

Article

Precise Asymptotics for Bifurcation Curve of Nonlinear Ordinary Differential Equation

Tetsutaro Shibata

Laboratory of Mathematics, School of Engineering, Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima 739-8527, Japan; tshibata@hiroshima-u.ac.jp

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Abstract: We study the following nonlinear eigenvalue problem $-u''(t) = \lambda f(u(t))$, $u(t) > 0$, $t \in I := (-1, 1)$, $u(\pm 1) = 0$, where $f(u) = \log(1 + u)$ and $\lambda > 0$ is a parameter. Then λ is a continuous function of $\alpha > 0$, where α is the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ . We establish the precise asymptotic formula for $\lambda = \lambda(\alpha)$ as $\alpha \rightarrow \infty$ up to the third term of $\lambda(\alpha)$.

Keywords: precise asymptotics; global bifurcation; time-map argument

1. Introduction

We consider the following nonlinear eigenvalue problem

$$-u''(t) = \lambda f(u(t)), \quad t \in I := (-1, 1), \quad (1)$$

$$u(t) > 0, \quad t \in I, \quad (2)$$

$$u(-1) = u(1) = 0, \quad (3)$$

where $\lambda > 0$ is a parameter. In this paper, we mainly consider the case $f(u) = \log(1 + u)$. We know from [1] that, if $f(u)$ is continuous in $u \geq 0$ and positive for $u > 0$, then for a given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Since λ is a continuous for $\alpha > 0$, we write as $\lambda = \lambda(\alpha)$ for $\alpha > 0$.

Nonlinear eigenvalue and bifurcation problems have been one of the main topics in the study of nonlinear equations, and many authors have investigated the global behavior of bifurcation diagrams intensively. We refer to [2–7] and the references therein.

The purpose of this paper is to establish the asymptotic expansion formula for $\lambda(\alpha)$ when $\alpha \gg 1$ up to the third term by time-map method. Our study is motivated by the following inverse bifurcation problem.

Problem A. Consider (1)–(3). Assume that $f(u)$ is a given function with bifurcation curve $\lambda(\alpha)$, and another function $f_1(u)$ is unknown. Let $\lambda_1(\alpha)$ be the bifurcation curve associated with $f_1(u)$. Suppose that $\lambda_1(\alpha) = \lambda(\alpha)(1 + \delta_0(\alpha))$ as $\alpha \rightarrow \infty$, where $\delta_0(\alpha) \rightarrow 0$ as $\alpha \rightarrow \infty$. Then can we conclude that $f_1(u) = f(u)(1 + \delta_1(u))$ as $u \rightarrow \infty$? Here $\delta_1(u)$ is a function of u satisfying $\delta_1(u) \rightarrow 0$ as $u \rightarrow \infty$.

Since there are a few studies of inverse bifurcation problems, even the standard formulation of the inverse problems has not been established yet. Indeed, Problem A is a modified formulation which was proposed in [8]. It is natural to expect that if $\lambda(\alpha)$ and $\lambda_1(\alpha)$ are closer asymptotically, then $f(u)$ and $f_1(u)$ are also closer. Therefore, the first approach to Problem A is to study the “direct problems” precisely. Namely, we investigate the precise asymptotic formula for $\lambda(\alpha)$ as $\alpha \rightarrow \infty$ for $f(u) = \log(1 + u)$ here. Then, if we obtain the asymptotic behavior of $\lambda_1(\alpha)$ corresponding to $f_1(u) = f(u) + g(u)$, which was perturbed by some $g(u)$, then we will obtain the good evidence to propose the improved formulation of Problem A.

We introduce some known results about global behavior of bifurcation curve. One of the most famous results was shown for the one-dimensional Gelfand problem, namely, the Equations (1)–(3) with $f(u) = e^u$. It was shown in [9] that it has the exact solution

$$u_\alpha(t) = \alpha + \log \left(\operatorname{sech}^2 \left(\frac{\sqrt{2\lambda(\alpha)}}{2} t e^{\alpha/2} \right) \right), \quad (4)$$

where $\operatorname{sech} x = 1/\cosh x$. Then by time-map method, we explicitly obtain that for $\alpha > 0$ (cf. [10]),

$$\lambda(\alpha) = \frac{1}{2e^\alpha} \left[\log \left(2e^\alpha + 2\sqrt{e^\alpha(e^\alpha - 1)} - 1 \right) \right]^2. \quad (5)$$

Unfortunately, however, such explicit expression of bifurcation curves as (5) cannot be expected in general. For example, we introduce the following result.

Theorem 1 ([11]). *Let $f(u) = u + \sin u$ and consider (1)–(3). Then as $\alpha \rightarrow \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2}} \alpha^{-3/2} \sin \left(\alpha - \frac{\pi}{4} \right) + o \left(\alpha^{-3/2} \right). \quad (6)$$

The case $f(u) = u + \sin u$ was introduced in [12] first, which was inspired by [13]. We see from Theorem 1 that the oscillatory property of the nonlinear term $f(u)$ gives explicit effect to its bifurcation curve, as expected. Moreover, we understand the convergent behavior of $\lambda(\alpha)$ to the line $\lambda = \pi^2/4$, which is the “bifurcation curve” of (1)–(3) with $f(u) = u$.

As far as the author knows, however, the exact solution $u_\alpha(t)$ of the equation in Theorem 1 is not known, although it is quite simple, and it also seems impossible to obtain the explicit formula for $\lambda(\alpha)$ as (5). From this point of view, one of the standard approaches for the better understanding of the global structure of $\lambda(\alpha)$ is to establish precise asymptotic expansion formula for $\lambda(\alpha)$ as $\alpha \rightarrow \infty$. In some cases, the asymptotic expansion formulas for $\lambda(\alpha)$ up to the second term like (6) have been obtained. We refer to [11,14–16] and the references therein. However, to obtain the third term of $\lambda(\alpha)$, we need a very long and complicated calculation in general. We overcome this difficulty and establish the following results.

