven-dimensional unit sphere. It is well-known that $\mathbb{S}^7(1)$ possesses the almost contact structure (φ, ξ, η), which is also contact and Sasakian. Having in mind the behaviour of the endomorphism φ, submanifolds in the Sasakian manifolds carrying a φ-invariant distribution such that its orthogonal complement is φ-anti-invariant, are called contact CR-submanifolds. This notion is the odd-dimensional analogue of CR-submanifolds in (almost) Kählerian manifolds, introduced by Bejancu in [1], who requested the existence of a differentiable holomorphic distribution such that its orthogonal complement is a totally real distribution. Also, CR-submanifolds of the nearly Kähler six-dimensional unit sphere have also been investigated (see [2,3], for example). As the φ-invariant distribution $H(M)$ is always even-dimensional, the lowest possible dimension for a proper contact CR-submanifold (i.e., suchthat the dimensions of both φ-invariant and anti-invariant distributions are different from zero) is four. In this paper we continue our study of certain contact CR-submanifolds in seven-dimensional unit sphere, which we started in [4] for the case of four-dimensional submanifolds and continued in [5], where we presented several examples of four and five-dimensional contact CR-submanifolds of $\mathbb{S}^7(1)$, which are of product and warped product type.

One of the natural problems in the theory of submanifolds is the condition of immersibility. For example, it is interesting to investigate totally geodesic submanifolds, that is, those submanifolds for which all geodesics—when the induced Riemannian metric is considered—are also geodesics on the ambient manifold. This property is equivalent to the vanishing of the second fundamental form. It is well-known that any contact CR-submanifold in a Sasakian manifold can never be totally geodesic.

Therefore, on this occasion we study those five-dimensional contact CR-submanifolds in a Sasakian sphere $\mathbb{S}^7(1)$ which are close to be totally geodesic, namely those whose second fundamental form restricted to both φ-invariant and φ-anti-invariant distributions vanishes identically. Calling them *nearly totally geodesic* contact CR-submanifolds, we prove that such submanifolds are (multiply) warped products of spheres and finding the immersions, we obtain their complete classification.

Theorem 1. *Let M be a five-dimensional proper nearly totally geodesic contact CR-submanifold of seven-dimensional unit sphere. Then M is locally congruent to (multiply) warped product via the immersions (86) and (114).*

Remark 1. *In [5] we presented several examples of four and five-dimensional contact CR-submanifolds of product and warped product type of seven-dimensional unit sphere, which are nearly totally geodesic, minimal and which satisfy the equality sign in some Chen type inequalities.*

2. Preliminaries

2.1. Sasakian Manifolds

Let $\widetilde{M}^{2n+1}$ be a Sasakian manifold with the structure tensors φ, ξ, η and $\tilde{g}$. If $\tilde{\nabla}$ denotes the Levi-Civita connection on $\widetilde{M}$, then the following relations

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0,$$
$$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = \tilde{g}(X,\xi), \quad d\eta(X,Y) = \tilde{g}(\varphi X, Y),$$
$$(\widetilde{\nabla}_X \varphi)Y = -\tilde{g}(X,Y)\xi + \eta(Y)X, \quad \widetilde{\nabla}_X \xi = \varphi X,$$

hold on $\widetilde{M}$ for all $X, Y \in \chi(\widetilde{M})$. For more details we refer to [6], although we use the convention of [7,8].

2.2. Submanifolds

Let (M, g) be a submanifold in the Riemannian manifold $(\widetilde{M}, \tilde{g})$, where g is the induced metric and let ∇ and $\widetilde{\nabla}$ be the Levi Civita connections on M and $\widetilde{M}$, respectively. Recall the formulae of Gauss and Weingarten

$$(\mathbf{G}) \quad \widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$(\mathbf{W}) \quad \widetilde{\nabla}_X N = -S_N X + \nabla^{\perp}_X N,$$

for $X, Y \in T(M)$ and $N \in T^{\perp}M$, where h and S_N are the second fundamental form and the shape operator corresponding to N, respectively, related by $\tilde{g}(h(X,Y),N) = g(S_N X, Y)$. Here $\nabla^{\perp}$ is a connection in the normal bundle of M and $R^{\perp}$ is its curvature tensor. If $\bar{\nabla}$ stands for the van der Waerden–Bortolotti connection, which is defined as

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for X, Y, Z tangent to M, the Codazzi Theorem can be written as

$$(\mathbf{EC}) \quad \mathrm{nor}(\widetilde{R}_{XY} Z) = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

where $\widetilde{R}$ is the curvature tensor on $\widetilde{M}$ defined by $\widetilde{R}_{XY} = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$, and by nor we mean the projection on the normal bundle. For submanifolds of real space forms, like spheres, this projection vanishes identically and therefore the covariant derivative $\bar{\nabla}h$ of the second fundamental form, defined by $\bar{\nabla}h(X,Y,Z) = (\bar{\nabla}_X h)(Y,Z)$, is totally symmetric.

The equations of Gauss and Ricci are given by

$$(\mathbf{EG}) \quad g(R_{XY}Z, W) = \tilde{g}(\widetilde{R}_{XY}Z, W) + \tilde{g}(h(Y,Z), h(X,W)) - \tilde{g}(h(X,Z), h(Y,W)),$$
$$(\mathbf{ER}) \quad \tilde{g}(R^{\perp}_{XY} N_a, N_b) = \tilde{g}(\widetilde{R}_{XY} N_a, N_b) + g([S_a, S_b]X, Y),$$

respectively, where R is the curvature on M, and S_a, S_b are the shape operators corresponding to the normal vectors N_a, N_b, respectively.

2.3. Contact CR-Submanifolds

The notion of *contact CR-submanifolds* in Sasakian manifolds is the odd-dimensional analogue of CR-submanifolds in (almost) Kählerian manifolds. See also [9]. Particularly, a contact CR-submanifold in the Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, \tilde{g})$ is a submanifold M carrying a φ-invariant distribution $\mathcal{D}$, that is, $\varphi_p \mathcal{D}_p \subseteq \mathcal{D}_p$, for any $p \in M$, such that the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is φ-anti-invariant, that is, $\varphi_p \mathcal{D}_p^\perp \subseteq T_p^\perp M$, for any $p \in M$. This notion was used by Bejancu and Papaghiuc in [10], using the terminology of *semi-invariant submanifold*. It is standard to require that ξ is tangent to M rather than normal, which is too restrictive (by Prop. 1.1 in [11], p. 43, M must be φ-anti-invariant, that is, $\varphi T_p M \subseteq T_p^\perp M$, for all $p \in M$), or oblique which leads to highly complicated embedding equations. The contact CR-submanifold is called *proper* if both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are non-trivial distributions.

For a contact CR-submanifold M of a Sasakian manifold $\widetilde{M}$, with ξ tangent to M, the tangent space at each point decomposes orthogonally as

$$T(M) = H(M) \oplus E(M) \oplus \mathbb{R}\xi,$$

where $\varphi H(M) = H(M)$, $\varphi^2 = -I$ along $H(M)$ and $\varphi E(M) \subseteq T^\perp M$. It should be remarked that the *Levi distribution* $H(M)$ is never integrable (see [12]), while $H(M) \oplus \mathbb{R}\xi$ can be, as in the case of contact CR-products (see [13]). On the other hand, the normal bundle of M can be decomposed as $T^\perp M = \varphi E(M) \oplus \nu(M)$, where $\nu(M)$ is the orthogonal complement of $\varphi E(M)$ in $T^\perp M$, invariant under the action of φ.

2.4. CR Warped Product Submanifolds in Sasakian Manifolds

The notion of warped product is the natural and very fruitful generalization of Riemannian products. It was introduced by Bishop and O'Neill in [14] in order to construct a large class of complete manifolds of negative curvature.

Let B, F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and let f be a smooth positive function on B. Considering the product manifold $B \times F$, let $\pi_1 : B \times F \to B$ and $\pi_2 : B \times F \to F$ be the canonical projections. The manifold $M = B \times_f F$ is called the *warped product* if it is equipped with the Riemannian structure such that

$$||X||^2 = ||\pi_{1,*}(X)||^2 + f^2(\pi_1(x))||\pi_{2,*}(X)||^2$$

for all $X \in T_x(M)$, $x \in M$, or, equivalently, $g = g_B + f^2 g_F$ with the usual meaning, while f is called the *warping function* on the warped product. For more details we refer to [15].

A contact CR-submanifold M in a Sasakian manifold $\widetilde{M}$, tangent to the structure vector field ξ, is called a *contact CR warped product*, with the warping function f, if it is the warped product $N^T \times_f N^\perp$ of an invariant submanifold N^T, tangent to ξ and a totally real submanifold $N^\perp$ of $\widetilde{M}$, where f is the warping function (see [13] for more details). It is notable to point out that there is no proper contact CR-submanifolds in Sasakian manifolds in the form $N^\perp \times_f N^T$. This fact was proved in [13,16].

2.5. The Sasakian Structure on $\mathbb{S}^{2m+1}(1)$

Identifying $\mathbb{R}^{2m+2}$ with $\mathbb{C}^{m+1}$, let J denote the multiplication with the imaginary unit $i = \sqrt{-1}$, on $\mathbb{R}^{2m+2}$. The $(2m+1)$-dimensional unit sphere $\mathbb{S}^{2m+1}(1) = \{\mathbf{p} \in \mathbb{R}^{2m+2} : \langle \mathbf{p}, \mathbf{p} \rangle = 1\}$, where $\langle\ ,\ \rangle$ is the usual scalar product in $\mathbb{R}^{2m+2}$, carries a canonical almost contact metric structure (φ, η, ξ, g) induced from $(J, \langle \cdot, \cdot \rangle)$. Strictly speaking, as at any point $\mathbf{p} \in \mathbb{S}^{2m+1}(1)$, the outward unit normal to sphere coincides with the position vector $\mathbf{p}$, putting $\xi = J\mathbf{p}$ to be the characteristic vector field, for X tangent to $\mathbb{S}^{2m+1}$, JX fails, in general, to be tangent and decomposing JX into the tangent and the normal part we have $JX = \varphi X - \eta(X)\mathbf{p}$. Moreover, this structure is Sasakian. For more details and proofs we refer to [6,17].

