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Some Remarks on Reich and Chatterjea Type Nonexpansive Mappings

Ovidiu Popescu *,† **and Gabriel Stan** †

Department of Mathematics and Computer Science, Transilvania University of Brasov, 500036 Brasov, Romania; gabriel.stan@unitbv.ro
* Correspondence: ovidiu.popescu@unitbv.ro
† These authors contributed equally to this work.

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Abstract: In the paper, we show that some results related to Reich and Chatterjea type nonexpansive mappings are still valid if we relax or remove some hypotheses.

Keywords: nonexpansive map; approximate fixed point sequence; Reich type nonexpansive mapping

1. Introduction

Let Y be a normed linear space. A mapping $S : Y \to Y$ is said to be nonexpansive if $\|Su - Sv\| \leq \|u - v\|$ for all $u, v \in Y$. A mapping $S : Y \to Y$ is said to be asymptotically nonexpansive if there exists a sequence $\{\alpha_n\}$ with $\alpha_n \geq 1$ and $\lim\limits_{n\to\infty} \alpha_n = 1$ such that $\|S^n u - S^n v\| \leq \alpha_n \|u - v\|$ for all $u, v \in Y$ and $n \in \mathbb{N}$. It is well known that every nonexpansive mapping or asymptotically nonexpansive mapping on a non-empty closed, bounded, convex subset of a uniformly convex Banach space has at least one fixed point, see [1–3].

A sequence $\{u_n\}$ in a normed linear space Y is said to be an approximate fixed point sequence (AFPS) for $S : Y \to Y$ if $\lim\limits_{n\to\infty} \|u_n - Su_n\| = 0$.

Many authors have studied the AFPS for different types of nonexpansive mappings (see [4–6]). Most of the results have been established in uniformly convex Banach spaces, smooth reflexive Banach spaces or Hilbert spaces. These results have become a major tool in solving various problems such as integral equations, differential equations, optimization problems (see [7,8]).

Recently, Som et al. [9] introduced two types of mappings, Reich type nonexpansive and Chatterjea type nonexpansive mappings, and gave sufficient conditions under which these classes of mappings possess an AFPS, in more general spaces, more specifically in Banach spaces. They checked some properties of the fixed point sets of these mappings: closedness, convexity, remotality, etc. and gave some sufficient conditions under which a Reich type nonexpansive mapping reduces to that of nonexpansive one. For more considerations on Reich contractions and Chatterjea contractions see [10–12]. To obtain the desired AFPS, Som et al. used the Schäefer iteration method:

$$u_{n+1} = (1 - \alpha)\, u_n + \alpha S u_n,$$

where $\alpha \in (0, 1)$.

Definition 1 ([9]). *Let Y be a normed linear space, E a non-empty subset of Y and $S : E \to E$ be a mapping. Mapping S is said to be a Reich type nonexpansive mapping if there exist non-negative real numbers p, q, r with $p + q + r = 1$ such that*

$$\|Su - Sv\| \leq p\, \|u - v\| + q\, \|u - Su\| + r\, \|v - Sv\|,$$

for all $u, v \in E$.

Definition 2 ([9]). *Let* Y *be a normed linear space,* E *a non-empty subset of* Y *and* $S : E \to E$ *be a mapping. Mapping* S *is said to be a Chatterjea type nonexpansive mapping if there exist non-negative real numbers* $p,\ q,\ r$ *with* $p + q + r = 1$ *such that*

$$\|Su - Sv\| \leq p\,\|u - v\| + q\,\|u - Sv\| + r\,\|v - Su\|\,,$$

for all $u, v \in E$.

To prove their main results, Som et al. used the following lemma:

Lemma 1 ([13]). *Let* $\{z_n\}$ *and* $\{w_n\}$ *be two bounded sequences in a Banach space* Y *and* $\alpha \in (0,1)$. *Let* $z_{n+1} = \alpha w_n + (1-\alpha) z_n$ *and suppose* $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$ *for all* $n \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} \|w_n - z_n\| = 0$.

