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The Reduced-Order Extrapolating Method about the Crank-Nicolson Finite Element Solution Coefficient Vectors for Parabolic Type Equation

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Received: 13 July 2020; Accepted: 31 July 2020; Published: 1 August 2020

Abstract: This study is mainly concerned with the reduced-order extrapolating technique about the unknown solution coefficient vectors in the Crank-Nicolson finite element (CNFE) method for the parabolic type partial differential equation (PDE). For this purpose, the CNFE method and the existence, stability, and error estimates about the CNFE solutions for the parabolic type PDE are first derived. Next, a reduced-order extrapolating CNFE (ROECNFE) model in matrix-form is established with a proper orthogonal decomposition (POD) method, and the existence, stability, and error estimates of the ROECNFE solutions are proved by matrix theory, resulting in an graceful theoretical development. Specially, our study exposes that the ROECNFE method has the same basis functions and the same accuracy as the CNFE method. Lastly, some numeric tests are shown to computationally verify the validity and correctness about the ROECNFE method.

Keywords: reduced-order extrapolating Crank-Nicolson finite element method; parabolic type partial differential equation; existence and stability as well as error estimates

MSC: 65M15; 65N35

1. Introduction

The finite element (FE) method has been widely used in scientific engineering computations since it was proposed by Turner, Clough, Martin, and Topp in 1956 in order to solve a structural problem (see [1]). It has emerged as a forceful approach for solving PDEs including heat conduction and hydrodynamics equations. The CNFE method for the parabolic type PDEs has the second-order accuracy about time-step and is more popular than the usual FE method. The FE method or the CNFE method includes hundreds of thousands or even tens of millions unknowns (degrees of freedom) to solve the real-world engineering problems. Even if they are computed on some advanced computers, it takes days or even tens of days to gain the numerical results. Owing to the FE or CNFE method containing a lot of unknowns, the round-off errors in the calculations are rapidly accumulated, resulting in that the gaining numerical solutions emerge with very large deviation and we could not gain the desired numerical solutions. Adopting the FE or CNFE method to solve the real-world engineering problems, a key thing is how to lessen the unknowns in the FE or CNFE method so as to slow down the accumulation of round-off errors in the computations, lessen CPU run time and the computational load, and enhance the calculating accuracy of numerical solutions.

The POD method is one of the most valid tools for lessening the unknowns in the numerical methods ([2–6]). Its precursor was the Principal Vector Analysis presented by Pearson in 1901 ([7]), which has still being used in data mining now. The term POD method was first employed by Sirovich in 1987 ([8]), but for a long time, it was mainly used to study the characteristics of turbulence ([9,10]).

It was not until 2001 that the POD method was used to concern with the reduced-order of the Galerkin method for the parabolic type PDEs and to discuss the error estimates of the reduced-order Galerkin solutions ([11]). The POD method is used to the reduced-order for the classical numerical methods such as the finite difference schemes ([12–14]), the FE methods ([15–17]), the finite volume element (FVE) methods ([18,19]), the collocation spectral (CS) methods ([20,21]), and the natural boundary element (NBE) method ([22,23]). It is also used in the reduced basis methods ([24,25]). It has played a key role in lessening the unknowns in the numerical methods.

Whether the reduced-order Galerkin methods in [11,26,27], the reduced-order FE methods in [15,17], the reduced-order FVE methods in [18,19], the reduced-order CS methods in [20,21], the reduced-order NBE methods in [22,23] and the reduced basis methods in [24,25] are constructed by the continuous POD basic functions. The constructing process of the continuous POD basic functions requires the knowledge of the optimization methods and functional analysis. The reduced-order numerical methods is difficult to be mastered and understood by the engineers with a weak mathematical foundation. The above reduced-order methods are established by replacing the basic functions in the classical numerical methods (Galerkin, CS, FE, FVE, NBE) with the few continuous POD basic functions, resulting in that these reduced-order methods would emerge with large errors. Moreover, the reduced-order Galerkin methods in [11,26,27] and the reduced-order CS method in [20,21] can only solve the problems defined on rectangular domains.