Theorem 2. *Let $f(u) = \log(1 + u)$ and consider (1)–(3). Then as $\alpha \rightarrow \infty$,*

$$\begin{aligned} \sqrt{\lambda(\alpha)} = & \sqrt{\frac{2\alpha}{\log(1+\alpha)}} \left\{ 1 - \frac{1}{2} (4\log 2 - 3) \frac{1}{\log(1+\alpha)} \right. \\ & \left. + \frac{3}{8} (5 - 8\log 2 + C_1) \frac{1}{(\log(1+\alpha))^2} \right\} + R_3, \end{aligned} \quad (7)$$

where R_3 is the remainder term satisfying

$$b_3(\log(1+\alpha))^{-3} \leq |R_3| \leq b_3^{-1}((\log(1+\alpha))^{-3}), \quad (8)$$

and $0 < b_3 < 1$ is a constant independent of $\alpha \gg 1$.

By Taylor expansion, it is easy to see that if $0 < u \ll 1$, then $\log(1 + u) = u + o(u)$. Therefore, (1) is approximated by $-u''_\alpha(t) = \lambda(\alpha)u_\alpha(t)$ when $\alpha = \|u_\alpha\|_\infty \ll 1$. So $\lambda(\alpha)$ starts from $(\alpha, \lambda) = (0, \pi^2/4)$.

Remark 1. *From a view point of asymptotic expansion formula for $\lambda(\alpha)$, it is natural to expect that the following asymptotic formula for $\sqrt{\lambda(\alpha)}$ holds.*

$$\sqrt{\lambda(\alpha)} = \sqrt{\frac{2\alpha}{\log(1+\alpha)}} \left\{ 1 + \sum_{n=1}^{\infty} b_n (\log(1+\alpha))^{-n} \right\}. \quad (9)$$

Here, $\{b_n\}$ ($n = 1, 2, \dots$) are expected to be constants, which are determined by induction. However, if the readers look at Section 3 below, then they understand immediately that it seems quite difficult to prove (9), since the calculation are quite long to obtain even the third term of (7).

The proof of Theorem 2 depends on the time-map method and Taylor expansion formula.

2. Second Term of $\lambda(\alpha)$ in Theorem 2

In this section, let $\alpha \gg 1$. In what follows, we denote by C the various positive constants independent of α . It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \leq t \leq 1, \quad (10)$$

$$u_\alpha(0) = \max_{-1 \leq t \leq 1} u_\alpha(t) = \alpha, \quad (11)$$

$$u'_\alpha(t) > 0, \quad -1 < t < 0. \quad (12)$$

By (1), we have

$$\{u''_\alpha(t) + \lambda(\log(1 + u_\alpha(t)))\} u'_\alpha(t) = 0.$$

By this, (11) and putting $t = 0$, we obtain

$$\begin{aligned} \frac{1}{2} u'_\alpha(t)^2 + \lambda \{u_\alpha(t) \log(1 + u_\alpha(t)) - u_\alpha(t) + \log(1 + u_\alpha(t))\} &= \text{const.} \\ &= \lambda \{\alpha \log(1 + \alpha) - \alpha + \log(1 + \alpha)\}. \end{aligned}$$

This along with (12) implies that for $-1 \leq t \leq 0$,

$$u'_\alpha(t) = \sqrt{2\lambda} \sqrt{\alpha \log(1 + \alpha) - u_\alpha(t) \log(1 + u_\alpha(t)) + \zeta(u_\alpha(t))}, \quad (13)$$

where

$$\zeta(u) := \log(1 + \alpha) - \log(1 + u) - (\alpha - u). \quad (14)$$

By this and putting $u_\alpha(t) = \alpha s^2$, we obtain

$$\begin{aligned} \sqrt{2\lambda} &= \int_{-1}^0 \frac{u'_\alpha(t)}{\sqrt{\alpha \log(1 + \alpha) - u_\alpha(t) \log(1 + u_\alpha(t)) + \zeta(u_\alpha(t))}} dt \\ &= \int_0^1 \frac{2\alpha s}{\sqrt{\alpha(1 - s^2) \log(1 + \alpha) + \alpha s^2 A_\alpha(s) + \zeta(\alpha s^2)}} ds \\ &:= \frac{2\sqrt{\alpha}}{\sqrt{\log(1 + \alpha)}} \int_0^1 \frac{s}{\sqrt{1 - s^2}} \frac{1}{\sqrt{1 + g_\alpha(s)}} ds, \end{aligned} \quad (15)$$

where

$$A_\alpha(s) := \log(1 + \alpha) - \log(1 + \alpha s^2), \quad (16)$$

$$g_\alpha(s) := \frac{1}{\log(1 + \alpha)} \frac{s^2}{(1 - s^2)} A_\alpha(s) + \frac{\zeta(\alpha s^2)}{\alpha(1 - s^2) \log(1 + \alpha)}. \quad (17)$$

For $0 \leq s \leq 1$, we have

$$\left| \frac{1}{\log(1+\alpha)} \frac{s^2}{(1-s^2)} A_\alpha(s) \right| \leq \frac{s^2}{1-s^2} \frac{1}{\log(1+\alpha)} \int_{\alpha s^2}^{\alpha} \frac{1}{1+x} dx \quad (18)$$

$$\begin{aligned} &\leq \frac{1}{\log(1+\alpha)} \frac{\alpha s^2}{1+\alpha s^2} \\ &\leq \frac{1}{\log(1+\alpha)} \ll 1, \end{aligned}$$

$$\left| \frac{\xi(\alpha s^2)}{\alpha(1-s^2)\log(1+\alpha)} \right| \leq \frac{2}{\log(1+\alpha)} \ll 1. \quad (19)$$

By this, (15) and Taylor expansion, we obtain

$$\begin{aligned} \sqrt{2\lambda} &= \frac{2\sqrt{\alpha}}{\sqrt{\log(1+\alpha)}} \int_0^1 \frac{s}{\sqrt{1-s^2}} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{n!2^n} g_\alpha(s)^n \right\} ds \\ &= \frac{2\sqrt{\alpha}}{\sqrt{\log(1+\alpha)}} \int_0^1 \frac{s}{\sqrt{1-s^2}} \left\{ 1 - \frac{1}{2} g_\alpha(s) + \sum_{n=2}^{\infty} (-1)^n \frac{(2n-1)!!}{n!2^n} g_\alpha(s)^n \right\} ds, \end{aligned} \quad (20)$$

where $(2n-1)!! = (2n-1)(2n-3) \cdots 3 \cdot 1$, $(-1)!! = 1$. We see from (20) that the second term of $\lambda(\alpha)$ in Theorem 2 follows from Lemma 1 below.

