Review

Differential Geometry of Identity Maps: A Survey

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Abstract: An identity map $id_M : M \rightarrow M$ is a bijective map from a manifold M onto itself which carries each point of M return to the same point. To study the differential geometry of an identity map $id_M : M \rightarrow M$, we usually assume that the domain M and the range M admit metrics g and g' , respectively. The main purpose of this paper is to provide a comprehensive survey on the differential geometry of identity maps from various differential geometric points of view.

Keywords: identity map; harmonic map; biharmonic map; Gauss map; Laplace map; harmonic metrics; Walker manifold; Gödel-type space time; index; nullity

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1. Introduction

An identity map $id_M : M \rightarrow M$ from a differentiable manifold M into itself, also known as an identity transformation, is a map that always returns to the same point that was used in as its argument. In other words, the identity map id_M on a manifold M is defined to be the map with domain and range M which satisfies

$$id_M(p) = p \quad \text{for any } p \in M. \quad (1)$$

Obviously, the identity map $id_M : M \rightarrow M$ is the simplest map which is both continuous and bijective.

The main purpose of studying the differential geometry of identity maps is to investigate the following problem:

“To what extent does the identity map of a manifold determine the geometry of the manifold?”

In general, to study the differential geometry of an identity map $id_M : M \rightarrow M$, we shall assume that the domain M and the range M admit two geometric structures of the same kind. In [1], Nagano proved that the identity map $id_M : (M, g) \rightarrow (M, g)$ of a compact Riemannian manifold (M, g) onto itself is a stable map if and only if the diagonal map in $M \times M$ is stable as a minimal submanifold. Further, the author and Nagano proved in [2] that a Riemannian manifold (M, g) admits a geodesic vector field v if and only if the identity map $id_M : (M, g) \rightarrow (M, \mathcal{L}_v g)$ is a harmonic map, where $\mathcal{L}_v$ denotes the Lie derivative with respect to v . For two linear connections Γ and Γ' on a manifold M , the identity map $id_M : (M, \Gamma) \rightarrow (M, \Gamma')$ is called affine if the identity map id_M carries every parallel vector field along each curve γ of (M, Γ) into a parallel vector field along γ in (M, Γ') . It is known from ([3] page 225) that the identity map $id_M : (M, \Gamma) \rightarrow (M, \Gamma')$ is affine if and only if it carries each geodesic of (M, Γ) into a geodesic of (M, Γ') .

To study the conformal geometry of the identity map on a manifold M , we assume that domain M and range M equipped with metrics g and g' , respectively. The identity map $id_M : (M, g) \rightarrow (M, g')$ is called a conformal change of metric if $g' = \rho^2 g$ for some positive function ρ . It is well-known that, in the case $\dim M \geq 4$, if the identity map $id_M : (M, g) \rightarrow (M, g')$ is a conformal change of metric, then (M, g) and (M, g') have the same Weyl tensor (cf. [4]); and in the case $\dim M = 3$, (M, g) and (M, g') have the same Cotton tensor (cf. e.g., [5,6]).

The main purpose of this paper is to provide a survey on known results about identity maps from various Riemannian geometric points of view.

This article is organized as follows. The basic materials on pseudo-Riemannian manifolds is given in the second section. Section 3 provides a brief introduction on harmonic maps including the first and the second variations of Dirichlet energy functionals as well as the stability, index and nullities of harmonic maps. In Section 4, we discuss the relationship between identity maps and (relative) harmonic metrics, (relative) harmonic tensor and geodesic vector fields. We also present a volume-decreasing phenomenon for co-closed harmonic metrics. We mention a necessary and sufficient conditions for the warped product metric of two Einstein manifolds to be harmonic as well as identity maps on Kaehler and hyper-Kaehler manifolds. In Section 5, we present the result for a Walker metric (with neutral signature) on a Walker 4-manifold W^4 to be harmonic with respect to another Walker metric. In Section 6, we discuss index, relative nullity and Killing nullity of identity maps for several important Riemannian manifolds. In addition, results on stabilities and nullities of compact symmetric spaces are presented in this section. In Section 7, we discuss identity maps on compact symmetric spaces. In Section 8, we survey results on the stability of some other important spaces; including homogeneous spaces, spheres, flat tori, and their products, as well as the stability of generalized Sasakian space forms. In Section 9, we present some known results on biharmonic identity maps; including biharmonic metrics, obstructions to the existence of biharmonic metrics on Einstein manifolds, and biharmonicity of Riemannian submersions. In Section 10, we provide the relationship between Gauss maps and identity maps, mainly for surfaces. In Section 11, we present results on Laplace maps and the related identity maps proved in the book [7]. In the last section, we present results on the identity maps of tangent bundles of manifolds equipped with Sasakian, lift-complete, or g -natural metrics. In addition, we discuss the identity maps on the tangent bundles of Walker 4-manifolds and of Gödel-type spacetimes.

2. Pseudo-Riemannian Manifolds and Submanifolds

We follow the notations given in [6,8–10].

2.1. Basics on Pseudo-Riemannian Manifolds

Consider a smooth n -manifold M covered by a system of coordinate neighborhoods $(U, \{x^i\})$, where U is a neighborhood and the $\{x^i\}$ denote local coordinates on U , where the indices $i, j, k, \ell, \dots$ take on values in the range $1, 2, \dots, n$. Then a Riemannian (or pseudo-Riemannian) metric g on M can be expressed as

$$g = g_{ij} dx^i dx^j$$

where we have applied the Einstein convention, i.e., repeated indices, with one upper index and one lower index, denoted summation over its range.

In the following, we denote by $\partial_i = \frac{\partial}{\partial x^i}$ the basis vectors on the coordinate neighborhood $\{U, (x^1, \dots, x^n)\}$. Let X and Y be two vector fields on M . We then have

$$X = X^h \partial_h, \quad Y = Y^h \partial_h,$$

where X^h and Y^h are the local components of the vector fields X and Y , respectively. With respect to the natural frame ∂_h , we have

$$g_{ij} = g(\partial_i, \partial_j),$$

which are the local components of the metric tensor field g . The Christoffel symbols of the Riemannian manifold (M, g) are given by

$$\Gamma_{ij}^k = \frac{g^{k\ell}}{2} (\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}), \quad (2)$$

where $(g^{k\ell})$ denotes the inverse matrix of (g_{ij}) . Thus, we have $g_{ij} g^{jk} = \delta_i^k$, where $\delta_i^k = 1$ or 0 according to $i = k$ or $i \neq k$.

Let ∇ denote the Levi-Civita connection of (M, g) . Then for vector fields X, Y on M the covariant derivative $\nabla_X Y$ has local components

$$X^j \nabla_j Y^h = X^j (\partial_j Y^h + \Gamma_{ji}^h Y^i). \quad (3)$$

The covariant derivative of a tensor T_{ij} of type $(0, 2)$ is defined by

$$\nabla_k T_{ij} = \partial_k T_{ij} - \Gamma_{ki}^h T_{hj} - \Gamma_{kj}^h T_{hi}. \quad (4)$$

Let X, Y and Z be three vector fields in M . Then

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (5)$$

defines a tensor field K of type $(1, 3)$. If we put

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (6)$$

then $K(X, Y)$ is a tensor field of type $(1, 1)$ which is linear both in X and Y . In terms of local components, (6) can be written as

$$K_{kji}^h Z^i = \nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h, \quad (7)$$

where

$$K_{kji}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{k\ell}^h \Gamma_{ji}^\ell - \Gamma_{j\ell}^h \Gamma_{ki}^\ell \quad (8)$$

are the local components of the curvature tensor K .

The Ricci tensor Ric of (M, g) is the $(0, 2)$ -tensor with local components given by

$$Ric_{ij} = K_{kij}{}^k = g^{kh} K_{kjih}. \quad (9)$$

The scalar curvature r is given by

$$r = g^{ij} Ric_{ij}. \quad (10)$$

Let $\mathcal{T}_s^r$ denote the space of all tensors of type (r, s) on (M, g) . Following [2], we denote by δ the codifferential of $\mathcal{T}_s^r$ defined by

$$(\delta T)_{i_1 \dots i_s}^{j_1 \dots j_r} = -\nabla^k T_{ki_1 \dots i_s}^{j_1 \dots j_r}. \quad (11)$$

Then $\delta : \mathcal{T}_{s+1}^r \rightarrow \mathcal{T}_s^r$. As in [2], a tensor T is called co-closed if it satisfies $\delta T = 0$.

2.2. Bianchi Identities

The curvature tensor K satisfies the first Bianchi identity

$$K(X, Y)Z + K(Y, Z)X + K(Z, X)Y = 0, \quad (12)$$

or in local components

$$K_{kjih} + K_{jikh} + K_{ikjh} = 0. \quad (13)$$

The curvature tensor K also satisfies the second Bianchi identity

$$(\nabla_X K)(Y, Z) + (\nabla_Y K)(Z, X) + (\nabla_Z K)(X, Y) = 0, \quad (14)$$

or in local components

$$\nabla_\ell K_{kjih} + \nabla_k K_{j\ell ih} + \nabla_j K_{\ell kih} = 0. \quad (15)$$

2.3. Gradient, Divergence and Laplacian

Assume (M, g) is a pseudo-Riemannian manifold and f is a smooth function on M . Then the gradient of f , denote by ∇f (or by $\text{grad } f$), is the vector field dual to the differential df . In other word, ∇f is defined by

$$\langle \nabla f, X \rangle = df(X) = Xf \quad \forall X \in T(M). \quad (16)$$

In terms of a local coordinate system $\{x^i\}$ of M , the divergence of a vector field $X = X^j \partial_j$, denoted by $\text{div } X$, is given by

$$\text{div } X = \sum_{j=1}^n \left\{ \frac{\partial X^j}{\partial x^j} + \sum_{k=1}^n \Gamma_{jk}^j X^k \right\}. \quad (17)$$

The Laplacian of f , denoted by Δf , is defined by $\Delta f = -\text{div}(\nabla f)$, we have

$$\Delta f = - \sum_{i,j=1}^n g^{ij} \left\{ \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right\}. \quad (18)$$

2.4. Basics on Submanifolds

Assume that $\phi : (M, g) \rightarrow (N, h)$ is an isometric immersion of a pseudo-Riemannian manifold into another. Denote by ∇ and ∇^N the Levi Civita connections on M and N , respectively. The formulas of Gauss and Weingarten are given respectively by

$$\nabla_X^N Y = \nabla_X Y + B(X, Y), \quad (19)$$

$$\nabla_X^N \xi = -A_\xi X + D_X \xi \quad (20)$$

for tangent vector fields X, Y and a normal vector field ξ , where B , A and D are the second fundamental form, the shape operator and the normal connection.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$.

The shape operator and the second fundamental form are related by

$$\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle \quad (21)$$

for X, Y tangent to M and ξ normal to M . The mean curvature vector is defined by $H = \frac{1}{2} \text{trace } h$.

For a vector $\tilde{X} \in T_p N$, $p \in M$, we denote by $\tilde{X}^\top$ and $\tilde{X}^\perp$ the tangential and the normal components of $\tilde{X}$, respectively. The equations of Gauss, Codazzi and Ricci are given respectively by

$$(R^N(X, Y)Z)^T = R(X, Y)Z + A_{B(X, Z)}Y - A_{B(Y, Z)}X, \quad (22)$$

$$(R^N(X, Y)Z)^\perp = (\bar{\nabla}_X B)(Y, Z) - (\bar{\nabla}_Y B)(X, Z), \quad (23)$$

$$(R^N(X, Y)\xi)^\perp = h(A_\xi X, Y) - h(X, A_\xi Y) + R^D(X, Y)\xi, \quad (24)$$

for vector fields X, Y, Z tangent to M , ξ normal to M , and $\bar{\nabla} B$ is given by

$$(\bar{\nabla}_X B)(Y, Z) = D_X B(Y, Z) - h(\nabla_X Y, Z) - B(Y, \nabla_X Z), \quad (25)$$

and R^D is the curvature tensor associated to the normal connection D .

The submanifold M is called totally geodesic if $B = 0$ holds identically; and parallel if $\bar{\nabla} B = 0$ identically. The submanifold is said to have parallel mean curvature vector if $DH = 0$ identically (cf. [11] for a detailed survey on submanifolds with parallel mean curvature vector).

