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The Heavy-Tailed Exponential Distribution: Risk Measures, Estimation, and Application to Actuarial Data

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Abstract: Modeling insurance data using heavy-tailed distributions is of great interest for actuaries. Probability distributions present a description of risk exposure, where the level of exposure to the risk can be determined by “key risk indicators” that usually are functions of the model. Actuaries and risk managers often use such key risk indicators to determine the degree to which their companies are subject to particular aspects of risk, which arise from changes in underlying variables such as prices of equity, interest rates, or exchange rates. The present study proposes a new heavy-tailed exponential distribution that accommodates bathtub, upside-down bathtub, decreasing, decreasing-constant, and increasing hazard rates. Actuarial measures including value at risk, tail value at risk, tail variance, and tail variance premium are derived. A computational study for these actuarial measures is conducted, proving that the proposed distribution has a heavier tail as compared with the alpha power exponential, exponentiated exponential, and exponential distributions. We adopt six estimation approaches for estimating its parameters, and assess the performance of these estimators via Monte Carlo simulations. Finally, an actuarial real data set is analyzed, proving that the proposed model can be used effectively to model insurance data as compared with fifteen competing distributions.

Keywords: risk measures; exponential distribution; parameter estimation; tail variance premium; VaR; TVaR

1. Introduction

Heavy-tailed distributions have been used in modeling data in several applied areas such as risk management, economic, and actuarial sciences. The insurance data sets are usually positive (Klugman et al. [1]), unimodal shaped (Cooray and Ananda [2]), right-skewed (Lane [3]), and with heavy tails (Ibragimov and Prokhorov [4]). Right-skewed data may be adequately modeled by skewed distributions (Bernardi et al. [5]). Modeling insurance data using heavy-tailed distributions is of a great interest for actuaries. Furthermore, actuaries and risk managers are often interested in “the chance of an adverse outcome”, which can be expressed through the value at risk (VaR) at a particular probability level. The VaR can also be utilized to determine the amount of capital required to withstand such adverse outcomes. Investors and rating agencies are particularly interested in the company’s ability to withstand such events.

Hence, several unimodal positively skewed distributions are utilized to model such data sets (Adcock et al. [6] and Bhati and Ravi [7]).

The heavy-tailed distributions have right tail probabilities which are heavier than the exponential one, that is, for any baseline with cumulative distribution function (CDF) $G(x)$, we have

$$\lim_{x \rightarrow \infty} \frac{\exp(-\eta x) - 1}{1 - G(x)} = 0, \quad \eta > 0.$$

More information about heavy-tailed distributions can be found in Resnick [8] and Beirlant et al. [9].

An empirical analysis for loss distributions to estimate the risk using some approaches is conducted by Dutta and Perry [10], who pointed out that the exponential, gamma, and Weibull models can not be used because of their poor results and stated that one would need to use a more flexible model. There are some methods that have been introduced to construct new distributions with heavier tails than the exponential distribution, called the transformation method, compounding of distributions, composition of two or more models and finite mixture distributions. The interested reader can refer to Eling [11], Kazemi and Noorizadeh [12], Bakar et al. [13], Punzo [14], Mazza and Punzo [15], Miljkovic and Grun [16], and Punzo et al. [17].

Aforementioned approaches may be very useful in constructing more flexible distributions; however, these methods are still subject to some sort of deficiencies, such as the inferences of transformation approach become difficult and require much computational work to derive the distributional characteristics (see Bagnato and Punzo [18]), and the approach of composition of two or more models based on fixed or pre-defined mixing weights, may be very restrictive (see Calderin-Ojeda and Kwok [19]).

Hence, it is important to develop more flexible models either from the existing classical distributions or a new class of distributions to model different insurance data including insurance loss data, unemployment insurance data, and financial returns, among others. In the present paper, we are motivated to propose a more flexible distribution called alpha power exponentiated exponential (APExE) distribution, which provides greater accuracy and flexibility in fitting actuarial data.

Furthermore, the aim of the paper is three-fold: first to study a new extension of the exponential (E) and exponentiated exponential (ExE) distributions based on an alpha power-G (AP-G) family proposed by Mahdavi and Kundu [20], called the APExE distribution. Some distributional properties of the APExE distribution are derived. The proposed model has some desirable properties such as

- The APExE model contains, as special cases, some lifetime distributions, called E, ExE [21], and alpha power exponential (APE) [20] distributions.
- The APExE distribution accommodates upside down bathtub, bathtub, decreasing, decreasing-constant and increasing hazard rates, and right-skewed, symmetrical, left-skewed, J-shape, reversed-J shape, and unimodal densities (see Figures 1 and 2).
- It provides a heavier tailed distribution than the APE, ExE, and E distributions based on computational results of risk measures (see also Section 4).
- The APExE has closed forms for its CDF and hazard rate function (HRF). Hence, it can be used conveniently in analyzing censored data. Furthermore, it can be used to model various data in actuarial applications.
- It can be utilized to model heavy-tailed insurance data from actuarial science than other competing models. The proposed APExE distribution provides better fits than other fifteen competing distributions in modeling unemployment insurance data (see Section 7).

One of the most important subjects of actuarial sciences is to evaluate the exposure to market risk in a portfolio of instruments, which arise from changes in underlying variables such as prices of equity, interest rates, or exchange rates. Hence, our second objective is to derive some important risk or actuarial measures including VaR, tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) for the APExE distribution, which play a crucial role in portfolio optimization under uncertainty. Third, we explore the estimation of the APExE parameters by six methods of

estimation. Such methods include the maximum likelihood estimators (MLE), ordinary least-squares estimators (OLSE), weighted least squares estimators (WLSE), Anderson–Darling estimators (ADE), Cramér–von Mises estimators (CVME), and percentile estimators (PE). We compare these estimators using an extensive computational study in order to develop a guideline for choosing the best method of estimation that provides better estimates for the APEXE parameters, which we think would be of a great interest to applied actuaries/statisticians/engineers.

The paper is organized as follows. In Section 2, we define the APEXE distribution. Its mathematical properties are derived in Section 3. In Section 4, we discuss four important actuarial measures based on the APEXE distribution and present some numerical results for them. Six methods of parameter estimation are explored in Section 5. The performance of estimation methods is adopted by simulation results in Section 6. In Section 7, we consider a heavy-tailed real data set from the insurance field to illustrate the usefulness of the APEXE distribution. Final remarks are presented in Section 8.

2. The APEXE Distribution

In this section, we present the APEXE model that can be specified by the following CDF and probability density function (PDF)

$$F(x) = \frac{\alpha^{(1-e^{-ax})^c} - 1}{\alpha - 1}, \quad x > 0, \quad (1)$$

$$f(x) = \frac{ace^{-ax} \log(\alpha) (1 - e^{-ax})^{c-1} \alpha^{(1-e^{-ax})^c}}{\alpha - 1}, \quad (2)$$

where $a > 0$ is a scale parameter and $\alpha > 0, \alpha \neq 1$, and $c > 0$ are shape parameters. By setting $c = 1$, the APEXE reduces to APE distribution (Mahdavi and Kundu [20]). The ExE distribution (Gupta and Kundu [21]) follows from the APEXE distribution with $\alpha = 1$, whereas the E follows with $\alpha = 1$ and $c = 1$.

The survival function (SF) and HRF of APEXE distribution have the forms

$$S(x) = \frac{\alpha - \alpha^{(1-e^{-ax})^c}}{\alpha - 1}, \quad (3)$$

$$h(x) = \frac{ac \log(\alpha) (1 - e^{-ax})^c \alpha^{(1-e^{-ax})^c}}{(1 - e^{-ax}) (\alpha^{(1-e^{-ax})^c} - \alpha)}. \quad (4)$$

Plots of the PDF and HRF of APEXE distribution are shown in Figures 1 and 2, respectively. These plots reveal that the APEXE distribution accommodates bathtub, upside-down bathtub, decreasing, decreasing-constant and increasing hazard rate functions as well as symmetrical, left-skewed, right-skewed, J-shape, and reversed-J shape densities.

Figure 1. Cont.

Figure 1. Plots of APExE PDFs for different parametric values.

Figure 2. Plots of APExE HRF for different parametric values.

3. Mathematical Properties

3.1. Quantile Function

The quantile function (QF) of the APExE distribution is derived by determining the inverse function of the CDF (1) as

$$Q(p) = \left[\log \left(1 - \left\{ \frac{\log[(\alpha - 1)p + 1]}{\log(\alpha)} \right\}^{1/c} \right) \right]^{-\frac{1}{a}}, \quad 0 < p < 1. \quad (5)$$

The first, second, and third quartiles of the APExE distribution are obtained by setting $p = 0.25, 0.5$, and 0.75 , respectively, in (5).

Let p follow uniform distribution $(0, 1)$, then the QF can be used for generating random data sets of size n from APExE distribution as follows:

$$x_i = \left[\log \left(1 - \left\{ \frac{\log[(\alpha - 1)p_i + 1]}{\log(\alpha)} \right\}^{1/c} \right) \right]^{-\frac{1}{a}}, \quad i = 1, 2, \dots, n.$$

3.2. Moments

The r th moments of the APExE distribution has the form

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x) dx = \frac{(-1)^r}{a^r(\alpha - 1)} \sum_{i=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} \int_0^1 [\log(1 - y^{\frac{1}{c}})]^r y^i dy.$$

Setting $r = 1, 2, 3$, and 4 , respectively, we obtain the first four moments about the origin of the APExE distribution.

The plots of the mean, variance, skewness, and kurtosis of the APExE model for various parametric values of α and c are displayed in Figure 3.

Figure 3. Plots of the mean, variance, skewness, and kurtosis of the APExE model with $a = 1$.