2.6. Problem

On this occasion, we consider the problem of finding all five-dimensional proper contact CR-submanifolds in $\mathbb{S}^7(1)$ such that

$$h(H(M), H(M)) = 0 \quad \text{and} \quad h(E(M), E(M)) = 0, \tag{1}$$

where, as we have already mentioned, $T(M) = H(M) \oplus E(M) \oplus \mathbb{R}\xi$ and $T^{\perp}M = \varphi E(M) \oplus \nu(M)$, with $\varphi H(M) = H(M)$, $\varphi E(M) \subseteq T^{\perp}M$ and $\nu(M)$ being the orthogonal complement of $\varphi E(M)$ in $T^{\perp}M$.

3. Five-Dimensional Nearly Totally Geodesic Contact CR-Submanifolds in $\mathbb{S}^7(1)$

In order to prove our results, we first select an appropriate frame on M^5 in such a way that equations, which are the consequences of (1), become satisfied. Then, we classify all 5-dimensional proper contact CR-submanifolds in the seven-dimensional unit sphere satisfying (1).

3.1. Essential Characteristics of Five-Dimensional Contact CR-Submanifolds in $\mathbb{S}^7(1)$

In this subsection, after choosing the appropriate basis, we introduce some smooth functions to describe the induced connection and we express the shape operators. Using Codazzi and Ricci equations, we obtain relations between these functions and we derive conditions on these functions, namely a system of algebraic and differential equations.

First, it is straightforward, using the formulae of Gauss and Weingarten, as well as the Sasakian structure of the 7-sphere, to prove the following:

Lemma 1. *If M is a contact CR-submanifold in the Sasakian manifold $\mathbb{S}^7(1)$ we have*

$$\tilde{g}(h(X,Z), \varphi W) = \tilde{g}(h(X,W), \varphi Z),$$

for every $X \in H(M)$, $Z, W \in E(M)$.

As M is a five-dimensional submanifold of $\mathbb{S}^7(1)$, having in mind the condition (1), we conclude that

$$\dim H(M) = 2,\ \dim E(M) = 2 \text{ and } \varphi E(M) = T^{\perp}M.$$

Further, starting with two arbitrary orthonormal bases $\{e_1, e_2 = \varphi e_1\}$ in $H(M)$ and $\{e_3, e_4\}$ in $E(M)$, respectively, we will choose a basis in $T(M)$ so that the second fundamental form will depend only on four smooth functions. In that direction, as a consequence of Lemma 1, we define, for each $X \in H(M)$, a symmetric operator

$$A(X) : E(M) \times E(M) \to C^{\infty}(M) \quad \text{by} \quad A(X)(Z,W) = \tilde{g}(h(X,Z), \varphi W).$$

Since $T^{\perp}M = \varphi E(M)$, there exist six smooth functions $a_1, a_2, a_3, b_1, b_2, b_3$ on M such that

$$\begin{aligned} h(e_1, e_3) &= a_1\varphi e_3 + a_2\varphi e_4, \qquad & h(e_2, e_3) &= b_1\varphi e_3 + b_2\varphi e_4, \\ h(e_1, e_4) &= a_2\varphi e_3 + a_3\varphi e_4, \qquad & h(e_2, e_4) &= b_2\varphi e_3 + b_3\varphi e_4. \end{aligned}$$

For an arbitrary (for the moment) angle $t \in C^{\infty}(M)$, we consider $X = \cos t\, e_1 + \sin t\, e_2$ and we compute $\operatorname{trace} A(X) = A(X)(e_3, e_3) + A(X)(e_4, e_4) = (a_1 + a_3)\cos t + (b_1 + b_3)\sin t$.

We distinguish the following situations:

(i) if $a_1 + a_3 = 0$ and $b_1 + b_3 = 0$ then $\operatorname{trace} A(X) = 0$ for any $X \in H(M)$;
(ii) if $a_1 + a_3 = 0$ and $b_1 + b_3 \neq 0$, then take $t = 0$ and denote $E_2 = e_1$ and $E_1 = -e_2$;
(iii) if $a_1 + a_3 \neq 0$ and $b_1 + b_3 = 0$, then take $t = \pi/2$ and denote $E_2 = e_2$ and $E_1 = e_1$;
(iv) if both $a_1 + a_3 \neq 0$ and $b_1 + b_3 \neq 0$, then take t such that $\tan t = -\frac{a_1+a_3}{b_1+b_3}$; denote the corresponding X by E_2 and set $E_1 = -\varphi E_2$.

It follows that in Cases (ii)–(iv) we can choose E_1 and $E_2 = \varphi E_1$ in $H(M)$ such that the operator $A(E_2)$ is traceless. Additionally, because the operator $A(E_1)$ is also symmetric, we will take the basis in $E(M)$ defined by the eigenvectors of this operator. Denote them by E_3 and E_4. Consequently we have that $h(E_1, E_3)$ is proportional to φE_3 and $h(E_1, E_4)$ is proportional to φE_4.

Concerning the Case (i), we (apparently) have the freedom of choosing E_2. Nevertheless, if we make a rotation about a certain angle s in $E(M)$, we set $E_3 = \cos s\, e_3 + \sin s\, e_4$ and $E_4 = -\sin s\, e_3 + \cos s\, e_4$. If $a_2 = 0$ take $s = 0$ and if $a_2 \neq 0$ take s such that $\cot 2s = \frac{a_1}{a_2}$. Consequently, we obtain $h(e_1, E_3) = \tilde{a}_1 \varphi E_3$ and $h(e_1, E_4) = -\tilde{a}_1 \varphi E_4$. Since s depends on a_1 and a_2 and hence on e_1 and e_2, we set $E_1 = e_1$ and $E_2 = e_2$.

For simplicity of notation, we continue to write E_5 for φE_3 and E_6 for φE_4. Moreover, since $\mathbb{S}^7(1)$ is Sasakian, using the Gauss formula, we can easily compute $h(E_i, \xi)$, $i = 1, \ldots, 4$.

Summarizing, we have thus proved

Proposition 1. *For a proper contact CR-submanifold M^5 in $\mathbb{S}^7(1)$ such that the condition (1) is satisfied, we can choose orthonormal differential vector fields E_1, E_2, E_3, E_4 defined locally on M, such that $\{E_1, E_2 = \varphi E_1\} \subset H(M)$, $\{E_3, E_4\} \subset E(M)$ and*

$$\begin{aligned} h(E_1, E_3) &= a_1 E_5, \qquad & h(E_2, E_3) &= b_1 E_5 + b_2 E_6, \\ h(E_1, E_4) &= a_3 E_6, \qquad & h(E_2, E_4) &= b_2 E_5 - b_1 E_6, \end{aligned} \tag{2}$$

where a_1, a_3, b_1, b_2 are smooth functions on M and consequently

$$h(E_1, \xi) = h(E_2, \xi) = h(\xi, \xi) = 0, \quad h(E_3, \xi) = E_5, \quad h(E_4, \xi) = E_6. \tag{3}$$

Further, let us introduce some smooth functions to describe the induced connection on M.

Proposition 2. *Under the conditions stated above, the Levi-Civita connection ∇ is given by*

$$\begin{array}{lll}
\nabla_{E_1} E_1 = pE_2, & \nabla_{E_1} E_2 = -pE_1 - \xi, & \\
\nabla_{E_1} E_3 = rE_4, & \nabla_{E_1} E_4 = -rE_3, & \\
\nabla_{E_1} \xi = E_2, & & \\
\nabla_{E_2} E_1 = -qE_2 + \xi, & \nabla_{E_2} E_2 = qE_1, & \nabla_{\xi} E_1 = \omega E_2, \\
\nabla_{E_2} E_3 = lE_4, & \nabla_{E_2} E_4 = -lE_3, & \nabla_{\xi} E_2 = -\omega E_1, \\
\nabla_{E_2} \xi = -E_1, & & \nabla_{\xi} E_3 = \theta E_4, \\
\nabla_{E_3} E_1 = aE_2 + b_1 E_3 + b_2 E_4, & \nabla_{E_3} E_2 = -aE_1 - a_1 E_3, & \nabla_{\xi} E_4 = -\theta E_3, \\
\nabla_{E_3} E_3 = -b_1 E_1 + a_1 E_2 + \alpha E_4, & \nabla_{E_3} E_4 = -b_2 E_1 - \alpha E_3, & \nabla_{\xi} \xi = 0, \\
\nabla_{E_3} \xi = 0, & & \\
\nabla_{E_4} E_1 = cE_2 + b_2 E_3 - b_1 E_4, & \nabla_{E_4} E_2 = -cE_1 - a_3 E_4, & \\
\nabla_{E_4} E_3 = -b_2 E_1 + \beta E_4, & \nabla_{E_4} E_4 = b_1 E_1 + a_3 E_2 - \beta E_3, & \\
\nabla_{E_4} \xi = 0, & &
\end{array} \tag{4}$$

for certain smooth functions $a, l, p, q, r, \alpha, \beta, c, \omega$ and θ on M.

Proof. Since $\widetilde{\nabla}_{E_i}\xi = \varphi E_i$, we immediately obtain that $\nabla_{E_i}\xi = 0$ for $i = 3, 4$ and this implies that $\nabla_{E_i}X$ has no component along ξ, for every $X \in H(M)$ and $i = 3, 4$.

Now let's prove one of the formulae in (4).

Considering $\widetilde{\nabla}_{E_3}E_2 = \widetilde{\nabla}_{E_3}(\varphi E_1) = \varphi\nabla_{E_3}E_1 + \varphi h(E_1, E_3)$ and identifying the tangent and the normal parts, respectively, we obtain

$$\nabla_{E_3}E_1 = aE_2 + b_1E_3 + b_2E_4, \quad \nabla_{E_3}E_2 = -aE_1 - a_1E_3,$$

for a certain function $a \in C^\infty(M)$. □

In order to have the complete description of the geometry of M, we write the expression of the normal connection, that is

$$\begin{aligned}
&\nabla^\perp_{E_1}E_5 = rE_6, \quad \nabla^\perp_{E_1}E_6 = -rE_5,\\
&\nabla^\perp_{E_2}E_5 = lE_6, \quad \nabla^\perp_{E_2}E_6 = -lE_5,\\
&\nabla^\perp_{E_3}E_5 = \alpha E_6, \quad \nabla^\perp_{E_3}E_6 = -\alpha E_5,\\
&\nabla^\perp_{E_4}E_5 = \beta E_6, \quad \nabla^\perp_{E_4}E_6 = -\beta E_5,\\
&\nabla^\perp_{\xi}E_5 = \theta E_6, \quad \nabla^\perp_{\xi}E_6 = -\theta E_5.
\end{aligned} \tag{5}$$

Lemma 2. *Under the above assumptions, the coefficient b_2 vanishes.*

Proof. Using the fact that $\bar{\nabla}h$ is totally symmetric, we obtain the equations given in Table 1.