2.5. Submanifolds of Finite Type

The theory of finite type submanifolds was initiated in [12–14]. The first results on finite type submanifolds were collected in the book [15]. Since that time, the subject has developed rapidly. A detailed survey on results up to 1996 was given in author's report [16]. The most recent comprehensive survey on this subject was given in author's book [9] (see also [17–19]).

A submanifold M of $\mathbb{E}^m$ is said to be of finite type if its immersion ϕ is a finite sum of eigenmaps of the Laplacian Δ , i.e.,

$$\phi = c_0 + \phi_1 + \cdots + \phi_k, \quad (26)$$

where c_0 is a constant map and $\phi_1, \dots, \phi_k$ are non-constant maps satisfying

$$\Delta \phi_i = \lambda_{t_i} \phi_i, \quad \lambda_{t_i} \in \mathbb{R}, \quad i = 1, \dots, k. \quad (27)$$

If the eigenvalues $\lambda_{t_1}, \dots, \lambda_{t_k}$ of Δ in (27) are distinct, then M (or the immersion ϕ) is said to be k -type. In particular, if one of $\lambda_{t_1}, \dots, \lambda_{t_k}$ is zero, then M is said to be of null k -type.

Analogously, a smooth map $\psi : M \rightarrow \mathbb{E}^m$ of a Riemannian manifold M is called a finite type map if ψ is a finite sum of $\mathbb{E}^m$ -valued eigenmaps of Δ . In the same way, we also have the notion of k -type

maps and null k -type maps. An isometric immersion (or a map) of a Riemannian manifold into $\mathbb{E}^m$ is said to be of infinite type if it is not of finite type.

For each ϕ_i in the spectral decomposition (27) of ϕ , we put

$$E_i = \text{Span}\{\phi_i(p) : p \in M\}. \quad (28)$$

Then each E_i is a linear subspace of $\mathbb{E}^m$.

2.6. Linearly Independent and Orthogonal Immersions

Consider a k -type isometric immersion $\phi : M \rightarrow \mathbb{E}^m$ whose spectral decomposition is given by (26). Then ϕ is said to be linearly independent if the subspaces $E_1, \dots, E_k$ defined by (28) are linearly independent, i.e., the dimension of the subspace spanned by all vectors in $E_1 \cup \dots \cup E_k$ is equal to $\dim(E_1) + \dots + \dim(E_k)$. The immersion ϕ is said to be orthogonal if the subspaces $E_1, \dots, E_k$ are mutually orthogonal in $\mathbb{E}^m$ (see [20,21]).

If we choose a Euclidean coordinate system $\{u_i\}$ on $\mathbb{E}^m$ with c as its origin, then the spectral decomposition (26) of ϕ reduces to

$$\phi = \phi_1 + \dots + \phi_k, \quad \Delta\phi_i = \lambda_i\phi_i, \quad \lambda_1 < \dots < \lambda_k.$$

For each $i \in \{1, \dots, k\}$, we choose a basis $\{c_{ij} : j = 1, \dots, m_i\}$ of E_i , where m_i denotes the dimension of E_i . Put $\ell = m_1 + \dots + m_k$ and let E^ℓ denote the subspace of $\mathbb{E}^m$ spanned by $E_1, \dots, E_k$. If the immersion ϕ is linearly independent, then $\{c_{ij} : i = 1, \dots, k; j = 1, \dots, m_i\}$ are ℓ linearly independent vectors in E^ℓ . Further, we may choose the Euclidean coordinate system $\{u_i\}$ on $\mathbb{E}^m$ in such way that E^ℓ is defined by $u_{\ell+1} = \dots = u_m = 0$.

We regard each c_{ij} as a column ℓ -vector and put

$$S = (c_{11}, \dots, c_{1m_1}, \dots, c_{k1}, \dots, c_{km_k}),$$

then S is a nonsingular $(\ell \times \ell)$ -matrix. Let D denote the diagonal $(\ell \times \ell)$ -matrix

$$D = \text{Diag}(\overbrace{\lambda_1, \dots, \lambda_1}^{m_1 \text{ times}}, \dots, \overbrace{\lambda_k, \dots, \lambda_k}^{m_k \text{ times}}),$$

where λ_i repeats m_i -times for $i = 1, \dots, k$. If we put $A = SDS^{-1}$, then we get $Ac_{ij} = \lambda_i c_{ij}$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, m_i\}$. Thus, we obtain

$$\Delta\phi = A\phi \quad (29)$$

for the immersion $\phi : M \rightarrow \mathbb{E}^\ell$ induced from $\phi : M \rightarrow \mathbb{E}^m$. We may regard the $(\ell \times \ell)$ -matrix A as an $(m \times m)$ -matrix in a natural way (with zeros for each of the additional entries). For this $(m \times m)$ -matrix A , we also have

$$\Delta\phi = A\phi, \quad A = (a_{ij}). \quad (30)$$

By using the $(m \times m)$ -matrix A in (30), the author introduced in [20] the notion of adjoint hyperquadric as follows.

Let $\phi : M \rightarrow \mathbb{E}^m$ be a non-minimal, linearly independent immersion whose spectral decomposition is given by (26). Let $\{u_i\}$ be a Euclidean coordinate system on $\mathbb{E}^m$ with c as its origin. Let A be the $(m \times m)$ -matrix given in (30). Then, for a given point $p \in M$,

$$\langle Au, u \rangle := \sum_{i,j}^m a_{ij} u_i u_j = c_p, \quad (c_p = \langle A\phi, \phi \rangle(p))$$

defines a hyperquadric Q_p in $\mathbb{E}^m$, called the adjoint hyperquadric of ϕ at p . In particular, if $\phi(M)$ lies in an adjoint hyperquadric Q_p of ϕ for some point $p \in M$, then all of the adjoint hyperquadrics $\{Q_p : p \in M\}$ give the same adjoint hyperquadric which is denoted by Q , called the adjoint hyperquadric of the linearly independent immersion ϕ .

3. Harmonic Maps

Let $\phi : (M, g) \rightarrow (N, \bar{g})$ be a map between Riemannian manifolds. We use $\phi^{-1}(TN)$ to denote the pull-back bundle of the tangent bundle TN by ϕ (or the induced bundle of ϕ). The pull-back bundle has base M and the fiber $T_{\phi(p)}N$ over $p \in M$. The sections of $\phi^{-1}(TN)$ are called vector fields along ϕ . The differential $d\phi$ of ϕ can be regarded as a section of

$$T^*M \otimes \phi^{-1}(TN) = \text{Hom}(TM, \phi^{-1}(TN)) \rightarrow M.$$

Let ∇^ϕ denote the pull-back connection on $\phi^{-1}(TN)$ defined by

$$\nabla^\phi : \Gamma(TM) \times \Gamma(\phi^{-1}(TN)) \rightarrow \Gamma(\phi^{-1}(TN)),$$

$$\nabla_X^\phi(\phi^*\eta) := \nabla_{d\phi(X)}^N(\eta)$$

where $\phi^*\eta = \eta \circ \phi \in \Gamma(\phi^{-1}(TN))$. Denote by $\Gamma(\phi^{-1}(TN))$ the space of all sections of $\phi^{-1}(TN)$.

The second fundamental form σ of a map $\phi : (M, g) \rightarrow (N, \bar{g})$ is given by the covariant differential $\nabla(d\phi)$. A map ϕ is called totally geodesic if its second fundamental form vanishes identically.

The energy density $e(\phi)$ of ϕ is a non-negative function given by

$$e(\phi)_u = \frac{1}{2} \|d\phi_u\|^2 = \frac{1}{2} \text{tr}(\phi^*\bar{g})_u, \quad u \in M, \quad (31)$$

where $d\phi$ denotes the differential of ϕ , $\|d\phi\|$ is the Hilbert–Schmidt norm of $d\phi$, and “tr” denotes the trace. The Dirichlet energy of ϕ over any compact domain D of M is defined by

$$E(\phi, D) = \int_D e(\phi) dV. \quad (32)$$

A variation of a map ϕ is a smooth map $F : M \times (-\varepsilon, \varepsilon) \rightarrow N$ such that $F(p, 0) = \phi(p)$, $p \in M$. A variation $\{\phi_t\}$ is said to be supported in a domain D if $\phi_t = \phi$ on $M \setminus \mathring{D}$ for all $t \in I$, where $\mathring{D}$ denotes the interior of D . Note that $\frac{\partial \phi_t}{\partial t}|_{t=0} = V(p)$ defines a vector field on the pull back bundle $\phi^{-1}(TN)$ and it is called the vector field of the variation $\{\phi_t\}$.

It was well-known that for a smooth variation $\{\phi_t\}$ of $\phi : (M, g) \rightarrow (N, \bar{g})$ supported in D , we have the first variation (cf. [22])

$$\frac{d}{dt} E(\phi_t, D)|_{t=0} = - \int_D \langle V, \tau(\phi) \rangle dv_g, \quad (33)$$

where $V = \frac{\partial \phi_t}{\partial t}|_{t=0}$ is the variation vector field, and $\tau(\phi)$ is the tension field of ϕ given by

$$\tau(\phi) = \text{div}(d\phi). \quad (34)$$

3.1. Harmonic Map and Tension Field

Definition 1. [22] A map $\phi : (M, g) \rightarrow (N, \bar{g})$ between two Riemannian manifolds is called a harmonic map if it is a critical point of the energy functional $E(\phi, D)$ defined by (32), where D is any compact domain of M .

It follows from Definition 1 that a harmonic map $\phi : (M, g) \rightarrow (N, \bar{g})$ is a solution of the corresponding Euler-Lagrange equation for the Dirichlet energy functional. Thus, a map ϕ is harmonic if and only if its tension field $\tau(\phi)$ vanishes identically [22].

For a given map $\phi : (M, g) \rightarrow (N, \bar{g})$, let $\{x^i\}$ and $\{y^\alpha\}$ be the local coordinates on M , and N , respectively. Then

$$\phi(x) = (\phi^1(x), \dots, \phi^n(x))$$

will be the local expression of the map ϕ . Let $\{\partial_i\}$ and $\{\partial_\alpha\}$ be the natural frames of the local coordinates on M and N respectively. For simplicity, we put $\phi_i^\alpha = \frac{\partial \phi^\alpha}{\partial x^i}$ and $\phi_{ij}^\alpha = \frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j}$. Then, in terms of local coordinates, we have

$$\tau(\phi) = g^{ij} \left(\phi_{ij}^\gamma - \Gamma_{ij}^k \phi_k^\gamma + \phi_i^\alpha \phi_j^\beta (\bar{\Gamma}_{\alpha\beta}^\gamma \circ \phi) \right) \partial_\gamma, \quad (35)$$

where Γ_{ij}^k and $\bar{\Gamma}_{\alpha\beta}^\gamma$ denote the Christoffel symbols of (M, g) and $(N, \bar{g})$ with respect to the local coordinate systems $\{x^i\}$ and $\{y^\alpha\}$, respectively.

Remark 1. Definition 1 makes sense for harmonic maps between pseudo-Riemannian manifolds as well.

3.2. Second Variation of Energy

Assume that $\phi : (M, g) \rightarrow (N, \bar{g})$ is a harmonic map from a compact manifold M into $(N, \bar{g})$. Let

$$F : I_\varepsilon \times I_\varepsilon \times M \rightarrow N, \quad F(s, t, p) = \phi_{s,t}(p), \quad \text{and} \quad \phi_{0,0}(p) = \phi(p),$$

be a smooth two parameter variation of ϕ .

Let $v(p) = \frac{\partial F}{\partial s}|_{(0,0,p)} \in T_{\phi(p)}N$, $w(p) = \frac{\partial F}{\partial t}|_{(0,0,p)} \in T_{\phi(p)}N$ be the variation vector fields. Then the Hessian, H_ϕ , of the energy at ϕ is defined by

$$H_\phi(v, w) = \frac{\partial^2}{\partial s \partial t} E(\phi_{s,t})|_{(s,t)=(0,0)}. \quad (36)$$

Then we have

Theorem 1. [23] If $\phi : (M, g) \rightarrow (N, \bar{g})$ is a harmonic map from a compact manifold. Then the Hessian of the energy at ϕ is given by

$$\begin{aligned} H_\phi(v, w) &= \int_M \{ \langle \nabla^\phi v, \nabla^\phi w \rangle - \text{tr}_g \langle K^N(d\phi(\cdot), v) d\phi(\cdot), w \rangle \} dv_g \\ &= \int_M \langle J_\phi v, w \rangle dv_g \end{aligned} \quad (37)$$

for vector fields v, w along ϕ , where K^N denotes the curvature tensor of $(N, \bar{g})$ and the Jacobi operator J_ϕ is given by

$$J_\phi v = -\text{tr}_g(\nabla^2 v) - \text{tr}_g K^N(d\phi, v) d\phi.$$

3.3. Stability, Index and Nullity

A vector field v along ϕ satisfying $J_\phi v = 0$ is called a Jacobi field. The index of ϕ , denoted by $\text{Ind}(\phi)$, is the dimension of the largest subspace of $\Gamma(\phi^{-1}(TN))$ on which H_ϕ is negative definite.