The moment generating function of the APExE distribution takes the form

$$M(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^r t^r [\log(\alpha)]^{i+1}}{a^r(\alpha - 1) i! r!} \int_0^1 [\log(1 - y^{\frac{1}{c}})]^r y^i dy.$$

The characteristic function of the APExE distribution follows from the above equation by replacing t with it .

3.3. Moments of Residual Life

Let $S(t)$ be the SF of the APEXE distribution, then its m th moment of residual life is given by

$$\begin{aligned} M_m &= \frac{1}{S(t)} \int_t^\infty (x-t)^m f(x) dx \\ &= \frac{1}{S(t)(\alpha-1)} \sum_{i=0}^{\infty} \sum_{k=0}^m \frac{(-1)^{m+2k} \binom{m}{k} t^{m-k} (\log \alpha)^{i+1}}{i! a^k} \int_{[1-\exp(-at)]^c}^1 [\log(1-y^{\frac{1}{c}})]^k y^i dy. \end{aligned}$$

The mean residual life function of the APEXE distribution is obtained by setting $m = 1$ in the previous equation and then by setting $t = 0$, we have the mean of the APEXE distribution.

3.4. Entropies

The entropy of a random variable X is a measure of variation of the uncertainty, and it has some applications in several applied areas such as statistics to test hypotheses in parametric models (see Morales et al. [22]), and information theory, engineering and physics for describing nonlinear chaotic or dynamical systems (see Kurths et al. [23]). Furthermore, Song [24] developed a log-likelihood-based distribution measure using the Rényi information which exists for all distributions and allows for meaningful comparisons between distributions than the traditional kurtosis measure. Song's measure can be used in exploring density shapes especially for heavy-tailed distributions, while the kurtosis measure does not exist for many of these distributions.

In this section, we derive the continuous Rényi, Tsallis, and Shannon entropies of the APEXE distribution. The Rényi, $K_X(r)$, and Tsallis, $L_X(r)$ entropies of order r , where $r > 0$, $r \neq 1$ of the APEXE distribution are given, respectively, by

$$\begin{aligned} K_X(r) &= \frac{1}{1-r} \log \int_{x=0}^{\infty} f^r(x) dx, \quad r > 0, r \neq 1 \\ &= \frac{1}{1-r} \log \left(a^{r-1} \left(\frac{c}{\alpha-1} \right)^r \sum_{i=0}^{\infty} \frac{r^i (\log \alpha)^{i+r} \Gamma(r) \Gamma[-r+c(i+r)+1]}{i! \Gamma[c(i+r)+1]} \right) \end{aligned}$$

and

$$\begin{aligned} L_X(r) &= \frac{1}{1-r} \left[\int_{x=0}^{\infty} f^r(x) dx - 1 \right] \\ &= \frac{1}{1-r} \left[a^{r-1} \left(\frac{c}{\alpha-1} \right)^r \sum_{i=0}^{\infty} \frac{r^i (\log \alpha)^{i+r} \Gamma(r) \Gamma[-r+c(i+r)+1]}{i! \Gamma[c(i+r)+1]} - 1 \right]. \end{aligned}$$

The Rényi entropy reduces to Shannon entropy, $H_X(1)$, as r approaches to 1. The Shannon entropy of APEXE distribution takes the form

$$\begin{aligned} H_X(1) &= \lim_{r \rightarrow 1} K_X(r) \\ &= \log \left(\frac{\alpha-1}{a c} \right) - \frac{1}{\alpha-1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1} \left\{ \log[\log(\alpha)] - \Phi(ic+c) + i - \frac{1}{i+1} - \gamma \right\}}{\Gamma(i+2)}, \end{aligned}$$

where $\Phi(z) = \frac{d}{dz} \log(\Gamma(z))$ and γ is the Euler Mascheroni constant.

Table 1 reports some numerical values for the Rényi, Tsallis, and Shannon entropies of the APEXE distribution.

Table 1. Numerical values of Rényi, Tsallis, and Shannon entropies for the APExE distribution.

r	α	a	c	$K_X(r)$	$L_X(r)$	$H_X(1)$
0.30	1.50	0.50	0.50	2.26779	5.55895	1.17257
			1.50	2.54614	7.06212	1.97560
			3.00	2.65667	7.74517	2.14846
1.50	2.00	1.00	0.50	0.02699	0.02682	0.59257
			1.50	1.20487	0.90505	1.32986
			3.00	1.37495	0.99431	1.48913
2.50	3.00	1.50	0.50	0.51623	0.35933	0.33243
			1.50	0.75144	0.45070	0.98347
			3.00	0.90294	0.49460	1.12484
0.75	3.50	0.75	0.50	1.29866	1.53427	1.07636
			1.50	1.79408	2.26397	1.69667
			3.00	1.92315	2.46939	1.83166

3.5. Inequality Curves

The most used and known important curves among inequality curves are Lorenz and Bonferroni that have useful applications in some applied areas including economics to study income and poverty, demography, reliability, medicine, and insurance.

The Lorenz and Bonferroni curves are defined for the APExE distribution as follows:

$$L(p) = \frac{1}{\mu} \int_0^{x_p} x f(x) dx = \frac{1}{\mu a (1-\alpha)} \sum_{i=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} \int_0^{[1-\exp(-ax_p)]^c} \log(1-y^{\frac{1}{c}}) y^i dy,$$

$$B(p) = \frac{L(p)}{F(x)},$$

respectively, where x_p is the QF of the APExE distribution.

3.6. Order Statistics

The PDF and CDF of the i th order statistic for the APExE distribution are

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} f(x)$$

$$= \frac{acn!e^{-ax} \log(\alpha) (1-e^{-ax})^{c-1} \alpha^{(1-e^{-ax})^c} \left[\frac{\alpha^{(1-e^{-ax})^c} - 1}{\alpha - 1} \right]^{i-1} \left[\frac{\alpha - \alpha^{(1-e^{-ax})^c}}{\alpha - 1} \right]^{n-i}}{(\alpha - 1)(i-1)!(n-i)!},$$

$$F_{i:n}(x) = \sum_{r=i}^n \binom{n}{r} (F(x))^r (1-F(x))^{n-r}$$

$$= \binom{n}{i} \left[\frac{\alpha^{(1-e^{-ax})^c} - 1}{\alpha - 1} \right]^i \left[\frac{\alpha - \alpha^{(1-e^{-ax})^c}}{\alpha - 1} \right]^{n-i} {}_2F_1 \left[1, i-n; i+1; \frac{\alpha^{(1-e^{-ax})^c} - 1}{\alpha^{(1-e^{-ax})^c} - \alpha} \right],$$

where ${}_2F_1 \left[1, i-n; i+1; \frac{\alpha^{(1-e^{-ax})^c} - 1}{\alpha^{(1-e^{-ax})^c} - \alpha} \right]$ is a hyper geometric function.

By setting $i = 1$, we have the PDF and CDF of minimum order statistics (W_n). The limit distribution for W_n reduces to (see Theorem 2.1.5 in Galambos [25])

$$\lim_{n \rightarrow +\infty} P(W_n < d_n x) = 1 - \exp(-x^c), \quad x > 0, d_n = F^{-1} \left(\frac{1}{n} \right).$$

By setting $i = n$, we have the PDF and CDF of maximum order statistics (Z_n). The limit distribution for Z_n takes the form (see Theorem 2.1.1 in Galambos [25])

$$\lim_{n \rightarrow +\infty} P(Z_n < b_n x) = \begin{cases} 0, & 0 < x < 1, \\ e^{-1}, & x = 1, \\ 1, & x > 1, \end{cases}$$

where $b_n = F^{-1}\left(1 - \frac{1}{n}\right)$.

4. Actuarial Measures

Probability distributions present a description of risk exposure. The level of risk exposure can be described by “key risk indicators” (numbers) that usually are functions of the model. Actuaries and risk managers often use such key risk indicators to determine the degree to which their companies are subject to particular aspects of risk, which arise from changes in underlying variables such as prices of equity, interest rates, or exchange rates.

In this section, we discuss the theoretical and computational aspects of some important risk measures including VaR, TVaR, TV, and TVP for the APEXE distribution, which play a crucial role in portfolio optimization under uncertainty.

4.1. VaR Measure

The VaR is also known as the quantile risk measure or quantile premium principle, and it is specified with a given degree of confidence say q (typically 90%, 95% or 99%). Furthermore, VaR is a quantile of the distribution of aggregate losses. Risk managers are often interested in “the chance of an adverse outcome” that can be expressed through the VaR at a particular probability level. The VaR can be used to evaluate exposure to risk, and hence it is used to determine the amount of capital required to withstand such adverse outcomes. Investors, regulators, and rating agencies are particularly interested in the company’s ability to withstand such events. The VaR of a random variable X is the q th quantile of its CDF, denoted by VaR_q , and it is defined by $VaR_q = Q(q)$ (see Artzner [26]).

If X has the PDF (2), then its VaR can be derived as

$$VaR_q = \left[\log \left(1 - \left\{ \frac{\log[(\alpha - 1)q + 1]}{\log(\alpha)} \right\}^{1/c} \right) \right]^{-\frac{1}{a}}.$$

4.2. TVaR Measure

Another important measure is the TVaR that has been given several names including conditional tail expectation, conditional-value at-risk, and expected shortfall. The TVaR is used to quantify the expected value of the loss given that an event outside a given probability level has occurred. The TVaR is defined by

$$TVaR_q = \frac{1}{(1 - q)} \int_{VaR_q}^{\infty} x f(x) dx.$$

The TVaR of the APEXE distribution is defined by

$$\begin{aligned} TVaR_q &= \frac{a c \log(\alpha)}{(\alpha - 1)(1 - q)} \int_{VaR_q}^{\infty} x e^{-ax} (1 - e^{-ax})^{c-1} \alpha^{(1-e^{-ax})^c} dx \\ &= \frac{1}{a(\alpha - 1)(1 - q)} \sum_{i=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{(i+1)!} \left[\sum_{j=1}^{c+ci} \frac{1}{j} + A^{i+1} \left(\sum_{k=0}^{\infty} \frac{A^{\frac{k+1}{c}}}{k + c + ci + 1} - a VaR_q \right) \right], \quad (6) \end{aligned}$$

where $A = [1 - \exp(-a VaR_q)]^c$.

