Strong Edge Coloring of Generalized Petersen Graphs

Ming Chen ¹, Lianying Miao ^{1,*} and Shan Zhou ²

¹ School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China; chenming314@cumt.edu.cn

² School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China; zhous@jsnu.edu.cn

* Correspondence: miaolianying@cumt.edu.cn; Tel.: +86-136-8514-9128

Received: 15 July 2020; Accepted: 31 July 2020; Published: 2 August 2020

Abstract: A strong edge coloring of a graph G is a proper edge coloring such that every color class is an induced matching. In 2018, Yang and Wu proposed a conjecture that every generalized Petersen graph $P(n, k)$ with $k \geq 4$ and $n > 2k$ can be strong edge colored with (at most) seven colors. Although the generalized Petersen graph $P(n, k)$ is a kind of special graph, the strong chromatic index of $P(n, k)$ is still unknown. In this paper, we support the conjecture by showing that the strong chromatic index of every generalized Petersen graph $P(n, k)$ with $k \geq 4$ and $n > 2k$ is at most 9.

Keywords: strong edge coloring; generalized Petersen graphs; partition

1. Introduction

All graphs in this paper are finite and simple. We denote the minimum and maximum degree of vertices in G by $\delta(G)$ and $\Delta(G)$, respectively. $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G . The distance between v_1 and v_2 , denote as $\text{dist}(v_1, v_2)$, is the length of shortest path from v_1 to v_2 . Let $\text{dist}(e_i, e_j) = \min\{\text{dist}(u, v) | u \in e_i, v \in e_j\}$ denote the distance of two edges e_i and e_j . We use (b, k) to denote the greatest common division between two integers b and k .

A *strong edge coloring* of a graph G is a mapping $\phi : E(G) \rightarrow \{1, 2, \dots, k\}$ such that any two edges e_i and e_j with distance at most one are colored by different colors, which is $\phi(e_i) \neq \phi(e_j)$. The minimum number of colors k required in a strong edge coloring of G is called the *strong chromatic index* denoted by $\chi'_s(G)$, and the coloring is a *strong k edge coloring*. The concept of strong edge coloring was introduced by Fouquet and Jolivet [1]. A significant amount of interesting papers were devoted to strong edge coloring of graphs.

In 1985, Erdős and Nešetřil [2] conjectured that $\frac{5}{4}\Delta^2(G)$ when $\Delta(G)$ is even and $\frac{5}{4}\Delta^2(G) - \frac{1}{2}\Delta(G) + \frac{1}{4}$ when $\Delta(G)$ is odd.

Andersen [3] and Horák, Qing and Trotter [4] independently showed that $\chi'_s(G) \leq 10$ for any graph G with $\Delta(G) = 3$. Cranston verified that $\chi'_s(G) \leq 22$ for graphs with $\Delta(G) = 4$ in [5] and Huang, Santana and Yu [6] improved the upper bound to 21. Molloy and Reed [7] proved that $\chi'_s(G) \leq 1.998\Delta^2(G)$. Recently, the coefficient was improved to 1.835 by Bonamy, Perrett and Postle [8]. Wang, Shiu, Wang and Chen [9] proved that every planar graph with a maximum degree of 4 can be strong edge colored with at most 19 colors.

In 1993, Brualdi and Massey [10] conjectured that every bipartite graph $G(A, B)$ can be strong edge colored with at most $\Delta(A)\Delta(B)$ colors.

Steger and Yu [11] confirmed that the chromatic index of any bipartite subcubic graph is at most 9. Nakprasit [12] confirmed the upper conjecture for $(2, \Delta)$ -bipartite graphs. Then, Bensmail, Lagoutte and Valicov [13] proved that if G is a $(3, \Delta)$ -bipartite graph, then $\chi'_s(G) \leq 4\Delta$. Recently, Huang, Yu and Zhou [14] confirmed the upper conjecture for $(3, \Delta)$ -bipartite graphs.

Generalized Petersen graph $P(n, k)$ has $2n$ vertices where $v_1, v_2, \dots, v_n$ are denoted as outer vertices and $u_1, u_2, \dots, u_n$ are denoted as inner vertices. $E(P(n, k)) = E_1 \cup E_2 \cup E_3$ where E_1 is the set of edges $v_i v_{i+1}$ for $i \in [1, n]$, named as outer edges; E_2 is the set of edges $u_i u_{i+k}$ for $i \in [1, n]$, named as inner edges; E_3 is the set of edges $v_i u_i$ for $i \in [1, n]$, named as called leg edges. Figure 1 is an illustration of generalized Petersen graph $P(n, k)$ with $n = 9$ and $k = 4$.

Figure 1. Illustrations of generalized Petersen graph $P(9, 4)$.

In 2018, Wu and Yang [15] conjectured that every generalized Petersen graph $P(n, k)$ with $n > 2k$ and $k \geq 4$ can be strong edge colored with 7 colors. In the same paper, they showed some upper bound of strong chromatic index of $P(n, k)$ with $k \in \{1, 2, 3\}$.

In this paper, we prove following theorem:

Theorem 1. Every generalized Petersen graph $P(n, k)$ with $n > 2k$ and $k \geq 4$ can be strong edge colored with at most 9 colors.