Table 1. Symmetries of $\bar{\nabla}h$.

The Symmetry We Use	The Result We Get
$(\bar{\nabla}h)(E_3, E_3, E_4)$	$a_1b_2 = 0$ $b_1(a_1 + a_3) = 0$
$(\bar{\nabla}h)(E_3, E_4, E_4)$	$a_3b_2 = 0$ $b_1(a_1 + a_3) = 0$
$(\bar{\nabla}h)(E_1, E_2, E_3)$	$E_1(b_1) + pa_1 + 1 - 2b_2r = E_2(a_1) + qb_1 - 1 = a_1^2 - b_1^2 - b_2^2$ $E_1(b_2) + 2b_1r = l(a_1 - a_3) + qb_2 = 0$
$(\bar{\nabla}h)(E_1, E_2, E_4)$	$-E_1(b_1) + pa_3 + 1 + 2b_2r = E_2(a_3) - qb_1 - 1 = a_3^2 - b_1^2 - b_2^2$ $E_1(b_2) + 2b_1r = l(a_1 - a_3) + qb_2 = 0$

If $b_2 \neq 0$ it follows that $a_1 = 0$ and $a_3 = 0$. Consequently, we get $q = 0$ and $b_1^2 + b_2^2 = 1$. Finally, $E_1(b_1) - 2b_2r = -2$ (on one hand) and $-E_1(b_1) + 2b_2r = -2$ (on the other hand). Hence we get a contradiction. □

So, from now on we will take $b_2 = 0$.

Let us develop all situations for the Codazzi quation. Due to the totally symmetry of $\bar{\nabla}h$ one has 30 non-trivial possibilities. Nevertheless, some of the equations are consequences of the other ones, or they are automatically satisfied. For example we have:

Lemma 3. *Under the same hypothesis as for Proposition 2, the Codazzi equations are automatically satisfied for the triple (ξ, ξ, Z) for $Z \in E(M)$, as well as for the triple (ξ, ξ, X) for $X \in H(M)$.*

Therefore, we emphasize only the non-trivial conditions we get from the Codazzi equations.

We remark, in the Table 2, two types of conditions, namely algebraic equations and differential equations, respectively.

Lemma 4. *Under the above conditions, the coefficient b_1 vanishes.*

Proof. Contrary, if $b_1 \neq 0$ in a point, it is different from zero on an open neighborhood. Looking to line L1 in Table 2, we deduce that $a_1 + a_3 = 0$.

Adding, side by side, the differential equations we have in lines L2 and L3 in Table 2, we get

$$p(a_1 + a_3) + 2 = E_2(a_1 + a_3) - 2 = a_1^2 + a_3^2 - 2b_1^2.$$

We obtain a contradiction and therefore b_1 vanishes. □

From now on we will distinguish two cases: **Case 1.** $a_1 = a_3$ and **Case 2.** $a_1 \neq a_3$.

We will obtain some more equations in each of the two cases and then we completely solve our problem in Sections 3.2 and 3.3.

Case 1. For the sake of simplicity, we make the following notation $a_1 = a_3 := A$

From line L2 in Table 2 we get

$$pA + 1 = E_2(A) - 1 = A^2, \tag{6}$$

which implies that A cannot vanish. Developing all the equations in the Table 2, we obtain:

$$p = A - \frac{1}{A}, \quad a = q = c = 0, \quad \omega = 1,$$

$$E_1(A) = 0, \quad E_2(A) = 1 + A^2, \quad E_3(A) = 0, \quad E_4(A) = 0, \quad \xi(A) = 0.$$

With respect to the orthonormal basis $\{E_1, E_2, E_3, E_4, \xi\}$ in $T(M)$ we may express the two shape operators as follows

$$S_{E_5} = \begin{pmatrix} 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \; S_{E_6} = \begin{pmatrix} 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

Straightforward computation shows that Ricci Equations (**ER**) imply new relations between the functions we have considered:

$$E_1(\theta) = \xi(r). \tag{8}$$

$$E_2(\theta) = \xi(l). \tag{9}$$

$$E_3(\theta) = \xi(\alpha) - \beta\,\theta. \tag{10}$$

$$E_4(\theta) = \xi(\beta) + \alpha\,\theta. \tag{11}$$

$$E_1(l) - E_2(r) + rp + 2\theta = 0. \tag{12}$$

$$E_1(\alpha) - E_3(r) - \beta r = 0. \tag{13}$$

$$E_1(\beta) - E_4(r) + \alpha r = 0. \tag{14}$$

$$E_2(\alpha) - E_3(l) - \beta l - \alpha A = 0. \tag{15}$$

$$E_2(\beta) - E_4(l) + \alpha l - \beta A = 0. \tag{16}$$

$$E_3(\beta) - E_4(\alpha) + \alpha^2 + \beta^2 + A^2 + 1 = 0. \tag{17}$$

Moreover, the normal curvature is completely determined by the following component

$$R^{\perp}_{E_3\,E_4}E_5 = -(A^2+1)E_6. \tag{18}$$

Table 2. Gauss equations and symmetries of $\bar{\nabla}h$.

	The Symmetry We Use	**The Result We Get**
L1	$(\bar{\nabla}h)(E_3,E_3,E_4)$	$b_1(a_1+a_3)=0$
L2	$(\bar{\nabla}h)(E_1,E_2,E_3)$	$E_1(b_1)+pa_1+1=E_2(a_1)+qb_1-1=a_1^2-b_1^2$ $2b_1r=l(a_1-a_3)=0$
L3	$(\bar{\nabla}h)(E_1,E_2,E_4)$	$-E_1(b_1)+pa_3+1=E_2(a_3)-qb_1-1=a_3^2-b_1^2$
L4	$(\bar{\nabla}h)(E_1,E_1,E_3)$	$E_1(a_1)-pb_1=-2a_1b_1$ $r(a_1-a_3)=0$
L5	$(\bar{\nabla}h)(E_1,E_1,E_4)$	$E_1(a_3)+pb_1=2a_3b_1$
L6	$(\bar{\nabla}h)(E_1,E_3,E_3)$	$E_3(a_1)-ab_1=0$ $\alpha(a_1-a_3)=0$
L7	$(\bar{\nabla}h)(E_1,E_4,E_4)$	$E_4(a_3)+cb_1=0$ $\beta(a_1-a_3)=0$
L8	$(\bar{\nabla}h)(E_2,E_2,E_3)$	$E_2(b_1)-qa_1=2a_1b_1$ $2lb_1=0$
L9	$(\bar{\nabla}h)(E_2,E_2,E_4)$	$E_2(b_1)+qa_3=2a_3b_1$
L10	$(\bar{\nabla}h)(E_2,E_3,E_3)$	$E_3(b_1)+aa_1=0$ $2\alpha b_1=0$
L11	$(\bar{\nabla}h)(E_2,E_4,E_4)$	$E_4(b_1)-ca_3=0$ $2\beta b_1=0$
L12	$(\bar{\nabla}h)(E_1,E_3,\xi)$	$\xi(a_1)=(\omega-1)b_1$ $\theta(a_1-a_3)=0$
L13	$(\bar{\nabla}h)(E_1,E_4,\xi)$	$\xi(a_3)=(1-\omega)b_1$
L14	$(\bar{\nabla}h)(E_2,E_3,\xi)$	$\xi(b_1)=(1-\omega)a_1$ $2\theta b_1=0$
L15	$(\bar{\nabla}h)(E_2,E_4,\xi)$	$\xi(b_1)=(\omega-1)a_3$
L16	$(\bar{\nabla}h)(E_1,E_3,E_4)$	$E_3(a_3)+ab_1=0$ $E_4(a_1)-cb_1=0$
L17	$(\bar{\nabla}h)(E_2,E_3,E_4)$	$E_3(b_1)-aa_3=0$ $E_4(b_1)+ca_1=0$

Case 2. As $a_1 \neq a_3$, we immediately obtain $\omega = 1$, $p = a_1 + a_3$ and a, c, q, l, r, θ, α and β vanish. Moreover, we should have $a_1a_3 = -1$. Again, for the sake of simplicity we denote $a_1 = A$ and hence $a_3 = -\frac{1}{A}$ and $p = A - \frac{1}{A}$. Obviously, A cannot vanish and satisfies the following partial differential equations

$$E_1(A)=0,\quad E_2(A)=1+A^2,\quad E_3(A)=0,\quad E_4(A)=0,\quad \xi(A)=0.$$

Similarly to the Case 1, we may express the two shape operators as follows

$$S_{E_5}=\begin{pmatrix} 0&0&A&0&0\\ 0&0&0&0&0\\ A&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix},\ S_{E_6}=\begin{pmatrix} 0&0&0&-1/A&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1/A&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}. \tag{19}$$

Straightforward computation shows that the normal connection is flat, so the normal bundle is parallel. Additionally, Ricci Equations (**ER**) imply no new relations.

3.2. The Case 1: M is Congruent to $\mathbb{S}^3 \times_f \mathbb{S}^2$

In this subsection we study in detail the case $a_1 = a_3$ and prove that then the contact nearly totally geodesic CR-submanifold M is congruent to $\mathbb{S}^3 \times_f \mathbb{S}^2$ and we determine the explicit immersion.