4.3. TV Measure

Landsman [27] introduced the TV risk that is defined by the variance of the loss distribution beyond some critical value. The TV is one of the most important risk measures which pay attention to the tail variance beyond the VaR. The TV of the APExE distribution can be defined as

$$TV_q(X) = E(X^2|X > x_q) - (TVaR_q)^2 = \frac{1}{(1-q)} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2, \quad (7)$$

where

$$E(X^2|X > x_q) = \frac{\frac{a}{(\alpha-1)(1-q)} \int_{VaR_q}^{\infty} x^2 e^{-ax} (1-e^{-ax})^{c-1} \alpha^{(1-e^{-ax})^c} dx}{\frac{\sum_{i=0}^{\infty} \frac{[\log(a)]^{i+1}}{(i+1)!}}{a^2(\alpha-1)(1-q)}} = \frac{\sum_{i=0}^{\infty} \frac{[\log(a)]^{i+1}}{(i+1)!}}{a^2(\alpha-1)(1-q)} \quad (8)$$

where $A = [1 - \exp(-a VaR_q)]$, B and C are generalized hyper geometric functions given by $B = {}_4F_3(1, 1, 1, -ci - c + 1; 2, 2, 2; 1 - A)$ and $C = {}_3F_2(1, 1, -ci - c + 1; 2, 2; 1 - A)$. Using Equation (6)–(8), we get the TV of APExE distribution.

4.4. TVP Measure

The TVP is another important measure which plays an essential role in insurance sciences. The TVP of the APExE distribution takes the form

$$TVP_q(x) = TVaR_q + \lambda TV_q, \quad (9)$$

where $0 < \lambda < 1$. Substituting the expressions (6) and (7) in Equation (9), one can obtain the TVP of the proposed distribution.

4.5. Numerical Simulations for Risk Measures

In this sub-section, we present some numerical results for the VaR, TVaR, TV, and TVP measures of the APExE, APE, ExE, and E distributions for different parametric values. The results are obtained through the following two steps:

1. Random sample of size $n = 100$ is generated from the APExE, APE, ExE and E distributions, and parameters have been estimated via the maximum likelihood method.
2. 1000 repetitions are made to calculate the VaR, TVaR, TV, and TVP of the four distributions.

Simulation results of the VaR, TVaR, TV, and TVP for the APExE, APE, ExE, and E distributions are provided in Tables 2 and 3. Furthermore, the results in these tables are depicted graphically in Figures 4 and 5.

The model with higher values of VaR, TVaR, TV, and TVP measures is said to have a heavier tail than other competing models. The results in Tables 2 and 3, and the plots in Figures 4 and 5 show that the proposed APExE model has higher values of the four risk measures than the APE, ExE, and E distributions. Hence, the APExE model has a heavier tail than other distributions and can be utilized accurately to model heavy-tailed insurance data.

Table 2. Simulation results for the VaR, TVaR, TV, and TVP of the APExE, APE, ExE, and E distributions.

Distribution	Parameters	Significance Level	VaR	TVaR	TV	TVP
APExE	$\alpha = 1.5, a = 0.75, c = 2$	0.70	2.59765	4.02514	2.07422	5.47709
		0.75	2.86601	4.28452	2.07659	5.84197
		0.80	3.18843	4.60024	2.08469	6.26798
		0.85	3.59781	5.00572	2.10132	6.79185
		0.90	4.16826	5.57646	2.13301	7.49617
		0.95	5.13813	6.55519	2.20077	8.64593
		0.99	7.41201	8.86152	2.37397	11.21175
APE	$\alpha = 1.5, a = 0.75$	0.70	1.78327	3.14166	1.86606	4.44790
		0.75	2.03446	3.38895	1.86355	4.78661
		0.80	2.33981	3.69078	1.86154	5.18001
		0.85	2.73111	4.07895	1.86007	5.66001
		0.90	3.27972	4.62485	1.85921	6.29814
		0.95	4.21345	5.55633	1.85903	7.32240
		0.99	6.37467	7.71611	1.85945	9.55697
ExE	$a = 0.75, c = 2$	0.70	0.90094	2.12174	1.67798	3.29633
		0.75	1.10500	2.34625	1.70330	3.62373
		0.80	1.36456	2.62552	1.72831	4.00817
		0.85	1.71117	2.99117	1.75300	4.48122
		0.90	2.21578	3.51416	1.77740	5.11381
		0.95	3.10488	4.42102	1.80148	6.13242
		0.99	5.22333	6.55326	1.82054	8.35560
E	$a = 0.75$	0.70	1.60895	2.94532	1.80352	4.20778
		0.75	1.85260	3.18897	1.80352	4.54161
		0.80	2.15080	3.48717	1.80352	4.92999
		0.85	2.53525	3.87162	1.80352	5.40461
		0.90	3.07710	4.41347	1.80352	6.03663
		0.95	4.00340	5.33977	1.80352	7.05311
		0.99	6.15420	7.49057	1.80352	9.27605

Table 3. Simulation results for the VaR, TVaR, TV, and TVP of the APExE, APE, ExE, and E distributions.

Distribution	Parameters	Significance Level	VaR	TVaR	TV	TVP
APExE	$\alpha = 2, a = 1, c = 1.5$	0.70	1.79510	2.86734	1.15530	3.67605
		0.75	1.99723	3.06212	1.15366	3.92737
		0.80	2.24023	3.29901	1.15430	4.22246
		0.85	2.54868	3.60288	1.15831	4.58744
		0.90	2.97791	4.02970	1.16818	5.08107
		0.95	3.70539	4.75910	1.19194	5.89144
		0.99	5.39794	6.46568	1.25596	7.70908
APE	$\alpha = 2, a = 1$	0.70	1.45073	2.48810	1.06452	3.23326
		0.75	1.64551	2.67664	1.05922	3.47105
		0.80	1.88083	2.90604	1.05413	3.74934
		0.85	2.18056	3.20012	1.04926	4.09199
		0.90	2.59816	3.61234	1.04461	4.55248
		0.95	3.30427	4.31331	1.04015	5.30145
		0.99	4.92820	5.93332	1.03673	6.95968
ExE	$a = 1, c = 1.5$	0.70	0.88021	1.82967	0.96444	2.50477
		0.75	1.04621	2.00352	0.97180	2.73237
		0.80	1.25321	2.21801	0.97899	3.00120
		0.85	1.52468	2.49665	0.98601	3.33476
		0.90	1.91341	2.89225	0.99289	3.78585
		0.95	2.58785	3.57329	0.99962	4.52293
		0.99	4.17394	5.16447	1.00491	6.15933
E	$a = 1$	0.70	1.20243	2.20114	1.00739	2.90632
		0.75	1.38451	2.38323	1.00739	3.13877
		0.80	1.60737	2.60609	1.00739	3.41200
		0.85	1.89468	2.89340	1.00739	3.74968
		0.90	2.29963	3.29834	1.00739	4.20499
		0.95	2.99189	3.99060	1.00739	4.94762
		0.99	4.59926	5.59797	1.00740	6.59530

Figure 4. Plots of the VaR, TVaR, TV, and TVP using the values in Table 2.

Figure 5. Plots of the VaR, TVaR, TV, and TVP using the values in Table 3.

5. Methods of Estimation

In this section, we discuss the estimation of the APExE parameters by different methods of estimation including the MLE, OLSE, WLSE, ADE, CVME, and PE. Parameter estimation using different classical estimators have been discussed by many researchers. For example, the alpha logarithmic transformed Weibull distribution [28], quasi xgamma-geometric distribution [29], Weibull Marshall–Olkin Lindley distribution [30], and APE distribution [31], among others.

5.1. Maximum Likelihood Estimation

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the PDF (2), then the log-likelihood function reduces to

$$L = n \log \left[\frac{ac \log(\alpha)}{\alpha - 1} \right] - a \sum_{i=1}^n x_i + \log(\alpha) \sum_{i=1}^n (1 - e^{-ax_i})^c + (c - 1) \sum_{i=1}^n \log(1 - e^{-ax_i}). \quad (10)$$

By differentiating Equation (10) with respect to α , a and c , respectively, and equating to zero, we have

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \sum_{i=1}^n \frac{(1 - e^{-ax_i})^c}{\alpha} + \frac{n[\alpha - \alpha \log(\alpha) - 1]}{(\alpha - 1)\alpha \log(\alpha)} = 0, \\ \frac{\partial L}{\partial a} &= \frac{n}{a} - \sum_{i=1}^n x_i + c \log(\alpha) \sum_{i=1}^n x_i e^{-ax_i} (1 - e^{-ax_i})^{c-1} + (c - 1) \sum_{i=1}^n \frac{x_i e^{-ax_i}}{1 - e^{-ax_i}} = 0, \\ \frac{\partial L}{\partial c} &= \frac{n}{c} + \log(\alpha) \sum_{i=1}^n (1 - e^{-ax_i})^c \log(1 - e^{-ax_i}) + \sum_{i=1}^n \log(1 - e^{-ax_i}) = 0. \end{aligned}$$

Solving the previous equations, we obtain estimators of the APExE parameters by the MLE.