The remainder of this paper is organized as follows. In Section 2, we give some lemmas. In Section 3, we show how to color generalized Petersen graphs $P(n, k)$ with $n > 2k$ and $k \geq 4$ with 9 colors.

2. Lemmas

Lemma 1. If $P(n, k)$ is a generalized Petersen graph with $n > 2k$, $k \geq 4$ and $n = ak + b$, then all inner vertices $u_1, u_2, \dots, u_n$ are covered by either one cycle when $(b, k) = 1$ or b cycles when $k = db$.

Proof. Since $P(n, k)$ is a generalized Petersen graph, all inner vertices are covered by some vertex disjoint cycles which are denoted as $C_1, C_2, \dots, C_k$. By symmetry, it is obvious that $|C_1| = |C_2| = \dots = |C_k|$ and $k = \frac{xn}{k}$ where x is the minimum positive integer satisfying $x \cdot b \equiv 0 \pmod{k}$.

- If $(b, k) = 1$, then $|C| = \frac{kn}{k} = n$. The cycle C cover all inner vertices;
- If $k \equiv 0 \pmod{b}$, then $x = d$ and $|C_i| = \frac{dn}{k} = \frac{d(ak+b)}{k} = da + 1$. The corresponding number of cycles which cover all inner vertices is $\frac{n}{|C_i|} = \frac{n}{da+1} = \frac{b(da+1)}{ad+1} = b$.

□

Lemma 2. If C is a cycle with $E(C) = \{e_1, e_2, \dots, e_n\}$, $|L_{e_i}| \geq 3$ for $i \in [1, n]$ and $|L_{e_{n-1}}| = 4$, $|L_{e_n}| = 5$, then cycle C can be strong edge colored properly.

Proof. We greedily color edges from e_1 to e_{n-2} by some color in their list. Since $|L_{e_i}| \geq 3$, it's easy to know $|A_{e_i}| \geq |L_{e_i}| - 2 = 1$ for $i \in [1, n-2]$. Hence, the former $n-2$ edges can be well strong edge colored.

Next, we check and show how to color e_{n-1} and e_n . For e_{n-1} , $A_{e_{n-1}} = L_{e_{n-1}} - \phi(e_{n-2}) - \phi(e_{n-3}) - \phi(e_1)$. Since $|L_{e_{n-1}}| = 4$, then $|A_{e_{n-1}}| \geq 1$, which means e_{n-1} can be colored by some color in $A_{e_{n-1}}$. By the same argument, we can color e_n by a color from A_{e_n} . $\square$

Lemma 3. *If C is a cycle with $E(C) = \{e_1, e_2, \dots, e_n\}$, $|L_{e_i}| \geq 3$ for $i \in [1, k-2] \cup [k+1, n-1]$, $|L_{e_{n-1}}| = 4$, $|L_{e_n}| = 5$, and $|L_{e_{k-1}}| = 4$, $|L_{e_k}| = 2$, then cycle C can be strong edge colored properly.*

Proof. By the argument of Lemma 2, we only need to show how to color e_k , and we have following two cases.

- If $A_{e_k} = L_{e_k} - \phi(e_{k-1}) - \phi(e_{k-2}) \neq \emptyset$, then e_k can be colored by some color in A_{e_k} ;
- If $A_{e_k} = L_{e_k} - \phi(e_{k-1}) - \phi(e_{k-2}) = \emptyset$, then we know $L_{e_k} = \phi(e_{k-1}) + \phi(e_{k-2})$. Next, we show how to color e_k by recoloring e_{k-1} . Since, $|L_{e_{k-1}}| = 4$, then $|A_{e_{k-1}}| = |L_{e_{k-1}} - \phi(e_{k-2}) - \phi(e_{k-3})| = 2$. W.l.o.g, $A_{e_{k-1}} = \{c, d\}$ and $\phi(e_{k-2}) = a$, $\phi(e_{k-3}) = b$, $\phi(e_{k-1}) = c$. By assumption, we can recolor e_{k-1} by d , and color e_k by c .

$\square$

Lemma 4. *If C is a cycle with $E(C) = \{e_1, e_2, \dots, e_n\}$, $|L_{e_i}| \geq 3$ for $i \in [1, n-2]$ and $|L_{e_{n-1}}| = 5$, $|L_{e_n}| = 4$, then C can be strong edge colored properly.*

Proof. By Lemma 2, we only need to show how to color e_n and have following two cases.

- If $A_{e_n} = L_{e_n} - \phi(e_{n-1}) - \phi(e_{n-2}) - \phi(e_1) - \phi(e_1) - \phi(e_2) \neq \emptyset$, then we can color e_n by some color in A_{e_n} ;
- If $A_{e_n} = L_{e_n} - \phi(e_{n-1}) - \phi(e_{n-2}) - \phi(e_1) - \phi(e_1) - \phi(e_2) = \emptyset$, then we know $L_{e_n} = \phi(e_{n-1}) + \phi(e_{n-2}) + \phi(e_1) + \phi(e_1) + \phi(e_2)$. Since $|L_{e_{n-1}}| = 5$, and $|A_{e_{n-1}}| = |L_{e_{n-1}} - \phi(e_{n-2}) - \phi(e_{n-3}) - \phi(e_1)| \geq 2$, then we can recolor edge e_{n-1} by another color of $A_{e_{n-1}}$. W.l.o.g, $A_{e_{n-1}} = \{c, d\}$, $L_{e_n} = \{c, a, e, f\}$, $\phi(e_{n-2}) = a$, $\phi(e_{n-3}) = b$, $\phi(e_1) = e$ and $\phi(e_{n-1}) = c$. We can recolor e_{n-1} by d , and color e_n by c . All edges can be strong edge colored properly.