A harmonic map ϕ is called stable if then index $\text{ind}(\phi) = 0$. Geometrically, a harmonic map ϕ is stable if its energy integral $E(\phi)$ is a local minimum of $E(\phi_t)$ for any variation (ϕ_t) with $\phi_0 = \phi$. Otherwise, it is called unstable.

The nullity of ϕ , denoted by $\text{Null}(\phi)$ is the dimension of the kernel of J_ϕ . It follows from the spectral properties of elliptic operators on compact manifolds that both $\text{Ind}(\phi)$ and $\text{Null}(\phi)$ are finite.

For a Riemannian manifold (M, g) , let $\mathbf{i}(M)$ denote the space of all infinitesimal isometries on (M, g) . Then the Killing nullity of a harmonic map $\phi : (M, g) \rightarrow (N, \bar{g})$ is defined by

$$\text{Null}_k(\phi) = \dim(\text{Span}(\mathbf{i}(N), d\phi(\mathbf{i}(M))), \quad (38)$$

where $d\phi(\mathbf{i}(M))$ is considered as variation vector fields along ϕ . The reduced nullity of ϕ , denoted by $\text{Null}_r(\phi)$, is defined by (see [23])

$$\text{Null}_r(\phi) = \text{Null}(\phi) - \text{Null}_k(\phi). \quad (39)$$

The following two results on stable harmonic maps are well-known.

Theorem 2. [24] For $n \geq 3$, there exists no non-constant stable harmonic map from any compact Riemannian manifold to S^n .

Theorem 3. [25] For $n \geq 3$, there exists no non-constant stable harmonic map from S^n to any Riemannian manifold.

4. Identity Maps, Harmonic Metrics and Harmonic Tensors

4.1. Relative Harmonic Metrics

Let $\phi : (M, g) \rightarrow (N, h)$ be an immersion and $g' = \phi^*h$ be the Riemannian metric on M induced via ϕ . Let $\varphi : (M, g') \rightarrow (N, h)$ be the associated isometric immersion, and $\text{id}_M : (M, g) \rightarrow (M, g')$ be the identity map. Then the tension field of the composition $\varphi \circ \text{id}_M$ of maps φ and id_M is given by

$$\tau(\varphi \circ \text{id}_M) = d\phi(\tau(\text{id}_M)) + \text{tr}_g B, \quad (40)$$

where “ tr_g ” is the trace with respect to g and B is the second fundamental form of the associated isometric immersion $\varphi : (M, g') \rightarrow (N, h)$. Thus, it follows from (40) that the composition $\varphi \circ \text{id}_M$ is a harmonic map if and only if the identity map $\text{id}_M : (M, g) \rightarrow (M, g')$ is harmonic and $\text{tr}_g B = 0$.

Partly motivated by this fact, the author and Nagano introduced in 1984 the following notion of (relative) harmonic metrics on a Riemannian manifold.

Definition 2. [2] A metric g' on a Riemannian manifold (M, g) is called harmonic with respect to g if the identity map $\text{id}_M : (M, g) \rightarrow (M, g')$ is a harmonic map.

Let g and g' be metrics on an n -manifold M . Denote by $\nabla, \Gamma_{ij}^k, \dots$, etc, the Levi-Civita connection, Christoffel symbols, $\dots$, etc of (M, g) ; and by $\nabla', \Gamma_{ij}^k, \dots$, etc, the corresponding quantities of (M, g') , respectively.

From (35) we find the following.

Lemma 1. A metric g' on M is harmonic with respect to a metric g on M if and only if

$$g^{ji}(\Gamma_{ji}^k - \Gamma_{ji}^k) = 0. \quad (41)$$

For the metric tensor g' , if we put

$$f = \text{tr}_g g' = g^{ij} g'_{ij}, \quad (42)$$

then we have

$$\nabla_j f = 2g_i'^k L_{jk}^i. \quad (43)$$

Put

$$L_{ji}^k = \Gamma_{ji}'^k - \Gamma_{ji}^k. \quad (44)$$

Then $L = (L_{ji}^k)$ is a tensor of type $(1, 2)$ satisfying $L_{ji}^k = L_{ij}^k$. Moreover, it follows from the definition of Christoffel symbols that

$$\nabla_k g'_{ji} = g_{ja} L_{ki}^a + g'_{ia} L_{kj}^a. \quad (45)$$

Now, we may state the following useful result from [2] which provides an easy way to determine whether a metric g' is harmonic with respect to a given one g on a manifold M .

Proposition 1. *Let g and g' be two Riemannian metrics on a manifold M . Then g' is harmonic with respect to g if and only if we have*

$$\nabla_j f = 2\omega_i, \quad (46)$$

where $f = \text{tr}_g g'$ and $\omega_i = \nabla^k g'_{ki}$.

4.2. Space of Harmonic Tensors

Let $\mathcal{S}$ denote the space of all symmetric covariant tensor fields of degree 2 on a Riemannian manifold (M, g) . If we regard a tensor $T \in \mathcal{S}$ as an energy-momentum tensor, the equation $\delta T = 0$ is known in [26] as the conservation law of energy momentum.

Let $\mathcal{M}$ denote the space of all Riemannian metrics on (M, g) . We put

$$\mathcal{H}_g = \{g' \in \mathcal{M} : g' \text{ is harmonic with respect to } g\}.$$

Then $\mathcal{H}_g$ is a star set centered at g in the space $\mathcal{S}$ (see [2]). The following notion of (relative) harmonic tensors was introduced in [2].

Definition 3. *Let (M, g) be a Riemannian manifold. A tensor $T \in \mathcal{S}$ on (M, g) is called a harmonic tensor with respect to g if it satisfies (46).*

The next two theorems describe the space of all harmonic tensors with respect to g .

Theorem 4. [2] *Let (M, g) be a Riemannian manifold of dimension ≥ 3 . Then we have the following linear isomorphism of vector spaces:*

$$\{g' \in \mathcal{S} : g' \text{ is harmonic w.r.t. } g\} \cong \ker(\delta) \equiv \{T \in \mathcal{S} : \delta T = 0\}.$$

This isomorphism is given by $\mathcal{S} \ni g' \mapsto g' - \frac{1}{2}(\text{tr}_g g')g \in \ker(\delta)$.

Theorem 5. [2] *Let (M, g) be a Riemannian 2-manifold. Then*

$$\{g' \in \mathcal{S} : g' \text{ is harmonic w.r.t. } g\} = \{\lambda g : \lambda \in \mathcal{F}(M)\} \oplus \{T \in \ker(\delta) : \text{tr } T = 0\},$$

where $\mathcal{F}(M)$ is the space of all smooth functions on M .

Remark 2. *Theorems 4 and 5 imply that the space of all harmonic tensors with respect to g is infinite-dimensional.*

The following theorem follows from Proposition 1 and the second Bianchi identity (see [2] page 399).

Theorem 6. Let (M, g) be Riemannian manifold. Then

- (a) The Ricci tensor of (M, g) is a harmonic tensor with respect to g .
- (b) The Ricci tensor of (M, g) is co-closed if and only if (M, g) has a constant scalar curvature.

In particular, Theorem 6 implies the following.

Corollary 1. Let (M, g) be a Riemannian manifold whose Ricci tensor Ric is definite. Then the identity map $\text{id}_M : (M, g) \rightarrow (M, \text{Ric})$ is harmonic.

For 2-dimensional manifolds, we also have the following.

Theorem 7. [2] If (M, g) is an orientable Riemannian 2-manifold, then $\{T \in \ker(\delta) : \text{tr}_g T = 0\}$ is linearly isomorphic with the space of holomorphic quadratic differentials on M with the natural complex structure.

From Theorem 7 and Riemann–Roch’s theorem, we have

Corollary 2. If M is a 2-sphere or a real projective plane, then we have $\{g' \in \mathcal{S} : g' \text{ is harmonic w.r.t. } g\} = \{\lambda g : \lambda \in \mathcal{F}(M)\}$.

Remark 3. One of the reviewers of this article pointed out that there is a discussion of the harmonicity of the identity map and its relation to DeTurck’s method for breaking gauge invariance by Graham and Lee [27] published in 1991.

4.3. Links between Geodesic Vector Fields and Harmonic Tensors

A Killing vector field on an orientable compact Riemannian manifold is characterized by the following two conditions (see [28]):

- (1) $g^{ji} \nabla_j \nabla_i v^k + \text{Ric}_i^k v^i = 0$ and
- (2) divergence $\nabla_i v^i = 0$.

On the other hand, the notion of geodesic vector fields was introduced by Yano and Nagano in [29] as follows.

Definition 4. A vector field $v = (v^i)$ on a Riemannian manifold (M, g) is called a geodesic vector field if it satisfies $g^{ji} \nabla_j \nabla_i v^k + \text{Ric}_i^k v^i = 0$.

The next theorem provides a simple link between geodesic vector fields and harmonic metrics.

Theorem 8. [2] A vector field v on a Riemannian manifold (M, g) is a geodesic vector field if and only if $g' = \mathcal{L}_v g$ is a harmonic tensor with respect to g , where $\mathcal{L}_v$ denotes the Lie derivative with respect to v .

Remark 4. Analogous to Definition 2, a pseudo-Riemannian metric g' on a manifold M is said to be harmonic with respect to another pseudo-Riemannian metric g on M if the identity map $\text{id}_M : (M, g) \rightarrow (M, g')$ is harmonic.

It was shown in [30] that the results given in Sections 4.2 and 4.3 hold true in pseudo-Riemannian setting. Some further results on harmonic metrics in pseudo-Riemannian setting can also be found in [31,32] by Bejan and Duggal.

4.4. A Volume-Decreasing Phenomenon

If g' is a co-closed harmonic metric with respect to a given metric g on a Riemannian manifold M , then we have

- (i) $\text{tr}_g g'$ is constant and
- (ii) for any positive number c , cg' is also a co-closed harmonic metric with respect to g .

For co-closed harmonic metrics, we have the following volume-decreasing phenomenon.

Theorem 9. [2] *Let (M, g) is Riemannian manifold. If g' is a co-closed harmonic metric w.r.t. g on M satisfying $\text{tr}_g g' = \text{tr}_g g$, then*

- (1) *The volume form $dv_{g'} \leq dv_g$ at each point of M ; hence the identity map $i_M : (M, g) \rightarrow (M, g')$ is volume-decreasing.*
- (2) *$dv_{g'} = dv_g$ on M if and only if $g' = g$ on M .*

4.5. Identity Maps on Warped Products

Let (B, g_1) and (N, g_2) be Riemannian manifolds and f a positive function on B . Then the product manifold $B \times N$ equipped with the warped product metric $g = g_1 + f^2 g_2$ is called a warped product manifold with f as its warping function. We denote this warped product manifold by $B \times_f N$.

Pyo, Kim and Park proved the following result in [33].

Theorem 10. *Let (B, g_1) and (N, g_2) be Einstein manifolds such that the Ricci tensors R^B and R^N of B and N satisfy $R^B = c_1 g_1$ and $R^N = c_2 g_2$, respectively. If the identity map*

$$id_{B \times N} : (B \times N, g_1 + g_2) \rightarrow (B \times N, g_1 + f^2 g_2)$$

is harmonic, then $B \times_f N$ is Einstein if and only if $c_1 = f^2 c_2$.