5.2. Ordinary Least-Squares and Weighted Least-Squares Estimators

Let $x_{1:n}, x_{2:n}, \dots, x_{2:n}$ be the order statistics of a random sample of size n from the APExE distribution. Hence, we have the OLSE of the APExE parameters by minimizing the following equation:

$$O = \sum_{i=1}^n \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2 = \sum_{i=1}^n \left\{ \frac{\alpha^{(1-e^{-ax_{i:n}})^c} - 1}{\alpha - 1} - \frac{i}{n+1} \right\}^2,$$

The OLSE of the APExE parameters can also be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^n \left\{ \frac{\alpha^{(1-e^{-ax_{i:n}})^c} - 1}{\alpha - 1} - \frac{i}{n+1} \right\} \Delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3,$$

where

$$\Delta_1(x_{i:n}) = \frac{\partial}{\partial \alpha} F(x_{i:n}) = \frac{(\alpha - 1)(1 - e^{-ax_{i:n}})^c \alpha^{(1-e^{-ax_{i:n}})^c - 1} - \alpha^{(1-e^{-ax_{i:n}})^c} + 1}{(\alpha - 1)^2}, \quad (11)$$

$$\Delta_2(x_{i:n}) = \frac{\partial}{\partial a} F(x_{i:n}) = \frac{c \log(\alpha) x_{i:n} e^{-ax_{i:n}} (1 - e^{-ax_{i:n}})^{c-1} \alpha^{(1-e^{-ax_{i:n}})^c}}{\alpha - 1}, \quad (12)$$

$$\Delta_3(x_{i:n}) = \frac{\partial}{\partial c} F(x_{i:n}) = \frac{\log(\alpha) (1 - e^{-ax_{i:n}})^c \log(1 - e^{-ax_{i:n}}) \alpha^{(1-e^{-ax_{i:n}})^c}}{\alpha - 1}. \quad (13)$$

The WLSE of the APExE parameters can be calculated by minimizing the following equation:

$$W = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2 = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{\alpha^{(1-e^{-ax_{i:n}})^c} - 1}{\alpha - 1} - \frac{i}{n+1} \right]^2.$$

Furthermore, the WLSE of the APExE parameters follow by solving the following nonlinear equations:

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right] \Delta_s(x_{i:n}) = 0,$$

where $\Delta_s(x_{i:n})$, $s = 1, 2, 3$ were defined in (11), (12), and (13), respectively.

5.3. Anderson–Darling Estimation

The ADE of the APExE parameters are obtained by minimizing the following equation:

$$A = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{i:n})].$$

The ADE can also be calculated by solving the following nonlinear equations:

$$\sum_{i=1}^n (2i-1) \left[\frac{\Delta_s(x_{i:n})}{F(x_{i:n})} - \frac{\Delta_s(x_{n+1-i:n})}{S(x_{n+1-i:n})} \right] = 0,$$

where $\Delta_s(x_{i:n})$, $s = 1, 2, 3$ were defined in (11), (12), and (13), respectively.

5.4. Cramér–von Mises Estimation

The CVME of APExE parameters are obtained by minimizing the following equation:

$$CV = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2 = \frac{1}{12n} + \sum_{i=1}^n \left[\frac{\alpha^{(1-e^{-ax_{i:n}})^c} - 1}{\alpha - 1} - \frac{2i-1}{2n} \right]^2,$$

or by solving the following nonlinear equations

$$\sum_{i=1}^n \left[\frac{\alpha^{(1-e^{-ax_{i:n}})^c} - 1}{\alpha - 1} - \frac{2i-1}{2n} \right] \Delta_s(x_{i:n}) = 0,$$

where $\Delta_s(x_{i:n})$, $s = 1, 2, 3$ were defined in (11), (12), and (13), respectively.

5.5. Percentile Estimation

Let $p_i = \frac{i}{n+1}$ be an estimate of $F(x_{i:n})$, then the PE of the APExE parameters are obtained by minimizing the following equation:

$$PE = \sum_{i=1}^n [x_{i:n} - Q(p_i)]^2 = \sum_{i=1}^n \left\{ x_{i:n} - \left[\log \left(1 - \left\{ \frac{\log[(\alpha-1)p_i + 1]}{\log(\alpha)} \right\}^{1/c} \right) \right]^{-\frac{1}{a}} \right\}^2,$$

or by solving the following nonlinear equations:

$$\sum_{i=1}^n \left\{ x_{i:n} - \left[\log \left(1 - \left\{ \frac{\log[(\alpha-1)p_i + 1]}{\log(\alpha)} \right\}^{1/c} \right) \right]^{-\frac{1}{a}} \right\} \Phi_s(x_{i:n}) = 0, \quad s = 1, 2, 3,$$

where

$$\begin{aligned}\Phi_1(x_{i:n}) &= \frac{\partial}{\partial \alpha} Q(p_i) = \frac{\left\{ \frac{\log[(\alpha-1)p_i+1]}{\alpha} - \frac{\log(\alpha)p_i}{(\alpha-1)p_i+1} \right\} \left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\}^{\frac{1}{c}-1}}{ac[\log(\alpha)]^2 \left(\left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\}^{1/c} - 1 \right)}, \\ \Phi_2(x_{i:n}) &= \frac{\partial}{\partial a} Q(p_i) = \frac{1}{a^2} \log \left(1 - \left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\}^{1/c} \right), \\ \Phi_3(x_{i:n}) &= \frac{\partial}{\partial c} Q(p_i) = \frac{\log \left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\} \left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\}^{1/c}}{ac^2 \left(\left\{ \frac{\log[(\alpha-1)p_i+1]}{\log(\alpha)} \right\}^{1/c} - 1 \right)}.\end{aligned}$$

6. Simulation Results

In this section, we explore the performance of the aforementioned estimation methods in estimating the APExE parameters using simulation results. We consider various sample sizes, $n = \{20, 50, 100, 200\}$, and various parametric values, $\alpha = (0.5, 2)$, $a = (0.5, 1.5)$ and $c = (0.5, 3)$. We generate $n = 1000$ random samples from the APExE distribution via Equation (6). We determine the average values of the estimates (AEs) along with their corresponding average absolute biases (ABs), average mean square error (MSEs), and average mean relative estimates (MREs) for all sample sizes and parameter combinations using the R software®.

The ABs, MSEs, and MREs can be calculated by the following respective equations:

$$ABs = \frac{1}{N} \sum_{i=1}^N |\hat{\theta} - \theta|, \quad MSEs = \frac{1}{N} \sum_{i=1}^N (\hat{\theta} - \theta)^2, \quad MREs = \frac{1}{N} \sum_{i=1}^N |\hat{\theta} - \theta|/\theta,$$

where $\theta = (\alpha, a, c)'$.

Tables A1–A8 report the simulation results including AEs, ABs, MSEs, and MREs of the APExE parameters using the six estimation approaches. These tables are given in Appendix A. One can note that the estimates of the APExE parameters obtained from all six estimation methods are entirely good, that is, these estimates are quite reliable and very close to the true values, showing small biases, MSEs and MREs in all parameter combinations. All six estimators show the consistency property, where the MSEs, ABs, and MREs decrease as sample size increases, for all parameter combinations. We conclude that the MLE, OLSE, WLSE, ADE, CVME, and PE methods perform very well in estimating the APExE parameters.

7. Modeling Insurance Data

In this section, we consider a heavy-tailed real data set from the insurance field to illustrate the usefulness of the APExE distribution. This data set represents monthly metrics on unemployment insurance from July 2008 to April 2013 including 58 observations, and it is reported by the Department of Labor, Licensing and Regulation, State of Maryland, USA. The data consist of 21 variables and we particularly analyze the variable number 12. The data are available at: <https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013>. We compare the goodness-of-fit results and some discrimination measures of the proposed distribution with some other well-known competing distributions, including the APE [20], ExE [21], beta exponential (BE) [32], E, extended odd Weibull exponential (EOWE) [33], exponentiated Weibull (ExW) [34], Weibull (W), transmuted generalized exponential (TGE) [35], Marshall–Olkin exponential (MOE) [36], generalized transmuted exponential (GTE) [37], transmuted exponentiated generalized exponential (TEGE) [38], complementary geometric transmuted exponential (CGTE) [39], gamma (G), Harris extended exponential (HEE) [40], and transmuted exponential (TE) [41] distributions.

The competing models can be compared using some discrimination measures such as Akaike information (AKI), consistent Akaike information (CAKI), Bayesian information (BAI), and Hannan–Quinn information (HAQUI) criteria. Further discrimination measures including Anderson Darling (ANDA), Cramér–von Mises (CRVMI), and Kolmogorov–Smirnov (KOSM) with its p -value.

The MLEs and the analytical measures are computed using the R software©. Table 4 gives the MLEs and their standard errors. The analytical measures are provided in Table 5. The results in Table 5 indicate that the APExE provides better fits than other competing models and could be chosen as an adequate model to analyze the studied heavy-tailed insurance data. The fitted PDF, CDF, SF, and probability-probability (P–P) plots of the APExE model are depicted in Figure 6. Furthermore, we use the six estimation approaches discussed before in Section 5 to estimate the APExE parameters. Table 6 reports the estimates of the APExE parameters using these approaches and the numerical values of goodness-of-fit for insurance data. Based on the values of KOSM and p -values listed in Table 6, we conclude that the performance ordering of six estimators, from best to worst, for insurance data are OLSE, CVME, WLSE, ADE, MLE, and PE. The P–P plots and histogram of insurance data with the fitted APExE density for various estimation methods are shown in Figure 7 that supports the results in Table 6.

Table 4. Estimated parameters with their standard errors of the APExE model and other fitted models.