Lemma 1. As $\alpha \rightarrow \infty$,

$$L := \int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s) ds = \frac{1}{\log(1+\alpha)} (4\log 2 - 3) + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \quad (21)$$

The proof of Lemma 1 is a conclusion of Lemmas 2 and 3 below. By (17), we have

$$L = L_1 + L_2, \quad (22)$$

where

$$L_1 := \frac{1}{\log(1+\alpha)} \int_0^1 \frac{s^3}{(1-s^2)^{3/2}} A_\alpha(s) ds, \quad (23)$$

$$L_2 := \frac{1}{\alpha \log(1+\alpha)} \int_0^1 \frac{s}{(1-s^2)^{3/2}} \xi(\alpha s^2) ds. \quad (24)$$

Lemma 2. As $\alpha \rightarrow \infty$,

$$L_1 = (4\log 2 - 2) \frac{1}{\log(1+\alpha)} + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \quad (25)$$

Proof. We put $s = \sin \theta$ in (23). Then by integration by parts,

$$\begin{aligned} L_1 &= \frac{1}{\log(1+\alpha)} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \left\{ \sin^3 \theta A_\alpha(\sin \theta) \right\} d\theta \\ &= \frac{1}{\log(1+\alpha)} \left[\tan \theta \left\{ \sin^3 \theta A_\alpha(\sin \theta) \right\} \right]_0^{\pi/2} \\ &\quad - \frac{1}{\log(1+\alpha)} \int_0^{\pi/2} \tan \theta \left\{ \sin^3 \theta A_\alpha(\sin \theta) \right\}' d\theta \\ &:= Q_1 + Q_2 + Q_3, \end{aligned} \quad (26)$$

where

$$Q_1 := \frac{1}{\log(1+\alpha)} \left[\tan \theta \left\{ \sin^3 \theta A_\alpha(\sin \theta) \right\} \right]_0^{\pi/2}, \quad (27)$$

$$Q_2 := -\frac{3}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta A_\alpha(\sin \theta) d\theta, \quad (28)$$

$$Q_3 := \frac{2}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta. \quad (29)$$

Then by l'Hôpital's rule, we obtain

$$\lim_{\theta \rightarrow \pi/2} \frac{A_\alpha(\sin \theta)}{\cos \theta} = \lim_{\theta \rightarrow \pi/2} \frac{2\alpha \cos \theta}{1 + \alpha \sin^2 \theta} = 0. \quad (30)$$

By this, we see that $Q_1 = 0$. We next calculate Q_2 . For $0 \leq s \leq 1$, by Taylor expansion, we obtain

$$\begin{aligned} A_\alpha(s) &= \log(1+\alpha) - \log \alpha - \log \left(\frac{1}{\alpha} + s^2 \right) + 2 \log s - 2 \log s \\ &= \frac{1}{\alpha} + O(\alpha^{-2}) - \log \left(\frac{1}{\alpha} + s^2 \right) + 2 \log s - 2 \log s. \end{aligned} \quad (31)$$

Then by direct calculation, we obtain the following (32) and (33).

$$0 \leq \sin \theta A_\alpha(\sin \theta) = -2 \sin \theta \log(\sin \theta) + O \left(\frac{1}{\sqrt{\alpha}} \right), \quad (32)$$

$$0 \leq \sin^2 \theta A_\alpha(\sin \theta) = -2 \sin^2 \theta \log(\sin \theta) + O \left(\frac{1}{\alpha} \right). \quad (33)$$

The proofs of (32) and (33) are elementary but long and complicated. Therefore, the precise proofs of (32) and (33) will be given in Appendix A. For $0 \leq \theta \leq \pi/2$, we have

$$\sin^2 \theta - \frac{1}{\alpha} = \sin^2 \theta \left(1 - \frac{1}{\alpha \sin^2 \theta} \right) \leq \sin^2 \theta \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right) \leq \sin^2 \theta. \quad (34)$$

By (34), we obtain

$$\begin{aligned} Q_2 &= -\frac{3}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta A_\alpha(\sin \theta) d\theta \\ &= -\frac{3}{\log(1+\alpha)} \int_0^{\pi/2} \sin \theta \left\{ -2 \sin^2 \theta \log(\sin \theta) + O \left(\frac{1}{\alpha} \right) \right\} d\theta \\ &= \frac{6}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta \log(\sin \theta) d\theta + O \left(\frac{1}{\alpha \log(1+\alpha)} \right) \\ &= \frac{1}{\log(1+\alpha)} \left(4 \log 2 - \frac{10}{3} \right) + O \left(\frac{1}{\alpha \log(1+\alpha)} \right). \end{aligned} \quad (35)$$

By (33), we have

$$\begin{aligned} Q_3 &= \frac{2}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\ &= \frac{2}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta d\theta + O \left(\frac{1}{\alpha \log(1+\alpha)} \right) \\ &= \frac{4}{3 \log(1+\alpha)} + O \left(\frac{1}{\alpha \log(1+\alpha)} \right). \end{aligned} \quad (36)$$

By this, (30) and (35), we obtain (25). Thus the proof is complete. \square

Lemma 3. As $\alpha \rightarrow \infty$,

$$L_2 = -\frac{1}{\log(1+\alpha)} + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \quad (37)$$

Proof. We have

$$\begin{aligned} L_2 &= \frac{1}{\alpha \log(1+\alpha)} \int_0^1 \frac{s\{A_\alpha(s) - \alpha(1-s^2)\}}{(1-s^2)^{3/2}} ds \\ &:= L_{21} + L_{22}. \end{aligned} \quad (38)$$

Then

$$L_{22} = -\frac{1}{\log(1+\alpha)} \int_0^1 \frac{s}{(1-s^2)^{1/2}} ds = -\frac{1}{\log(1+\alpha)}. \quad (39)$$