Theorem 1 ([9]). *Let* Y *be a Banach space and* E *be a non-empty closed, convex, bounded subset of* Y. *Let* $S : E \to E$ *be a Reich type nonexpansive mapping with coefficients* (p, q, r) *such that* $r < 1$. *Furthermore, assume that for* $u, v \in E$

$$\frac{1-r}{6}\,\|u - Su\| \leq \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\|\,.$$

Then S *has an AFPS in* E. *Moreover, the AFPS is asymptotically regular.*

Theorem 2 ([9]). *Under the assumptions of Theorem 1,* S *has a fixed point, provided* $p < 1$.

Theorem 3 ([9]). *Let* Y *be a Banach space and* E *be a non-empty subset of* Y. *If* S *is Reich type nonexpansive mapping on* E *with coefficients* (p, q, r) *such that* $r < 1$, *then* $Fix(S)$ *is a closed subset of* E.

Theorem 4 ([9]). *Let* Y *be a Hilbert space and* E *be a nonempty convex subset of* Y. *Let* S *be Reich type nonexpansive mapping on* E *with coefficients* (p, q, r) *such that* $r < 1$. *Assume that* $q \leq r$. *Then* $Fix(S)$ *is a convex subset of* E.

Theorem 5 ([9]). *Let* Y *be a Banach space and* E *be a non-empty closed, convex, bounded subset of* Y. *Let* $S : E \to E$ *be a Chatterjea type nonexpansive mapping with coefficients* (p, q, r) *such that* $r < 1$. *Furthermore, assume that for* $u, v \in E$

$$\frac{1-q}{7}\,\|u - Sv\| \leq \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\|\,.$$

Then S *has an AFPS in* E. *Moreover, the AFPS is asymptotically regular.*

Theorem 6 ([9]). *Suppose that all conditions of Theorem 5 are satisfied. Further, assume that for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\|u - v\| + \|u - Sv\| + \|v - Su\| < 3\varepsilon + \delta \Rightarrow \|Su - Sv\| \leq \frac{\varepsilon}{2}.$$

Then S *has a fixed point in* E.

Theorem 7 ([9]). *Let* Y *be a finite dimensional Banach space, and* E *be a non-empty subset of* Y. *Let* $S : E \to E$ *be a Reich type nonexpansive mapping with coefficients* (p, q, r) *and assume that* $diam\,(Fix\,(S)) > 0$. *If* $Fix\,(S)$ *is centerable and contains its Chebyshev center, then* S *becomes nonexpansive.*

In this paper, we show that in Theorem 1 and Theorem 5 S does not need to be a Reich type nonexpansive mapping or Chatterjea type nonexpansive mapping, and the additional conditions can be replaced with some weaker conditions. Moreover, our proofs are very simple. Next, we generalize Theorem 2, Theorem 3 and Theorem 6 and make some usefull remarks on Theorem 4 and Theorem 7. Some examples will validate our results. We mention that some of these results can be extended to Hardy-Rogers nonexpansive mapings.

2. Main Results

The first result is the following generalization of Theorem 1.

Theorem 8. *Let Y be a Banach space, E be a non-empty closed, convex, bounded subset of Y and $S : E \to E$ be a mapping. Assume that there exists $\alpha \in (0,1)$ such that for $u, v \in E$*

$$\alpha \|u - Su\| = \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\| .$$

Then S has an AFPS in E and the AFPS is asymptotically regular.

Proof. Fix $u_0 \in E$ and consider the sequence $\{u_n\}$ in Y defined by $u_{n+1} = (1-\alpha) u_n + \alpha S u_n$ for all $n \geq 0$. Obviously, since E is convex and bounded, it follows that $\{u_n\}$ is a bounded sequence in E. We have

$$\alpha \|u_n - Su_n\| = \|u_n - u_{n+1}\| .$$

By hypothesis, taking $u = u_n$, $v = u_{n+1}$ we get

$$\|Su_n - Su_{n+1}\| \leq \|u_n - u_{n+1}\| .$$

Hence, using Lemma 1, we obtain $\lim\limits_{n\to\infty} \|u_n - Su_n\| = 0$, i.e., $\{u_n\}$ is an AFPS of S. Moreover, we have

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = \alpha \lim_{n\to\infty} \|u_n - Su_n\| = 0.$$

Therefore, the AFPS $\{u_n\}$ is asymptotically regular. □

The next result is a generalization of Theorem 2.