$\square$

Lemma 5. *If C is a cycle with $E(C) = \{e_1, e_2, \dots, e_n\}$, $|L_{e_i}| \geq 3$ for $i \in [5, n-2]$, $|L_{e_1}| = 4$, $|L_{e_2}| = 2$ and $|L_{e_3}| = 3$, $|L_{e_4}| = 2$, $|L_{e_{n-1}}| = 4$, $|L_{e_n}| = 5$, then cycle C can be strong edge colored properly.*

Proof. By Lemma 2, we only need to show how to color e_4 properly and have following two cases.

- If $L_{e_4} - \phi(e_2) - \phi(e_3) \neq \emptyset$, then we can color e_4 by some color of A_{e_4} ;
- If $L_{e_4} - \phi(e_2) - \phi(e_3) = \emptyset$, then we consider the available colors of e_3 .
 - If $L_{e_3} - \phi(e_2) - \phi(e_1) - \phi(e_3) \neq \emptyset$, then we can recolor e_3 by some color of A_{e_3} . In this case, we have $L_{e_4} - \phi(e_2) - \phi(e_3) \neq \emptyset$, which means we can color e_4 by another color differ from $\phi(e_3)$;
 - If $L_{e_3} - \phi(e_2) - \phi(e_1) - \phi(e_3) = \emptyset$, then we can recolor e_1 by a color differ from $\phi(e_2)$, recolor e_3 by the former color of e_1 and color edge e_4 by the former color of e_3 .

$\square$

Lemma 6. *If C is a cycle with $E(G) = \{e_1, e_2, \dots, e_n\}$, $|L_{e_i}| \geq 3$ for $i \in [1, j-3] \cup [j+1, n-1]$, $|L_{e_{j-2}}| = 5$, $|L_{e_{j-1}}| = 3$, $|L_{e_j}| = 2$, and $|L_{e_{n-1}}| = 4$, $|L_{e_n}| = 5$, then cycle C can be strong edge colored properly.*

Proof. By Lemma 2, we only need to show how to color e_j , and we have following two cases.

- If $L_{e_j} - \phi(e_{j-1}) - \phi(e_{j-2}) \neq \emptyset$, then we can color e_j by some color of A_{e_j} ;

- If $L_{e_j} - \phi(e_{j-1}) - \phi(e_{j-2}) = \emptyset$. Since $|L_{e_{j-2}}| = 5$, then we have $|L_{e_{j-2}} - \phi(e_{j-1}) - \phi(e_{j-3}) - \phi(e_{j-4})| \geq 2$ hold. Hence, we can recolor e_{j-2} by another color which differ from $\phi(e_{j-2})$. After recounting A_{e_j} , we have $L_{e_j} - \phi(e_{j-1}) - \phi(e_{j-2}) \neq \emptyset$ holds. Next, we can color e_j by the former color of e_{j-2} and all edges can be strong edge colored properly.

□

3. Coloring

Theorem 2. If G is a generalized Petersen graph with $n > 2k$, $k \geq 4$ and $n \equiv 0 \pmod{k}$, then $\chi'_s(G) \leq 9$.

Proof. We prove this theorem by following progresses.

First, we pre-color all inner edges by following progresses.

- If $a \equiv 0 \pmod{3}$, then we color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ with “1, 2, 3” in order for $i \in \{1, k\}$;
- If $a \equiv 1 \pmod{3}$, then we color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ for $i \in \{1, k\}$ with “1, 2, 3, 4, 1, 2, 3, $\dots$, 1, 2, 3” in order;
- If $a \equiv 2 \pmod{3}$, then we color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ for $i \in \{1, k\}$ with “1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, $\dots$, 1, 2, 3” in order.

Second, we pre-color all leg-edges by following four cases, and show that every generalized Petersen graph $P(n, k)$ can be strong edge colored with 9 colors in each cases.

Case 1. If k is even and a is even, then we pre-color all leg edges by following four steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6” for $\{i | i \in [1, a]$ and i is odd $\}$ and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 2: Color $e_{L_{(i-1)k+j}}$ with color “7” for $\{i | i \in [1, a]$ and i is odd $\}$ and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 3: Color $e_{L_{(i-1)k+j}}$ with color “8” for $\{i | i \in [1, a]$ and i is even $\}$ and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 4: Color $e_{L_{(i-1)k+j}}$ with color “9” for $\{i | i \in [1, a]$ and i is even $\}$ and $\{j | j \in [1, k]$ and j is even $\}$.