Remark 5. *Eells and Sampson proved in [22] that the identity map id_{S^m} of an m -sphere S^m ($m \geq 3$) is deformable to maps of arbitrary small energy. For further results on identity maps from homotopic points of view, we refer to the two reports [34,35] by Eells and Lemaire.*

4.6. Identity Maps on Kaehler and Hyper-Kaehler Manifolds

Let (M, J, g) be a Kaehler manifold. Then, with respect to a system of local coordinates (x^h) , we have

$$g_{rs} J_j^r J_i^s = g_{ji}, \quad \nabla_k J_j^i = 0.$$

The following proposition was proved in [36] by Watanabe and Dohira.

Proposition 2. *Let (M, J, g) be a Kaehler manifold. If a symmetric tensor T on (M, J, g) is represented by $T_{ji} = J_j^k (\nabla_i \eta_k - \nabla_k \eta_i)$ for some 1-form $\eta = (\eta_i)$, then T is a harmonic tensor with respect to g .*

Watanabe and Dohira also considered in [36] two Kaehler structures (J, g) and (J, g') on a complex manifold M . Let Ω and Ω' denote the Kaehler forms corresponding to g and g' , respectively. By applying Proposition 2, they proved following.

Theorem 11. *If Ω' belongs to the Kaehler class $[\Omega]$, then the identity map $id_M : (M, J, g) \rightarrow (M, J, g')$ is harmonic, i.e., g' is harmonic with respect to g .*

Proposition 2 was extended to hyper-Kaehler manifolds by A. D. Vilcu and G. E. Vilcu as follows.

Theorem 12. [37] *Let (M, g, I, J, K) be a hyper-Kaehler manifold. If a symmetric tensor T is represented by*

$$T_{ji} = \sum_{\Phi=I,J,K} \Phi_j^k (\nabla_i \varphi_k - \nabla_k \varphi_i).$$

for some 1-form $\varphi = (\varphi_i)$, then T is a harmonic tensor with respect to g .

5. Walker Metrics and Identity Maps

5.1. Walker Manifolds

Definition 5. A Walker manifold $(W, g, \mathcal{D})$ is a pseudo-Riemannian manifold (W^n, g) equipped with a lightlike distribution $\mathcal{D}$ (i.e., g restricted to $\mathcal{D}$ is zero) which is parallel with respect to the Levi-Civita connection of g (cf. [38,39]).

In the 4-dimensional case, where the distribution $\mathcal{D}$ has the maximum dimension (i.e., when $\dim \mathcal{D} = 2$), the metric g is of neutral signature and there exists an adapted local coordinates (x, y, z, t) such that the matrix Q associated with the metric g takes the form

$$Q = \begin{pmatrix} 0 & I \\ I & S \end{pmatrix}, \quad (47)$$

where 0 and I denote 2×2 zero and identity matrices, respectively; and

$$S = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad (48)$$

with a, b, c being functions of (x, y, z, t) .

5.2. Equivalence Classes of Walker 4-Manifolds

In [40], Bejan and Druță-Romaniuc introduced an equivalence relation on the set of all Walker metrics on a 4-dimensional manifold as follows: Two Walker metrics g and $\hat{g}$ on a Walker 4-manifold W^4 are said to be equivalent, denoted by $g \sim \hat{g}$, if around any point there exist some adapted local coordinates (x, y, z, t) with respect to which the entries of the associated matrices $Q_{a,b,c}$ of g and $\hat{Q}_{\hat{a},\hat{b},\hat{c}}$ of $\hat{g}$ satisfy the following relation:

$$\begin{cases} \hat{a} = a + \int \lambda_y dx + \alpha(y, z, t), \\ \hat{b} = b + \int \lambda_x dy + \beta(x, z, t), \\ \hat{c} = c - \lambda(x, y, z, t), \end{cases} \quad (49)$$

for some local smooth functions $\alpha(y, z, t)$, $\beta(x, z, t)$ and $\lambda(x, y, z, t)$.

Example 1. Let $\mathcal{M}$ denote the set of all Walker metrics on $\mathbb{R}^4$, whose associated matrices with respect to the coordinate frame are given by (47). Then we may identify $\mathcal{M}$ with $Q_{a,b,c}$ ($a, b, c \in \mathcal{F}(\mathbb{R}^4)$). Consider the group

$$G = \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^4), \quad (50)$$

with respect to the additive operation. Then G acts on $\mathcal{M}$ by the relation (49), and “ $\sim$ ” is the equivalence relation whose classes are the orbits.

On the canonical Walker space $(\mathbb{R}^4, g_0, \mathcal{D})$ with a canonical neutral metric g_0 , the set of all Walker metrics harmonic with respect to g_0 is in one-to-one correspondence with the transformation group G from (50) (see [40]).

5.3. Relative Harmonic Metrics on Walker 4-Manifolds

Bejan and Drctă-Romaniuc determined all Walker metrics which are harmonic with respect to a given Walker metric on a Walker 4-manifold.

Theorem 13. [40] Let $(W^4, g, \mathcal{D})$ be a Walker 4-manifold. Then, for any Walker metric $\hat{g}$ on W^4 , the identity map $id_{W^4} : (W^4, g, \mathcal{D}) \rightarrow (W^4, \hat{g}, \mathcal{D})$ is harmonic if and only if $g \sim \hat{g}$.

6. Index and Nullity of Identity Maps

In this section we present results on nullity and stability on identity maps.

6.1. Nullity of Identity Maps

Let us consider the identity map $id_M : (M, g) \rightarrow (M, g)$. Lemma 1 shows that id_M is always harmonic. In this case, the second variation of the harmonic $id_M : (M, g) \rightarrow (M, g)$ is given by

$$H_{id_M}(v, v) = \int_M \{ \langle \nabla v, \nabla v \rangle - Ric(v, v) \} dv \quad (51)$$

for vector fields v on M . Thus, the nullity of $id_M : (M, g) \rightarrow (M, g)$ is the dimension of the space of the vector fields v satisfying

$$H_{id_M}(u, v) = \int_M \{ \langle \nabla u, \nabla v \rangle - Ric(u, v) \} dv = 0 \quad (52)$$

for any vector field u with compact support in M . The following useful formula can be found in ([28] page 57).

$$H_{id_M}(v, v) = \int_M \left\{ \frac{1}{2} |\mathcal{L}_v g|^2 - (\operatorname{div} v)^2 \right\} dv_g, \quad (53)$$

where $\operatorname{div} v$ denotes the divergence of v .

We have the following general result.

Theorem 14. [2] Let (M, g) be a Riemannian manifold. Then the nullity of the identity map $id_M : (M, g) \rightarrow (M, g)$ is equal to the dimension of $\ker(\delta)^\perp = \{G \in \mathcal{S} : G = \mathcal{L}_v g \text{ for some geodesic vector field } v\}$.

Theorem 14 implies the following.

Corollary 3. If (M, g) be a compact Riemannian manifold, then $\{G \in \mathcal{S} : G \text{ is harmonic w.r.t. } g \text{ and } G \perp \ker(\delta)\}$ is finite-dimensional.

Corollary 4. Let (M, g) be a compact irreducible symmetric space. Then the nullity of the identity map $id_M : (M, g) \rightarrow (M, g)$ is greater than the dimension of the isometry group if and only if (M, g) is either a Hermitian symmetric space or the exceptional simple group manifold G_2 .

Corollary 4 follows from combining Theorem 14 and a result in [41] given by of Yano and Nagano.

6.2. Index and Relative Nullity of Identity Maps

In view of (53), Smith [23] obtained the following.

Proposition 3. Let (M, g) be a compact oriented Riemannian n -manifold with $n \geq 3$. Then the index of $id_M : (M, g) \rightarrow (M, g)$ satisfies

$$\operatorname{index}(id_M) \geq \dim \mathbf{ci}(M),$$

where $\mathbf{ci}(M)$ denotes the space of infinitesimally conformal vector fields on a Riemannian manifold (M, g) .

Let Δ_H be the Hodge Laplacian acting on vector fields via duality. Then the Jacobi operator J_{id_M} satisfies

$$J_{id_M} v = \Delta_H v - Ric(v), \quad (54)$$

where the Ricci tensor Ric of M is regarded as a linear map on the tangent bundle TM . In particular, if (M, g) is Einstein, (54) becomes

$$J_{id_M} = \Delta_H - 2cI, \quad (55)$$

where I denotes the identity operator and c is a constant. By applying this formula, Smith [23] was able to relate the index and nullity of the identity map id_M to the eigenvalues of the Laplacian Δ acting on the space of smooth functions $\mathcal{F}(M)$.

By putting $\lambda(r) = \#\{\text{eigenvalues } \lambda : 0 < \lambda < r\}$ and $m(r)$ the multiplicity of r with $m(0) = 0$, Smith obtained the following result in [23].

Theorem 15. *Let (M, g) be a compact oriented Einstein manifold with Ricci tensor $Ric = cg$ for some constant c . Then the index, nullity and the relative nullity of the identity map $id_M : (M, g) \rightarrow (M, g)$ satisfy*

- (1) $index(id_M) = \lambda(2c)$,
- (2) $Null(id_M) = \dim \mathbf{i}(M) + m(2c)$,
- (3) $Null_r(id_M) = m(2c)$,

In particular, Theorem 15 implies the following.

Corollary 5. *Let (M, g) be a compact Einstein manifold with Ricci tensor $Ric = cg$ for some constant c . Then, the identity map id_M of (M, g) is stable if and only if the first positive eigenvalue λ_1 of the Laplacian Δ acting on $\mathcal{F}(M)$ satisfies $\lambda_1 \geq 2c$.*

6.3. Index and Nullity of Spheres and Flat Torus

Smith determined in [23] the index, relative nullity and Killing nullity of the identity maps of S^n :

$$\begin{aligned} index(id_{S^n}) &= \begin{cases} 0 & \text{if } n = 1, 2; \\ n + 1 & \text{if } n = 3, \end{cases} \\ Null_r(id_{S^n}) &= \begin{cases} 0 & \text{if } n \neq 2; \\ n + 1 & \text{if } n = 2; \end{cases} \\ Null_k(id_{S^n}) &= \frac{n(n+1)}{n} \quad (n \geq 1); \end{aligned}$$

and for the flat torus T^n , he obtained

$$\begin{aligned} index(id_{T^n}) &= 0 & \text{for all } n \geq 2, \\ Null_r(id_{T^n}) &= 0 & \text{for all } n \geq 2, \\ Null_k(id_{T^n}) &= n & \text{for all } n \geq 2. \end{aligned}$$

6.4. Index and Nullity of Product Maps

Let $\phi_i : (M_i, g_i) \rightarrow (N_i, \bar{g}_i)$ $i = 1, 2$, be two harmonic maps. Then the product map

$$\phi : (M_1 \times M_2, g_1 \times g_2) \rightarrow (N_1 \times N_2, \bar{g}_1 \times \bar{g}_2)$$

of ϕ_1, ϕ_2 is the map which carries (x, y) in $M_1 \times M_2$ to $(\phi_1(x), \phi_2(y))$ in $N_1 \times N_2$. The product map of two harmonic maps is also harmonic.

Fardoun and Ratto studied the index of nullity of product maps and obtained the following.

Proposition 4. [42] *Let (M, g) be a compact oriented Einstein manifold with $Ric = cg$ for some constant c . Then the index, nullity and the relative nullity of the identity map $id_M : (M, g) \rightarrow (M, g)$ satisfy*

- (1) $index(\phi) = index(\phi_1) + index(\phi_2)$,

$$(2) \quad \text{Null}(\phi) = \text{Null}(\phi_1) + \text{Null}(\phi_2).$$

Additionally, in [42], Fardoun and Ratto determined the index and nullity of the identity maps for $S^n \times S^m$ and $S^n \times T^m$.

For $S^n \times T^m$, they obtained the following.

Theorem 16. [42] For the identity map $\text{id}_{S^n \times T^m}$ of the product manifolds $S^n \times T^m$, we have:

$$\begin{aligned} (a) \quad & \text{index}(\text{id}_{S^2 \times T^m}) = 0, \quad m \geq 2; \\ (b) \quad & \text{index}(\text{id}_{S^n \times T^m}) = (n+1) \sum_{k=1}^N \#\{z \in \mathbb{Z}^m : |z|^2 = k^2\}, \quad m \geq 2, \quad n > 2; \\ (c) \quad & \text{Null}_k(\text{id}_{S^n \times T^m}) = m + \frac{1}{2}n(n+1), \quad m, n \geq 2; \\ (d) \quad & \text{Null}_r(\text{id}_{S^n \times T^m}) = \begin{cases} 3 & \text{if } n = 2, \quad m \geq 2, \\ 0 & \text{if } n > 2, \quad m \geq 2, \end{cases} \end{aligned}$$

where $N = \max\{k \in \mathbb{N} - \{0\} : 4\pi^2 k^2 < n - 2\}$.