Theorem 9. *Let Y be a Banach space and E be a non-empty closed, convex, bounded subset of Y. Let $S : E \to E$ a mapping and p, q, r non-negative real numbers with $p < 1$ and $\min\{q, r\} < 1$ such that condition*

$$\|Su - Sv\| \leq p \|u - v\| + q \|u - Su\| + r \|v - Sv\| \tag{1}$$

holds for all u, $v \in E$. Under the assumption of Theorem 1, S has a unique fixed point.

Proof. By Theorem 1, we know that S has an AFPS $\{u_n\}$ which is asymptotically regular.

Taking $u = u_n$ and $v = u_m$ in (1), we get

$$\begin{aligned} \|u_n - u_m\| &\leq \|u_n - Su_n\| + \|Su_n - Su_m\| + \|Su_m - u_m\| \\ &\leq \|u_n - Su_n\| + p \|u_n - u_m\| + q \|u_n - Su_n\| \\ &\quad + r \|u_m - Su_m\| + \|u_m - Su_m\| . \end{aligned}$$

Since $p < 1$, it follows that

$$\|u_n - u_m\| \leq \frac{1+q}{1-p} \|u_n - Su_n\| + \frac{1+q}{1-p} \|u_m - Su_m\| ,$$

which implies $\|u_n - u_m\| \to 0$ as $n, m \to \infty$.

Hence, $\{u_n\}$ is a Cauchy sequence in E and then, is convergent to some $z \in E$. Moreover, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} Su_n = z.$$

Suppose $\min\{q, r\} = r$. Then we have $r < 1$.

Now, taking $u = u_n$ and $v = z$ in (1), we get for all $n \geq 1$

$$\|Su_n - Sz\| \leq p\|u_n - z\| + q\|u_n - Su_n\| + r\|z - Sz\|.$$

Taking limit as $n \to \infty$, we obtain

$$\|z - Sz\| \leq r\|z - Sz\|.$$

It follows that $\|z - Sz\| = 0$, i.e., $z = Sz$.

Similarly, if $\min\{q, r\} = q$ taking $u = z$ and $v = u_n$ in (1) we obtain, for all $n \geq 1$

$$\|Sz - Su_n\| \leq p\|z - u_n\| + q\|z - Sz\| + r\|u_n - Su_n\|.$$

Letting $n \to \infty$, we get

$$\|Sz - z\| \leq q\|z - Sz\|,$$

by where $\|z - Sz\| = 0$, i.e., $z = Sz$. □

The following theorem is a generalization of Theorem 3.

Theorem 10. *Let Y be a Banach space and E be a non-empty subset of Y. If $S : E \to E$ satisfies (1) with $\min\{q, r\} < 1$, then $Fix(S)$ is a closed subset of E.*

Proof. Due to symmetry, we can suppose $q < 1$. Let $\{z_n\}$ be a sequence in $Fix(S)$ such that $\{z_n\}$ converges to $z \in E$. Taking $u = z$ and $v = z_n$ in (1) we get

$$\|Sz - Sz_n\| \leq p\|z - z_n\| + q\|z - Sz\| + c\|z_n - Sz_n\|,$$

for all $n \in \mathbb{N}$.

It follows that

$$\|Sz - z_n\| \leq p\|z - z_n\| + q\|z - Sz\|.$$

Taking limit as $n \to \infty$, we obtain

$$\|z - Sz\| \leq q\|z - Sz\|,$$

which gives $z = Sz$, i.e., $z \in Fix(S)$. Hence, $Fix(S)$ is a closed subset of E. □

Our next result shows that in Theorem 5 it is not necessary that S be a Chatterjea type nonexpansive mapping. Moreover, we relax the additional condition satisfying by S.

Theorem 11. *Let Y be a Banach space and E be a non-empty closed, convex, bounded subset of Y. Let $S : E \to E$ be a mapping such that there exists $\mu > 0$ with*

$$\mu\|u - Sv\| = \|u - v\| \Rightarrow$$

$$\|Su - Sv\| \leq \|u - v\|.$$

Then S has an AFPS in E. Moreover, the AFPS is asymptotically regular.