In Case 1, we have three subcases separately as $a \equiv 0 \pmod{3}$, $a \equiv 2 \pmod{3}$ and $a \equiv 2 \pmod{3}$. For the sake of description, we only consider subcase $a \equiv 0 \pmod{3}$, and the other two subcases can be done by same arguments.

By upper steps, we have that all leg edges are colored by $\{6, 7, 8, 9\}$ and inner edges are colored by $\{1, 2, 3\}$. Hence, all leg edges can be distinguished from all inner edges. Next, we only need to show that all leg edges and inner edges can be well strong edge colored separately.

For two leg edges, we have following two cases.

- If two leg edges are $e_{L_{ik+j}}$ and $e_{L_{ik+j+1}}$, then they can be distinguished by two different colors of $\{6, 7\}$;
- If two leg edges are $e_{L_{ik+j}}$ and $e_{L_{(i+1)k+j}}$, then they can be distinguished by two different color sets $\{6, 7\}$ and $\{8, 9\}$.

For any two inner edges with distance at most 1, since they were colored in some order, it is obvious that all inner edges can be strong edge colored properly.

By upper arguments, all leg edges and inner edges have been pre-strong edge colored properly. It's easy to count and get $|A_{e_{ik+1}}| = 4$, $|A_{e_{ik+2}}| = 5, \dots, |A_{e_{ik+k-2}}| = 5$, $|A_{e_{ik+k-1}}| = 4$ and $|A_{e_{ik+k}}| = 2$ for $i \in \{0, a-1\}$. By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly.

Hence, every generalized Petersen graph $P(n, k)$ with $n > 2k$, $k \geq 4$, $n = ak$, k is even and a is even can be strong edge colored properly with at most 9 colors.

Case 2. k is even and a is odd. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a-1]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 2: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a-1]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 3: Color $e_{L_{(i-1)k+j}}$ with color “8”, for $\{i | i \in [1, a-1]$ and i is even $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;

- Step 4: Color $e_{L_{(i-1)k+j}}$ with color “9”, for $\{i \mid i \in [1, a-1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color $e_{L_{(a-1)k+j}}$ with color “7” for $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 6: Color $e_{L_{(a-1)k+j}}$ with color “8”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$.

By similar argument of Case 1, we only consider the subcase $a \equiv 1 \pmod{3}$. By counting A_{e_i} for $i \in [0, ak]$, we have

- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+2}}| = 5, \dots, |A_{e_{ik+k-2}}| = 5, |A_{e_{ik+k-1}}| = 4$ and $|A_{e_{ik+k}}| = 2$ for $i \in [0, a-3] \cup \{a-1\}$;
- $|A_{e_{(a-2)k+1}}| = 4, |A_{e_{(a-2)k+2}}| = \dots |A_{e_{(a-1)k-2}}| = 5, |A_{e_{(a-1)k-1}}| = 4$ and $|A_{e_{(a-1)k}}| = 3$.

By Lemmas 2 and 3, all outer edges can be strong edge colored properly. Hence, every *generalized Petersen graph* $P(n, k)$ with $n > 2k, k \geq 4, n = ak, k$ is even and a is odd can be strong edge colored with 9 colors.

Case 3. k is odd and a is even. First, we pre-color all leg edges by following four steps.

- Step 1: Color $e_{L_{ik+j}}$ with color “6”, for $\{i \mid i \in [1, a-1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L_{ik+j}}$ with color “7”, for $\{i \mid i \in [1, a-1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L_{ik+j}}$ with color “7”, for $\{i \mid i \in [1, a-1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L_{ik+j}}$ with color “6”, for $\{i \mid i \in [1, a-1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$.

By similar argument of Case 1, we only consider the subcase $a \equiv 1 \pmod{3}$. By counting A_{e_i} for $i \in [0, a-1]$, we have $|A_{e_{ik+1}}| = |A_{e_{ik+2}}| = \dots = |A_{e_{ik+k-1}}| = 5, |A_{e_{ik+k}}| = 4$ hold.

By Lemma 2, all outer edges can be strong edge colored properly. Hence, *generalized Petersen graph* $P(n, k)$ with $n > 2k, k \geq 4, n = ak, k$ is odd and a is even can be strong edge colored with 9 colors.

Case 4. k is odd and a is odd. First, we pre-color all leg edges by following three steps.

- Step 1: Color $e_{L_{ik+j}}$ with color “6”, for $\{i \mid i \in [1, a-2] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L_{ik+j}}$ with color “7”, for $\{i \mid i \in [1, a-2] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is even}\}$;
- Step 4: Color $e_{L_{ik+j}}$ with color “7”, for $\{i \mid i \in [1, a-2] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 5: Color $e_{L_{ik+j}}$ with color “6”, for $\{i \mid i \in [1, a-2] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 6: Color $e_{L_{(a-1)k+j}}$ with color “8”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 7: Color $e_{L_{(a-1)k+j}}$ with color “9”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$.