In order to rise the above drawbacks in the reduced-order methods based on the continuous POD basic functions, we take the CNFE method for the parabolic type PDE as an example to elucidate how to structure the ROECNFE method by reducing the order of unknown coefficient vectors of CNFE solutions, resulting in that the ROECNFE method (Problem 5) has the same FE basic functions and the same accuracy as the classical CNFE method (Problem 4), whose basic ideas can be easily generalized to other numerical methods (Galerkin, CS, FVE, and NBE) and other types of PDEs including the nonlinear PDEs. The ROECNFE method here is not constructed with the continuous POD basic functions. It is established by means of the POD basic vectors formed by the initial few known coefficient vectors of CNFE solutions, which is different from the reduced-order methods based on the continuous POD basic functions. It may keep away from the abstract mathematical knowledge such as the functional analysis and optimization theory. The ROECNFE method (Problem 5) based on the POD basic vectors has a matrix-form, its theoretical analysis adopts matrix tool easily understood and mastered by the engineers with a weak mathematical foundation.

For convenience, we take into account the following parabolic type PDE.

Problem 1. For a given function $g(\mathbf{x}, t)$, find $u : \Omega \times J \rightarrow \mathbb{R}$ such that

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = g(\mathbf{x}, t), & \mathbf{x} \in \Omega, t \in J, \\ u(\mathbf{x}, t) = u_0(\mathbf{x}, t), & \mathbf{x} \in \partial\Omega, t \in \bar{J}, \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), & \mathbf{x} \in \bar{\Omega}, \end{cases} \quad (1)$$

where u stands for the unknown function, $\Omega \subset \mathbb{R}^s$ ($s = 2, 3$) is an interconnected bounded domain, $J = (0, T)$, T stands for the final moment, $\mathbf{x} = (x_1, x_2, \dots, x_s)$, $\partial_t u = \partial u / \partial t$, $g(\mathbf{x}, t)$ is the given source function, $u_0(\mathbf{x}, t)$ and $u^0(\mathbf{x})$ stand for the given boundary and initial value functions, respectively and $\bar{\Omega} = \Omega \cup \partial\Omega$.

For convenience, we assume that $u_0(\mathbf{x}, t) = 0$ in the following theoretical analysis.

The parabolic type PDE (Problem 1) holds very momentous physical meaning. It is an important mathematical and physical equation describing the processes of heat conduction, diffusion, and seepage ([28–33]). However, if it includes complex initial value or source functions, or the shape of computing domain is irregular, it has no analytical solution, so that we can only find its numeric solutions ([29–33]).

The CNFE method is one of the best numerical methods for solving Problem 1 (see [28]), but it also contains lots unknowns. When it is used to calculate the real-world engineering problems, it has even more than tens of millions unknowns. It brings a lot of difficulties to practical applications, so we will adopt the POD method to lessen the unknowns in the CNFE method and the computed load, save CPU run time and retard the accumulation of rounded-off errors in the computations.

As far as we know, there has been no report that the matrix-form ROECNFE model for the parabolic type PDE is established by reducing the order of coefficient vectors of CNFE solutions via the POD basic vecctors, which keeps the same basic functions as the CNFE model. Here we develop the matrix-form ROECNFE model for the parabolic type PDE based on the POD basic vectors, set up by reducing the order of the solution coefficient vectors in the CNFE method. Although some reduced-order FE methods for the parabolic type PDE have been developed in [31–33], they are absolutely different from the ROECNFE methods in this paper, technically and theoretically. The reduced-order FE methods in [31–33] were established by replacing the FE subspaces with subspaces spanned with few continuous POD basic functions, resulting that their accuracy had been affected by the POD reduced-order. The matrix-form ROECNFE model in this paper consists of the basic functions in the FE subspace firstly absorbed into the stiffness matrix of the matrix-form CNFE model. The unknown solution coefficient vectors in the matrix-form CNFE model is reduced by means of the linear combinations the few POD basic vectors, so that the matrix-form ROECNFE model possesses the same basic functions and accuracy as the CNFE model. The stability and error estimates of the ROECNFE solutions are argued by the matrix analysis, resulting in that theoretical analysis becomes very laconic.