Proof. Fix $u_0 \in E$ and consider the sequence $\{u_n\}$ in Y defined by $u_{n+1} = (1-\alpha)x_n + \alpha S x_n$, with $\alpha = \frac{1}{1+\mu}$. Since $\alpha \in (0,1)$ and E is convex and bounded, it follows that $\{u_n\}$ is a bounded sequence in E. We have for all $n \geq 1$

$$\|u_{n+1} - u_n\| = \alpha \|u_n - Su_n\|$$

and

$$\|u_{n+1} - Su_n\| = (1-\alpha)\|u_n - Su_n\|.$$

Hence,

$$\|u_{n+1} - Su_n\| = \frac{1-\alpha}{\alpha}\|u_{n+1} - u_n\| = \mu \|u_{n+1} - u_n\|.$$

By hypothesis, we get for all $n \geq 1$

$$\|Su_n - Su_{n+1}\| \leq \|u_n - u_{n+1}\|.$$

Therefore, using Lemma 1, we obtain $\lim_{n\to\infty} \|u_n - Su_n\| = 0$, i.e., $\{u_n\}$ is an AFPS of S and

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = \alpha \lim_{n\to\infty} \|u_n - Su_n\| = 0.$$

Thus, the AFPS $\{u_n\}$ is asymtotically regular. □

Now, using Theorem 11, we prove a generalization of Theorem 6.

Theorem 12. *Suppose that all the conditions of Theorem 11 are satisfied. Further, assume that for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|u - v\| + \|u - Sv\| + \|v - Su\| < 3\varepsilon + \delta \Rightarrow \|Su - Sv\| \leq \varepsilon. \tag{2}$$

Then S has a fixed point in E.

Proof. By Theorem 11, S has an AFPS $\{u_n\}$, where

$$u_{n+1} = (1-\alpha)u_n + \alpha S u_n$$

and

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = \alpha \lim_{n\to\infty} \|u_n - Su_n\| = 0 \tag{3}$$

We suppose that $\{u_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that

$$\limsup_{n,m\to\infty} \|u_n - u_m\| \geq 2\varepsilon.$$

By hypothesis, there exists a $\delta > 0$ such that (2) holds. We can assume $\delta \leq \varepsilon$.
By (3) we can find M so that

$$\|u_n - u_{n+1}\| \leq \|u_n - Su_n\| < \frac{\delta}{12},$$

for all $n \geq M$.
Pick $m,n > M$, $m < n$ so that $\|u_n - u_m\| > 2\varepsilon$.
For j in $[m,n]$ we have

$$\left| \|u_m - u_j\| - \|u_m - u_{j+1}\| \right| \leq \|u_j - u_{j+1}\| < \frac{\delta}{12}.$$

This implies that, since $\|u_m - u_{m+1}\| < \varepsilon$ and $\|u_m - u_n\| > \varepsilon + \delta$, there exists j in $[m, n]$ with

$$\varepsilon + \frac{\delta}{6} \leq \|u_m - u_j\| \leq \varepsilon + \frac{\delta}{4} \tag{4}$$

Hence, we have

$$\begin{aligned} & \|u_m - u_j\| + \|u_m - Su_j\| + \|u_j - Su_m\| \\ \leq \; & \|u_m - u_j\| + \|u_m - u_j\| + \|u_j - Su_j\| + \|u_j - u_m\| + \|u_m - Su_m\| \\ < \; & 3\left(\varepsilon + \frac{\delta}{4}\right) + 2\frac{\delta}{12} = 3\varepsilon + \frac{11\delta}{12} < 3\varepsilon + \delta. \end{aligned}$$

Therefore, we get

$$\|Su_m - Su_j\| \leq \varepsilon.$$

It follows that

$$\begin{aligned} \|u_m - u_j\| & \leq \|u_m - Su_m\| + \|Su_m - Su_j\| + \|Su_j - u_j\| \\ & < \varepsilon + 2\frac{\delta}{12} = \varepsilon + \frac{\delta}{6}, \end{aligned}$$

which contradicts (4).

This contradiction proves that $\{u_n\}$ must be a Cauchy sequence and hence is convergent to some $z \in E$. Obviously, $Su_n \to z$ as $n \to \infty$.