By similar argument of Case 1, we only consider the subcase $a \equiv 2 \pmod{3}$. By counting A_{e_i} for $i \in [0, ak]$, we have

- $|A_{e_1}| = 4, |A_{e_2}| = \dots = |A_{e_{k-1}}| = 5, |A_{e_k}| = 4$;
- $|A_{e_{ik+1}}| = \dots = |A_{e_{ik+k-1}}| = 5, |A_{e_{ik+k}}| = 4$ for $i \in [1, a-3]$;
- $|A_{e_{(a-2)k+1}}| = \dots = |A_{e_{(a-1)k-2}}| = 5, |A_{e_{(a-1)k-1}}| = 4$ and $|A_{e_{(a-1)k}}| = 2$;
- $|A_{e_{(a-1)k+1}}| = 4, |A_{e_{(a-1)k+2}}| = \dots = |A_{e_{(a-1)k+k-2}}| = 5, |A_{e_{(a-1)k+k-1}}| = 4$ and $|A_{e_{ak}}| = 2$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. Hence, every *generalized Petersen graph* $P(n, k)$ with $n > 2k, k \geq 4, n = ak, k$ is odd and a is odd can be strong edge colored with 9 colors.

By upper four cases, we have Theorem 2 hold. $\square$

Theorem 3. If G is a generalized Petersen graph with $n > 2k, k \geq 4, a \geq 2$ and $n = ak + b$, then $\chi'_{sl}(G) \leq 9$.

Proof. To prove the theorem, we divide the proof into following eight cases. Case 1: k is even, a is even and b is even; Case 2: k is even, a is even and b is odd; Case 3: k is even, a is odd and b is even; Case 4: k is even, a is odd and b is odd; Case 5: k is odd, a is even and b is even; Case 6: k is odd, a is even and b is odd; Case 7: k is odd, a is odd and b is odd; Case 8: k is odd, a is odd and b is even. First, we pre-color all *inner edges* by following progresses.

- If $a - 1 \equiv 0 \pmod{3}$, then we color edges by following four steps.
 - Color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ for $i \in \{1, k\}$ with “1, 2, 3” in order;
 - Color edges $u_{i+(a-1)k} u_{i+ak}$ for $i \in \{1, b\}$ with “4”;
 - Color edges $u_i u_{(a-1)k+b+i}$ for $i \in \{1, b\}$ with “5”;
 - Color edges $u_{ak+i} u_{k-b+i}$ for $i \in \{1, b\}$ with “5”.
- If $a - 1 \equiv 1 \pmod{3}$, then we color edges by following four steps.
 - Color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ for $i \in \{1, k\}$ with “1, 4, 2, 3, 1, 2, 3, $\dots$, 1, 2, 3” in order;
 - Color edges $u_{i+(a-1)k} u_{i+ak}$ for $i \in \{1, b\}$ with “4”;
 - Color edges $u_i u_{(a-1)k+b+i}$ for $i \in \{1, b\}$ with “5”;
 - Color edges $u_{ak+i} u_{k-b+i}$ for $i \in \{1, b\}$ with “5”.
- If $a - 1 \equiv 2 \pmod{3}$, then we color edges by following four steps.
 - Color edges $u_i u_{i+k}, u_{i+k} u_{i+2k}, \dots, u_{i+(a-2)k} u_{i+(a-1)k}$ for $i \in \{1, k\}$ with “1, 4, 2, 1, 3, 2, 1, $\dots$, 2, 1, 3” in order;
 - Color edges $u_{i+(a-1)k} u_{i+ak}$ for $i \in \{1, b\}$ with “4”;
 - Color edges $u_i u_{(a-1)k+b+i}$ for $i \in \{1, b\}$ with “5”;
 - Color edges $u_{ak+i} u_{k-b+i}$ for $i \in \{1, b\}$ with “5”.

Next, we color all *leg edges* by following eight cases, and show every *generalized Petersen graph* $P(n, k)$ can be strong edge colored with 9 colors in each case.

Case 1. k is even, a is even and b is even. First, we pre-color all leg edges e_{L_i} for $i \in \{1, n\}$ by following six steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j | j \in [1, a] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j | j \in [1, a] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L_{(i-1)k+j}}$ with color “8”, for $\{i | i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j | j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L_{(i-1)k+j}}$ with color “9”, for $\{i | i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j | j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color $e_{L_{ak+j}}$ with color “7”, for $\{j | j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 6: Color $e_{L_{ak+j}}$ with color “8”, for $\{j | j \in [1, k] \text{ and } j \text{ is even}\}$.

It is obvious that all inner edges and all leg edges can be distinguished separately, and all inner edges can be distinguished from all leg edges. Hence, we only need to color all outer edges properly. Similarly, we only show the case $a - 1 \equiv 1 \pmod{3}$, and the other two cases $a - 1 \equiv 0 \pmod{3}$ and $a - 1 \equiv 2 \pmod{3}$ can be done by same arguments.