Model	Estimated Parameters (Standard Errors)			
APExE	$\alpha = 0.0394(0.0554)$	$a = 0.0319(0.0072)$	$c = 11.5252(3.7413)$	
TGE	$\alpha = 13.5952(4.4275)$	$\lambda = 0.0405(0.0063)$	$\theta = 0.6225(0.2603)$	
TE _x GE	$a = 0.6498(1.2631)$	$b = 13.6027(4.4443)$	$\lambda = 0.6222(0.2613)$	$\theta = 0.0623(0.1209)$
HEE	$\alpha = 3.5443(0.9610)$	$\lambda = 0.0401(0.0067)$	$\theta = 951.7063(976.1832)$	
BE	$a = 33.5316(25.4898)$	$b = 0.5119(0.2359)$	$\lambda = 0.0767(0.02505)$	
GTE	$a = 16.0078(5.2511)$	$b = 0.0010(0.5567)$	$\lambda = 0.0657(116.1412)$	$\theta = 0.0484(0.0056)$
ExE	$a = 16.0062(5.2159)$	$c = 0.0484(0.0056)$		
APE	$\alpha = 144176100(8388)$	$a = 0.0050(0.0002)$		
ExW	$\beta = 0.5822(0.2089)$	$\lambda = 0.4825(0.5860)$	$\theta = 157.8721(298.9929)$	
MOE	$\alpha = 75.3745(44.0188)$	$\lambda = 0.066(0.0084)$		
G	$a = 6.6558(1.2051)$	$b = 0.0942(0.0177)$		
CGTE	$a = 0.0386(0.0146)$	$\lambda = 0.9676(0.0233)$	$\theta = 0.0100(0.0175)$	
W	$\beta = 2.2499(79.8433)$	$\lambda = 0.1954(4.9612)$		
EOWE	$\alpha = 16.728(12.1169)$	$\beta = 160.1196(99.8510)$	$\lambda = 0.1344(0.0588)$	
TE	$\lambda = -1.0000(0.6535)$	$\theta = 0.0207(0.0032)$		
E	$a = 0.0142(0.0018)$			

Table 5. Discrimination measures of the APExE model and other competing models.

Model	AKI	CAKI	BAI	HAQUI	CRVMI	ANDA	KOSM	p -Value
APExE	536.6166	537.0610	542.7979	539.0244	0.12743	0.70096	0.09917	0.61833
TGE	538.7118	539.1563	544.8932	541.1196	0.16417	0.89289	0.10256	0.57523
TE _x GE	540.7119	541.4666	548.9536	543.9222	0.16414	0.89279	0.10264	0.57418
HEE	537.6917	538.1362	543.8731	540.0995	0.12755	0.71904	0.10325	0.56659
BE	539.2951	539.7395	545.4764	541.7028	0.20575	1.11650	0.11309	0.44841
GTE	542.9731	543.7278	551.2148	546.1834	0.20153	1.09101	0.11331	0.44582
ExE	538.9731	539.1913	543.0940	540.5782	0.20154	1.09102	0.11336	0.44533
APE	539.4181	539.6363	543.5390	541.0233	0.20404	1.10685	0.11356	0.44309
ExW	538.3406	538.7850	544.5219	540.7483	0.16188	0.89393	0.11738	0.40120
MOE	552.6302	552.8483	556.7510	554.2353	0.39977	2.21493	0.13766	0.22173
G	546.5680	546.7862	550.6889	548.1731	0.32475	1.76169	0.14778	0.15870
CGTE	546.0846	546.5290	552.2659	548.4923	0.25008	1.38560	0.15902	0.10643
W	562.8610	563.0790	566.9820	564.4660	0.60409	3.35661	0.17806	0.05056
EOWE	619.4344	619.8789	625.6158	621.8422	0.33115	1.79688	0.39919	0.00001
TE	580.8023	581.0205	584.9232	582.4075	0.30278	1.64113	0.29592	0.00001
E	611.9344	612.0058	613.9949	612.7370	0.32364	1.75539	0.38643	0.00001

Figure 6. The fitted APExE PDF, CDF, SF, and P–P plots for insurance data.

Figure 7. The P–P plots and histogram of insurance data with the fitted APExE density for estimation methods.

Table 6. The estimates of the APExE parameters along with goodness-of-fit measures for insurance data.

Method	α	b	c	CRVMI	ANDA	KOSM	p -Value
OLSE	0.07212	0.04361	22.08913	0.11186	0.63023	0.06228	0.97805
CVME	0.13154	0.05054	30.46968	0.12014	0.68067	0.06540	0.96517
WLSE	0.04363	0.03666	15.43835	0.11500	0.63903	0.07948	0.85719
ADE	0.03780	0.03370	12.99506	0.12089	0.66759	0.09101	0.72273
MLE	0.03943	0.03191	11.52520	0.12743	0.70096	0.09917	0.61833
PE	0.04501	0.02207	5.592190	0.17006	0.92493	0.17028	0.06922

8. Conclusions

In this paper, we propose a new heavy-tailed distribution to model heavy-tailed insurance data, called alpha power exponentiated exponential (APExE) distribution that extends the exponential (E), exponentiated exponential (ExE), and alpha power exponential (APE) distributions. Its associated hazard rate function can be bathtub, unimodal, decreasing, decreasing-constant, increasing, and reversed-J shaped. Some of its statistical properties are derived. The risk measures such as value at risk, tail value at risk, tail variance, and tail variance premium are derived for the APExE distribution along with a conducted simulation study for these actuarial measures, proving that the APExE distribution has a heavier tail than the APE, ExE, and E distributions. Its unknown parameters are estimated by six frequentist estimation approaches. The practical applicability of the APExE distribution has been illustrated by an insurance real-life data, proving its superiority over fifteen competing models.

The research in this article can be extended in some ways. For example, the APExE distribution can be utilized for analyzing and modeling data in other fields such as reliability engineering, medicine, economics, survival analyses, and life testing.

Bayesian estimation of the APExE parameters based on complete and several types of censored samples under different types of loss functions could be discussed. Furthermore, a bivariate extension of the APExE distribution may also be studied.

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Abbreviations

CDF	Cumulative distribution function
PDF	Probability density function
SF	Survival function
QF	Quantile function
HRF	Hazard rate function
VaR	Value at risk
TVar	Tail value at risk
TV	Tail variance
TVP	Tail variance premium
MLE	Maximum likelihood estimators
OLSE	Ordinary least-squares estimators
WLSE	Weighted-least-squares estimators
ADE	Anderson–Darling estimators
CVME	Cramér–von Mises estimators
PE	Percentile estimators

AEs	Average values of the estimates
ABs	Average absolute biases
MSEs	Average mean square error
MREs	Average mean relative estimates
AKI	Akaike information
CAKI	consistent Akaike information
BAI	Bayesian information
HAQUI	Hannan–Quinn information
ANDA	Anderson Darling
CRVMI	Cramér–von Mises
KOSM	Kolmogorov–Smirnov
P–P	probability-probability
APExE	Alpha power exponentiated exponential
ExE	Exponentiated exponential
E	exponential
APE	Alpha power exponential
BE	Beta exponential
EOWE	Extended odd Weibull exponential
ExW	Exponentiated Weibull
W	Weibull
TGE	Transmuted generalized exponential
MOE	Marshall–Olkin exponential
GTE	Generalized transmuted exponential
TExGE	Transmuted exponentiated generalized exponential
CGTE	Complementary geometric transmuted exponential
G	Gamma
HEE	Harris extended exponential
TE	Transmuted exponential

Appendix A

Table A1. The AEs, ABs, MSEs, and MREs for ($\alpha = 0.5, a = 0.5, c = 0.5$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	0.49429	0.51330	0.51074	0.09209	0.07908	0.06605	0.00895	0.00723	0.00554	0.18418	0.15816	0.13209
	50	0.50291	0.51300	0.51255	0.08907	0.06948	0.05312	0.00855	0.00602	0.00391	0.17815	0.13897	0.10624
	100	0.50643	0.50955	0.50638	0.08728	0.06027	0.04149	0.00830	0.00482	0.00259	0.17455	0.12054	0.08298
	200	0.50175	0.50658	0.50543	0.08667	0.05194	0.03405	0.00823	0.00374	0.00178	0.17335	0.10388	0.06809
ADE	20	0.50232	0.49713	0.50131	0.04771	0.04409	0.04123	0.00235	0.00211	0.00191	0.09542	0.08819	0.08246
	50	0.50331	0.49744	0.50121	0.04631	0.04161	0.03597	0.00226	0.00195	0.00157	0.09262	0.08322	0.07194
	100	0.49648	0.50209	0.50032	0.04561	0.03896	0.03088	0.00221	0.00177	0.00125	0.09122	0.07792	0.06175
	200	0.49756	0.50190	0.50126	0.04426	0.03583	0.02618	0.00212	0.00155	0.00095	0.08852	0.07167	0.05236
CVME	20	0.44678	0.48931	0.52387	0.14734	0.08400	0.07143	0.02989	0.00785	0.00627	0.29468	0.16800	0.14285
	50	0.43848	0.48734	0.52188	0.14614	0.07719	0.06257	0.02924	0.00696	0.00510	0.29228	0.15437	0.12513
	100	0.44831	0.48581	0.51659	0.13633	0.07128	0.05391	0.02558	0.00625	0.00400	0.27266	0.14255	0.10782
	200	0.45154	0.48000	0.51335	0.13280	0.06533	0.04154	0.02414	0.00541	0.00258	0.26561	0.13065	0.08309
OLSE	20	0.50862	0.48615	0.49541	0.09172	0.08181	0.06842	0.00887	0.00759	0.00581	0.18344	0.16361	0.13683
	50	0.50814	0.48762	0.49785	0.08968	0.07586	0.05534	0.00862	0.00674	0.00418	0.17936	0.15173	0.11067
	100	0.50062	0.49167	0.49679	0.08878	0.06592	0.04527	0.00850	0.00552	0.00294	0.17755	0.13184	0.09054
	200	0.50256	0.49153	0.49772	0.08858	0.05547	0.03584	0.00848	0.00417	0.00197	0.17716	0.11094	0.07169
WLSE	20	0.50335	0.48766	0.49226	0.09088	0.08232	0.06513	0.00879	0.00766	0.00547	0.18177	0.16464	0.13025
	50	0.50023	0.49563	0.49867	0.08936	0.07407	0.05241	0.00859	0.00655	0.00382	0.17871	0.14814	0.10481
	100	0.49677	0.49706	0.50227	0.08852	0.06238	0.04387	0.00846	0.00505	0.00281	0.17705	0.12476	0.08773
	200	0.49767	0.50182	0.50412	0.08813	0.05208	0.03267	0.00841	0.00376	0.00165	0.17626	0.10415	0.06535
PE	20	0.52555	0.47740	0.51429	0.09335	0.08291	0.07667	0.18670	0.16581	0.15334	0.18670	0.16581	0.15334
	50	0.51727	0.46863	0.49572	0.09196	0.07780	0.06644	0.18392	0.15561	0.13288	0.18392	0.15561	0.13288
	100	0.50209	0.47350	0.48650	0.09026	0.07242	0.06021	0.18053	0.14483	0.12043	0.18053	0.14483	0.12043
	200	0.49351	0.47594	0.48729	0.08957	0.06429	0.05581	0.17914	0.12857	0.11161	0.17914	0.12857	0.11161