7. Identity Maps on Compact Symmetric Spaces

If the identity map $\text{id}_M : (M, g) \rightarrow (M, g)$ is stable, we simply say that the Riemannian manifold M is stable. Otherwise, M is said to be unstable. It is well-known that M is stable whenever M is Kaehlerian [23] (see also [1] page 131). Theorems 2 and 3 imply that the standard n -sphere S^n is unstable for $n \geq 3$.

7.1. Stability of Compact Symmetric Spaces

The following result was obtained by combining Theorem 14 and Nagano's table of spectra given in [43].

Theorem 17. [23] The classical compact irreducible symmetric spaces are stable with the following exceptions: (1) $S^n, n \geq 3$, (2) $Sp(p+q)/Sp(p) \times Sp(q)$ and (3) $SU(2m)/Sp(m), m > 1$.

For exceptional symmetric spaces, Nagano proved in [1] that the Cayley projective plane $F_4/Spin(9)$ is also unstable.

In [44], Ohnita proved the following two theorems.

Theorem 18. Let M be an n -dimensional compact irreducible symmetric space. Denote by λ_1 and c the first eigenvalue of the Laplacian on functions and the scalar curvature of M respectively. Then the identity map id_M is unstable if and only if one of the following three conditions holds:

- (a) $\lambda_1 < 2c/n$.
- (b) There exists no non-constant stable harmonic map from M to any Riemannian manifold.
- (c) There exists no non-constant stable harmonic map from any compact Riemannian manifold to M .

Theorem 19. Let M be a simply connected compact irreducible symmetric space. Then the identity map id_M is unstable if and only if M belongs to the following list:

- (i) Simple Lie groups of type $A_n (n \geq 2), B_2$ and $C_n (n \geq 3)$;
- (ii) $SU(2n)/Sp(n) (n \geq 3)$;
- (iii) $S^n (n \geq 3)$;
- (iv) $Sp(p+q)/Sp(p) \times Sp(q) (p \geq q \geq 1)$;
- (v) E_6/F_4 ;
- (vi) $F_4/Spin(9)$.

Remark 6. Theorem 18 shows that a compact irreducible symmetric space is simply-connected if its identity map is unstable.

For stable compact symmetric space, Nagano [1] proved the following.

Theorem 20. A compact symmetric space M is stable if and only if the Casimir operator (in the sense of [45]) acting on the tangent bundle TM or on the functions has no eigenvalue < -1 .

Theorem 21. A compact simple, simply-connected Lie group M is stable if and only if it is the exceptional group E_8 .

Theorem 22. A compact connected simple Lie group M with trivial center is stable if and only if it is the adjoint group of $SU(n)$, $SO(2n)$, E_6 , E_7 or E_8 .

7.2. Instability of Compact Symmetric Spaces

Nagano proved the following properties for unstable symmetric spaces.

Theorem 23. Let $M = G/K$ be a unstable simply-connected symmetric space with compact and simple Lie group G . Then M has the following properties:

- (1) M admits a totally geodesic immersion φ into another symmetric space $M' = G'/K'$ such that (i) $K' = G$ or $SO(2) \times G$ and (ii) $\varphi(M)$ is a connected component of the fixed point set M'_+ of the symmetry s_x at some point x of M' or this is true if one projects $\varphi(M)$ into a symmetric space of which M' is a covering space;
- (2) The non-compact form G'' of G' with respect to G'/M' can act on M and as such contains G properly;
- (3) The submanifold $\varphi(M)$ is unstable as a minimal submanifold of M' .

Remark 7. For geometric theory of the polars (M_+) as well as of the meridians (M_-) of compact symmetric spaces, see [46–48].

8. Identity Maps of Other Spaces

8.1. Identity Maps on Homogeneous Space $(SU(3)/T(k, \ell), g)$

Let $i = \sqrt{-1}$ and put

$$T(k, \ell) = \{\text{diag}[e^{2\pi i k \theta}, e^{2\pi i \ell \theta}, e^{-2\pi i (k+\ell) \theta}] : \theta \in \mathbf{R}, |k| + |\ell| \neq 0 (k, \ell \in \mathbb{Z})\}.$$

Then $SU(3)/T(k, \ell)$ is a non-symmetric 7-dimensional homogeneous space which admit a positively curved Riemannian metric (cf. [49]).

Park [50] proved the following result for $SU(3)/T(k, \ell)$.

Theorem 24. Let $SU(3)/T(k, \ell)$ have the $SU(3)$ -invariant metric g which is canonically induced from an $Ad(SU(3))$ -invariant inner product on the Lie algebra $\mathfrak{su}(3)$. If k and ℓ are relatively prime, then the identity map of $(SU(3)/T(k, \ell), g)$ is stable.

8.2. Stability of Spheres with Deformed Metrics

From Section 6, we know that the standard unit m -sphere (S^m, g_0) is unstable for $m \geq 3$. In [51], Tanno studied the stability of a family of deformed metrics $g(t)$ on S^m with $m = 2n + 1$ as follows:

For $m = 2n + 1$, we have the Hopf fibration $\pi : (S^m, g_0) \rightarrow (CP^n, h_0)$, where (CP^n, h_0) is the complex projective n -space of constant holomorphic sectional curvature 4. Let ξ be a vector field on S^m which is tangent to the fibers and of unit length. Then ξ is a Killing vector field with respect to the standard metric g_0 and let η be 1-form dual to ξ with respect to g_0 .

Consider the one-parameter family of metrics $g(t)$ on S^m is defined by

$$g(t) = t^{-1}g_0 + t^{-1}(t^m - 1)\eta \otimes \eta, \quad (56)$$

where $0 < t < \infty$. It is known that the metric given by (56) on S^{2n+1} are homogeneous Riemannian metrics which are not symmetric nor Einstein. With respect to these metrics $(g(t))$, the volume elements remain unchanged (cf. [51,52]).

Tanno proved the following.

Theorem 25. [51] For $m = 2n + 1 \geq 3$ and $t \in (t_0(m), \infty)$, the m -sphere $(S^m, g(t))$ is unstable, where

$$t_0(m)^m = [(m^2 - 4)^2 - 1]/(m^2 - 5) \text{ and } t_0(3) = 0.67 \cdots < t_0(m) < 1.$$

Analogously, for $m = 4r + 3$, S. Tanno [53] applied the Hopf fibration

$$\pi : (S^{4r+3}, g_0) \rightarrow (HP^r, h_0)$$

with the corresponding Sasakian three-structure $\{\eta_1, \eta_2, \eta_3\}$ to define a one-parameter family of metric $g(t)$ on S^{4r+3} by

$$g(t) = t^{-1}g_0 + t^{-1}(t^{m/3} - 1) \sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}. \quad (57)$$

Then he proved the following instability result.

Theorem 26. [51] We have:

- (a) For $m = 4r + 3 = 7, 11$, the sphere $(S^{4r+3}, g(t))$ is unstable.
- (b) For $m = 4r + 3 \geq 15$ and for $t \in (0, t_0(m))$ or $t \in (t_1(m), \infty)$, the sphere $(S^{4r+3}, g(t))$ is unstable, where $t_0(m)$ and $t_1(m)$ are defined in §4 of [53].

8.3. Identity Maps and Riemannian Submersions

In Theorem 3, Xin gave an important result which says that each non-constant harmonic map from the unit n -sphere S^n , $n \geq 3$, into a Riemannian manifold is always unstable. Thus, Urakawa asked in [54] the following

Question 1. Does there exist a deformation $g_t, 0 < t < \infty$, of the standard metric g_1 on S^n such that if g_t is far from g_1 , then (S^n, g_t) admits a stable harmonic map?

To provide an answer to this question, Urakawa [54] investigated a Riemannian submersion $\phi : (M, g) \rightarrow (N, h)$ with totally geodesic fibers. Then he considered the canonical variation $g_t, 0 < t < \infty$, of the metric g which also gives a Riemannian submersion $\phi : (M, g_t) \rightarrow (N, h)$ with totally geodesic fibers. More precisely, Urakawa [54] proved the following theorem which gave an answer to this question.

Theorem 27. Assume that the identity map id_N of (N, h) is stable. Then there exists a small number ε such that, for every $0 < t < \varepsilon$, the Riemannian submersion $\phi : (M, g_t) \rightarrow (N, h)$ is stable.

Since the identity map id_{CP^n} of the complex projective space (CP^n, h) is stable, the Hopf fibering $\pi : (S^{2n+1}, g_t) \rightarrow (CP^n, h)$ is stable for $0 < t < \varepsilon$, for the canonical variation $g_t, 0 < t < \varepsilon$, with $g_1 =$ canonical metric. This provides an example which contrasts with the instability theorem of Xin (Theorem 3). Some further results on identity maps were also obtained by Urakawa in [54].

8.4. Identity Maps on Generalized Sasakian Space Forms

A Riemannian $(2n + 1)$ -manifold $(\bar{M}^{2n+1}, g)$ is called an almost contact metric manifold [55] if there exist a $(1, 1)$ tensor field ϕ , a vector field ξ (called the structure vector field), and a 1-form η on $\bar{M}$ such that

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad (58)$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \quad (59)$$

for any vector fields X, Y on $\bar{M}$.

An almost contact metric manifold $(\bar{M}^{2n+1}, \phi, \xi, \eta, g)$ is called a Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (60)$$

for X, Y tangent to $\bar{M}^{2n+1}$, where ∇ is the Levi-Civita connection of $\bar{M}^{2n+1}$. A Sasakian manifold with constant ϕ -sectional curvature c is called Sasakian space form. In this case, the curvature tensor of $\bar{M}^{2n+1}$ is given by

$$\begin{aligned} R(X, Y)Z = & \frac{c+3}{4} \{g(Y, Z)X - g(X, Z)Y\} \\ & + \frac{c-1}{4} \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + \frac{c-1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned}$$

for vector fields X, Y, Z tangent to $\bar{M}^{2n+1}$. Sasakian space forms can be modeled based on $c > -3$, $c = -3$ or $c < -3$.

Alegre, Blair and Carriazo introduced and studied in [56] the notion of a generalized Sasakian space form. An odd-dimensional manifold $\bar{M}^{2n+1}$ equipped with an almost contact metric structure (ϕ, ξ, η, g) is called generalized Sasakian space form if there exist three functions f_1, f_2, f_3 on $\bar{M}^{2n+1}$ such that

$$\begin{aligned} \bar{R}(X, Y)Z = & f_1 \{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2 \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned} \quad (61)$$

We denote such a manifold by $\bar{M}^{2n+1}(f_1, f_2, f_3)$. Obviously, a generalized Sasakian space form $\bar{M}^{2n+1}(f_1, f_2, f_3)$ is a Sasakian space form if $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

Rehman investigated in [57] the stability of generalized Sasakian space forms and obtained the following results.

Theorem 28. Let $\bar{M}^{2n+1}(f_1, f_2, f_3)$ be a compact generalized Sasakian space form. If $(3f_2 + (2n - 1)f_3) \geq 0$ and $(3f_2 + 2nf_1 - f_3) \leq 0$, then the identity map $id_{\bar{M}}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ is stable.

Theorem 29. Let $\bar{M}^{2n+1}(f_1, f_2, f_3)$ be a compact domain of a generalized Sasakian space form such that $(3f_2 + (2n - 1)f_3) \geq 0$. If the first eigenvalue λ_1 of the Laplacian Δ acting on $\mathcal{F}(\bar{M})$ satisfies $\lambda_1 < 2(3f_2 + 2nf_1 - f_3)$, then the identity map $id_{\bar{M}}$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ is unstable.

Theorems 28 and 29 imply the following corollaries, respectively.

Corollary 6. Let D be a compact domain of a Kenmotsu space form M^{2n+1} of constant φ -sectional curvature c . If c lies in the open interval $(-1, \frac{3n-1}{n+1})$, then the identity map id_D of D is stable.

Corollary 7. Let M^{2n+1} be a compact Sasakian space form of constant φ -sectional curvature $c \leq 1$. If the first eigenvalue of the Laplacian satisfies $\lambda_1 < (n+1)c + 3n - 1$, then the identity map id_M is unstable.