By putting $s = \sin \theta$, (32), l'Hôpital's rule and integration by parts, we have

$$\begin{aligned} L_{21} &= \frac{1}{\alpha \log(1+\alpha)} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \{\sin \theta A_\alpha(\sin \theta)\} d\theta \\ &= \frac{1}{\alpha \log(1+\alpha)} [\tan \theta \sin \theta A_\alpha(\sin \theta)]_0^{\pi/2} \\ &\quad - \frac{1}{\alpha \log(1+\alpha)} \int_0^{\pi/2} \sin \theta A_\alpha(\sin \theta) d\theta + \frac{2}{\alpha \log(1+\alpha)} \int_0^{\pi/2} \frac{\alpha \sin^3 \theta}{1 + \alpha \sin^2 \theta} d\theta \\ &= O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \end{aligned} \quad (40)$$

By this, (38) and (39), we obtain (37). Thus the proof is complete. \square

By Lemmas 2 and 3, we obtain Lemma 1.

3. The Third Term of $\lambda(\alpha)$ in Theorem 2

By (20), to obtain the third term of $\lambda(\alpha)$, we calculate

$$M := \int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s)^2 ds. \quad (41)$$

We have

$$\begin{aligned} g_\alpha(s)^2 &= \frac{1}{(\log(1+\alpha))^2} \frac{s^4}{(1-s^2)^2} A_\alpha(s)^2 + \frac{1}{\alpha^2 (\log(1+\alpha))^2} \frac{\xi(\alpha s^2)^2}{(1-s^2)^2} \\ &\quad + 2 \frac{1}{\alpha (\log(1+\alpha))^2} \frac{s^2}{(1-s^2)^2} A_\alpha(s) \xi(\alpha s^2). \end{aligned} \quad (42)$$

We put

$$I_1 := \frac{1}{(\log(1+\alpha))^2} J_1 = \frac{1}{(\log(1+\alpha))^2} \int_0^1 \frac{s^5}{(1-s^2)^{5/2}} A_\alpha(s)^2 ds, \quad (43)$$

$$I_2 := \frac{1}{\alpha^2 (\log(1+\alpha))^2} J_2 = \frac{1}{\alpha^2 (\log(1+\alpha))^2} \int_0^1 \frac{s \xi(\alpha s^2)^2}{(1-s^2)^{5/2}} ds, \quad (44)$$

$$I_3 := \frac{2}{\alpha (\log(1+\alpha))^2} J_3 = \frac{2}{\alpha (\log(1+\alpha))^2} \int_0^1 \frac{s^3}{(1-s^2)^{5/2}} A_\alpha(s) \xi(\alpha s^2) ds. \quad (45)$$

Then $M = I_1 + I_2 + I_3$.

Lemma 4. As $\alpha \rightarrow \infty$,

$$I_1 = C_1 \frac{1}{(\log(1+\alpha))^2} + O\left(\frac{1}{\alpha(\log(1+\alpha))^2}\right), \quad (46)$$

where

$$C_1 := -\frac{4}{3}C_{11} + \frac{1}{3}(C_{12} + C_{13} + C_{14} + C_{15} + C_{16}), \quad (47)$$

$$C_{11} := 20 \int_0^{\pi/2} \sin^7 \theta (\log(\sin \theta))^2 d\theta - 8 \int_0^{\pi/2} \sin^7 \theta \log(\sin \theta) d\theta, \quad (48)$$

$$C_{12} := 100 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) (\log(\sin \theta))^2 d\theta, \quad (49)$$

$$C_{13} := 40 \int_0^{\pi/2} \sin^7 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta, \quad (50)$$

$$C_{14} := 8 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) d\theta, \quad (51)$$

$$C_{15} := 56 \int_0^{\pi/2} \sin^7 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta, \quad (52)$$

$$C_{16} := -16 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta. \quad (53)$$

Proof. We recall that

$$\frac{1}{\cos^4 \theta} = \frac{d}{d\theta} \left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right). \quad (54)$$

We put $s = \sin \theta$. Then we have

$$\begin{aligned} J_1 &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin^5 \theta A_\alpha(\sin \theta)^2 d\theta \\ &= \int_0^{\pi/2} \frac{d}{d\theta} \left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right) \sin^5 \theta A_\alpha(\sin \theta)^2 d\theta \\ &= \left[\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \sin^5 \theta A_\alpha(\sin \theta)^2 \right]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \frac{\sin \theta}{3 \cos^2 \theta} (2 \cos^2 \theta + 1) \left\{ 5 \sin^4 \theta A_\alpha(\sin \theta)^2 - 4 \alpha A_\alpha(\sin \theta) \sin^6 \theta \frac{1}{1 + \alpha \sin^2 \theta} \right\} d\theta. \end{aligned} \quad (55)$$

By using l'Hôpital's rule, we easily see that the first term of (55) is equal to 0. Then

$$\begin{aligned}
 J_1 &= -\frac{1}{3} \int_0^{\pi/2} (\tan \theta)' (2 \cos^2 \theta + 1) \\
 &\quad \times \left\{ 5 \sin^5 \theta A_\alpha(\sin \theta)^2 - 4\alpha A_\alpha(\sin \theta) \sin^7 \theta \frac{1}{1 + \alpha \sin^2 \theta} \right\} d\theta \\
 &= -\frac{1}{3} \left[\tan \theta (2 \cos^2 \theta + 1) \left\{ 5 \sin^5 \theta A_\alpha(\sin \theta)^2 - 4\alpha A_\alpha(\sin \theta) \sin^7 \theta \frac{1}{1 + \alpha \sin^2 \theta} \right\} \right]_0^{\pi/2} \\
 &\quad - \frac{4}{3} \int_0^{\pi/2} \sin^2 \theta \left\{ 5 \sin^5 \theta A_\alpha(\sin \theta)^2 - 4\alpha A_\alpha(\sin \theta) \sin^7 \theta \frac{1}{1 + \alpha \sin^2 \theta} \right\} d\theta \\
 &\quad + \frac{1}{3} \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) \left\{ 25 \sin^4 \theta A_\alpha(s)^2 - 20\alpha A_\alpha(\sin \theta) \sin^6 \theta \frac{1}{1 + \alpha \sin^2 \theta} \right. \\
 &\quad \left. + 8\alpha^2 \sin^8 \theta \frac{1}{(1 + \alpha \sin^2 \theta)^2} - 28\alpha \sin^6 \theta A_\alpha(\sin \theta) \frac{1}{1 + \alpha \sin^2 \theta} \right. \\
 &\quad \left. + 8\alpha^2 \sin^8 \theta A_\alpha(\sin \theta) \frac{1}{(1 + \alpha \sin^2 \theta)^2} \right\} d\theta \\
 &:= -\frac{1}{3} J_{10} - \frac{4}{3} J_{11} + \frac{1}{3} (J_{12} + J_{13} + J_{14} + J_{15} + J_{16}).
 \end{aligned} \tag{56}$$