Table A2. The AEs, ABs, MSEs, and MREs for ($\alpha = 2, a = 0.5, c = 0.5$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	1.99030	0.51376	0.51136	0.19184	0.07348	0.07082	0.03787	0.00655	0.00613	0.09592	0.14696	0.14163
	50	1.99100	0.50721	0.51217	0.19077	0.06323	0.05859	0.03767	0.00519	0.00458	0.09538	0.12645	0.11718
	100	1.99119	0.50688	0.51000	0.18885	0.05173	0.04920	0.03720	0.00373	0.00344	0.09442	0.10345	0.09840
	200	1.99023	0.50758	0.50585	0.18696	0.04040	0.03619	0.03669	0.00248	0.00202	0.09348	0.08079	0.07238
ADE	20	1.99942	0.49763	0.49587	0.19195	0.07481	0.06944	0.03781	0.00668	0.00603	0.09597	0.14962	0.13888
	50	2.00634	0.50083	0.50082	0.18989	0.06477	0.05838	0.03738	0.00535	0.00458	0.09495	0.12954	0.11676
	100	1.99476	0.50006	0.50195	0.18923	0.05279	0.04969	0.03710	0.00391	0.00348	0.09461	0.10558	0.09939
	200	2.00669	0.50318	0.50208	0.18745	0.04215	0.03882	0.03664	0.00265	0.00228	0.09373	0.08430	0.07764
CVME	20	2.03250	0.51031	0.50510	0.32277	0.07458	0.07195	0.11214	0.00666	0.00627	0.16138	0.14915	0.14390
	50	2.04455	0.50869	0.50601	0.32238	0.06709	0.06177	0.11189	0.00565	0.00499	0.16119	0.13418	0.12355
	100	2.03861	0.50413	0.50273	0.32058	0.05783	0.05006	0.11092	0.00443	0.00359	0.16029	0.11566	0.10012
	200	2.03448	0.50510	0.50315	0.31826	0.04682	0.04269	0.10969	0.00314	0.00267	0.15913	0.09364	0.08539
OLSE	20	2.03997	0.48375	0.49568	0.44086	0.07665	0.07316	0.21238	0.00692	0.00647	0.22043	0.15331	0.14632
	50	2.02901	0.49152	0.49704	0.43799	0.06783	0.06209	0.20881	0.00577	0.00506	0.21899	0.13566	0.12419
	100	2.01371	0.49639	0.50005	0.43696	0.05853	0.05481	0.20852	0.00459	0.00411	0.21848	0.11706	0.10961
	200	2.01453	0.49809	0.50472	0.43660	0.04640	0.04498	0.20791	0.00314	0.00295	0.21830	0.09280	0.08996
WLSE	20	2.01427	0.49289	0.49113	0.36843	0.07652	0.07104	0.14320	0.00688	0.00621	0.18421	0.15304	0.14209
	50	1.99966	0.49676	0.49939	0.36484	0.06656	0.05919	0.14084	0.00558	0.00467	0.18242	0.13312	0.11839
	100	2.03515	0.49557	0.50172	0.35806	0.05540	0.04953	0.13756	0.00420	0.00350	0.17903	0.11079	0.09906
	200	2.01349	0.50073	0.50401	0.35687	0.04338	0.04212	0.13694	0.00277	0.00261	0.17843	0.08675	0.08424
PE	20	2.12964	0.46124	0.47823	0.30129	0.12978	0.12389	0.10217	0.02141	0.01994	0.15065	0.25955	0.24779
	50	2.07355	0.46240	0.47181	0.28488	0.10534	0.11068	0.09188	0.01514	0.01672	0.14244	0.21068	0.22137
	100	2.03807	0.47166	0.47057	0.27639	0.08607	0.08974	0.08608	0.01075	0.01158	0.13870	0.17213	0.17947
	200	2.04661	0.48058	0.47571	0.27303	0.06186	0.06792	0.08588	0.00584	0.00706	0.13601	0.12371	0.13583

Table A3. The AEs, ABs, MSEs, and MREs for ($\alpha = 2, a = 1.5, c = 0.5$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	1.95092	1.47863	0.50578	0.40619	0.12256	0.07696	0.17873	0.01812	0.00783	0.20309	0.08171	0.15392
	50	1.91835	1.48625	0.50869	0.40320	0.11155	0.06099	0.17724	0.01561	0.00511	0.20160	0.07437	0.12198
	100	1.88769	1.48459	0.51262	0.40008	0.10198	0.04925	0.17619	0.01356	0.00354	0.20004	0.06799	0.09850
	200	1.91756	1.48379	0.51005	0.38549	0.09322	0.04186	0.16672	0.01146	0.00261	0.19275	0.06215	0.08373
ADE	20	2.00856	1.49588	0.51393	0.47681	0.09381	0.10027	0.23439	0.00916	0.01578	0.23841	0.06254	0.20054
	50	1.98429	1.49783	0.51354	0.46765	0.08704	0.06865	0.22837	0.00828	0.00754	0.23382	0.05803	0.13731
	100	2.00514	1.49510	0.50690	0.44871	0.08462	0.05472	0.21522	0.00796	0.00468	0.22436	0.05641	0.10944
	200	2.02087	1.49116	0.50548	0.42898	0.07587	0.04227	0.20225	0.00681	0.00279	0.21449	0.05058	0.08454
CVME	20	1.93732	1.51731	0.53416	0.46717	0.09339	0.10712	0.22935	0.00912	0.01808	0.23359	0.06226	0.21425
	50	1.97703	1.50451	0.51809	0.46599	0.08872	0.07371	0.22814	0.00853	0.00911	0.23299	0.05915	0.14742
	100	1.97302	1.50208	0.51472	0.45597	0.08701	0.05791	0.22088	0.00825	0.00551	0.22799	0.05801	0.11582
	200	1.98525	1.50327	0.51014	0.43489	0.07891	0.04666	0.20753	0.00721	0.00343	0.21744	0.05261	0.09332
OLSE	20	2.06512	1.48082	0.51193	0.47487	0.09266	0.10378	0.23473	0.00902	0.01685	0.23744	0.06177	0.20756
	50	2.04507	1.48920	0.50330	0.47155	0.08888	0.07536	0.23163	0.00853	0.00934	0.23577	0.05925	0.15071
	100	2.01887	1.49448	0.50216	0.45121	0.08430	0.05381	0.21848	0.00792	0.00462	0.22560	0.05620	0.10761
	200	2.02143	1.49272	0.50187	0.43887	0.07957	0.04476	0.20909	0.00727	0.00313	0.21943	0.05305	0.08953
WLSE	20	2.11441	1.49127	0.49304	0.37496	0.09275	0.06988	0.15622	0.00906	0.00605	0.18748	0.06183	0.13975
	50	2.10743	1.49499	0.49564	0.36414	0.08797	0.05684	0.15084	0.00840	0.00433	0.18207	0.05864	0.11367
	100	2.08991	1.50291	0.49575	0.36317	0.08268	0.04622	0.14792	0.00768	0.00309	0.18159	0.05512	0.09244
	200	2.10289	1.50092	0.49829	0.35583	0.07770	0.03921	0.14608	0.00706	0.00229	0.17792	0.05180	0.07843
PE	20	2.21283	1.47193	0.51166	0.40805	0.09383	0.07742	0.18164	0.00914	0.00702	0.20402	0.06255	0.15485
	50	2.20788	1.46869	0.50735	0.40236	0.08975	0.06643	0.17841	0.00863	0.00562	0.20118	0.05984	0.13285
	100	2.17682	1.47167	0.49897	0.39531	0.08702	0.05316	0.17213	0.00829	0.00392	0.19766	0.05801	0.10633
	200	2.13365	1.47332	0.49006	0.38478	0.08416	0.04545	0.16350	0.00785	0.00296	0.19239	0.05611	0.09091