9. Biharmonic Identity Maps

9.1. Biharmonic Maps and Biharmonic Submanifolds

A biharmonic map is a map $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional

$$E_2(\phi, g) = \frac{1}{2} \int_D |\tau(\phi)|^2 dv_g,$$

where D is any compact domain D of M and $\tau(\phi)$ is the tension field of the map ϕ . The Euler–Lagrange equation of the bienergy functional was computed by Jiang in [58]. The first variational formula for the bienergy functional is given by

$$\left. \frac{dE_2(\phi_t)}{dt} \right|_{t=0} = \int_D \langle \tau_2(\phi), \eta \rangle dV_g, \quad (62)$$

where η is the variational vector field along ϕ and the bitension field $\tau_2(\phi)$ is given by

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \text{tr}_g K^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot), \quad (63)$$

with $\Delta^\phi \tau(\phi) = \text{tr}_g(\nabla_{\nabla_M}^\phi - \nabla^\phi \nabla^\phi) \tau(\phi)$ and K^N being the curvature tensor of (N, h) . Hence, ϕ is biharmonic if and only if its bitension field $\tau_2(\phi)$ vanishes identically. A biharmonic submanifold of a Riemannian manifold is one whose defining isometric immersion is a biharmonic map. On the other hand, in his program to study finite type submanifolds, the author defined in the 1980s the notion of biharmonic submanifolds M of Euclidean spaces as those whose position vector field x satisfies $\Delta^2 x = 0$ (cf. [16,59,60]).

In 2002, Caddeo, Montaldo and Oniciuc shown in [61] that the condition $\tau_2(\phi) = 0$ is equivalent to $\Delta^2 x = 0$ for Euclidean submanifolds. Thus, both definitions of biharmonicity coincide for Euclidean submanifolds.

9.2. Biharmonic Maps and Relative Biharmonic Metrics

The following definition was made in [62] (see also [63] page 449).

Definition 6. A Riemannian metric $\bar{g}$ on M is called a (relative) biharmonic metric with respect to another metric g if the identity map $id_M : (M, g) \rightarrow (M, \bar{g})$ is biharmonic. A biharmonic metric $\bar{g}$ of (M, g) is called proper biharmonic metric if it is non-harmonic.

Example 2. For $\lambda = |x|^{-1}, |x|^{-2}, \frac{2}{1 \pm |x|^2}$, the conformally flat metrics $\lambda^2 g_0$ is biharmonic with respect to Euclidean metric g_0 on an open set of $U \subset \mathbb{E}^4$, i.e., each of the following identity maps

$$\begin{aligned} id_{\mathbb{E}^4} : (\mathbb{E}^4 \setminus \{0\}, g_0) &\rightarrow (\mathbb{E}^4 \setminus \{0\}, |x|^{-2} g_0), \\ id_{\mathbb{E}^4} : (\mathbb{E}^4 \setminus \{0\}, g_0) &\rightarrow (\mathbb{E}^4 \setminus \{0\}, |x|^{-4} g_0), \\ id_{\mathbb{E}^4} : (\mathbb{E}^4, g_0) &\rightarrow (\mathbb{E}^4, 4(1 + |x|^2)^{-2} g_0), \\ id_{\mathbb{E}^4} : (\mathbb{B}^4, g_0) &\rightarrow (\mathbb{B}^4, 4(1 - |x|^2)^{-2} g_0), \end{aligned}$$

is a proper biharmonic map.

The purpose of this section is to present known results on biharmonicity of identity maps arisen from conformal change of metrics.

Theorem 30. [31] Let M be an m -manifold. Then the identity map $id_M : (M, g) \rightarrow (M, \bar{g} = e^{2\rho}g)$ is biharmonic if and only if

$$0 = \operatorname{tr}_g \nabla^2(\nabla \rho) + (-2\Delta \rho + (2-m)|\nabla \rho|^2)\nabla \rho + \frac{6-m}{2}\nabla(|\nabla \rho|^2) + \operatorname{Ric}^g(\nabla \rho).$$

Proposition 5. (cf. [63, Corollary 12.17]) Let M be an m -manifold. Then

(a) The identity map

$$id_M : (M, g) \rightarrow (M, \bar{g} = \lambda^2 g), \quad (64)$$

is a biharmonic map if and only if

$$\lambda \nabla \Delta \lambda - 3(\Delta \lambda) \nabla \lambda - \frac{m-4}{2} \nabla |\nabla \lambda|^2 + 2\lambda \operatorname{Ric}^g(\nabla \lambda) = 0, \quad (65)$$

where Ric^g is the Ricci operator of (M, g) . In particular, for $m = 4$, the identity map (64) is biharmonic if and only if

$$\nabla(\lambda \Delta \lambda) - 4(\Delta \lambda) \nabla \lambda + 2\lambda \operatorname{Ric}^g(\nabla \lambda) = 0. \quad (66)$$

(b) The identity map

$$id_M : (M, \bar{g} = \lambda^2 g) \rightarrow (M, g) \quad (67)$$

is a biharmonic map if and only if

$$\lambda \nabla \Delta \lambda - 3(\Delta \lambda) \nabla \lambda - 3(m-6)\lambda^{-1}|\nabla \lambda|^2 \nabla \lambda + \frac{m-8}{2} \nabla |\nabla \lambda|^2 + 2\lambda \operatorname{Ric}^g(\nabla \lambda) = 0. \quad (68)$$

In particular, for $m = 6$, the identity map (67) is biharmonic if and only if

$$\lambda \nabla \Delta \lambda - 3(\Delta \lambda) \nabla \lambda - \nabla |\nabla \lambda|^2 + 2\lambda \operatorname{Ric}^g(\nabla \lambda) = 0. \quad (69)$$

From Definition 6 and Proposition 5, we have

Corollary 8. A conformally related metric $\bar{g} = \lambda^2 g$ on a Riemannian manifold (M^m, g) with $m \geq 3$ is biharmonic with respect to g if and only if the function λ is solution of (65). In particular, a conformally related metric $\bar{g} = \lambda^2 g$ on a 4-manifold (M^4, g) is biharmonic with respect to g if and only if the function λ solves (66).

A conformally related metric $\bar{g} = e^{2\gamma}g$ is called biharmonic in [62] if the identity map $id_M : (M, \bar{g}) \rightarrow (M, g)$ is biharmonic. Thus, by statement (b) of Proposition 5, the equation for the “biharmonic metric” defined in [62] is Equation (68).

Corollary 9. [64] A conformally flat metric $h = \lambda^2 g_0$ on an open set $U \subseteq \mathbb{E}^4$ is biharmonic if and only if the conformal factor is a solution of the Yamabe equation or, equivalently, the Riemannian manifold $(U, h = \lambda^2 g_0)$, has constant scalar curvature ρ_h and $\Delta \lambda = -\frac{\rho_h}{6} \lambda^3$.

The 4-dimensional conformally flat space $(\mathbb{E}^4 \setminus \{0\}, |x|^{-2}g_0)$ appeared in a study of Yang–Mills equation in [65]. The following corollary shows that this conformally flat metric is one of the proper biharmonic metrics in a Euclidean 4-space.

Corollary 10. *The conformally flat metric $h = |x|^{2\alpha}g_0$ on $\mathbb{E}^4 \setminus \{0\}$ is proper biharmonic with respect to Euclidean metric g_0 if and only if $\alpha = -1$, or -2 .*

9.3. Obstructions to Biharmonic Metrics on Einstein Manifolds

The next three results of Baird, Fardoun and Ouakkas provide obstructions to the existence of proper biharmonic metric on compact Einstein manifolds.

Theorem 31. [66] *Let (M^m, g) be a compact Einstein manifold (M^m, g) with $\text{Ric} = c g$. If $m \geq 6$ and $c < 0$, or $m > 6$ and $c \leq 0$, then the identity map $\text{id}_{M^m} : (M^m, g) \rightarrow (M^m, e^{2\rho}g)$ is biharmonic if and only if it is harmonic, i.e., ρ is constant.*

Theorem 32. [66] *Let (M^m, g) be a compact Einstein manifold with $m > 2$ and $\text{Ric} = c g$. Assume that $c \geq 0$ and the 1-form $\alpha = d\rho$ satisfies*

$$\frac{10-m}{2} \int_M |\alpha|^2 \delta \alpha dv_g + (2-m) \int_M |\alpha|^4 dv_g + 2c \int_M |\alpha|^2 dv_g \leq 0. \quad (70)$$

Then $\text{id}_{M^m} : (M^m, g) \rightarrow (M^m, e^{2\rho}g)$ is biharmonic if and only if id_{M^m} is harmonic.

Remark 8. *Given an arbitrary 1-form α , one can always find a constant k such that $k\alpha$ satisfies condition (70).*

Theorem 33. [66] *Let (M, g) be a compact Einstein 10-manifold (M, g) with $\text{Ric} = c g$, $c > 0$. Then*

- (1) *If $c \int_M |\alpha|^2 dv_g \leq 4 \int_M |\alpha|^4 dv_g$, then $\text{id}_{M^m} : (M^m, g) \rightarrow (M^m, e^{2\rho}g)$ is biharmonic if and only if it is harmonic.*
- (2) *If $\frac{c}{4}(\text{vol}(M))^{1/2} \leq (\int_M |\alpha|^4 dv_g)^{1/2}$, then $\text{id}_{M^m} : (M, g) \rightarrow (M, e^{2\rho}g)$ cannot be biharmonic.*

9.4. A Biharmonicity Reduction Theorem For Submersions

The next result was obtained by Balmuş.

Theorem 34. [31] *Let $\pi : (M, g) \rightarrow (N, h)$ be a harmonic Riemannian submersion and $\tilde{h}$ a Riemannian metric on N . Denote by $\tilde{\pi} = \text{id}_N \circ \pi : (M, g) \rightarrow (N, \tilde{h})$, where $\text{id}_N : (N, h) \rightarrow (N, \tilde{h})$ is the identity map. Then we have*

$$\tau(\tilde{\pi}) = \tau(\text{id}_N) \quad \text{and} \quad \tau_2(\tilde{\pi}) = \tau_2(\text{id}_N),$$

where $\tau(\phi)$ and $\tau_2(\phi)$ denote tension field and bitension field of a map ϕ .

This theorem implies that the study of biharmonicity of a harmonic Riemannian submersion $\pi : (M, g) \rightarrow (N, h)$ reduces to the study of biharmonicity of the identity map $\text{id}_N : (N, h) \rightarrow (N, \tilde{h})$.

10. Gauss and Identity Maps

Let Q_{m-n} be the Grassmann manifold consisting of oriented linear n -subspaces of the Euclidean m -space $(\mathbb{E}^m, \tilde{g}_0)$ equipped with the standard Euclidean metric $\tilde{g}_0$. Then Q_{m-n} admits a standard Riemannian metric $\tilde{G}_0$ which makes Q_{m-n} a compact symmetric space. In particular, Q_{m-2} with a natural complex structure is holomorphically isometric to the complex quadric of complex dimension $m-2$.

10.1. Gauss Images

Consider an immersion $\phi : M^n \rightarrow (\mathbb{E}^m, \bar{g}_0)$ from an n -manifold M^n into a Euclidean m -space $\mathbb{E}^m$. Let g_0 denote the induced metric on M^n via $\phi : M^n \rightarrow (\mathbb{E}^m, \bar{g}_0)$. The Gauss map $\nu : M^n \rightarrow (Q_{m-n}, \tilde{G}_0)$ is the map obtained by parallel displacement of the tangent plane $d\phi(T_p M^n)$ ($p \in M^n$) in $\mathbb{E}^m$. In this section, we always assume that the Gauss map ν is a regular map so that the Gauss map ν induces a metric G_0 on M^n . In this way, M^n admits two metrics g_0 and G_0 induced via ϕ and ν , respectively.

Definition 7. A submanifold M^n of $\mathbb{E}^m$ is said to have totally geodesic Gauss image if geodesics of (M, G_0) are carried to geodesics of $(Q_{m-n}, \tilde{G}_0)$ by the Gauss map.

The following result was proved by the author and Yamaguchi.