The rest of the paper is planed as follows. The functional-form CNFE model and the existence, stability, and error estimates of the CNFE solutions as well as the matrix-form CNFE model for the parabolic type PDE are offered in Section 2. In Section 3, the matrix-form ROECNFE model is developed with a set of POD basic vectors produced by the initial few coefficient vectors of the CNFE solutions and the stability and error estimates of the ROECNFE solutions are proven by the matrix analysis. Some numerical tests are supplied in Section 4 in order to confirm the superiority for the matrix-form ROECNFE model. The main conclusions are given in Section 5.

2. The CNFE Method for the Parabolic Type PDE

This paper adopts the classical Sobolev spaces and norms [34,35].

Let $\mathbb{V} := H_0^1(\Omega) = \{v : v|_{\partial\Omega} = 0, \int_{\Omega} [|\nabla v|^2 + |v|^2] dx_1 dx_2 \dots dx_s < \infty\}$. Using Green's formulas, we can deduce the weak form for Problem 1 in the following.

Problem 2. For $0 < t < T$, find $u \in \mathbb{V}$ such that

$$(u_t, v) + a(u, v) = (g, v), \quad \forall v \in \mathbb{V}, \quad (2)$$

$$u(x, 0) = u^0(x), \quad x \in \overline{\Omega}, \quad (3)$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$ and $a(u, v) = (\nabla u, \nabla v)$.

The existence and uniqueness of solution in Problem 2 have been given in [28,34].

Let $\mathfrak{S}_h$ be an uniformly regular partition of triangulation on $\overline{\Omega}$ [28,34]. The M -dimensional FE subspace $\mathbb{V}_h$, spanned by the orthonormal basis $\{\xi_j(x)\}_{j=1}^M$ under the inner product in $L^2(\Omega)$ where $\xi_j(x)$ can be achieved by the standard orthogonalization in ([35], Section 6.3), is defined as follows:

$$\mathbb{V}_h = \left\{ v_h \in H_0^1(\Omega) \cap C(\Omega) : v_h|_K \in \mathbb{P}_l(K), K \in \mathfrak{S}_h \right\}, \quad (4)$$

in which $\mathbb{P}_l(K)$ is formed by l -th degree polynomials on $K \in \mathfrak{S}_h$.

For integer $N > 0$, let $\Delta t = T/N$, u_h^n be the CNFE approximations to solutions u for Problem 2 at $t_n = n\Delta t$ ($0 \leq n \leq N$), $\bar{\partial}u_h^n = (u_h^n - u_h^{n-1})/\Delta t$, $\bar{u}_h^n = (u_h^n + u_h^{n-1})/2$, and $g^{n-\frac{1}{2}} = g(x, t_{n-\frac{1}{2}})$. Thus, the functional-form CNFE model for Problem 2 is stated as follows.

Problem 3. Find $u_h^n \in \mathbb{V}_h$ ($1 \leq n \leq N$) such that

$$(u_h^n, v_h) + \Delta t a(\bar{u}_h^n, v_h) = (u_h^{n-1}, v_h) + \Delta t (g^{n-\frac{1}{2}}, v_h), \forall v_h \in \mathbb{V}_h, 1 \leq n \leq N, \quad (5)$$

$$u_h^0(x) = \Pi_h u^0(x), \quad x \in \bar{\Omega}, \quad (6)$$

in which $\Pi_h : H_0^1(\Omega) \rightarrow \mathbb{V}_h$ is an interpolation operator such that $u_h^0(x) \in \mathbb{V}_h$ and $u_h^0(x)|_K \in \mathbb{P}_l(K)$ if $K \in \mathfrak{S}_h$.

If $u^0 \in H^{l+1}(\Omega) \cap H_0^1(\Omega)$, then the above interpolation operator satisfies the following boundedness and error estimates [28,34]

$$\|\Pi_h u^0\|_0 \leq \sigma \|u^0\|_0 \quad \text{and} \quad \|u^0 - \Pi_h u^0\|_0 \leq \sigma h^{l+1} \|u^0\|_{l+1}, \quad (7)$$

in which $\|\cdot\|_0$ and $\|\cdot\|_l$ are, respectively, the norms in $L^2(\Omega)$ and $H^l(\Omega)$ and σ used in the context stands for generic positive constant independent of Δt and h , which may be unequal at different sites.

The following conclusion of the existence, stability, and error estimates about solutions to Problem 3 has been supplied in [28,33,34].