Table A4. The AEs, ABs, MSEs, and MREs for ($\alpha = 2, a = 1.5, c = 3$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	1.90575	1.48684	3.08247	0.41613	0.10653	0.39385	0.18518	0.01446	0.17911	0.20806	0.07102	0.13128
	50	1.90493	1.48784	3.06355	0.40262	0.09259	0.34392	0.17748	0.01142	0.14746	0.20131	0.06173	0.11464
	100	1.91548	1.49199	3.05693	0.38741	0.08492	0.31594	0.16844	0.00994	0.12890	0.19370	0.05662	0.10531
	200	1.93326	1.49271	3.04026	0.38561	0.07056	0.26763	0.16605	0.00704	0.10052	0.19280	0.04704	0.08921
ADE	20	1.89539	1.46164	2.94206	0.41643	0.11170	0.38176	0.18611	0.01602	0.17197	0.20821	0.07447	0.12725
	50	1.91628	1.47359	2.99873	0.39949	0.10039	0.36059	0.17510	0.01318	0.15568	0.19975	0.06693	0.12020
	100	1.91697	1.48110	2.99628	0.38777	0.08968	0.31907	0.16843	0.01101	0.13048	0.19388	0.05979	0.10636
	200	1.90373	1.48471	3.02791	0.38224	0.07732	0.27984	0.16523	0.00841	0.10665	0.19112	0.05155	0.09328
CVME	20	1.91625	1.47001	3.00302	0.40681	0.11092	0.39620	0.17987	0.01582	0.18053	0.20341	0.07395	0.13207
	50	1.90316	1.47808	3.04251	0.39691	0.10233	0.36276	0.17391	0.01373	0.15978	0.19846	0.06822	0.12092
	100	1.90716	1.48511	3.04471	0.38571	0.09095	0.33434	0.16658	0.01130	0.14176	0.19285	0.06063	0.11145
	200	1.91176	1.48817	3.04623	0.38045	0.08156	0.29805	0.16331	0.00922	0.11812	0.19022	0.05437	0.09935
OLSE	20	1.93345	1.49679	3.07456	0.43800	0.08428	0.27356	0.20861	0.00790	0.09833	0.21900	0.05619	0.09119
	50	1.92097	1.49647	3.05617	0.41169	0.07800	0.25254	0.19175	0.00704	0.08694	0.20584	0.05200	0.08415
	100	1.89309	1.49499	3.07546	0.39370	0.07387	0.25216	0.18033	0.00654	0.08690	0.19685	0.04925	0.08409
	200	1.92904	1.49433	3.05546	0.38635	0.06789	0.23046	0.17403	0.00575	0.07405	0.19317	0.04526	0.07682
WLSE	20	1.99071	1.47470	2.95573	0.04954	0.10447	0.18157	0.00247	0.01440	0.03510	0.02477	0.06964	0.06052
	20	1.99049	1.48456	2.96394	0.04944	0.08428	0.17261	0.00246	0.00983	0.03273	0.02472	0.05619	0.05754
	100	1.99205	1.49169	2.97633	0.04925	0.07023	0.16445	0.00245	0.00701	0.03076	0.02463	0.04682	0.05482
	200	1.99681	1.50024	2.99031	0.04916	0.05478	0.15674	0.00244	0.00452	0.02857	0.02458	0.03652	0.05225
PE	20	1.99682	1.43802	2.88990	0.04957	0.12478	0.38735	0.00247	0.01947	0.17476	0.02479	0.08319	0.12912
	50	1.99214	1.44187	2.85397	0.04938	0.11444	0.35925	0.00246	0.01706	0.15601	0.02469	0.07630	0.11975
	100	1.98632	1.45081	2.84300	0.04927	0.09965	0.34272	0.00245	0.01343	0.14548	0.02464	0.06643	0.11424
	200	1.98246	1.45661	2.86540	0.04921	0.08777	0.31131	0.00245	0.01073	0.12629	0.02460	0.05851	0.10377

Table A5. The AEs, ABs, MSEs, and MREs for ($\alpha = 0.5, a = 1.5, c = 0.5$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	0.44893	1.47999	0.51836	0.13319	0.12530	0.07178	0.02105	0.01872	0.00677	0.26639	0.08354	0.14355
	50	0.45794	1.47563	0.51667	0.12486	0.12045	0.05686	0.01885	0.01763	0.00446	0.24971	0.08030	0.11371
	100	0.45934	1.46441	0.51168	0.11909	0.12021	0.04455	0.01752	0.01785	0.00294	0.23817	0.08014	0.08909
	200	0.46604	1.46776	0.50855	0.11252	0.10795	0.03515	0.01581	0.01514	0.00195	0.22504	0.07197	0.07030
ADE	20	0.47160	1.44672	0.50761	0.12836	0.13625	0.07003	0.01953	0.02199	0.00660	0.25672	0.09084	0.14006
	50	0.45994	1.44124	0.51046	0.12349	0.13442	0.05475	0.01847	0.02172	0.00431	0.24699	0.08962	0.10951
	100	0.45391	1.44934	0.51429	0.12071	0.12459	0.04646	0.01806	0.01917	0.00317	0.24143	0.08306	0.09291
	200	0.46225	1.44600	0.50939	0.11033	0.11731	0.03735	0.01560	0.01771	0.00212	0.22067	0.07821	0.07469
CVME	20	0.49987	1.50186	0.50531	0.04746	0.09380	0.06752	0.00234	0.00917	0.00571	0.09493	0.06253	0.13504
	50	0.49791	1.50297	0.50347	0.04737	0.08949	0.05080	0.00233	0.00859	0.00366	0.09474	0.05966	0.10161
	100	0.49712	1.50287	0.50397	0.04649	0.08801	0.04002	0.00227	0.00843	0.00245	0.09298	0.05867	0.08004
	200	0.50254	1.49432	0.50172	0.04574	0.08278	0.02875	0.00222	0.00774	0.00131	0.09148	0.05518	0.05750
OLSE	20	0.50660	1.48350	0.49639	0.04816	0.09448	0.06709	0.00237	0.00926	0.00566	0.09632	0.06298	0.13417
	50	0.50422	1.48646	0.49435	0.04749	0.09052	0.05175	0.00233	0.00874	0.00377	0.09497	0.06034	0.10350
	100	0.50265	1.49215	0.49890	0.04713	0.08775	0.04013	0.00231	0.00838	0.00242	0.09426	0.05850	0.08025
	200	0.50096	1.49265	0.50177	0.04594	0.08330	0.02898	0.00223	0.00776	0.00133	0.09189	0.05554	0.05795
WLSE	20	0.50113	1.49225	0.49513	0.04782	0.09373	0.06854	0.00236	0.00915	0.00579	0.09565	0.06249	0.13708
	50	0.50101	1.49421	0.50093	0.04723	0.09024	0.05114	0.00232	0.00868	0.00364	0.09445	0.06016	0.10228
	100	0.50148	1.49185	0.50120	0.04644	0.08814	0.03809	0.00226	0.00841	0.00223	0.09287	0.05876	0.07618
	200	0.49816	1.49748	0.50185	0.04603	0.08102	0.02886	0.00224	0.00748	0.00130	0.09205	0.05402	0.05772
PE	20	0.51263	1.47425	0.51912	0.04891	0.09520	0.07880	0.00243	0.00935	0.00721	0.09782	0.06347	0.15759
	50	0.51506	1.47151	0.52122	0.04845	0.09249	0.06937	0.00240	0.00900	0.00595	0.09690	0.06166	0.13874
	100	0.51296	1.46755	0.51050	0.04784	0.08986	0.05750	0.00236	0.00867	0.00447	0.09569	0.05990	0.11500
	200	0.51032	1.47434	0.50326	0.04766	0.08812	0.04423	0.00235	0.00840	0.00285	0.09533	0.05875	0.08846

Table A6. The AEs, ABs, MSEs, and MREs for ($\alpha = 0.5, a = 1.5, c = 3$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	0.48970	1.47264	2.97345	0.09109	0.11684	0.21614	0.00882	0.02013	0.05247	0.18219	0.07789	0.07205
	50	0.49660	1.48197	2.98771	0.09098	0.09904	0.19932	0.00880	0.01447	0.04639	0.18196	0.06603	0.06644
	100	0.49528	1.49105	2.98522	0.08872	0.08547	0.18765	0.00849	0.01032	0.04282	0.17744	0.05698	0.06255
	200	0.49668	1.49662	3.00523	0.08743	0.07363	0.16368	0.00835	0.00771	0.03374	0.17486	0.04909	0.05456
ADE	20	0.44427	1.45604	2.99165	0.13571	0.11229	0.25374	0.02180	0.01640	0.07139	0.27143	0.07486	0.08458
	50	0.43290	1.46031	2.98915	0.13431	0.10361	0.23943	0.02170	0.01439	0.06587	0.26861	0.06880	0.07981
	100	0.44713	1.44637	2.99080	0.12588	0.10224	0.22959	0.01954	0.01416	0.06133	0.25175	0.06843	0.07653
	200	0.45308	1.45530	2.99265	0.12436	0.09061	0.19993	0.01904	0.01166	0.05018	0.24872	0.06041	0.06664
CVME	20	0.51399	1.53061	3.01520	0.09074	0.10916	0.09525	0.00879	0.01538	0.00937	0.18147	0.07277	0.03175
	50	0.51267	1.52197	3.01816	0.08946	0.09118	0.09375	0.00859	0.01163	0.00917	0.17892	0.06079	0.03125
	100	0.51477	1.50753	3.00556	0.08894	0.07651	0.09123	0.00847	0.00836	0.00883	0.17788	0.05100	0.03041
	200	0.50357	1.50638	3.00538	0.08776	0.06793	0.08950	0.00838	0.00665	0.00861	0.17551	0.04528	0.02983
OLSE	20	0.50027	1.48817	2.94953	0.04766	0.08339	0.13944	0.00235	0.00779	0.02263	0.09532	0.05559	0.04648
	50	0.50134	1.48874	2.94830	0.04617	0.07580	0.13327	0.00225	0.00676	0.02127	0.09234	0.05053	0.04442
	100	0.50172	1.48537	2.94841	0.04561	0.06822	0.12900	0.00223	0.00581	0.02027	0.09221	0.04548	0.04300
	200	0.50267	1.48848	2.96272	0.04507	0.05864	0.12064	0.00221	0.00463	0.01797	0.09113	0.03909	0.04021
WLSE	20	0.49794	1.49819	2.98235	0.04814	0.08016	0.17681	0.00237	0.00743	0.03373	0.09628	0.05344	0.05894
	50	0.50058	1.49326	3.00066	0.04721	0.07442	0.16369	0.00231	0.00660	0.03033	0.09441	0.04961	0.05456
	100	0.49673	1.49743	2.99070	0.04575	0.06733	0.15784	0.00222	0.00571	0.02871	0.09159	0.04489	0.05261
	200	0.49916	1.49503	2.99735	0.04556	0.06103	0.14676	0.00221	0.00491	0.02605	0.09102	0.04069	0.04892
PE	20	0.50975	1.47135	3.00965	0.04745	0.08714	0.17785	0.00233	0.00829	0.03403	0.09490	0.05810	0.05928
	50	0.50683	1.47543	2.98678	0.04722	0.08359	0.16633	0.00232	0.00780	0.03101	0.09444	0.05573	0.05544
	100	0.50181	1.47720	2.97307	0.04520	0.07904	0.15933	0.00218	0.00721	0.02912	0.09040	0.05270	0.05311
	200	0.49846	1.47443	2.96308	0.04402	0.07222	0.14971	0.00210	0.00636	0.02674	0.08804	0.04815	0.04990