By (4) we get for all $u, v \in E$

$$\|Su - Sv\| \leq \frac{1}{3}\left(\|u - v\| + \|u - Sv\| + \|v - Su\|\right).$$

Taking $u = u_n$ and $v = z$ in the above inequality we obtain

$$\|Su_n - Sz\| \leq \frac{1}{3}\left(\|u_n - z\| + \|u_n - Sz\| + \|z - Su_n\|\right).$$

Letting $n \to \infty$ we get

$$\|z - Sz\| \leq \frac{1}{3}\|z - Sz\|,$$

which gives $\|z - Sz\| = 0$, i.e., $z = Sz$, so z is a fixed point of S. □

Remark 1. *In Theorem 4 it is necessary that E be convex. If $p < 1$, it follows that $Fix(S) = \{z\}$ which is convex, and if $p = 1$ then S is nonexpansive mapping and it is well known that $Fix(S)$ is convex.*

Remark 2. *In Theorem 7 by condition $diam(Fix(S)) > 0$ there exist $z_1, z_2 \in Fix(S)$, $z_1 \neq z_2$. Since S is a Reich type nonexpansive mapping we get $p = 1$, $q = r = 0$, so S is nonexpansive mapping. The other hypotheses are superfluous.*

Example 1. *Let us consider the Banach space $\mathbb{R}$ equipped with the usual norm. Let $E = [0, 2]$ and define $S : E \to E$ by*

$$Su = \begin{cases} 2, & \text{if } u < \frac{1}{3}; \\ \frac{5}{3} & \text{if } u \geq \frac{1}{3}. \end{cases}$$

Som et al. proved that S is a Chartterjea type nonexpansive mapping with coefficients $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (see Example 3.11, [9]). It is easy to prove that S is also a Reich type nonexpansive mapping with coefficients $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

If $u = \frac{1}{3}$ and $v = \frac{1}{9}$ we have

$$\frac{1}{6}\|u - Su\| = \frac{2}{9} = \|u - v\|$$

and

$$\|Su - Sv\| = \frac{1}{3} > \|u - v\|,$$

so S does not satisfy the hypothesis of Theorem 1.

Furthermore, if $u = \frac{1}{15}$ and $v = \frac{1}{3}$ we have

$$\frac{1}{6}\|u - Sv\| = \frac{4}{15} = \|u - v\|$$

and

$$\|Su - Sv\| = \frac{1}{3} > \|u - v\|,$$

so S does not satisfy the hypothesis of Theorem 5.

Now we prove that S satisfies property

$$\frac{1}{2}\|u - Su\| = \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\|.$$

If $u < \frac{1}{3}$, $v < \frac{1}{3}$ or $u \geq \frac{1}{3}$, $v \geq \frac{1}{3}$ we have $Su - Sv$ and the property is obvious.

If $u < \frac{1}{3}$ and $v \geq \frac{1}{3}$ we have to prove that

$$\frac{1}{2}(2 - u) = v - u \Rightarrow \frac{1}{3} \leq v - u.$$

Since $2 - u > \frac{5}{3}$ we have

$$v - u > \frac{5}{6} \geq \frac{1}{3}.$$

If $u \geq \frac{1}{3}$ and $v < \frac{1}{3}$, then

$$\frac{1}{2}\|u - Su\| = \|u - v\| \Longleftrightarrow \frac{1}{2}\left|u - \frac{5}{3}\right| = u - v$$

and

$$\|Su - Sv\| \leq \|u - v\| \Longleftrightarrow \frac{1}{3} \leq u - v.$$

For $u < \frac{2}{3}$ we have

$$\frac{1}{2}\left|u - \frac{5}{3}\right| > \frac{1}{2} > \frac{1}{3},$$

so $u - v \geq \frac{1}{3}$.

For $u \geq \frac{2}{3}$ we have $u - v > \frac{1}{3}$.

Hence $\|Su - Sv\| \leq \|u - v\|$.

Therefore, S satisfies the hypothesis of Theorem 8. Next we prove that

$$2\|u - Sv\| = \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\|.$$

If $u < \frac{1}{3}$, $v < \frac{1}{3}$ or $u \geq \frac{1}{3}$, $v \geq \frac{1}{3}$ we have $Su = Sv$ and the property is obvious.