Let us consider the following distributions on M: $\mathcal{D} = \text{span}\{E_1, E_2, \xi\}$ and $\mathcal{D}^\perp = \text{span}\{E_3, E_4\}$. Let P and Q be the orthogonal projections from $T(M)$ to $\mathcal{D}$, respectively to $\mathcal{D}^\perp$. From Proposition 2 we write the expression of the Levi-Civita connection on M for the case $a_1 = a_3 = A$:

$$
\begin{array}{lll}
\nabla_{E_1} E_1 = pE_2, & \nabla_{E_1} E_2 = -pE_1 - \xi, & \\
\nabla_{E_1} E_3 = rE_4, & \nabla_{E_1} E_4 = -rE_3, & \\
\nabla_{E_1} \xi = E_2, & & \\
\nabla_{E_2} E_1 = \xi, & \nabla_{E_2} E_2 = 0, & \nabla_{\xi} E_1 = E_2, \\
\nabla_{E_2} E_3 = lE_4, & \nabla_{E_2} E_4 = -lE_3, & \nabla_{\xi} E_2 = -E_1, \\
\nabla_{E_2} \xi = -E_1, & & \nabla_{\xi} E_3 = \theta E_4, \\
\nabla_{E_3} E_1 = 0, & \nabla_{E_3} E_2 = -AE_3, & \nabla_{\xi} E_4 = -\theta E_3, \\
\nabla_{E_3} E_3 = AE_2 + \alpha E_4, & \nabla_{E_3} E_4 = -\alpha E_3, & \nabla_{\xi} \xi = 0. \\
\nabla_{E_3} \xi = 0, & & \\
\nabla_{E_4} E_1 = 0, & \nabla_{E_4} E_2 = -AE_4, & \\
\nabla_{E_4} E_3 = \beta E_4, & \nabla_{E_4} E_4 = AE_2 - \beta E_3, & \\
\nabla_{E_4} \xi = 0, & &
\end{array}
\tag{20}
$$

As a consequence, we find that the following relations are true:

(a) $\mathsf{P}\nabla_Z W = g(Z, W)\mathsf{X}_0$, for all Z, W in $\mathcal{D}^\perp$;
(b) $\mathsf{P}\nabla_Z \mathsf{X}_0 = 0$, for all Z in $\mathcal{D}^\perp$;
(c) $\mathsf{Q}\nabla_X Y = 0$, for all X, Y in $\mathcal{D}$;

where $\mathsf{X}_0 = AE_2$. The statements (a) and (c) imply that $\mathcal{D}$ and $\mathcal{D}^\perp$ are both involutive. The statements (a) and (b) mean that the maximal integral manifolds of $\mathcal{D}^\perp$ are extrinsic spheres. Finally, the statement (c) says that the integral manifolds of $\mathcal{D}$ are totally geodesic.

Now, applying a famous result of Hiepko ([18], p. 213): *Let (M, g) be a (pseudo-)Riemannian manifold endowed with a pair (L, N) of non-degenerate foliations. This determines a local warped product structure with N as a normal factor, if and only if, the foliations are orthogonal, L is geodesic, and N is spherical,* we have:

For every point $p \in M$, there exists an isometry Φ from a warped product $N_1 \times_{\mathsf{f}} N_2$ to a neighborhood of p in M with the property

- $\Phi(N_1 \times \{p_2\})$ is an integral manifold for $\mathcal{D}$ for every $p_2 \in N_2$;
- $\Phi(\{p_1\} \times N_2)$ is an integral manifold for $\mathcal{D}^\perp$ for every $p_1 \in N_1$;

where $\mathsf{f} : N_1 \to (0, \infty)$ is the warping function on $N_1 \times_{\mathsf{f}} N_2$.

In order to find the warping function on M, let us consider the following vector field in $\mathcal{D}$: $\bar{E}_1 = \frac{A}{1+A^2} E_1 + \frac{1-A^2}{2(1+A^2)} \xi$. One can immediately prove that

$$[\bar{E}_1, E_2] = [E_2, \xi] = [\bar{E}_1, \xi] = 0.$$

Thus, we choose local coordinates x, y, z on M (in fact on $\Phi(N_1)$) such that $\bar{E}_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y}$ and $\xi = \frac{\partial}{\partial z}$. Using (6), after a possible translation in y-coordinate, we obtain

$$A = \tan y. \tag{21}$$

Taking $y \in (0, \pi/2)$, we obtain $p = -2\cot 2y$ and $E_1 = \frac{2}{\sin 2y}\frac{\partial}{\partial x} - \cot 2y \frac{\partial}{\partial z}$.

The restriction of the metric g to $\mathcal{D}$ can be expressed in terms of the coordinates x, y and z as follows:

$$g|_{\mathcal{D}} = \frac{1}{4}dx^2 + dy^2 + dz^2 + \cos 2y\, dxdz.$$

Moreover, from ([13], Theorem 3.2), we know that $S_{\varphi Z}X = \left[\eta(X) - (\varphi X)(\log f)\right]Z$, for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^{\perp}$. Using the expression (7) for the shape operator, we get that $E_2(\log(\Phi \circ \mathrm{f})) = -A$ and combining it with (21) we find

$$f \equiv \Phi \circ \mathrm{f} = \cos y,$$

which is the warping function on M.

Since $\Phi(N_1 \times \{p_2\})$ is totally geodesic in M and M is nearly totally geodesic in $\mathbb{S}^7$, it follows that $N_1 \times \{p_2\}$ is (isometrically) immersed in $\mathbb{S}^7$ as a totally geodesic submanifold. With a similar argument, $\{p_1\} \times N_2$ is immersed in $\mathbb{S}^7$ as a totally umbilical submanifold. Hence $\Phi(N_1 \times \{p_2\})$ can be considered to be a (portion of) $\mathbb{S}^3(1)$. Additionally, $\Phi(\{p_1\} \times N_2)$, being totally umbilical in $\mathbb{S}^7$, can be taken as a (portion of) 2-sphere of a certain radius. Looking back to the expression (20) of the covariant derivative ∇, we conclude that the mean curvature vector field of $\Phi(\{p_1\} \times N_2)$ in $\mathbb{S}^7$ is AE_2. Thus, the curvature of the 2-sphere above is $1 + A^2$, and hence its radius is $\frac{1}{\sqrt{1+A^2}} = \cos y$. Consequently, we will consider $M = \mathbb{S}^3 \times_f \mathbb{S}^2$ with the warping function $f = \cos y$, $y \in (0, \frac{\pi}{2})$.

Proposition 3. *Under the conditions stated for the case 1, it follows that M is locally congruent to a contact CR warped product $\mathbb{S}^3 \times_f \mathbb{S}^2$.*

Remark 2. *Defining the 1-form Ω on M by $\Omega(U) = -g(AE_2, U)$, we conclude it is closed and therefore, locally, there exists a smooth function on M such that its differential is equal to Ω. We may notice that this function is constant on the leaves of $\mathcal{D}^{\perp}$, that is, on $\Phi(\{p_1\} \times N_2)$. This function is nothing but $\log f$. It can be easily checked that $AE_2 = -\operatorname{grad} \log f$.*

Now, let us determine the metric. Choosing isothermal coordinates u, v on $\Phi(N_2)$, we have

$$g_{uu} = g_{vv} = \cos^2 y\, T^2(u, v), \quad g_{uv} = 0,$$

where T is a smooth positive function on M, depending on u and v. Here we made the following notations: $g_{uu} := g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)$ and similar for g_{vv} and g_{uv}.

As both $\{E_3, E_4\}$ and $\left\{\frac{1}{T\cos y}\frac{\partial}{\partial u}, \frac{1}{T\cos y}\frac{\partial}{\partial v}\right\}$ are orthonormal bases in $E(M)$, then one is obtained from the other by a (not necessary positively oriented) rotation.

Finally, due to the orthogonality of the two distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$, we have

$$g_{xu} = g_{yu} = g_{zu} = 0, \quad g_{xv} = g_{yv} = g_{zv} = 0.$$

We adopted similar notations as before for g_{xu}, g_{yu} and so on.

Hence the metric g is completely determined.

Now, let us find a more appropriate basis. Taking an arbitrary unit vector Z in $E(M)$, it can be expressed as

$$Z = \cos s E_3 + \sin s E_4, \quad \text{where } s \in C^{\infty}(M).$$

It follows that

$$h(E_1, Z) = A(\cos sE_5 + \sin sE_6) = A\varphi Z \quad \text{and} \quad h(E_2, Z) = 0.$$

Moreover, as the function $\tilde{g}(h(E_1, Z), h(E_1, Z))$ is equal to A^2, it is independent of Z. Hence, E_3 is not uniquely defined and it could be replaced by any other unit vector Z in $E(M)$.

So, because of this freedom, we choose

$$E_3 = \frac{1}{T(u,v)\cos y}\frac{\partial}{\partial u}, \quad E_4 = \frac{1}{T(u,v)\cos y}\frac{\partial}{\partial v}. \tag{22}$$

Now, we need to do some additional computations. In what follows, by subscripts we mean the partial derivatives; for example, $T_u = \frac{\partial T}{\partial u}$. Using (22) and (20), we conclude

- on one hand $[E_3, E_4] = \frac{1}{T^2 \cos y}(T_v E_3 - T_u E_4)$,
- and on the other hand $[E_3, E_4] = -\alpha E_3 - \beta E_4$.

Thus, $\alpha = -\frac{T_v}{T^2 \cos y}$ and $\beta = \frac{T_u}{T^2 \cos y}$.

Next we have to calculate the Levi-Civita connection of the metric g in terms of the coordinates x, y, z, u and v and then to compare the results with the relations (20). Being a straightforward computation, we present only one situation, namely we compute $\nabla_{E_2} E_3$:

$$\nabla_{E_2} E_3 = \nabla_{\partial_y}\left(\frac{1}{T\cos y}\partial_u\right) = \frac{1}{T}\frac{\sin y}{\cos^2 y}\partial_u + \frac{1}{T\cos y}\partial_y(\log(\cos y))\partial_u = 0.$$

Comparing with $\nabla_{E_2} E_3 = rE_4 = \frac{r}{T\cos y}\partial_v$, we get

$$r = 0. \tag{23}$$

In a similar way we find

$$l = 0 \quad \text{and} \quad \theta = 0. \tag{24}$$

Finally, the Ricci Equations (8)–(16) are automatically fulfilled, while the Equation (17) leads to the following partial differential equation for T

$$\frac{T_{uu} + T_{vv}}{T^3} - \frac{T_u^2 + T_v^2}{T^4} + 1 = 0. \tag{25}$$

Further, our aim is to find the isometric immersion $F : M^5 \longrightarrow \mathbb{S}^7$. Let $\iota : \mathbb{S}^7 \longrightarrow \mathbb{E}^8 = \mathbb{C}^4$ be the canonical inclusion of the 7-dimensional unit sphere in the 4-dimensional complex space. Denoting by $\langle\ ,\ \rangle$ the scalar product on $\mathbb{E}^8$ and by $\overset{\circ}{\nabla}$ the corresponding flat connection, we have

$$\tilde{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \langle X, Y\rangle \mathbf{p},$$

for all X and Y tangent to $\mathbb{S}^7(1)$, where $\mathbf{p}$ denotes the position vector of a point of the sphere. Using the Gauss formula $(\mathbf{G})$ we have

$$\overset{\circ}{\nabla}_X F_* Y = F_* \nabla_X Y + h(X, Y) - \langle X, Y\rangle F,$$

for all X and Y tangent to M.