If $a - 1 \equiv 0 \pmod{3}$, then we know that

- $|A_{e_1}| = 4, |A_{e_2}| = |A_{e_3}| = \dots = |A_{e_{k-2}}| = 5, |A_{e_{k-1}}| = 4$ and $|A_{e_k}| = 2$;
- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+2}}| = |A_{e_{ik+3}}| = \dots = |A_{e_{(i+1)k-2}}| = 5, |A_{e_{(i+1)k-1}}| = 4$ and $|A_{e_{(i+1)k}}| = 2$ for $i \in [0, a - 2]$;
- $|A_{e_{(a-1)k+1}}| = 4, |A_{e_{(a-1)k+2}}| = |A_{e_{(a-1)k+3}}| = \dots = |A_{e_{(a-1)k+b-2}}| = 5, |A_{e_{(a-1)k+b-1}}| = 4, |A_{e_{(a-1)k+b}}| = |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5, |A_{e_{ak-1}}| = 4$ and $|A_{e_{ak}}| = 3$;
- $|A_{e_{ak+1}}| = 4, |A_{e_{ak+2}}| = |A_{e_{ak+3}}| = \dots = |A_{e_{ak+b-2}}| = 5, |A_{e_{ak+b-1}}| = 4$ and $|A_{e_{ak+b}}| = 3$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. The other two subcases $a - 1 \equiv 1 \pmod{3}$ and $a - 1 \equiv 2 \pmod{3}$ can be done by similar counting progresses and arguments. For sake of describe, we omit the proof of the left two subcases.

Case 2. k is even, a is even and b is odd. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j | j \in [1, k] \text{ and } j \text{ is odd}\}$;

- Step 2: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L(i-1)k+j}$ with color “8”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L(i-1)k+j}$ with color “9”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color e_{Lak+j} with color “2”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is odd}\}$;
- Step 6: Color e_{Lak+j} with color “8”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is even}\}$.

For sake of describe, we only show the subcase $a - 1 \equiv 1 \pmod{3}$, and the other two subcases can be done by similar arguments. If $a - 1 \equiv 1 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+2}}| = |A_{e_{ik+3}}| = \dots = |A_{e_{(i+1)k-2}}| = 5, |A_{e_{(i+1)k-1}}| = 4$ and $|A_{e_{(i+1)k}}| = 2$ for $i \in [0, a - 2]$;
- $|A_{e_{(a-1)k+1}}| = 4, |A_{e_{(a-1)k+2}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5, |A_{e_{(a-1)k+b}}| \geq 3, |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5, |A_{e_{ak-1}}| \geq 3$ and $|A_{e_{ak}}| = 3$;
- $|A_{e_{ak+1}}| = 4, |A_{e_{ak+2}}| = \dots = |A_{e_{ak+b-2}}| = 5, |A_{e_{ak+b-1}}| = 4$ and $|A_{e_{ak+b}}| = 2$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. We are done.

Case 3. k is even, a is odd and b is even. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L(i-1)k+j}$ with color “8”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L(i-1)k+j}$ with color “9”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color $e_{L(a-1)k+j}$ with color “7”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 6: Color $e_{L(a-1)k+j}$ with color “6”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 7: Color e_{Lak+j} with color “8”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is odd}\}$;
- Step 8: Color e_{Lak+j} with color “9”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is even}\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a - 1 \equiv 0 \pmod{3}$, $a - 1 \equiv 1 \pmod{3}$, $a - 1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a - 1 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a - 1 \equiv 2 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+2}}| = |A_{e_{ik+3}}| = \dots = |A_{e_{(i+1)k-2}}| = 5, |A_{e_{(i+1)k-1}}| = 4$ and $|A_{e_{(i+1)k}}| = 2$ for $i \in [0, a - 2]$;
- $|A_{e_{(a-1)k+1}}| = 3, |A_{e_{(a-1)k+2}}| = |A_{e_{(a-1)k+3}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5, |A_{e_{(a-1)k+b}}| = 4, |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5, |A_{e_{ak-1}}| = 4, |A_{e_{ak}}| = 2$;
- $|A_{e_{ak+1}}| = 4, |A_{e_{ak+2}}| = \dots = |A_{e_{ak+b-2}}| = 5, |A_{e_{ak+b-1}}| = 4, |A_{e_{ak+b}}| = 2$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. We are done.

Case 4. k is even, a is odd and b is odd. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L(i-1)k+j}$ with color “8”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L(i-1)k+j}$ with color “9”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color e_{Lak+j} with color “8”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is odd}\}$;
- Step 6: Color e_{Lak+j} with color “9”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is even}\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a - 1 \equiv 0 \pmod{3}$, $a - 1 \equiv 1 \pmod{3}$, $a - 1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a - 2 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a - 1 \equiv 2 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+2}}| = |A_{e_{ik+3}}| = \dots = |A_{e_{(i+1)k-2}}| = 5, |A_{e_{(i+1)k-1}}| = 4$ and $|A_{e_{(i+1)k}}| = 2$ for $i \in [0, a-2]$;
- $|A_{e_{(a-1)k+1}}| = 4, |A_{e_{(a-1)k+2}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5, |A_{e_{(a-1)k+b}}| \geq 3, |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5, |A_{e_{ak-1}}| = 4, |A_{e_{ak}}| \geq 2$;
- $|A_{e_{ak+1}}| = 4, |A_{e_{ak+2}}| = \dots = |A_{e_{ak+b-2}}| = 5, |A_{e_{ak+b-1}}| = 4, |A_{e_{ak+b}}| \geq 2$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. We are done.