By l'Hôpital's rule, we see that $J_{10} = 0$. By (33), (34) and (56), we have

$$\begin{aligned}
 J_{11} &= 5 \int_0^{\pi/2} \sin^7 \theta A_\alpha(\sin \theta)^2 d\theta - 4 \int_0^{\pi/2} \sin^5 \theta (\sin^2 \theta A_\alpha(\sin \theta)) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
 &= 20 \int_0^{\pi/2} \sin^3 \theta (\sin^2 \theta (\log(\sin \theta))^2) d\theta - 8 \int_0^{\pi/2} \sin^7 \theta \log(\sin \theta) d\theta + O(\alpha^{-1}) \\
 &:= C_{11} + O(\alpha^{-1}).
 \end{aligned} \tag{57}$$

By (33), (34) and (56), we obtain

$$\begin{aligned}
 J_{12} &= 25 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta)^2 d\theta \\
 &= 100 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) (\log(\sin \theta))^2 d\theta + O(\alpha^{-1}) \\
 &:= C_{12} + O(\alpha^{-1}),
 \end{aligned} \tag{58}$$

$$\begin{aligned}
 J_{13} &= -20 \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) \left(\sin^2 \theta \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right) \right) d\theta \\
 &= 40 \int_0^{\pi/2} \sin^7 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta + O(\alpha^{-1}) := C_{13} + O(\alpha^{-1}),
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 J_{14} &= 8 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 d\theta \\
 &= 8 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) d\theta + O(\alpha^{-1}) := C_{14} + O(\alpha^{-1}),
 \end{aligned} \tag{60}$$

$$\begin{aligned}
 J_{15} &= -28 \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right) d\theta \\
 &= 56 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta + O(\alpha^{-1}) \\
 &:= C_{15} + O(\alpha^{-1}).
 \end{aligned} \tag{61}$$

By (33) and (34), we have

$$\begin{aligned}
 J_{16} &= 8 \int_0^{\pi/2} (2 \cos^2 \theta + 1) \sin^4 \theta \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 (\sin \theta A_\alpha(\sin \theta)) d\theta \\
 &= 8 \int_0^{\pi/2} (2 \cos^2 \theta + 1) (\sin^4 \theta + O(\alpha^{-1})) (\sin \theta A_\alpha(\sin \theta)) d\theta \\
 &= -16 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta + O(\alpha^{-1}) \\
 &:= C_{16} + O(\alpha^{-1}).
 \end{aligned} \tag{62}$$

By (55)–(62), we obtain (46). Thus the proof is complete. \square

Lemma 5. As $\alpha \rightarrow \infty$,

$$I_2 = \frac{1}{(\log(1 + \alpha))^2} + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right). \tag{63}$$

Proof. By (14) and (44), we have

$$\begin{aligned}
 J_2 &= \int_0^1 \frac{s}{(1 - s^2)^{5/2}} (A_\alpha(s) - \alpha(1 - s^2))^2 ds \\
 &= J_{21} + J_{22} + J_{23} \\
 &:= \int_0^1 \frac{s}{(1 - s^2)^{5/2}} A_\alpha(s)^2 ds - 2\alpha \int_0^1 \frac{s}{(1 - s^2)^{3/2}} A_\alpha(s) ds + \alpha^2 \int_0^1 \frac{s}{\sqrt{1 - s^2}} ds.
 \end{aligned} \tag{64}$$

It is clear that $J_{23} = \alpha^2$. By (54), putting $s = \sin \theta$, integration by parts and l'Hôpital's rule, we obtain

$$\begin{aligned}
 J_{21} &= \int_0^{\pi/2} \frac{\sin \theta}{\cos^4 \theta} A_\alpha(\sin \theta)^2 d\theta \\
 &= \left[\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \sin \theta A_\alpha(\sin \theta)^2 \right]_0^{\pi/2} \\
 &\quad - \int_0^{\pi/2} \frac{\sin \theta}{3 \cos^2 \theta} (2 \cos^2 \theta + 1) \left\{ A_\alpha(\sin \theta)^2 - 4 A_\alpha(\sin \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right\} d\theta \\
 &= \left[-\frac{1}{3} \tan \theta \sin \theta (2 \cos^2 \theta + 1) \left\{ A_\alpha(\sin \theta)^2 - 4 A_\alpha(\sin \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right\} \right]_0^{\pi/2} \\
 &\quad + \frac{1}{3} \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta - 4 \sin^2 \theta + 1) \left\{ A_\alpha(\sin \theta)^2 - 4 A_\alpha(\sin \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right\} d\theta \\
 &\quad + \frac{1}{3} \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) \left\{ -4 \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} A_\alpha(\sin \theta) \right. \\
 &\quad \quad \left. + 8 \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 - 8 A_\alpha(\sin \theta) \frac{\alpha \sin^2 \theta}{(1 + \alpha \sin^2 \theta)^2} \right\} d\theta \\
 &= O(1).
 \end{aligned} \tag{65}$$

By integration by parts and l'Hôpital's rule, we obtain

$$\begin{aligned}
 J_{22} &= -2\alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin \theta A_\alpha(\sin \theta) d\theta \\
 &= [-2\alpha \tan \theta \sin \theta A_\alpha(\sin \theta)]_0^{\pi/2} \\
 &\quad + 2\alpha \int_0^{\pi/2} \left\{ \sin \theta A_\alpha(\sin \theta) - 2 \sin \theta \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right\} d\theta \\
 &= O(\alpha).
 \end{aligned} \tag{66}$$