If $u < \frac{1}{3}$ and $v \geq \frac{1}{3}$, we have

$$2\|u - Sv\| = \|u - v\| \Rightarrow 2\left|u - \frac{5}{3}\right| = v - u.$$

Since $u < \frac{1}{3}$ we have

$$v - u > 2 \cdot \frac{4}{3} = \frac{8}{3} > \frac{1}{3},$$

so $\|Su - Sv\| = \frac{1}{3} \leq \|u - v\|$.

If $u \geq \frac{1}{3}$ and $v < \frac{1}{3}$ we have

$$2\,\|u - Sv\| = \|u - v\| \Rightarrow 2\,|u - 2| = u - v.$$

For $u \leq 2$ we have $v = 3u - 4 < \frac{1}{3}$ when $u < \frac{13}{9}$. Hence

$$\|Su - Sv\| = \frac{1}{3} \leq \|u - v\| \Leftrightarrow \frac{1}{3} \leq 4 - 2u \Leftrightarrow u \leq \frac{11}{6}.$$

Thus

$$2\,\|u - Sv\| = \|u - v\| \Rightarrow \|Su - Sv\| \leq \|u - v\|\,.$$

For $u > 1$ we have $v = 4 - u < \frac{1}{3}$ when $u > \frac{11}{3}$. Then

$$\|u - v\| > \frac{11}{3} - \frac{1}{3} = \frac{10}{3} \geq \frac{1}{3} = \|Su - Sv\|\,.$$

Therefore, S satisfies the hypothesis of Theorem 12.

Example 2. *Let us consider the Banach space $\mathbb{R}$ equipped with the usual norm and take $E = [0,1]$. We define $S : E \to E$ by*

$$Su = \begin{cases} \frac{2}{3}u, & \text{if } u \in A; \\ 0, & \text{if } u \in B, \end{cases}$$

where $A = E \cap \mathbb{Q}$ and $B = E \setminus \mathbb{Q}$.

Choose $p = q = r = \frac{2}{3}$.

Let $u, v \in E$ be arbitrary. Then the following three cases may arise:

Case I: Let $u, v \in A$. Then, $Su = \frac{2}{3}u$ and $Sv = \frac{2}{3}v$.

Therefore,

$$\|Su - Sv\| = \frac{2}{3}\,\|u - v\|$$

and

$$p\,\|u - v\| + q\,\|u - Su\| + r\,\|v - Sv\| = \frac{2}{3}\left(\|u - v\| + \frac{u}{3} + \frac{v}{3}\right).$$

Thus,

$$\|Su - Sv\| \leq p\,\|u - v\| + q\,\|u - Su\| + r\,\|v - Sv\|$$

Case II: Let $u, v \in B$. Then, $Su = Sv = 0$.

Therefore,

$$\|Su - Sv\| = 0 \leq p\,\|u - v\| + q\,\|u - Su\| + r\,\|v - Sv\|\,.$$

Case III: Let $u \in A$ and $v \in B$. Then, $Su = \frac{2}{3}u$ and $Sv = 0$.

Therefore,

$$\|Su - Sv\| = \frac{2}{3}u$$

and

$$p\,\|x - y\| + q\,\|u - Su\| + r\,\|v - Sv\| = \frac{2}{3}\left(\|u - v\| + \frac{u}{3} + v\right).$$

Since $u \le \|u-v\| + v$, it follows that

$$\|Su - Sv\| \le p\,\|u-v\| + q\,\|u-Su\| + r\,\|v-Sv\|\,.$$

Thus, we see that

$$\|Su - Sv\| \le \frac{2}{3}\,\|u-v\| + \frac{2}{3}\,\|u-Su\| + \frac{2}{3}\,\|v-Sv\|\,,$$

for all $u, v \in E$.

Similarly, we can prove that

$$\|Su - Sv\| \le \frac{2}{3}\,\|u-v\| + \frac{2}{3}\,\|u-Sv\| + \frac{2}{3}\,\|v-Su\|\,,$$

for all $u, v \in E$.

Suppose S is a Reich type nonexpansive mapping with coefficients (p, q, r).