Theorem 35. [67,68] A Euclidean submanifold M^n in $\mathbb{E}^m$ has totally geodesic Gauss image if and only if the second fundamental form B of M^n in $\mathbb{E}^m$ satisfies

$$(\bar{\nabla}_X B)(Y, Z) = B(\nabla_X^{G_0} Y - \nabla_X Y, Z) \quad (71)$$

for vector fields X, Y, Z tangent to M^n , where the Gauss image ∇^{G_0} is the Levi-Civita connection of (M^n, G_0) .

10.2. Geometry of Gauss Transformations

In this subsection, we discuss the identity map $id_M : (M, G_0) \rightarrow (M, g_0)$ associated with an isometric immersion $\phi : (M, g_0) \rightarrow \mathbb{E}^m$. We call this identity map the Gauss transformation of ϕ . More precisely, we present results for a surface in $\mathbb{E}^m$ to have harmonic, conformal, homothetic, or affine Gauss transformation. Here the identity map $id_M : (M, G_0) \rightarrow (M, g_0)$ is called affine if it carries geodesics into geodesics.

Theorem 36. [2] An isometric immersion $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ of a surface is harmonic if and only if $\phi : (M, G_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ is harmonic.

Theorem 36 implies the following.

Corollary 11. [2] Let $\phi : M \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an immersion of a surface M into $\mathbb{E}^m$. Then the Gauss map $\nu : (M, G_0) \rightarrow (Q_{m-2}, \tilde{G}_0)$ is harmonic and the identity map $id_M : (M, G_0) \rightarrow (M, g_0)$ is homothetic if and only if

- (1) (M, g_0) has constant Gauss curvature and
- (2) there exists a hypersphere S^{m-1} of $\mathbb{E}^m$ such that $\phi : (M, g_0) \rightarrow S^{m-1}$ is harmonic.

Remark 9. Theorem 36 is false in general if $\dim M \geq 3$. In fact, for $n \geq 3$, there exists a hypersurface in $\mathbb{E}^{n+1}$ such that $\phi : (M, g_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ is harmonic, but $\phi : (M, G_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ is not harmonic.

Theorem 37. [2] Let $\phi : M \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an immersion of a surface M into $\mathbb{E}^m$. Then the identity map $id_M : (M, G_0) \rightarrow (M, g_0)$ is conformal and the Gauss map $\nu : (M, G_0) \rightarrow (Q_{m-2}, \tilde{G}_0)$ is harmonic if and only if either

- (a) $\phi : (M, g_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ is harmonic, or
- (b) there exists a hypersphere S^{m-1} of $\mathbb{E}^m$ such that $\phi : (M, g_0) \rightarrow S^{m-1}$ is harmonic.

Theorem 38. [2] Let $\phi : M \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an immersion of a surface M into $\mathbb{E}^m$. If $id_M : (M, G_0) \rightarrow (M, g_0)$ is affine, then either $id_M : (M, G_0) \rightarrow (M, g_0)$ is homothetic or both (M, g) and (M, G) are flat surfaces.

Theorem 39. [2] Let $\phi : M \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an immersion of a surface M into $\mathbb{E}^m$. Then $id_M : (M, G_0) \rightarrow (M, g_0)$ is affine and the Gauss map $\nu : (M, G_0) \rightarrow (Q_{m-2}, \tilde{G}_0)$ is harmonic if and only if (M, g_0) has constant

Gauss curvature and either (1) (M, g_0) is immersed in a hypersphere of $\mathbb{E}^m$ as a minimal surface via ϕ , or (2) (M, g_0) is immersed as an open portion of the product surface of two planar circles via ϕ .

11. Laplace and Identity Maps

Consider an isometric immersion $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ of a Riemannian n -manifold M into a Euclidean m -space $\mathbb{E}^m$. Then the Laplace operator Δ of (M, g_0) gives rise to a differentiable map

$$L_\phi : M \rightarrow (\mathbb{E}^m, \bar{g}_0); \quad p \mapsto (\Delta\phi)(p), \quad (72)$$

called the Laplace map, where ϕ is regarded as the immersion as well as the position vector field of M in $\mathbb{E}^m$. The image $L_\phi(M)$ of M under map L_ϕ is called the Laplace image of M .

In this section, we assume that the Laplace map L_ϕ is a regular map, so that the Laplace image $L_\phi(M)$ admits an induced metric g_{L_ϕ} via L_ϕ . In this way, M admits two metrics g_0 and g_{L_ϕ} . Hence, we may consider the differential geometry of the identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$. Note that the identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ was called as the Laplace transformation in the book [7] of the author and Verstraelen.

The following question was asked and initially studied in [7].

Question 2. *To what extent does the identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ determine the immersion $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$?*

In this section we present some answers for this question from [7].

11.1. Submanifolds with Homothetic Laplace Transformations

Let $M = G/H$ be a homogeneous Riemannian manifold and $I(M)$ be the group of isometries on M . Then an isometric immersion $\psi : M \rightarrow M$ is said to be G -equivariant if there exists a homomorphism $\zeta : G \rightarrow I(M)$ such that $\psi(g(p)) = \zeta(g)\psi(p)$ for each $g \in G$ and $p \in M$.

Theorem 40. *Let $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an equivariant isometric immersion of a compact irreducible Riemannian homogeneous manifold into $\mathbb{E}^m$. Then the identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ is homothetic.*

For an isometric immersion of (M, g) into a Euclidean m -space $\mathbb{E}^m$ with homothetic identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$, we have the following results from [7].

Theorem 41. *Let $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an isometric immersion. If $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ is homothetic, then we have:*

- (1) *If ϕ is of finite type, the Laplace map $L_\phi : M \rightarrow \mathbb{E}^m$ is of finite type.*
- (2) *If M is compact, then the Laplace map L_ϕ is of k -type if and only if the immersion ϕ is of k -type.*
- (3) *If ϕ is of finite type, then the Laplace map is of non-null finite type; in particular, if x is of non-null k -type, then L_ϕ is of non-null k -type; and if ϕ is of null k -type, then L_ϕ is of non-null $(k - 1)$ -type.*

Theorem 42. *Let $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an isometric immersion. If $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ is homothetic, then the Laplace image lies in a hypersphere of $\mathbb{E}^m$ as a minimal submanifold if and only if either (1) M is a minimal submanifold of a hypersphere of $\mathbb{E}^m$ or (2) ϕ is of null 2-type.*

Theorem 43. *If $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ is an isometric immersion with parallel mean curvature vector, then $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ is homothetic if and only if M is immersed as a minimal submanifold of a hypersphere of $\mathbb{E}^m$ via ϕ .*

Theorem 44. *Let $\phi : (M^n, g_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ be a hypersurface of $\mathbb{E}^{n+1}$ with $n = 2$ or 3 . Then $id_{M^n} : (M^n, g_0) \rightarrow (M^n, g_{L_\phi})$ is homothetic if and only if M^n is an open part of a hypersphere in $\mathbb{E}^{n+1}$ via ϕ .*

Theorem 45. Let $\phi : (M^2, g_0) \rightarrow (\mathbb{E}^4, \bar{g}_0)$ be an isometric immersion with constant mean curvature. Then $id_{M^2} : (M^2, g_0) \rightarrow (M^2, g_{L_\phi})$ is homothetic if and only if M^2 is a minimal surface in a hypersphere of $\mathbb{E}^4$ via ϕ .

11.2. Geometry of Conformal or Harmonic Laplace Transformations

Theorem 46. Let $\phi : (M^n, g_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ be a hypersurface with $n \geq 4$. If the identity map $id_{M^n} : (M^n, g_0) \rightarrow (M^n, g_{L_\phi})$ is conformal and (M^n, g) has positive semi-definite Ricci tensor, then (M^n, g) is conformally flat.

Theorem 47. Let $\phi : (M^2, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an isometric immersion of a surface M^2 into $\mathbb{E}^m$. If $id_{M^2} : (M^2, g_0) \rightarrow (M^2, g_{L_\phi})$ is conformal, then the Laplace image $L_\phi(M)$ is a minimal surface of $\mathbb{E}^m$ via L_ϕ if and only if ϕ is biharmonic.

Theorem 48. Let $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an isometric immersion with parallel mean curvature vector. Then the identity map $id_M : (M, g_0) \rightarrow (M, g_{L_\phi})$ is harmonic.

11.3. Geometry of Laplace-Gauss Identity Maps

Suppose $\phi : (M^n, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ is an isometric immersion from a Riemannian n -manifold M^n into $\mathbb{E}^m$. For simplicity, we assume that the Gauss map and the Laplace map associated with ϕ are regular. Thus we have the metrics G_0 and g_{L_ϕ} on M^n via Gauss and Laplace maps, respectively. Therefore, we may consider the identity map

$$i_{LG_\phi} : (M^n, g_{L_\phi}) \rightarrow (M^n, G_0) \quad (73)$$

between the Laplace and Gauss images. We call this map the Laplace–Gauss identity map, or the LG-identity map of ϕ for short.

The following results on LG-identity maps are also obtained in [7].

Theorem 49. Let $\phi : (M, g_0) \rightarrow (\mathbb{E}^m, \bar{g}_0)$ be an isometric immersion. Then the LG-identity map $i_{LG_\phi} : (M, g_{L_\phi}) \rightarrow (M, G_0)$ is conformal if and only if locally either ϕ has constant mean curvature function or M is immersed as product of a planar curve and an affine $(n - 1)$ -space $\mathbb{E}^{n-1}$ via ϕ .

Remark 10. A surface in $\mathbb{E}^3$ is called a surface of Delaunay if it has constant mean curvature function. Theorem 49 implies that every surface of Delaunay in $\mathbb{E}^3$ has homothetic LG-identity map.

Theorem 50. Let $\phi : (M^n, g_0) \rightarrow (\mathbb{E}^{n+1}, \bar{g}_0)$ be a hypersurface. Then the LG-identity map $i_{LG_\phi} : (M, g_{L_\phi}) \rightarrow (M, G_0)$ is homothetic if and only if locally either M has constant mean curvature function in $\mathbb{E}^{n+1}$ or M is the product of an affine $(n - 1)$ -space $\mathbb{E}^{n-1}$ with a planar curve whose curvature function is given by

$$\kappa(s) = \frac{n}{2} \left(\sqrt{a^2 e^{-s^2} + 4(c + c^2)} - a e^{-s} \right)$$

for some positive constants a, c .

Theorem 51. Let $\phi : M^n \rightarrow S^{n+1} \subset \mathbb{E}^{n+2}$ be a hypersurface of a hypersphere S^{n+1} of $\mathbb{E}^{n+2}$. Then M has constant mean curvature function and $i_{LG_\phi} : (M, g_{L_\phi}) \rightarrow (M, G_0)$ is homothetic if and only if either

- (1) M^n is an open part of the product of two spheres $M^k(a) \times M^{n-k}(b)$ with some suitable radii a and b , or
- (2) M^n is an open part of a hypersphere of S^{n+1} .

Theorem 52. Let $\phi : M^n \rightarrow S^{n+1} \subset \mathbb{E}^{n+2}$ be a hypersurface of a hypersphere S^{n+1} of $\mathbb{E}^{n+2}$. If M has non-constant mean curvature function and $i_{LG_\phi} : (M, g_{L_\phi}) \rightarrow (M, G_0)$ is conformal, then

- (1) The gradient of the mean curvature function is an eigenvector of the shape operator A of M^n in S^{n+1} .
- (2) The shape operator A has at most three distinct eigenvalues.
- (3) The eigenvalue corresponding to the eigenvector given by the gradient of the mean curvature function is of multiplicity one.

Theorem 53. Let $\phi : M^2 \rightarrow S^3 \subset \mathbb{E}^4$ be an isometric immersion. Then M^2 has constant mean curvature in S^3 and the LG-identity map $i_{LG_\phi} : (M^2, g_{L_\phi}) \rightarrow (M^2, G_0)$ is conformal if and only if M^2 is either a totally umbilical surface or a minimal surface in S^3 .

Theorem 54. Let $\phi : M^2 \rightarrow S^3 \subset \mathbb{E}^4$ be an isometric immersion. Then M^2 has constant Gauss curvature and $i_{LG_\phi} : (M^2, g_{L_\phi}) \rightarrow (M^2, G_0)$ is homothetic if and only if locally M^2 is a totally umbilical surface or a Clifford torus in S^3 .

For further results in this respect, see the book [7].

12. Identity Maps of Tangent Bundles

For general information on geometry of tangent bundles, we refer to [69].