By this and (65), we obtain (63). Thus the proof is complete. \square

Lemma 6. As $\alpha \rightarrow \infty$,

$$I_3 = \frac{4}{(\log(1 + \alpha))^2} (-2 \log 2 + 1) + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right). \tag{67}$$

Proof. By putting $s = \sin \theta$, we have

$$\begin{aligned}
 J_3 &= \int_0^1 \frac{s^3}{(1 - s^2)^{5/2}} A_\alpha(s) \zeta(\alpha s^2) ds \\
 &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin^3 \theta A_\alpha(\sin \theta)^2 d\theta - \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin^3 \theta A_\alpha(\sin \theta) d\theta \\
 &:= J_{31} - J_{32}.
 \end{aligned} \tag{68}$$

By (54) and integration by parts, we obtain

$$\begin{aligned}
 J_{31} &= \left[\left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right) \sin^3 \theta A_\alpha(\sin \theta)^2 \right]_0^{\pi/2} \\
 &\quad - \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (2 \cos^2 \theta + 1) \sin^3 \theta A_\alpha(\sin \theta)^2 d\theta \\
 &\quad + \frac{4}{3} \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (2 \cos^2 \theta + 1) \sin^5 \theta \frac{1}{1 + \alpha \sin^2 \theta} A_\alpha(\sin \theta) d\theta \\
 &= J_{310} - J_{311} + J_{312}.
 \end{aligned} \tag{69}$$

By l'Hôpital's rule, we see that $J_{310} = 0$. By integration by parts, (32)–(34), we obtain

$$\begin{aligned}
 J_{311} &= \left[\tan \theta (2 \cos^2 \theta + 1) \sin^3 \theta A_\alpha(\sin \theta)^2 \right]_0^{\pi/2} \\
 &\quad - \int_0^{\pi/2} \tan \theta \{ \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta)^2 \}' d\theta \\
 &= \int_0^{\pi/2} (-6 \sin \theta \cos^2 \theta + 4 \sin^3 \theta - 3 \sin \theta) (\sin \theta A_\alpha(\sin \theta))^2 d\theta \\
 &\quad + 4 \int_0^{\pi/2} (2 \cos^2 \theta + 1) (\sin \theta A_\alpha(\sin \theta)) \sin^2 \theta \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
 &= 4 \int_0^{\pi/2} (-6 \sin \theta \cos^2 \theta + 4 \sin^3 \theta - 3 \sin \theta) (\sin \theta \log(\sin \theta))^2 d\theta + O(\alpha^{-1/2}) \\
 &\quad - 8 \int_0^{\pi/2} \sin^2 \theta (2 \cos^2 \theta + 1) (\sin \theta \log(\sin \theta)) d\theta + O(\alpha^{-1}) \\
 &= O(1).
 \end{aligned} \tag{70}$$

By (32)–(34) and integration by parts and l'Hôpital's rule, we have

$$\begin{aligned}
 J_{312} &= \frac{4}{3}\alpha \left[\tan \theta \sin^5 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) \frac{1}{1 + \alpha \sin^2 \theta} \right]_0^{\pi/2} \\
 &\quad - \frac{4}{3}\alpha \int_0^{\pi/2} \tan \theta \left\{ \sin^5 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) \frac{1}{1 + \alpha \sin^2 \theta} \right\}' d\theta \\
 &= -\frac{4}{3} \int_0^{\pi/2} (10 \sin^2 \theta \cos^2 \theta - 4 \sin^4 \theta + 5 \sin^2 \theta) (\sin \theta A_\alpha(\sin \theta)) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
 &\quad + \frac{8}{3} \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 d\theta \\
 &\quad + \frac{8}{3} \int_0^{\pi/2} \sin^2 \theta (2 \cos^2 \theta + 1) (\sin \theta A_\alpha(\sin \theta)) \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 d\theta \\
 &= O(1).
 \end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}
 J_{32} &= \alpha \left[\tan \theta \sin^3 \theta A_\alpha(\sin \theta) \right]_0^{\pi/2} - \alpha \int_0^{\pi/2} \tan \theta \{ \sin^3 \theta A_\alpha(\sin \theta) \}' d\theta \\
 &:= J_{320} + J_{321}.
 \end{aligned} \tag{71}$$

By l'Hôpital's rule, we obtain that $J_{320} = 0$. By (33) and (34), we obtain

$$\begin{aligned}
 J_{321} &= -3\alpha \int_0^{\pi/2} \sin \theta (\sin^2 \theta A_\alpha(\sin \theta)) d\theta + 2\alpha \int_0^{\pi/2} \sin^3 \theta \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
 &= 6\alpha \int_0^{\pi/2} \sin^3 \theta \log(\sin \theta) d\theta + 2\alpha \int_0^{\pi/2} \sin^3 \theta d\theta + O(1).
 \end{aligned} \tag{72}$$

By this and (69)–(71), we obtain

$$\begin{aligned}
 I_3 &= \frac{-4}{(\log(1 + \alpha))^2} \left(3 \int_0^{\pi/2} \sin^3 \theta \log(\sin \theta) d\theta + \int_0^{\pi/2} \sin^3 \theta d\theta \right) + O \left(\frac{1}{\alpha(\log(1 + \alpha))^2} \right) \\
 &= \frac{4}{(\log(1 + \alpha))^2} (-2 \log 2 + 1) + O \left(\frac{1}{\alpha(\log(1 + \alpha))^2} \right).
 \end{aligned} \tag{73}$$

This implies (20). Thus the proof is complete. \square

By Lemmas 4–6, we obtain

$$M = (C_1 + 5 - 8 \log 2) \frac{1}{(\log(1 + \alpha))^2} + O \left(\frac{1}{\alpha(\log(1 + \alpha))^2} \right). \tag{74}$$

This along with (20) and Lemma 1, we obtain (7).