For example, if we put $X = E_2 = \partial_y$ and $Y = \xi = \partial_z$ we get

$$F_{yz} = -F_* E_1 = -\frac{2}{\sin 2y}F_x + \cot 2y F_z. \tag{26}$$

We obtain that F also satisfies, simultaneously, the following partial differential equations:

$$F_y = \frac{2}{\sin 2y} F_{xz} - \cot 2y F_{zz}, \tag{27}$$

$$F_{zz} + F = 0, \tag{28}$$

$$\frac{1}{T\cos y}\left(\frac{2}{\sin 2y} F_{ux} - \cot 2y F_{uz}\right) = \tan y E_5, \tag{29}$$

$$F_{uy} = -\tan y F_u, \tag{30}$$

$$F_{uz} = T\cos y E_5, \tag{31}$$

$$F_{vy} = -\tan y F_v, \tag{32}$$

$$F_{yy} + F = 0, \tag{33}$$

$$2F_{xy} - 2\cot 2y F_x + \frac{1}{\sin 2y} F_z = 0, \tag{34}$$

$$\frac{1}{T\cos y}\left(\frac{2}{\sin 2y} F_{vx} - \cot 2y F_{vz}\right) = \tan y E_6, \tag{35}$$

$$F_{vz} = T\cos y E_6, \tag{36}$$

$$4F_{xx} + F = 0, \tag{37}$$

$$F_{uv} = \frac{T_u}{T} F_v + \frac{T_v}{T} F_u, \tag{38}$$

$$F_{uu} = \frac{T_u}{T} F_u - \frac{T_v}{T} F_v + T^2 \sin y \cos y F_y - T^2 \cos^2 y F, \tag{39}$$

$$F_{vv} = \frac{T_v}{T} F_v - \frac{T_u}{T} F_u + T^2 \sin y \cos y F_y - T^2 \cos^2 y F. \tag{40}$$

Remark 3. *Observe that Equation (26) also follows from (27) and (28). Moreover, using (37) and (28), we conclude that Equations (27) and (34) are equivalent. So, not all the previous PDEs are independent.*

Then, combining (29) with (31) and (35) with (36), we get

$$(2F_x - F_z)_u = 0, \quad (2F_x - F_z)_v = 0,$$

respectively, that is $2F_x - F_z$ depends neither on u, nor on v.

Considering Equations (28), (33) and (37), we deduce

$$\begin{aligned} &F(x,y,z,u,v) \\ &= \cos z \cos y \cos \tfrac{x}{2} u_1 + \cos z \cos y \sin \tfrac{x}{2} v_1 + \cos z \sin y \cos \tfrac{x}{2} u_2 + \cos z \sin y \sin \tfrac{x}{2} v_2 \\ &+ \sin z \cos y \cos \tfrac{x}{2} u_3 + \sin z \cos y \sin \tfrac{x}{2} v_3 + \sin z \sin y \cos \tfrac{x}{2} u_4 + \sin z \sin y \sin \tfrac{x}{2} v_4, \end{aligned} \tag{41}$$

where $u_1, \ldots, u_4, v_1, \ldots, v_4$ are vectors in $\mathbb{R}^8$ which do not depend on x, y and z, but they do depend on u and v.

Using (41) we compute

$$\begin{aligned} 2F_x - F_z = {}& \cos y \sin(z - \tfrac{x}{2})(u_1 + v_3) - \cos y \cos(z - \tfrac{x}{2})(u_3 - v_1) \\ & - \sin y \sin(z - \tfrac{x}{2})(u_2 + v_4) - \sin y \cos(z - \tfrac{x}{2})(u_4 - v_2). \end{aligned} \tag{42}$$

Remark 4. *Since vectors $u_1, \ldots, u_4, v_1, \ldots, v_4$ do not depend on x, y and z and $2F_x - F_z$ does not depend on u and v, using Equation (42) we conclude that $u_1 + v_3$, $u_2 + v_4$, $u_3 - v_1$ and $u_4 - v_2$ are constant vectors in $\mathbb{R}^8$.*

Equations (30) and (32) imply

$$\left(\frac{F_u}{\cos y}\right)_y = 0, \quad \left(\frac{F_v}{\cos y}\right)_y = 0, \tag{43}$$

respectively. Using (41) and the first equation in (43), we get

$$\begin{aligned} 0 = \frac{\partial}{\partial y}\Big\{ &\Big[\cos z \cos\frac{x}{2}\partial_u u_1 + \cos z \sin\frac{x}{2}\partial_u v_1 + \sin z \cos\frac{x}{2}\partial_u u_3 + \sin z \sin\frac{x}{2}\partial_u v_3\Big] \\ + \tan y &\Big[\cos z \cos\frac{x}{2}\partial_u u_2 + \cos z \sin\frac{x}{2}\partial_u v_2 + \sin z \cos\frac{x}{2}\partial_u u_4 + \sin z \sin\frac{x}{2}\partial_u v_4\Big]\Big\}. \end{aligned} \tag{44}$$

Since $u_1, \ldots, u_4, v_1, \ldots, v_4$ do not depend on y, from (44) we conclude $\partial_u u_2 = 0$, $\partial_u v_2 = 0$, $\partial_u u_4 = 0$, $\partial_u v_4 = 0$. In the same manner, using the second equation in (43), we obtain $\partial_v u_2 = 0$, $\partial_v v_2 = 0$, $\partial_v u_4 = 0$, $\partial_v v_4 = 0$. Hence, u_2, v_2, u_4, v_4 are constant vectors in $\mathbb{R}^8$.

Using Equations (27) and (28), we get

$$2F_{xz} + \cos 2yF - \sin 2yF_y = 0.$$

Combining now with (41) we obtain

$$\begin{aligned} 0 &= \cos z \cos\frac{x}{2}\left(\cos y u_1 - \sin y u_2 + \cos y v_3 + \sin y v_4\right) \\ &+ \cos z \sin\frac{x}{2}\left(-\cos y u_3 - \sin y u_4 + \cos y v_1 - \sin y v_2\right) \\ &+ \sin z \cos\frac{x}{2}\left(-\cos y v_1 - \sin y v_2 + \cos y u_3 - \sin y u_4\right) \\ &+ \sin z \sin\frac{x}{2}\left(\cos y u_1 + \sin y u_2 + \cos y v_3 - \sin y v_4\right). \end{aligned} \tag{45}$$

Since (45) is satisfied for all x, y, z, it follows

$$u_1 + v_3 = 0, \quad u_2 - v_4 = 0, \quad u_3 - v_1 = 0, \quad u_4 + v_2 = 0. \tag{46}$$

Replacing (46) in (41) yields

$$\begin{aligned} F(x,y,z,u,v) = &\cos y \cos\left(z + \tfrac{x}{2}\right) u_1 + \sin y \cos\left(z - \tfrac{x}{2}\right) u_2 \\ &+ \cos y \sin\left(z + \tfrac{x}{2}\right) u_3 + \sin y \sin\left(z - \tfrac{x}{2}\right) u_4, \end{aligned} \tag{47}$$

where u_2, u_4 are constant vectors in $\mathbb{R}^8$ and u_1, u_3 may depend on u and v. Now, using (39), we compute

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial u^2} = \dfrac{T_u}{T}\dfrac{\partial u_1}{\partial u} - \dfrac{T_v}{T}\dfrac{\partial u_1}{\partial v} - T^2 u_1, \\ \dfrac{\partial^2 u_3}{\partial u^2} = \dfrac{T_u}{T}\dfrac{\partial u_3}{\partial u} - \dfrac{T_v}{T}\dfrac{\partial u_3}{\partial v} - T^2 u_3. \end{cases}$$

Using (38), it follows

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial v^2} = \dfrac{T_v}{T}\dfrac{\partial u_1}{\partial v} - \dfrac{T_u}{T}\dfrac{\partial u_1}{\partial u} - T^2 u_1, \\ \dfrac{\partial^2 u_3}{\partial v^2} = \dfrac{T_v}{T}\dfrac{\partial u_3}{\partial v} - \dfrac{T_u}{T}\dfrac{\partial u_3}{\partial u} - T^2 u_3 \end{cases}$$

and using (40), we get

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial u \partial v} = \dfrac{T_u}{T}\dfrac{\partial u_1}{\partial v} + \dfrac{T_v}{T}\dfrac{\partial u_1}{\partial u}, \\ \dfrac{\partial^2 u_3}{\partial u \partial v} = \dfrac{T_u}{T}\dfrac{\partial u_3}{\partial v} + \dfrac{T_v}{T}\dfrac{\partial u_3}{\partial u}. \end{cases}$$

Therefore, we proceed solving the system

$$P_{uu} = \frac{T_u}{T}P_u - \frac{T_v}{T}P_v - T^2 P, \tag{48}$$

$$P_{vv} = \frac{T_v}{T}P_v - \frac{T_u}{T}P_u - T^2 P, \tag{49}$$

$$P_{uv} = \frac{T_u}{T}P_v + \frac{T_v}{T}P_u,, \tag{50}$$

having in mind $P \in \{u_1, u_3\}$.

Considering the two vector-valued functions $U(u,v) = \frac{1}{T^2}P_u$ and $V(u,v) = \frac{1}{T^2}P_v$ and using (48)–(50), we conclude that $U(u,v)$ and $V(u,v)$ satisfy the Cauchy-Riemann equations for each component of the vector valued function $U(u,v) + iV(u,v)$. This means that $U(u,v) + iV(u,v)$ depends only on $w = u + iv$ and not on $\bar{w} = u - iv$. Therefore, the function

$$U + iV = \frac{1}{T^2}(P_u + iP_v) = \frac{2}{T^2}\partial_{\bar{w}} P \tag{51}$$

is holomorphic. Denoting it by

$$\zeta(w) = \zeta(u + iv) = a(u,v) + ib(u,v),\ a, b \in \mathbb{R}^8, \tag{52}$$

where the functions a and b satisfy the Cauchy-Riemann equations, we conclude that the function P has to satisfy the equation

$$\partial_{\bar{w}} P = \frac{T^2}{2}\zeta(w). \tag{53}$$

Up to now, we have kept the conformal factor T in the general form, without thinking at any possible concrete expression. Our motivation has been a possibility to use this technique for solving another problem of the same type.