12.1. Tangent Bundles

Let (M, g) be a pseudo-Riemannian manifold with Levi–Civita connection ∇ . The tangent bundle TM of M consists of pairs (x, X) , where x is a point in M and X a tangent vector to M at x . The map $\pi : TM \mapsto M : (x, X) \mapsto x$ is the natural projection.

The tangent space $T_{(x,X)}TM$ at a point (x, X) in TM is a direct sum of the vertical subspace $\mathcal{V}_{(x,X)} = \ker(d\pi|_{(x,X)})$ and the horizontal subspace $\mathcal{H}_{(x,X)}$ with respect to the Levi–Civita connection ∇ of M :

$$T_{(x,X)}TM = \mathcal{H}_{(x,X)} \oplus \mathcal{V}_{(x,X)}.$$

For any vector $Y \in T_xM$, there exists a unique vector $Y^h \in \mathcal{H}_{(x,X)}$ at $(x, X) \in TM$ which is called the horizontal lift of Y to the point (x, X) such that $d\pi(Y^h) = Y$; and a unique vector $Y^v \in \mathcal{V}_{(x,X)}$ which is called the vertical lift of Y to the point (x, X) such that $Y^v(df) = Y(f)$ for all $f \in \mathcal{F}(M)$. Hence, every vector $\tilde{Y} \in T_{(x,X)}TM$ can be decomposed as

$$\tilde{Y} = Y_1^h + Y_2^v$$

for uniquely determined vectors $Y_1, Y_2 \in T_xM$. The horizontal lift (respectively, vertical lift) of a vector field X on M to TM is the vector field X^h (respectively, the vector field X^v) on TM whose value at (x, X) is the horizontal lift (respectively, vertical lift) of X_x to (x, X) .

The tangent bundle TM of (M, g) equips with a metric g^S in a natural way; called the Sasaki metric, which depends only on the metric g of the base manifold M . The Sasaki metric g^S on TM is defined by the relations:

$$\begin{cases} g^S(X^h, Y^h) = g(X, Y), \\ g^S(X^h, Y^v) = 0, \\ g^S(X^v, Y^v) = g(X, Y), \end{cases} \quad (74)$$

for vector fields X, Y on TM . Intuitively, the Sasaki metric g^S is constructed in such a way that the vertical and horizontal sub-bundles are orthogonal and the bundle map $\pi : (TM, g^S) \mapsto (M, g)$ is a pseudo-Riemannian submersion.

12.2. Identity Maps of Tangent Bundles with Lift-Complete Metrics

Let (M, g) be a Riemannian n -manifold. Oniciuc defined in [70] a pseudo-Riemannian metric g^c on TM , called the lift-complete of g , defined by the following relations:

$$\begin{cases} g^c(X^v, Y^v) = 0, \\ g^c(X^h, Y^v) = g(X, Y), \\ g^c(X^h, Y^h) = g(X, Y). \end{cases} \quad (75)$$

Via a non-linear connection on the tangent bundle TM , Oniciuc defined on TM in [70] the tensor field G of type $(0, 2)$, called of lift-complete type, by

$$\begin{cases} G(X^v, Y^v) = 0, \\ G(X^h, Y^v) = g(X, Y), \\ G(X^h, Y^h) = g(X, Y). \end{cases} \quad (76)$$

Oniciuc proved the following.

Theorem 55. [70] *We have:*

- (a) *The identity map $id_{TM} : (TM, G) \rightarrow (TM, g^c)$ is totally geodesic if and only if the projection $\pi : (TM, G) \rightarrow (M, g)$ is totally geodesic.*
- (b) *The identity map $id_{TM} : (TM, G) \rightarrow (TM, g^c)$ is biharmonic, i.e., g^c is biharmonic with respect to G .*

For detailed results in this respect, see [70].

12.3. Identity Maps of Tangent Bundles with g -Natural Metrics

For a Riemannian manifold M , the Sasaki metric g^S is only one possible choice inside a very large family of Riemannian metrics on the tangent bundle TM of M , known as Riemannian g -natural metrics. Those metrics are constructed in a “natural” way from the Riemannian metric g over M .

The introduction of g -natural metrics by Kowalski and Sekizawa in [71] converts the classification of second order natural transformations of Riemannian metrics on manifolds to that of metrics on tangent bundles.

The set of g -natural metrics which depend on six smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$ has been completely described by Abbassi and Serih in [72] as follows.

Theorem 56. *Let (M, g) be a Riemannian manifold and G be a g -natural metric on TM . Then there are six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \rightarrow \mathbb{R}$, $i = 1, 2, 3$, such that for every $u, X, Y \in T_x M$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g_x(X, Y) + (\beta_1 + \beta_3)(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g_x(X, Y) + \beta_1(r^2)g_x(X, u)g_x(Y, u), \end{cases}$$

where $r^2 = g_x(u, u)$. For $n = 1$, the same holds with $\beta_i = 0$, $i = 1, 2, 3$.

To state the next two theorems, we put

$$\begin{aligned} \phi_i(t) &= \alpha_i(t) + t\beta_i(t), \\ \alpha(t) &= \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t), \\ \phi(t) &= \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t) \end{aligned}$$

for all $t \in \mathbb{R}^+$. The following two results were proved by Abbassi and Calvaruso in [73].

Theorem 57. Let (M, g) be a Riemannian n -manifold and let G be an arbitrary g -natural metric on TM . Then the identity map $id_{TM} : (TM, g^S) \rightarrow (TM, G)$ is harmonic if and only if the equation

$$\phi'_3 = (n-1)[\beta_1 - \alpha'_1 - (\alpha_1 + \alpha_3)']$$

holds and either (1) the horizontal and vertical distributions are orthogonal with respect to G , or (2) (M, g) is an Einstein manifold with $\text{Ric}(u) = cu$, and $2\phi'_2 + (n-1)\beta_2 = c\alpha_2$ holds.

Theorem 58. Let (M, g) be a Riemannian manifold and G an arbitrary g -natural metric on TM . Then $id_{TM} : (TM, G) \rightarrow (TM, g^S)$ is harmonic if and only if the following two conditions are satisfied:

$$0 = \frac{1}{\phi} \{ \phi_2 \phi_1 (\phi_1 + \phi_3)' + \phi_2 (\phi + \phi_3) \phi'_1 - 2\phi_1 (\phi_1 + \phi_3) \phi'_2 \} \\ + \frac{n-1}{\alpha} \{ \phi_2 \alpha' - \phi_1 [\alpha_2 (\beta_1 + \beta_3) - (\alpha_1 + \alpha_3) \beta_2] - \phi_2 [(\alpha_1 + \alpha_3) \beta_1 - \alpha_2 \beta_2] \},$$

and

$$0 = \frac{1}{\phi} \{ (\phi_2^2 - \phi) (\phi_1 + \phi_3)' + (\phi_1 + \phi_3)^2 \phi'_1 - 2\phi_2 (\phi_1 + \phi_3) \phi'_2 \} \\ + \frac{n-1}{\alpha} [-(\phi_1 + \phi_3) \alpha' + \phi_2 [\alpha_2 (\beta_1 + \beta_3) - (\alpha_1 + \alpha_3) \beta_2] + (\phi_1 + \phi_3) [(\alpha_1 + \alpha_3) \beta_1 - \alpha_2 \beta_2]].$$

12.4. Harmonic Metrics on Non-Reductive Homogeneous Manifolds

Assume g and $\hat{g}$ are G -invariant pseudo-Riemannian metrics on a non-reductive homogeneous 4-manifold $M^4 = G/K$. Let $\hat{g}^S$ (respectively, $\hat{g}^h$ and $\hat{g}^c$) denote the Sasaki metric (respectively, the horizontal lift and the complete lift) of $\hat{g}$ on the tangent bundle TM^4 . In addition, let g^S (respectively, g^h and g^c) denote the Sasaki metric (respectively, the horizontal lift and the complete lift) of g .

In [74] by Zaeim and Atashpeykar proved the following.

Theorem 59. Let (M^4, g) be a non-reductive homogeneous 4-manifold. Then the following four statements are equivalent:

- (a) The identity map $id_{M^4} : (M^4, g) \rightarrow (M^4, \hat{g})$ is harmonic, i.e., the pseudo-Riemannian metric $\hat{g}$ on M^4 is harmonic with respect to the pseudo-Riemannian metric g .
- (b) The identity map $id_{TM^4} : (TM^4, g^S) \rightarrow (TM^4, \hat{g}^S)$ is harmonic.
- (c) The identity map $id_{TM^4} : (TM^4, g^h) \rightarrow (TM^4, \hat{g}^h)$ is harmonic.
- (d) The identity map $id_{TM^4} : (TM^4, g^c) \rightarrow (TM^4, \hat{g}^c)$ is harmonic.

Remark 11. A complete classification of non-reductive pseudo-Riemannian homogeneous spaces up to dimension four was obtained earlier in [75] by Fels and Renner.

12.5. Identity Maps of Tangent Bundles of Walker 4-Spaces

Bejan and Drăgă-Romaniuc investigated the total space of the tangent bundle TW^4 of a Walker 4-manifold $(W^4, g, \mathcal{D})$ and proved the following.

Theorem 60. [40] Let $(W^4, g, \mathcal{D})$ be a Walker 4-manifold. Then the following three statements are equivalent:

- (a) The identity map $id_{W^4} : (W^4, g, \mathcal{D}) \rightarrow (W^4, \hat{g}, \mathcal{D})$ is harmonic, i.e., a Walker metric $\hat{g}$ on W^4 is harmonic with respect to g .
- (b) The Sasaki metric $\hat{g}^S$ of a Walker metric $\hat{g}$ is harmonic with respect to the Sasaki metric g^S of the Walker metric g .

- (c) The horizontal lift $\hat{g}^h$ of a Walker metric $\hat{g}$ is harmonic with respect to the horizontal lift g^h of the Walker metric g .

12.6. Identity Maps of Gödel-Type Spacetimes

The Gödel metric, introduced by Kurt Gödel in [76], is an exact solution of the Einstein field equations such that the stress-energy tensor, which contains two terms, the first term representing the matter density of a homogeneous distribution of swirling dust particles (dust solution), and the second associated with a nonzero cosmological constant. Gödel spacetimes have been deeply investigated in physics (cf. e.g., [77–79]).

Gödel-type 4D spacetimes can be described by the Lorentzian metrics

$$g = [dt + H(r)d\varphi]^2 - dr^2 - P^2(r)d\varphi^2 - dz^2, \quad (77)$$

where t is the time variable and (r, φ, z) for $r \geq 0, \varphi, z \in \mathbf{R}$ (undetermined for $r = 0$). Further, $H(r)$ and $P(r)$ are arbitrary smooth functions on M and g is non-degenerate where $P(r) \neq 0$.

The following three theorems were proved by Zaeim, Jafari and Yaghoubi in [80].

Theorem 61. Let g and $\hat{g}$ be Gödel-type metrics on a 4-manifold M^4 with the metric g given by (77) and $\hat{g}$ given by

$$\hat{g} = [dt + \hat{H}(r)d\varphi]^2 - dr^2 - \hat{P}^2(r)d\varphi^2 - dz^2. \quad (78)$$

Then the identity map $id_{M^4} : (M^4, g) \rightarrow (M^4, \hat{g})$ is harmonic (equivalently, the metric tensor $\hat{g}$ is harmonic with respect to g) if and only if

$$\hat{H}'(r)(\hat{H}(r) - H(r)) - \hat{P}(r)\hat{P}'(r) + P(r)P'(r) = 0.$$

Theorem 62. Let (M^4, g) be homogeneous Gödel-type spacetime. In this case, a metric tensor $\hat{g}$ is harmonic with respect to g if and only if $g = \hat{g}$.

Theorem 63. Let (M^4, g) be a Gödel-type spacetime with the metric g given by (77) and let $\hat{g}$ be an arbitrary Gödel-type metric of Equation (78). Then the following four statements are equivalent:

- The identity map $id_{M^4} : (M^4, g) \rightarrow (M^4, \hat{g})$ is harmonic, i.e., $\hat{g}$ is harmonic with respect to g .
- The Sasaki metric $\hat{g}^S$ is harmonic with respect to the Sasaki metric g^S .
- The horizontal lift $\hat{g}^h$ is harmonic with respect to the horizontal lift g^h .
- The complete lift $\hat{g}^c$ is harmonic with respect to the complete lift g^c .

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