4. Remainder Estimate

To complete the proof of Theorem 2, in this section, we prove (8). Let $n \geq 3$. By (17), we have

$$g_\alpha(s)^n := \frac{1}{(\log(1 + \alpha))^n (1 - s^2)^n} \eta_\alpha(s)^n, \tag{75}$$

where

$$\eta_\alpha(s) := s^2 A_\alpha(s) + \frac{1}{\alpha} \tilde{\zeta}(\alpha s^2). \tag{76}$$

By direct calculation, we see that $\eta'_\alpha(s) = 2sA_\alpha(s) \geq 0$ and find that $\eta_\alpha(s)$ is increasing in $0 \leq s \leq 1$. We have $\eta_\alpha(0) = \frac{1}{\alpha} \log(1 + \alpha) - 1 = -1 + o(1)$, $\eta_\alpha(1) = 0$. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Then there exists a constant $\delta > 0$ independent of $\alpha \gg 1$ such that for $s \in [0, \epsilon]$ and $\alpha \gg 1$,

$$\begin{aligned} -1 &\leq \eta_\alpha(s) \leq \eta_\alpha(\epsilon) \\ &= (\epsilon^2 - 1) + \frac{\alpha\epsilon^2 + 1}{\alpha} \log \frac{1 + \alpha}{1 + \alpha\epsilon^2} \\ &= (\epsilon^2 - 1) - (\epsilon^2 + o(1)) \log(\epsilon^2 + o(1)) \\ &< -\delta < 0. \end{aligned} \quad (77)$$

By this, if n is odd, then we have

$$\begin{aligned} \int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s)^n ds &= \frac{1}{(\log(1+\alpha))^n} \int_0^1 \frac{s}{\sqrt{1-s^2}} \frac{1}{(1-s^2)^n} \eta_\alpha(s)^n ds \\ &\leq \frac{1}{(\log(1+\alpha))^n} \int_0^\epsilon \frac{s}{\sqrt{1-s^2}} \frac{1}{(1-s^2)^n} (-\delta)^n ds \\ &\leq -\delta^n \frac{1}{(\log(1+\alpha))^n}. \end{aligned} \quad (78)$$

If n is even, then by the same argument as that in (78), we have

$$\int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s)^n ds \geq \delta^n \frac{1}{(\log(1+\alpha))^n}. \quad (79)$$

By (16) and (19), for $n = 3, 4, \dots$, we have

$$|g_\alpha(s)|^n \leq \frac{3^n}{(\log(1+\alpha))^n}. \quad (80)$$

By this, (78) and (79), we obtain

$$C \frac{1}{(\log(1+\alpha))^n} \leq \left| \sum_{n=3}^{\infty} \int_0^1 \frac{s}{\sqrt{1-s^2}} (-1)^n \frac{(2n-1)!!}{n!2^n} g_\alpha(s)^n ds \right| \leq C^{-1} \frac{1}{(\log(1+\alpha))^n}. \quad (81)$$

This implies (8). Thus the proof is complete.

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Appendix A

Proof of (32). Let $\alpha \gg 1$. For $0 \leq s \leq 1$, we consider the maximum of the function

$$m(s) := s \left(\log \left(\frac{1}{\alpha} + s^2 \right) - 2 \log s \right). \quad (A1)$$

We first show that $m(s)$ attains its maximum in $0 \leq s \leq 1$ at $s_0 = \sqrt{t_0/\alpha}$ with $\epsilon \leq t_0 \leq \alpha - \epsilon$, where $\epsilon > 0$ is a small constant independent of $\alpha \gg 1$, and

$$0 = m(0) \leq m(s) \leq m(s_0) = \sqrt{t_0} \log \left(\frac{1+t_0}{t_0} \right) \alpha^{-1/2}. \quad (A2)$$

Indeed, we have

$$m'(s) = \log(1 + \alpha s^2) - \log \alpha - 2 \log s - \frac{2}{1 + \alpha s^2}, \quad (\text{A3})$$

$$m''(s) = \frac{2}{s(1 + \alpha s^2)} (2\alpha^2 s^4 + \alpha s^2 - 1) \quad (\text{A4})$$

Therefore,

$$m''(s) = 0 \Leftrightarrow s = \frac{1}{\sqrt{2\alpha}}. \quad (\text{A5})$$

We have

$$m'(0) = +\infty, \quad (\text{A6})$$

$$m'(1) = \log(1 + \alpha) - \log \alpha - \frac{1}{1 + \alpha} = \log \left(1 + \frac{1}{\alpha} \right) - \frac{1}{1 + \alpha} \quad (\text{A7})$$

$$= \frac{\alpha - 1}{2\alpha^2(\alpha + 1)} + \frac{1}{3\alpha^3} + O(\alpha^{-4}) > 0,$$

$$m' \left(\frac{1}{\sqrt{2\alpha}} \right) = \log 3 - \frac{4}{3} < 0. \quad (\text{A8})$$

Therefore, there exists $s_0 \in [0, 1/\sqrt{2\alpha})$ and $s_1 \in (1/\sqrt{2\alpha}, 1]$ such that $m'(s_0) = m'(s_1) = 0$. Namely,

$$\begin{aligned} m'(s) &> 0 \quad (0 \leq s < s_0), \quad m'(s_0) = 0, \quad m'(s) < 0 \quad (s_0 < s < s_1), \\ m'(s_1) &= 0, \quad m'(s) > 0 \quad (s_1 < s \leq 1). \end{aligned} \quad (\text{A9})$$

So, $m(s)$ is increasing in $[0, s_0]$, $[s_1, 1]$ and decreasing in $[s_0, s_1]$. We have

$$m(0) = 0, \quad (\text{A10})$$

$$m(s_1) < m(1/\sqrt{2\alpha}) = \frac{1}{\sqrt{2\alpha}} \log 3 < m(s_0), \quad (\text{A11})$$

$$m(1) = \log \frac{1 + \alpha}{\alpha} = \frac{1}{\alpha} + O(\alpha^{-2}). \quad (\text{A12})$$

We choose a small constant $0 < \epsilon < \frac{1}{\sqrt{2}}$ such that the following (A13) holds.