Recall that $N_2 = \mathbb{S}^2(1)$. There are several ways to consider isothermal coordinates u and v on the 2-sphere, that is, to write the metric as $T(u,v)^2(du^2 + dv^2)$. Recall two of them:

- $T(u,v) = \frac{2}{1+u^2+v^2}$, associated to the parametrization

$$(u,v) \mapsto \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right) \tag{54}$$

  obtained from the stereographic projection;
- $T(u,v) = \frac{1}{\cosh v}$, associated to the parametrization

$$(u,v) \mapsto \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, \tanh v\right) \tag{55}$$

  obtained as a surface of revolution.

After setting $T = \frac{2}{1+u^2+v^2}$ and solving the Equation (53), we obtain

$$P(w, \bar{w}) = -\frac{2\zeta(w)}{w(1 + w\bar{w})} + a_0(w) + ib_0(w),$$

where a_0 and b_0 also satisfy the Cauchy-Riemann equations. Since $P(w, \bar{w}) \in \mathbb{R}^8$, we conclude

$$P(u,v) = -\frac{2(ua(u,v) + vb(u,v))}{(u^2+v^2)(1+u^2+v^2)} + a_0(u,v), a_0 \in \mathbb{R}^8, \tag{56}$$

$$b_0(u,v) = \frac{2(ub(u,v) - va(u,v))}{(u^2+v^2)(1+u^2+v^2)}. \tag{57}$$

On the other hand, since $\partial_w \partial_{\bar{w}} P = \frac{1}{4}(P_{uu} + P_{vv}) \in \mathbb{R}^8$, using (53) and the information that a and b satisfy the Cauchy-Riemann equations, we compute

$$\frac{\partial a(u,v)}{\partial v} = \frac{2(va(u,v) - ub(u,v))}{1+u^2+v^2}. \tag{58}$$

Consequently, using (57), we conclude

$$b_0(u,v) = -\frac{1}{u^2+v^2}\frac{\partial a(u,v)}{\partial v}. \tag{59}$$

Let us express $ub - va$ from the Equation (57):

$$ub(u,v) - va(u,v) = \frac{b_0(u,v)}{2}(u^2+v^2)(1+u^2+v^2). \tag{60}$$

For simplicity of notation, we write b_{0v} instead of $\frac{\partial b_0(u,v)}{\partial v}$, for example.

Taking the partial derivatives of (60), with respect to u and with respect to v, multiplying the obtained equations respectively by u and v and adding them, we get

$$a_v = -\frac{ub_{0u} + vb_{0v}}{2}(1+u^2+v^2) - \frac{b_0}{2}(1+3u^2+3v^2). \tag{61}$$

Moreover, after computing b_{0u} and b_{0v} (using (59)) and replacing it in (61), together with b_0 from (59), we get

$$ua_{uv} + va_{vv} = a_v. \tag{62}$$

Further, taking the partial derivatives of (60), with respect to u and with respect to v, multiplying the obtained equations respectively by v and u and subtracting them, we get

$$vb + ua - (u^2+v^2)a_u = \frac{vb_{0u} - ub_{0v}}{2}(u^2+v^2)(1+u^2+v^2). \tag{63}$$

Using (63) and the partial derivatives of b_{0u} and b_{0v} (using (59)), we compute

$$\frac{2(ua+vb)}{(u^2+v^2)(1+u^2+v^2)} = \frac{2a_u}{1+u^2+v^2} + \frac{ua_{vv} - va_{uv}}{u^2+v^2}. \tag{64}$$

Taking the partial derivative of (62), with respect to v, we compute

$$ua_{uvv} + va_{vvv} = 0. \tag{65}$$

Having in mind that a is a harmonic function ($a_{uu} + a_{vv} = 0$) and taking the partial derivative of (62), with respect to u, we conclude

$$ua_{uuv} + va_{uvv} = 0. \tag{66}$$

Taking the partial derivatives of the equation $a_{uu} + a_{vv} = 0$ with respect to u and with respect to v, we get

$$\begin{aligned} a_{uuu} + a_{uvv} &= 0, \quad (67) \\ a_{uuv} + a_{vvv} &= 0.. \quad (68) \end{aligned}$$

Combining Equations (65) and (68), we obtain

$$ua_{uvv} - va_{uuv} = 0.. \quad (69)$$

Multiplying Equations (66) and (69) respectively by v and u and adding them, we get $a_{uvv} = 0$, which, together with (67), implies $a_{uuu} = 0$. From (66) and (65) we deduce $a_{uuv} = 0$ and $a_{vvv} = 0$. Since all the third order derivatives of the function $a(u, v)$ vanish, we set

$$a(u, v) = c_0 + c_1 u + c_2 v + l_1(u^2 - v^2) + 2l_2 uv, \quad (70)$$

where c_0, c_1, c_2, l_1, l_2 are constant vectors in $\mathbb{R}^8$. Using (62) and (70), it follows $c_2 = 0$ and therefore

$$a(u, v) = c_0 + c_1 u + l_1(u^2 - v^2) + 2l_2 uv. \quad (71)$$

From (59) we compute

$$b_0 = \frac{2(l_1 v - l_2 u)}{u^2 + v^2}. \quad (72)$$

The Cauchy-Riemann equations for a_0 and b_0, using (72), are

$$\begin{aligned} a_{0u} &= b_{0v} = \frac{2l_1}{u^2 + v^2} - \frac{4v(l_1 v - l_2 u)}{(u^2 + v^2)^2}, \quad (73) \\ -a_{0v} &= b_{0u} = -\frac{2l_2}{u^2 + v^2} - \frac{4u(l_1 v - l_2 u)}{(u^2 + v^2)^2}, \quad (74) \end{aligned}$$

Using (73) and (74), we compute

$$a_0 = \frac{-2(l_1 u + l_2 v)}{u^2 + v^2} + l_0, \ l_0 \in \mathbb{R}^8. \quad (75)$$

Using (60), (71) and (72), we get

$$l_2 u^3 - 2l_1 u^2 v - l_2 uv^2 - c_1 uv + (b + l_2)u - (c_0 + l_1)v = 0. \quad (76)$$

Since $a(u, v)$ and $b(u, v)$ satisfy the Cauchy-Riemann equations, from $b_u = -a_v$, using (71), we conclude

$$b(u, v) = 2l_1 uv - l_2 u^2 + \phi(v)$$

and from $b_v = a_u$ we get $\phi(v) = d_0 + c_1 v + l_2 v^2$, $d_0 \in \mathbb{R}^8$, which implies

$$b(u, v) = d_0 + c_1 v + 2l_1 uv + l_2(v^2 - u^2). \quad (77)$$

Using (76) and (77) we get $d_0 = -l_2$ and $c_0 = -l_1$. Consequently, (71) and (77) become

$$\begin{aligned} a(u, v) &= l_1(-1 + u^2 - v^2) + c_1 u + 2l_2 uv, \quad (78) \\ b(u, v) &= l_2(-1 - u^2 + v^2) + c_1 v + 2l_1 uv. \quad (79) \end{aligned}$$

Remark 5. *Equation (64) is also satisfied.*

Replacing (78), (79), (75) in (56), we compute

$$P(u,v) = l_0 - 2\frac{c_1 + 2l_1 u + 2l_2 v}{1 + u^2 + v^2} = l_0 - (c_1 + 2l_1 u + 2l_2 v)T. \tag{80}$$

From (48) and (80) we get $l_0 = c_1$ and conclude

$$P(u,v) = c_1(1-T) - 2(l_1 u + l_2 v)T. \tag{81}$$

Recalling that $P \in \{u_1, u_3\}$, it follows that there exist six constant vectors ϵ_{10}, ϵ_{11}, ϵ_{12}, ϵ_{30}, ϵ_{31}, $\epsilon_{32} \in \mathbb{R}^8$ such that

$$u_1 = (1-T)\epsilon_{10} + T(u\epsilon_{11} + v\epsilon_{12}), \tag{82}$$
$$u_3 = (1-T)\epsilon_{30} + T(u\epsilon_{31} + v\epsilon_{32}). \tag{83}$$

Using (47), (82), (83), we compute

$$\begin{aligned} F(x,y,z,u,v) &= (1-T)\cos y\left(\cos\left(z+\frac{x}{2}\right)\epsilon_{10} + \sin\left(z+\frac{x}{2}\right)\epsilon_{30}\right) \\ &+ Tu\cos y\left(\cos\left(z+\frac{x}{2}\right)\epsilon_{11} + \sin\left(z+\frac{x}{2}\right)\epsilon_{31}\right) \\ &+ Tv\cos y\left(\cos\left(z+\frac{x}{2}\right)\epsilon_{12} + \sin\left(z+\frac{x}{2}\right)\epsilon_{32}\right) \\ &+ \sin y\left(\cos\left(z-\frac{x}{2}\right)u_2 + \sin\left(z-\frac{x}{2}\right)u_4\right). \end{aligned} \tag{84}$$

Further, set p_0 to be the initial point on M corresponding to $x = 0$, $y = \frac{\pi}{4}$, $z = 0$, $u = 0$, $v = 0$. Then $T(0,0) = 2$, $T_u(0,0) = 0$, $T_v(0,0) = 0$. Set also the following initial conditions satisfied by F and the first partial derivatives, meaning that we fix the initial point on $\mathbb{S}^7$ and the initial tangent space at p_0 as a subspace in $T_{F(p_0)}\mathbb{S}^7$:

$$\begin{aligned} &F(p_0) = \tfrac{1}{\sqrt{2}}(1,0,0,0,1,0,0,0) \\ &F_z(p_0) = \xi(p_0) = JF(p_0) = \tfrac{1}{\sqrt{2}}(0,1,0,0,0,1,0,0) \\ &F_x(p_0) = \tfrac{1}{2}E_1(p_0) = \tfrac{1}{2\sqrt{2}}(0,1,0,0,0,-1,0,0) \\ &F_y(p_0) = E_2(p_0) = JE_1(p_0) = \tfrac{1}{\sqrt{2}}(-1,0,0,0,1,0,0,0) \\ &F_u(p_0) = \sqrt{2}E_3(p_0) = \sqrt{2}(0,0,1,0,0,0,0,0) \\ &F_v(p_0) = \sqrt{2}E_4(p_0) = \sqrt{2}(0,0,0,0,0,0,1,0). \end{aligned}$$

Here J is the complex structure on $\mathbb{R}^8$ locally defined by

$$J(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4) = (-y_1,x_1,-y_2,x_2,-y_3,x_3,-y_4,x_4). \tag{85}$$

Using (84) we compute

$$\begin{aligned} &F(p_0) = \tfrac{\sqrt{2}}{2}(-\epsilon_{10} + u_2) && F_y(p_0) = \tfrac{\sqrt{2}}{2}(\epsilon_{10} + u_2) \\ &F_z(p_0) = \tfrac{\sqrt{2}}{2}(-\epsilon_{30} + u_4) && F_u(p_0) = \sqrt{2}\epsilon_{11} \\ &F_x(p_0) = \tfrac{\sqrt{2}}{4}(-\epsilon_{30} - u_4) && F_v(p_0) = \sqrt{2}\epsilon_{12}. \end{aligned}$$

Therefore we deduce $\epsilon_{10} = -e_1$, $u_2 = e_5$, $\epsilon_{30} = -e_2$, $u_4 = e_6$, $\epsilon_{11} = e_3$ and $\epsilon_{12} = e_7$.