Case 5. k is odd, a is even and b is even. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 2: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 3: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a]$ and i is even $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 4: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a]$ and i is even $\}$, and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 5: Color $e_{L_{ak+j}}$ with color “6”, for $\{j | j \in [1, b]$ and j is odd $\}$;
- Step 6: Color $e_{L_{ak+j}}$ with color “7”, for $\{j | j \in [1, b]$ and j is even $\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a-1 \equiv 0 \pmod{3}$, $a-1 \equiv 1 \pmod{3}$, $a-1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a-1 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a-1 \equiv 2 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_1}| = 5, |A_{e_2}| = \dots = |A_{e_{k-1}}| = 5, |A_{e_k}| = 4$;
- $|A_{e_{ik+1}}| = 4, |A_{e_{ik+1}}| = \dots = |A_{e_{(i+1)k-1}}| = 5, |A_{e_{(i+1)k}}| = 4$, for $i \in \{1, a-2\}$;
- $|A_{e_{(a-1)k+1}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5, |A_{e_{(a-1)k+b}}| = 4, |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-1}}| = 5, |A_{e_{ak}}| = 4$;
- $|A_{e_{ak+1}}| = \dots = |A_{e_{ak+b-1}}| = 5, |A_{e_{ak+b}}| = 4$.

By Lemma 3 and Lemma 4, all outer edges can be strong edge colored properly. We are done.

Case 6. k is odd, a is even and b is odd. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a-1]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 2: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a-1]$ and i is odd $\}$, and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 3: Color $e_{L_{(i-1)k+j}}$ with color “7”, for $\{i | i \in [1, a-1]$ and i is even $\}$, and $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 4: Color $e_{L_{(i-1)k+j}}$ with color “6”, for $\{i | i \in [1, a-1]$ and i is even $\}$, and $\{j | j \in [1, k]$ and j is even $\}$;
- Step 5: Color $e_{L_{(a-1)k+j}}$ with color “8”, for $\{j | j \in [1, k]$ and j is odd $\}$;
- Step 6: Color $e_{L_{(a-1)k+j}}$ with color “9”, for $\{j | j \in [1, k]$ and j is even $\}$;
- Step 7: Color $e_{L_{ak+j}}$ with color “7”, for $\{j | j \in [1, b]$ and j is odd $\}$;
- Step 8: Color $e_{L_{ak+j}}$ with color “8”, for $\{j | j \in [1, b]$ and j is even $\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a-1 \equiv 0 \pmod{3}$, $a-1 \equiv 1 \pmod{3}$, $a-1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a-1 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a-1 \equiv 1 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = \dots = |A_{e_{(i+1)k-1}}| = 5, |A_{e_{(i+1)k}}| = 4$, for $i \in [0, a-3]$;
- $|A_{e_{(a-2)k+1}}| = \dots = |A_{e_{(a-1)k-2}}| = 5, |A_{e_{(a-1)k-1}}| = 4, |A_{e_{(a-1)k}}| = 2$;
- $|A_{e_{(a-1)k+1}}| = 4, |A_{e_{(a-1)k+2}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5, |A_{e_{(a-1)k+b}}| = 4, |A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5, |A_{e_{ak-1}}| = 4, |A_{e_{ak}}| = 3$;
- $|A_{e_{ak+1}}| = \dots |A_{e_{ak+b-2}}| = 5, |A_{e_{ak+b-1}}| = 4, |A_{e_{ak+b}}| = 3$.

By Lemma 2 and Lemma 3, all outer edges can be strong edge colored properly. We are done.

Case 7. k is odd, a is odd and b is odd. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color e_{Lak+j} with color “7”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 6: Color e_{Lak+j} with color “8”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a - 1 \equiv 0 \pmod{3}$, $a - 1 \equiv 1 \pmod{3}$, $a - 1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a - 1 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a - 1 \equiv 1 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = \dots = |A_{e_{(i+1)k-1}}| = 5$, $|A_{e_{(i+1)k}}| = 4$, for $i \in [0, a - 2]$;
- $|A_{e_{(a-1)k+1}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5$, $|A_{e_{(a-1)k+b}}| = 4$, $|A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-1}}| = 5$, $|A_{e_{ak}}| = 3$;
- $|A_{e_{ak+1}}| = 4$, $|A_{e_{ak+2}}| = \dots = |A_{e_{ak+b-2}}| = 5$, $|A_{e_{ak+b-1}}| = 4$, $|A_{e_{ak+b}}| = 3$;

By Lemma 2, all outer edges can be strong edge colored properly. We are done.

Case 8. k is odd, a is odd and b is even. First, we pre-color all leg edges by following six steps.

- Step 1: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is odd}\}$;
- Step 2: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is odd}\}$, and $\{j \mid j \in [1, a] \text{ and } j \text{ is even}\}$;
- Step 3: Color $e_{L(i-1)k+j}$ with color “7”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 4: Color $e_{L(i-1)k+j}$ with color “6”, for $\{i \mid i \in [1, a - 1] \text{ and } i \text{ is even}\}$, and $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 5: Color $e_{L(a-1)k+j}$ with color “9”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is odd}\}$;
- Step 6: Color $e_{L(a-1)k+j}$ with color “8”, for $\{j \mid j \in [1, k] \text{ and } j \text{ is even}\}$;
- Step 7: Color e_{Lak+j} with color “6”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is odd}\}$;
- Step 8: Color e_{Lak+j} with color “7”, for $\{j \mid j \in [1, b] \text{ and } j \text{ is even}\}$.