$$m' \left(\sqrt{\frac{\epsilon}{\alpha}} \right) = \log(1 + \epsilon) - \log \epsilon - \frac{2}{1 + \epsilon} > 0. \quad (\text{A13})$$

By (A13), we see that $\frac{\epsilon}{\sqrt{\alpha}} < s_0 < \frac{1}{\sqrt{2\alpha}}$. This implies that there exists a constant $\epsilon < t_0 < 1/\sqrt{2}$ such that $s_0 = \sqrt{t_0/\alpha}$. By this, we obtain (A2). By (31) and (A2), we have

$$\begin{aligned} 0 &\leq \sin \theta A_\alpha(\sin \theta) = \sin \theta \left(\frac{1}{\alpha} + O(\alpha^{-2}) - m(\sin \theta) - 2 \log(\sin \theta) \right) \\ &= -2 \sin \theta \log(\sin \theta) + O \left(\frac{1}{\sqrt{\alpha}} \right). \end{aligned} \quad (\text{A14})$$

Thus we get (32). \square

Proof of (33). By (31), we have

$$\begin{aligned} \sin^2 \theta A_\alpha(\sin \theta) &= O(\alpha^{-1}) - \sin^2 \theta \left(\log \left(\frac{1}{\alpha} + \sin^2 \theta \right) - 2 \log \sin \theta \right) \\ &\quad - 2 \sin^2 \theta \log \sin \theta. \end{aligned} \quad (\text{A15})$$

For $0 \leq t \leq 1$, we put

$$k(t) := t(\log(1 + \alpha t) - \log(\alpha t)). \quad (\text{A16})$$

Then

$$k'(t) = \log(1 + \alpha t) - \log(\alpha t) - \frac{1}{1 + \alpha t}, \quad (\text{A17})$$

$$k''(t) = -\frac{1}{t(1 + \alpha t)^2} < 0 \quad (t \neq 0). \quad (\text{A18})$$

So, $k'(t)$ is decreasing and by Taylor expansion and (A7), we have

$$k'(1) = \log(1 + \alpha) - \log \alpha - \frac{1}{1 + \alpha} > 0. \quad (\text{A19})$$

Therefore, $k'(t) > 0$ for $0 < t \leq 1$, and $k(t)$ is increasing in for $0 \leq t \leq 1$. So, for $0 \leq t \leq 1$, we obtain

$$0 = k(0) \leq k(t) \leq k(1) = \log(1 + \alpha) - \log \alpha = \log \left(1 + \frac{1}{\alpha} \right) = \frac{1}{\alpha} + O(\alpha^{-2}). \quad (\text{A20})$$

By (A15) and (A20), for $0 \leq \theta \leq \pi/2$, we obtain

$$0 \leq \sin^2 \theta A_\alpha(\sin \theta) = O(\alpha^{-1} - k(\sin^2 \theta) - 2 \sin^2 \theta \log(\sin \theta) = -2 \sin^2 \theta \log(\sin \theta) + O(\alpha^{-1}). \quad (\text{A21})$$

This implies (33). Thus the proof is complete. \square

References

1. Laetsch, T. The number of solutions of a nonlinear two point boundary value problem. *Indiana Univ. Math. J.* **1970**, *20*, 1–13. [\[CrossRef\]](#)
2. Cano-Casanova, S.; López-Gómez, J. Existence, uniqueness and blow-up rate of large solutions for a canonical class of one-dimensional problems on the half-line. *J. Differ. Equ.* **2008**, *244*, 3180–3203. [\[CrossRef\]](#)
3. Cano-Casanova, S.; López-Gómez, J. Blow-up rates of radially symmetric large solutions. *J. Math. Anal. Appl.* **2009**, *352*, 166–174. [\[CrossRef\]](#)
4. Chiappinelli, R. A-priori bounds and asymptotics on the eigenvalues in bifurcation problems for perturbed self-adjoint operators. *J. Math. Anal. Appl.* **2009**, *354*, 263–272. [\[CrossRef\]](#)
5. Chiappinelli, R. Upper and lower bounds for higher order eigenvalues of some semilinear elliptic equations. *Appl. Math. Comput.* **2010**, *216*, 3772–3777. [\[CrossRef\]](#)
6. Chiappinelli, R. Approximation and convergence rate of nonlinear eigenvalues: Lipschitz perturbations of a bounded self-adjoint operator. *J. Math. Anal. Appl.* **2017**, *455*, 1720–1732. [\[CrossRef\]](#)
7. Fraile, J.M.; López-Gómez, J.; Delis, J.C. On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems. *J. Differ. Equ.* **1995**, *123*, 180–212. [\[CrossRef\]](#)
8. Shibata, T. Inverse bifurcation problems for diffusive logistic equation of population dynamics. *J. Math. Anal. Appl.* **2014**, *413*, 495–501. [\[CrossRef\]](#)
9. Gelfand, I.M. Some problems in the theory of quasilinear equations. *Am. Math. Soc. Transl.* **1965**, *29*, 295–381.
10. Huang, S.-Y.; Wang, S.-H. Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory. *Arch. Ration. Mech. Anal.* **2016**, *222*, 769–825. [\[CrossRef\]](#)

11. Shibata, T. Global and local structures of oscillatory bifurcation curves. *J. Spectr. Theory* **2019**, *9*, 991–1003. [[CrossRef](#)]
12. Cheng, Y.J. On an open problem of Ambrosetti, Brezis and Cerami. *J. Differ. Integral Equ.* **2002**, *15*, 1025–1044.
13. Ambrosetti, A.; Brezis, H.; Cerami, G. Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **1994**, *122*, 519–543. [[CrossRef](#)]
14. Shibata, T. Asymptotic length of bifurcation curves related to inverse bifurcation problems. *J. Math. Anal. Appl.* **2016**, *438*, 629–642. [[CrossRef](#)]
15. Shibata, T. Oscillatory bifurcation for semilinear ordinary differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *44*, 1–13. [[CrossRef](#)]
16. Shibata, T. Global and local structures of oscillatory bifurcation curves with application to inverse bifurcation problem. *Topol. Methods Nonlinear Anal.* **2017**, *50*, 603–622. [[CrossRef](#)]



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