Using (31) and (36), with $E_5 = JE_3$ and $E_6 = JE_4$, we compute $\epsilon_{31} = e_4$, $\epsilon_{32} = e_8$, where $e_1, \ldots, e_8$ is the canonical basis in $\mathbb{R}^8$.

Consequently, we conclude

$$\begin{aligned} F(x,y,z,u,v) = &-\cos y(1-T)\left(\cos\left(z+\tfrac{x}{2}\right),\sin\left(z+\tfrac{x}{2}\right),0,0,0,0,0,0\right) \\ &+\cos yTu\left(0,0,\cos\left(z+\tfrac{x}{2}\right),\sin\left(z+\tfrac{x}{2}\right),0,0,0,0,\right) \\ &+\cos yTv\left(0,0,0,0,0,0,\cos\left(z+\tfrac{x}{2}\right),\sin\left(z+\tfrac{x}{2}\right)\right) \\ &+\sin y\left(0,0,0,0,\cos\left(z-\tfrac{x}{2}\right),\sin\left(z-\tfrac{x}{2}\right),0,0\right). \end{aligned} \tag{86}$$

Remark 6. *Recall that we used T that corresponds to the stereographic projection. The parametrization (54) can be re-written as $(u,v) \mapsto (\mathbf{u} = Tu,\ \mathbf{v} = Tv,\ \mathbf{w} = 1-T) \in \mathbb{S}^2 \subset \mathbb{R}^3$.*

Besides, we can consider the parametrization $(x,y,z) \mapsto (\mathbf{x_1},\mathbf{y_1},\mathbf{x_2},\mathbf{y_2})$, where

$$\begin{cases} \mathbf{x_1} = \cos y\cos(z+\frac{x}{2}), & \mathbf{y_1} = \cos y\sin(z+\frac{x}{2}), \\ \mathbf{x_2} = \sin y\cos(z-\frac{x}{2}), & \mathbf{y_2} = \sin y\sin(z-\frac{x}{2}). \end{cases}$$

With these notations, the immersion $F : \mathbb{S}^3 \times_f \mathbb{S}^2 \to \mathbb{S}^7$ given in (86) becomes

$$F(\mathbf{x_1},\mathbf{y_1},\mathbf{x_2},\mathbf{y_2};\mathbf{u},\mathbf{v},\mathbf{w}) = (-\mathbf{x_1}\mathbf{w},-\mathbf{y_1}\mathbf{w},\mathbf{x_1}\mathbf{u},\mathbf{y_1}\mathbf{u},\mathbf{x_1}\mathbf{v},\mathbf{y_1}\mathbf{v},\mathbf{x_2},\mathbf{y_2}). \tag{87}$$

Remark that F is nothing but the immersion given in (Reference [5], Equation (3.8)) up to some permutation of coordinates and orientation of $\mathbb{S}^7$. The warping function is $f = \sqrt{\mathbf{x_1}^2+\mathbf{y_1}^2} = \cos y$.

Remark 7. *Let us see what happens in the case when we work with $T(u,v) = \frac{1}{\cosh v}$. From the Equation (50) we immediately obtain that $\frac{P_u}{T}$ does not depend on v. Hence, there exist functions $\mathsf{A} = \mathsf{A}(u)$ and $\mathsf{B} = \mathsf{B}(v)$ such that $P(u,v) = \frac{\mathsf{A}(u)}{\cosh v} + \mathsf{B}(v)$. Using (48) and (49) we find that $\mathsf{A}(u) = c + c_1\cos u + c_2\sin u$ and $\mathsf{B}(v) = -\frac{c}{\cosh v} + c_3\frac{\sinh v}{\cosh v}$, for some constants c, c_1, c_2 and c_3. Hence*

$$P(u,v) = c_1\frac{\cos u}{\cosh v} + c_2\frac{\sin u}{\cosh v} + c_3\frac{\sinh v}{\cosh v}.$$

As before, we recall that $P \in \{u_1,u_3\}$; it follows that there exist six constant vectors ϵ_{10}, ϵ_{11}, ϵ_{12}, ϵ_{30}, ϵ_{31}, $\epsilon_{32} \in \mathbb{R}^8$ such that

$$\left[\begin{array}{l} u_1 = \dfrac{\cos u}{\cosh v}\epsilon_{10} + \dfrac{\sin u}{\cosh v}\epsilon_{11} + \dfrac{\sinh v}{\cosh v}\epsilon_{12} \\ u_3 = \dfrac{\cos u}{\cosh v}\epsilon_{30} + \dfrac{\sin u}{\cosh v}\epsilon_{31} + \dfrac{\sinh v}{\cosh v}\epsilon_{32}. \end{array}\right. \tag{88}$$

We now replace u_1 and u_3 from (88) in (47) to obtain

$$F(\mathbf{x_1},\mathbf{y_1},\mathbf{x_2},\mathbf{y_2};\mathbf{u},\mathbf{v},\mathbf{w})$$

$$= \mathbf{x_1}\mathbf{u}\epsilon_{10} + \mathbf{x_1}\mathbf{v}\epsilon_{11} + \mathbf{x_1}\mathbf{w}\epsilon_{12} + \mathbf{x_2}u_2 + \mathbf{y_1}\mathbf{u}\epsilon_{30} + \mathbf{y_1}\mathbf{v}\epsilon_{31} + \mathbf{y_1}\mathbf{w}\epsilon_{32} + \mathbf{y_2}u_4,$$

where $\mathbf{x_1}$, $\mathbf{y_1}$, $\mathbf{x_2}$, $\mathbf{y_2}$ are as in the Remark 6 and $\mathbf{u} = \frac{\cos u}{\cosh v}$, $\mathbf{v} = \frac{\sin u}{\cosh v}$ and $\mathbf{w} = \tanh v$ are obtained using the isothermal coordinates u,v on the 2-sphere. We note that, for an appropriate choice of initial conditions, the immersion is the same as (87).

This confirms that the choice of isothermal coordinates on $\mathbb{S}^2$ is not so important (in our problem) to arrive at the result. However, the most important fact is the ability of the reader in solving (explicitly) the system of PDE equations.

3.3. The Case 2: M is Congruent to $\mathbb{S}^3 \times_{f_1} \mathbb{S}^1 \times_{f_2} \mathbb{S}^1$

In this subsection we continue the study of Case 2, introduced in Section 3.1. Recall that in this case: $a_1 = A$, $a_3 = -\frac{1}{A}$, $p = A - \frac{1}{A}$ and

$$E_1(A) = 0,\ E_2(A) = A^2 + 1,\ \xi(A) = 0,\ E_3(A) = 0,\ E_4(A) = 0. \tag{89}$$

The only non-zero components of the second fundamental form are

$$h(E_1, E_3) = AE_5, \quad h(E_1, E_4) = -\frac{1}{A}E_6, \quad h(E_3, \xi) = E_5, \quad h(E_4, \xi) = E_6.$$

In order to obtain the expression of the isometric immersion $F : M^5 \longrightarrow \mathbb{S}^7$ in local coordinates, we write the Lie brackets

$$[E_1, E_2] = \frac{1 - A^2}{A}\, E_1 - 2\xi, \quad [E_2, E_3] = A\, E_3, \quad [E_2, E_4] = -\frac{1}{A}E_4, \tag{90}$$

all other being zero. Considering the following vector fields:

$$\bar{E}_1 = \frac{A}{1 + A^2}E_1 + \frac{1 - A^2}{2(1 + A^2)}\xi, \quad \bar{E}_3 = \frac{1}{\sqrt{1 + A^2}}E_3, \quad \bar{E}_4 = \frac{A}{\sqrt{1 + A^2}}E_4,$$

we can easily prove that the Lie brackets of any two vectors from the set $\{\bar{E}_1, E_2, \bar{E}_3, \bar{E}_4, \xi\}$ vanish. Therefore, we can set (local) coordinates on M, call them x, y, z, u and v, such that

$$\bar{E}_1 = \frac{\partial}{\partial x},\ E_2 = \frac{\partial}{\partial y},\ \bar{E}_3 = \frac{\partial}{\partial u},\ \bar{E}_4 = \frac{\partial}{\partial v},\ \xi = \frac{\partial}{\partial z}.$$

Using (89) we conclude $A = \tan y$, with $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$ (after a translation in the y-coordinate) and consequently we compute

$$E_1 = \frac{2}{\sin 2y}\frac{\partial}{\partial x} - \cot 2y\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = \frac{1}{\cos y}\frac{\partial}{\partial u}, \quad E_4 = \frac{1}{\sin y}\frac{\partial}{\partial v}, \quad \xi = \frac{\partial}{\partial z}.$$

We can write now the expression of the metric g in terms of the (local) coordinates

$$g = \frac{1}{4}dx^2 + dy^2 + dz^2 + \cos 2y\, dxdz + \cos^2 y\, du^2 + \sin^2 y\, dv^2.$$

Let F be the isometric immersion of M in $\mathbb{S}^7$. Analogously to Case 1, we obtain the system of partial differential equations satisfied by F:

$$F_{uu} = \sin y \cos y F_y - \cos^2 y F, \tag{91}$$

$$F_{uv} = 0, \tag{92}$$

$$F_{vv} = -\sin y \cos y F_y - \sin^2 y F, \tag{93}$$

$$F_{yy} = -F, \tag{94}$$

$$F_{zz} = -F, \tag{95}$$

$$F_{yz} = -\frac{2}{\sin 2y}F_x + \cot 2y F_z, \tag{96}$$

$$F_{uy} = -\tan y F_u, \tag{97}$$

$$F_{vy} = \cot y F_v, \tag{98}$$

$$F_{xz} = \frac{\sin 2y}{2} F_y - \frac{\cos 2y}{2} F, \tag{99}$$

$$F_{xy} = \cot 2y F_x - \frac{1}{2\sin 2y} F_z, \tag{100}$$

$$F_{xx} = -\frac{1}{4} F. \tag{101}$$

$$F_{uz} = \cos y E_5, \tag{102}$$

$$F_{vz} = \sin y E_6, \tag{103}$$

$$F_{xu} = \frac{\cos y}{2} E_5, \tag{104}$$

$$F_{xv} = -\frac{\sin y}{2} E_6. \tag{105}$$

Further, we solve these partial differential equations satisfied by F.