It is easy to see that the following progress, counting available colors for each outer edge, of three subcases $a - 1 \equiv 0 \pmod{3}$, $a - 1 \equiv 1 \pmod{3}$, $a - 1 \equiv 2 \pmod{3}$ are similar. Hence, we only show the counting progress of subcase $a - 1 \equiv 2 \pmod{3}$, and the other two subcases can be done by a same argument.

If $a - 1 \equiv 0 \pmod{3}$, then by upper coloring progress, we know that

- $|A_{e_{ik+1}}| = \dots = |A_{e_{(i+1)k-1}}| = 5$, $|A_{e_{(i+1)k}}| = 4$, for $i \in [0, a - 3]$;
- $|A_{e_{(a-2)k+1}}| = \dots = |A_{e_{(a-1)k-2}}| = 5$, $|A_{e_{(a-1)k-1}}| = 4$, $|A_{e_{(a-1)k}}| = 2$;
- $|A_{e_{(a-1)k+1}}| = 4$, $|A_{e_{(a-1)k+2}}| = \dots = |A_{e_{(a-1)k+b-1}}| = 5$, $|A_{e_{(a-1)k+b}}| = 4$, $|A_{e_{(a-1)k+b+1}}| = \dots = |A_{e_{ak-2}}| = 5$, $|A_{e_{ak-1}}| = 4$, $|A_{e_{ak}}| = 2$;
- $|A_{e_{ak+1}}| = 4$, $|A_{e_{ak+2}}| = \dots = |A_{e_{ak+b-1}}| = 5$, $|A_{e_{ak+b}}| = 4$.

By Lemma 2, all outer edges can be strong edge colored properly. We are done.

After proving Theorem 2 and Theorem 3, it is easy to know that Theorem 1 is correct. $\square$

4. Conclusions

By showing how to use 9 colors to color generalized Petersen graphs $P(n, k)$ with $n > 2k$ and $k \geq 4$, we got our main result.

Author Contributions: M.C. designed the problem and wrote the paper. L.M. and S.Z. validate the results. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Future Scientists Program of “Double First Class” of China University of Mining and Technology (2019WLKXJ047).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Fouquet, J.L.; Jolivet, J.L. Strong edge-colorings of graphs and applications to multi-k-gons. *Ars Comb. A* **1983**, *16*, 141–150.
2. Erdős, P.; Nešetřil, J. *Irregularities of Partitions*; Springer: Berlin, Germany, 1989, Volume 8, pp. 162–163.
3. Andersen, L.D. The strong chromatic index of a cubic graph is at most 10. *Discret. Math.* **1992**, *108*, 231–252. [[CrossRef](#)]
4. Horák, P.; Qing, H.; Trotter, T.W. Induced matchings in cubic graphs. *J. Graph Theory* **1993**, *17*, 151–160. [[CrossRef](#)]
5. Cranston, D.W. Strong edge-coloring of graphs with maximum degree 4 using 22 colors. *Discret. Math.* **2006**, *306*, 2772–2778. [[CrossRef](#)]
6. Huang, M.; Santana, M.; Yu, G. Strong chromatic index of graphs with maximum degree four. *Electron. J. Comb.* **2018**, *25*, 3–10. [[CrossRef](#)]
7. Molloy, M.; Reed, B. A bound on the strong chromatic index of a graph. *J. Comb. Theory Ser. B* **1997**, *69*, 103–109. [[CrossRef](#)]
8. Bonamy, M.; Perrett, T.; Postle, L. Colouring graphs with sparse neighbourhoods. *Bounds Appl.* **2018**, arXiv:1810.06704.
9. Wang, Y.; Shiu, W.C.; Wang, W.; Chen, M. Planar graphs with maximum degree 4 are strongly 19-edge-colorable. *Discret. Math.* **2018**, *341*, 1629–1635. [[CrossRef](#)]
10. Rualdi, R.A.; Massey, J.J.Q. Incidence and strong edge-colorings of graphs. *Discret. Math.* **1993**, *122*, 51–58. [[CrossRef](#)]
11. Steger, A.; Yu, M.L. On induced matchings. *Discret. Math.* **1993**, *120*, 291–295. [[CrossRef](#)]
12. Nakprasit, K. A note on the strong chromatic index of bipartite graphs. *Discret. Math.* **2008**, *308*, 3726–3728. [[CrossRef](#)]
13. Lužar, B.; Mockovčiaková, M.; Soták, R.; Škrekovski, R. Strong edge-coloring of $(3, \Delta)$ -bipartite graphs. *Discret. Math.* **2016**, *339*, 391–398.
14. Huang, M.; Yu, G.; Zhou, X. The strong chromatic index of $(3, \Delta)$ -bipartite graphs. *Discret. Math.* **2017** *340*, 1143–1149. [[CrossRef](#)]
15. Yang, Z.; Wu, B. Strong edge chromatic index of the generalized Petersen graphs. *Appl. Math. Comput.* **2018**, *321*, 431–441. [[CrossRef](#)]

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).