If $u = 1$ and $v = \frac{\sqrt{3}}{3}$ we have

$$\|Su - Sv\| = \frac{2}{3}$$

and

$$\begin{aligned} & p\,\|u-v\| + q\,\|u-Su\| + r\,\|v-Sv\| \\ = \;& p\frac{3-\sqrt{3}}{3} + \frac{q}{3} + \frac{r\sqrt{3}}{3} = \frac{p\left(3-\sqrt{3}\right) + q + r\sqrt{3}}{3}. \end{aligned}$$

Since $p + q + r = 1$ it follows that

$$\begin{aligned} 2 &\le p\left(3-\sqrt{3}\right) + q + r\sqrt{3} \Leftrightarrow \\ 2 &\le 1 + p\left(2-\sqrt{3}\right) + r\left(\sqrt{3}-1\right) \Longleftrightarrow \\ 1 &\le p\left(2-\sqrt{3}\right) + r\left(\sqrt{3}-1\right) < p + r \le 1, \end{aligned}$$

which is a contradiction.

Therefore, S is not a Reich type nonexpansive mapping.

Now we can suppose that T is a Chatterjea type nonexpansive mapping.

If $u = 1$ and $v = \frac{\sqrt{3}}{3}$ we have

$$\|Su - Sv\| = \frac{2}{3}$$

and

$$\begin{aligned} & p\,\|u-v\| + q\,\|u-Sv\| + r\,\|v-Su\| \\ = \;& p\frac{3-\sqrt{3}}{3} + q + \frac{r\left(2-\sqrt{3}\right)}{3} = \frac{p\left(3-\sqrt{3}\right) + 3q + r\left(2-\sqrt{3}\right)}{3}. \end{aligned}$$

Since $p + q + r = 1$ it follows that

$$\begin{aligned} 2 &\le p\left(3-\sqrt{3}\right) + 3q + r\left(2-\sqrt{3}\right) \Leftrightarrow \\ 2 &\le 2 + p\left(1-\sqrt{3}\right) + q - r\sqrt{3} \Longleftrightarrow \end{aligned}$$

$$(p + r)\sqrt{3} \le p + q. \tag{5}$$

Similarly, taking $u = \frac{\sqrt{3}}{3}$ *and* $v = 1$ *we get*

$$(p+q)\sqrt{3} \leq p+r. \tag{6}$$

By (5) *and* (6)*we have*

$$\begin{aligned} (p+r)\sqrt{3} &\leq p+q \leq \frac{p+r}{\sqrt{3}} \Longleftrightarrow \\ p+r &= p+q = 0 \Longleftrightarrow p=q=r=0, \end{aligned}$$

which contradicts $p+q+r=1$. *Therefore, S is not a Chatterjea type nonexpansive mapping.*

Now, we prove that

$$\frac{3}{4}\|u - Su\| = \|u-v\| \Rightarrow \|Su - Sv\| \leq \|u-v\|.$$

We distinguish four cases:

Case I: Let $u, v \in A$. *Then,*

$$\|Su - Sv\| = \frac{2}{3}\|u-v\| \leq \|u-v\|.$$

Case II: Let $u, v \in B$. *Then,*

$$\|Su - Sv\| = 0 \leq \|u-v\|.$$

Case III: Let $u \in A$, $v \in B$. *Then,*

$$\frac{3}{4}\|u - Su\| = \|u-v\| \Rightarrow$$

$$\frac{3}{4}\cdot\frac{u}{3} = |u-v| \Rightarrow v = \frac{3u}{4} \text{ or } v = \frac{5u}{4}.$$

If $v = \frac{3u}{4}$, *since* $u \in A$ *it follows that* $v \in A$ *which is a contradiction.*

If $v = \frac{5u}{4}$ *it follows that* $v \in A$ *or* $v \notin E$, *which is a contradiction.*

Case IV: Let $u \in B$, $v \in A$. *Then,*

$$\frac{3}{4}\|u - Su\| = \|u-v\| \Rightarrow$$

$$\frac{3}{4}u = |u-v| \Rightarrow v = \frac{u}{4} or v = \frac{7u}{4}.$$

If $v = \frac{u}{4}$ *it follows that* $v \in B$ *which is a contradiction.*

If $v = \frac{7u}{4}$ *it follows that* $v \in B$ *or* $v \notin E$, *which is a contradiction.*

Therefore, S satisfies the hypotheses of Theorem 8 *and Theorem* 9. *S has an AFPS* $\left\{\frac{1}{2^n}\right\}_{n\in\mathbb{N}}$ *and* $Fix(S) = \{0\}$.

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