Using Equation (94) we conclude

$$F = \sin y U + \cos y V, \tag{106}$$

where $U, V \in \mathbb{R}^8$ and U, V do not depend on y. Equations (97) and (98) imply that $\frac{\partial U}{\partial u} = 0$ and $\frac{\partial V}{\partial v} = 0$, that is $U = U(x, z, v)$ and $V = V(x, z, u)$.

Using (91) and (93), we get

$$V_{uu} = -V, \quad U_{vv} = -U, \tag{107}$$

and therefore

$$\begin{cases} U = \cos v\, u_1 + \sin v\, u_2, \\ V = \cos u\, v_1 + \sin u\, v_2, \end{cases} \tag{108}$$

where $u_1, u_2, v_1, v_2 \in \mathbb{R}^8$ depend on x and z.

The two functions U and V satisfy also other PDEs, namely

- from (99) we get

$$\begin{cases} U_{xz} - \frac{1}{2} U = 0, \\ V_{xz} + \frac{1}{2} V = 0; \end{cases} \tag{109}$$

- from (96) we obtain

$$\begin{cases} U_z + 2U_x = 0, \\ V_z - 2V_x = 0; \end{cases} \tag{110}$$

- from (101) we have

$$\begin{cases} U_{xx} + \frac{1}{4} U = 0, \\ V_{xx} + \frac{1}{4} V = 0; \end{cases} \tag{111}$$

- from (95) we find

$$\begin{cases} U_{zz} + U = 0, \\ V_{zz} + V = 0. \end{cases} \tag{112}$$

Then, combining (110) with (108) we deduce that

$$u_1 = u_1\left(\frac{x}{2} - z\right), u_2 = u_2\left(\frac{x}{2} - z\right), v_1 = v_1\left(\frac{x}{2} + z\right), v_2 = v_2\left(\frac{x}{2} + z\right).$$

Finally, using (109), (111) and (112) we obtain the last differential equations $u_1'' + u_1 = 0$, $u_2'' + u_2 = 0$, $v_1'' + v_1 = 0$ and $v_2'' + v_2 = 0$. Consequently, we get

$$\begin{cases} u_1 = \cos(\frac{x}{2} - z)e_1 + \sin(\frac{x}{2} - z)e_2, \\ u_2 = \cos(\frac{x}{2} - z)e_3 + \sin(\frac{x}{2} - z)e_4, \\ v_1 = \cos(\frac{x}{2} + z)e_5 + \sin(\frac{x}{2} + z)e_6, \\ v_2 = \cos(\frac{x}{2} + z)e_7 + \sin(\frac{x}{2} + z)e_8, \end{cases} \tag{113}$$

for some constant vectors $e_1, \ldots, e_8$ in $\mathbb{R}^8$.

Moreover, since F lies on $\mathbb{S}^7$, we conclude $\|U\| = \|V\| = 1$ and $\langle U, V\rangle = 0$ and, consequently, u_1, u_2, v_1 and v_2 are unitary and mutually orthogonal.

Further, set p_0 to be the initial point on M corresponding to $x = 0$, $y = \frac{\pi}{4}$, $z = 0$, $u = 0$, $v = 0$ and set the following initial conditions satisfied by F and its first partial derivatives, meaning that we fix the initial point on $\mathbb{S}^7$ and the initial tangent space at p_0 as a subspace in $T_{F(p_0)}\mathbb{S}^7$:

$$\begin{aligned} &F(p_0) = \tfrac{1}{\sqrt{2}}(1,0,0,0,1,0,0,0) \\ &F_z(p_0) = \xi(p_0) = JF(p_0) = \tfrac{1}{\sqrt{2}}(0,1,0,0,0,1,0,0) \\ &F_x(p_0) = \tfrac{1}{2}E_1(p_0) = \tfrac{1}{2\sqrt{2}}(0,1,0,0,0,-1,0,0) \\ &F_y(p_0) = E_2(p_0) = JE_1(p_0) = \tfrac{1}{\sqrt{2}}(-1,0,0,0,1,0,0,0) \\ &F_u(p_0) = \tfrac{1}{\sqrt{2}}(0,0,0,0,0,0,1,0) \\ &F_v(p_0) = \tfrac{1}{\sqrt{2}}(0,0,1,0,0,0,0,0). \end{aligned}$$

Here J is the complex structure on $\mathbb{R}^8$ locally defined by (85). Finally, set also the initial normal space (at p_0) as a subspace in $T_{F(p_0)}\mathbb{S}^7$:

$$\begin{aligned} &E_5(p_0) = JE_3(p_0) = \sqrt{2}JF_u(p_0) = (0,0,0,0,0,0,0,1) \\ &E_6(p_0) = JE_4(p_0) = \sqrt{2}JF_v(p_0) = (0,0,0,1,0,0,0,0). \end{aligned}$$

Using (106), (108) and (113), we conclude

$$\begin{aligned} F(x,y,z,u,v) = \; &\Big(\cos y \cos u \cos(z + \tfrac{x}{2}), \cos y \cos u \sin(z + \tfrac{x}{2}), \sin y \sin v \cos(z - \tfrac{x}{2}), \\ &\sin y \sin v \sin(z - \tfrac{x}{2}), \sin y \cos v \cos(z - \tfrac{x}{2}), \sin y \cos v \sin(z - \tfrac{x}{2}), \\ &\cos y \sin u \cos(z + \tfrac{x}{2}), \cos y \sin u \sin(z + \tfrac{x}{2})\Big) \end{aligned} \tag{114}$$

for $y \in (0, \frac{\pi}{2})$, $x, z, u, v \in \mathbb{R}$.

We will show that M can be expressed in terms of (multiply) warped products. Consider the following mutually orthogonal distributions on M:

$$D_0 = \text{span}\{E_1, E_2, \xi\}, \;\; D_3 = \text{span}\{E_3\} \;\; D_4 = \text{span}\{E_4\}.$$

The key of the proof is to apply a generalization of Hiepko's theorem given by Nölker in 1996 in (Reference [19], Theorem 4). The following conditions are satisfied; they are analogue to the previous conditions (a)–(c):

(i) the decomposition $T(M) = D_0 \textcircled{\perp} D_3 \textcircled{\perp} D_4$ is orthogonal;
(here $\textcircled{\perp}$ means the orthogonal decomposition);

(ii) the distributions D_3 and D_4 are spherical;
(iii) the distributions $D_3^\perp$ and $D_4^\perp$ are autoparallel, that is $\nabla_Z W \in D_k^\perp$, $(k = 3, 4)$, for any $Z, W \in D_k^\perp$.

Let us focus on the second condition: for example, the distributions D_3 and D_4 are spherical since they are totally umbilical and the corresponding mean curvature vector fields, X_0 and $\hat{\mathrm{X}}_0$, respectively, are parallel. From the Equation (20) we obtain $\mathrm{X}_0 = AE_2$ and $\hat{\mathrm{X}}_0 = -\frac{1}{A}E_2$, which are parallel with respect to the corresponding normal connections.

Thus, for any point $p \in M$, there exists an isometric immersion Φ of a warped product $N_0 \times_{\mathrm{f}} N_3 \times_{\hat{\mathrm{f}}} N_4$ onto a neighborhood of p in M such that

- $\Phi(N_0 \times \{p_3\} \times \{p_4\})$ is an integral manifold for D_0 for every $p_3 \in N_3$, $p_4 \in N_4$;
- $\Phi(\{p_0\} \times N_3 \times \{p_4\})$ is an integral manifold for D_3 for every $p_0 \in N_0$, $p_4 \in N_4$;
- $\Phi(\{p_0\} \times \{p_3\} \times N_4)$ is an integral manifold for D_4 for every $p_0 \in N_0$, $p_3 \in N_3$.

Similar computations as in the case $a_1 = a_3$ imply that the warping functions are given by $\mathrm{f} = \cos y$ and $\hat{\mathrm{f}} = \sin y$.

Proposition 4. *Under the conditions stated for the case 2, it follows that M is locally congruent to a contact CR multiply warped product $\mathbb{S}^3 \times_{f_1} \mathbb{S}^1 \times_{f_2} \mathbb{S}^1$.*

In accordance to the case 1 ($a_1 = a_3$), we consider the same parametrization $(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2})$ on $\mathbb{S}^3$. Then, on the two circles we set $(\mathbf{u} = \cos u, \mathbf{v} = \sin u) \in \mathbb{S}^1$ and $(\mathbf{a} = \cos v, \mathbf{b} = \sin v) \in \mathbb{S}^1$, respectively. Thus, the immersion F can be thought (see also Reference [5]) as the following map

$$\begin{aligned} &F : \mathbb{S}^3 \times_{f_1} \mathbb{S}^1 \times_{f_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^7 \\ &F(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}; \mathbf{u}, \mathbf{v}; \mathbf{a}, \mathbf{b}) = (\mathbf{ux_1}, \mathbf{uy_1}, \mathbf{vx_1}, \mathbf{vy_1}, \mathbf{ax_2}, \mathbf{ay_2}, \mathbf{bx_2}, \mathbf{by_2}), \end{aligned} \tag{115}$$

where the warping functions $f_1, f_2 : N_0 \to (0, \infty)$ are given by $f_1(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}) = \sqrt{\mathbf{x}_1^2 + \mathbf{y}_1^2}$ and $f_2(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}) = \sqrt{\mathbf{x}_2^2 + \mathbf{y}_2^2}$.

4. Conclusions and Further Research

We proved that a five-dimensional proper nearly totally geodesic contact CR-submanifold of seven-dimensional unit sphere is locally congruent to $S^3 \times_f S^2$ or to $S^3 \times_{f_1} S^1 \times_{f_2} S^1$, via the immersions (86) and (114). Thus, the list of five-dimensional *nearly totally geodesic* contact CR-submanifolds in the seven-sphere is now complete. So, to finalize the research in this direction, we have to investigate hypersurfaces in $\mathbb{S}^7$ which are nearly totally geodesic contact CR-submanifolds. This will be done in a future paper.

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