Table A7. The AEs, ABs, MSEs, and MREs for ($\alpha = 0.5, a = 0.5, c = 3$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	0.64736	0.50453	2.98921	0.39520	0.09017	0.23920	0.17490	0.01209	0.06795	0.79041	0.18034	0.07973
	50	0.61230	0.49620	2.95488	0.37197	0.07935	0.23248	0.16095	0.00867	0.06666	0.74394	0.15870	0.07749
	100	0.61383	0.49630	2.95650	0.35090	0.07053	0.21340	0.14808	0.00710	0.05843	0.70181	0.14105	0.07113
	200	0.57303	0.49160	2.96619	0.31240	0.06276	0.18750	0.12419	0.00576	0.04678	0.62479	0.12551	0.06250
ADE	20	0.59594	0.48656	2.91129	0.38417	0.09516	0.26195	0.16650	0.01324	0.08256	0.76834	0.19032	0.08732
	50	0.57415	0.48272	2.91905	0.36896	0.08415	0.25176	0.15699	0.00945	0.07699	0.73791	0.16829	0.08392
	100	0.58554	0.48275	2.91575	0.36026	0.07695	0.22938	0.15197	0.00794	0.06743	0.72053	0.15391	0.07646
	200	0.57580	0.48474	2.92652	0.33971	0.07094	0.20520	0.13922	0.00683	0.05524	0.67942	0.14187	0.06840
CVME	20	0.50455	0.49727	2.95329	0.04752	0.03779	0.23739	0.00234	0.00170	0.06951	0.09505	0.07558	0.07913
	50	0.50676	0.49459	2.95007	0.04740	0.03280	0.22266	0.00233	0.00138	0.06257	0.09479	0.06560	0.07422
	100	0.50380	0.49706	2.96360	0.04717	0.02965	0.21293	0.00231	0.00116	0.05821	0.09435	0.05930	0.07098
	200	0.50391	0.49619	2.97183	0.04668	0.02602	0.19631	0.00228	0.00094	0.05004	0.09335	0.05204	0.06544
OLSE	20	0.50087	0.49094	2.87601	0.04801	0.03837	0.33254	0.00236	0.00174	0.13163	0.09602	0.07674	0.11085
	50	0.50395	0.49354	2.93578	0.04787	0.03399	0.27946	0.00235	0.00145	0.09894	0.09575	0.06798	0.09315
	100	0.50389	0.49207	2.93704	0.04733	0.03197	0.26648	0.00232	0.00131	0.09044	0.09467	0.06394	0.08883
	200	0.50689	0.49556	2.95854	0.04661	0.02792	0.23050	0.00227	0.00105	0.07058	0.09321	0.05584	0.07683
WLSE	20	0.50019	0.49544	2.97812	0.04761	0.03932	0.30865	0.00235	0.00179	0.11154	0.09522	0.07864	0.10288
	50	0.49742	0.49665	2.98502	0.04736	0.03483	0.28284	0.00232	0.00150	0.09798	0.09472	0.06966	0.09428
	100	0.49704	0.49945	3.00371	0.04693	0.03117	0.26000	0.00230	0.00126	0.08544	0.09387	0.06234	0.08667
	200	0.49687	0.49870	3.00274	0.04692	0.02749	0.22710	0.00229	0.00104	0.06920	0.09385	0.05499	0.07570
PE	20	0.51243	0.48365	2.93035	0.04788	0.04018	0.31427	0.00236	0.00186	0.11887	0.09577	0.08036	0.10476
	50	0.51014	0.48577	2.91254	0.04716	0.03853	0.28111	0.00231	0.00174	0.10079	0.09431	0.07706	0.09370
	100	0.50613	0.48596	2.88352	0.04650	0.03576	0.27995	0.00226	0.00156	0.10035	0.09299	0.07153	0.09332
	200	0.49612	0.48696	2.89119	0.04620	0.03201	0.26422	0.00225	0.00132	0.09045	0.09240	0.06403	0.08807

Table A8. The AEs, ABs, MSEs, and MREs for ($\alpha = 2, a = 0.5, c = 3$).

Method	n	AEs			ABs			MSEs			MREs		
		α	a	c	α	a	c	α	a	c	α	a	c
MLE	20	2.01611	0.50174	2.97926	0.28273	0.03736	0.23811	0.08320	0.00166	0.06825	0.14136	0.07472	0.07937
	50	2.03662	0.49735	2.98232	0.27283	0.03179	0.22155	0.07937	0.00131	0.06067	0.13642	0.06359	0.07385
	100	2.02276	0.49908	2.99057	0.27276	0.02723	0.20989	0.07926	0.00102	0.05464	0.13638	0.05446	0.06996
	200	2.03926	0.50030	2.98488	0.27144	0.02266	0.19129	0.07873	0.00075	0.04804	0.13572	0.04533	0.06376
ADE	20	1.96129	0.49123	2.92514	0.35253	0.03728	0.25122	0.13961	0.00165	0.07641	.17626	0.07455	0.08374
	50	1.97187	0.49524	2.94914	0.33427	0.03234	0.23479	0.12782	0.00134	0.06854	0.16714	0.06467	0.07826
	100	1.97514	0.49249	2.94603	0.32915	0.02901	0.22715	0.12567	0.00113	0.06389	0.16457	0.05803	0.07572
	200	1.99259	0.49515	2.96885	0.32491	0.02361	0.20108	0.12127	0.00082	0.05286	0.16245	0.04722	0.06703
CVME	20	1.97750	0.49501	2.96586	0.34121	0.03715	0.24342	0.13258	0.00166	0.07096	0.17060	0.07429	0.08114
	50	2.00867	0.49594	2.99050	0.33137	0.03216	0.22747	0.12440	0.00134	0.06290	0.16569	0.06433	0.07582
	100	2.00671	0.49525	2.95567	0.32929	0.02837	0.22696	0.12424	0.00110	0.06397	0.16464	0.05674	0.07565
	200	1.98458	0.49610	2.97665	0.32584	0.02559	0.20441	0.12150	0.00092	0.05363	0.16292	0.05119	0.06814
OLSE	20	1.95478	0.49048	2.91470	0.34805	0.03802	0.26080	0.13781	0.00171	0.08143	0.17402	0.07604	0.08693
	50	1.97065	0.49115	2.92102	0.33617	0.03335	0.24500	0.12852	0.00140	0.07430	0.16808	0.06669	0.08167
	100	1.98205	0.49310	2.94491	0.32891	0.03085	0.23563	0.12412	0.00123	0.06847	0.16445	0.06170	0.07854
	200	1.98150	0.49308	2.95659	0.32288	0.02632	0.21036	0.12049	0.00097	0.05661	0.16144	0.05264	0.07012
WLSE	20	1.94604	0.49353	2.91373	0.35632	0.03728	0.25976	0.14211	0.00165	0.08039	0.17816	0.07456	0.08659
	50	1.96691	0.49330	2.92956	0.34036	0.03306	0.24669	0.13152	0.00138	0.07472	0.17018	0.06612	0.08223
	100	1.98704	0.49448	2.95539	0.32669	0.02892	0.22438	0.12369	0.00112	0.06279	0.16334	0.05784	0.07479
	200	1.98095	0.49537	2.97746	0.32612	0.02488	0.19588	0.12274	0.00088	0.04978	0.16306	0.04977	0.06529
PE	20	1.90149	0.47874	2.96831	0.09851	0.04815	0.09650	0.01470	0.00316	0.00952	0.04925	0.09630	0.03217
	50	1.90422	0.48293	2.96947	0.09578	0.03627	0.09532	0.01423	0.00193	0.00938	0.04789	0.07255	0.03177
	100	1.90583	0.48909	2.97441	0.09417	0.02799	0.09432	0.01401	0.00116	0.00925	0.04709	0.05599	0.03144
	200	1.90921	0.49130	2.98109	0.09079	0.02204	0.09294	0.01348	0.00076	0.00906	0.04539	0.04408	0.03098

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