Theorem 1. Problem 3 has a unique set of unconditionally stable ([6], Definition 1.1.2) CNFE solutions $\{u_h^n\}_{n=1}^N \subset \mathbb{V}_h$. When the solution u in Problem 3 is smooth enough such that $u \in H^3(0, T; H^{l+1}(\Omega))$, the CNFE solutions u_h^n meet the following error estimates

$$\|u(t_n) - u_h^n\|_0 \leq \sigma \left(\Delta t^2 + h^{l+1} \right), \quad n = 1, 2, \dots, N. \quad (8)$$

By using the orthonormal basis $\{\xi_j(x)\}_{j=1}^M$, the CNFE solutions in Problem 3 can be denoted by the scalar multiplication of two vectors:

$$u_h^n = \sum_{j=1}^M c_j^n \xi_j(x) = C^n \cdot \xi,$$

in which $C^n = (c_1^n, c_2^n, \dots, c_M^n)^T$ and $\xi = (\xi_1, \xi_2, \dots, \xi_M)^T$. Hence, Problem 3 can be rewritten into the following matrix-model.

Problem 4. Find $C^n \in \mathbb{R}^M$ and $u_h^n \in \mathbb{V}_h$ ($1 \leq n \leq N$) such that

$$\left(I + \frac{\Delta t}{2} B \right) C^n = \left(I - \frac{\Delta t}{2} B \right) C^{n-1} + \Delta t F^{n-\frac{1}{2}}, \quad n = 1, 2, \dots, N, \quad (9)$$

$$u_h^n = \sum_{j=1}^M c_j^n \xi_j(x) = C^n \cdot \xi, \quad n = 1, 2, \dots, N, \quad (10)$$

where $C^0 = (u^0(P_1), u^0(P_2), \dots, u^0(P_M))^T$ is the known vector formed by the values of $u^0(x)$ at l -th degree interpolation nodes P_j 's, I stands for $M \times M$ unit matrix, $B = (a(\xi_i, \xi_j))_{M \times M}$ is a positive definite matrix due to the coerciveness of $a(\cdot, \cdot)$, and $F^{n-\frac{1}{2}} = \left((g^{n-\frac{1}{2}}, \xi_1), (g^{n-\frac{1}{2}}, \xi_2), \dots, (g^{n-\frac{1}{2}}, \xi_M) \right)^T$.

Remark 1. There has a unique set of CNFE solution for Problem 3 or Problem 4 since B is the positive definite matrix and so is $\left(I + \frac{\Delta t}{2} B \right)$. When the meshes in $\mathfrak{S}_h$ are sufficiently refined, Problem 3 or

Problem 4 would have lots unknowns so that the rounded off errors in the calculations are quickly accumulated, leading to floating point overflow so as to be unable to achieve desired numerical solutions (see the numerical tests in Section 4). Therefore, it is necessary to lessen the unknowns for Problem 4 by the POD method.

3. The ROECNFE Method for the Parabolic Type PDE

3.1. Generation of POD Bases

We firstly formulate a snapshot matrix $A = (C^1, C^2, \dots, C^L)_{M \times L}$ with the initial L coefficient vectors C^n ($n = 1, 2, \dots, L$) in Problem 4. We then find out the positive eigenvalues χ_i ($1 \leq i \leq r := \text{rank}(A)$) with $\chi_1 \geq \chi_2 \geq \dots \geq \chi_r > 0$ and the corresponding orthonormal eigenvectors $\tilde{\Psi} = (\phi_1, \phi_2, \dots, \phi_r) \in \mathbb{R}^{M \times r}$ of AA^T . We lastly make up a group of POD basis $\Psi = (\phi_1, \phi_2, \dots, \phi_d)$ ($d \leq r$) from the foremost d vectors in $\tilde{\Psi}$ that meets the following formula [6,36,37]:

$$\|A - \Psi\Psi^T A\|_{2,2} = \sqrt{\chi_{d+1}}, \quad (11)$$

in which $\|A\|_{2,2} = \sup_{\eta \neq 0} \|A\eta\| / \|\eta\|$ and $\|\eta\|$ is Euclid's norm for vector η . In addition, we have

$$\|C^n - \Psi\Psi^T C^n\| = \|(A - \Psi\Psi^T A)e^n\| \leq \|A - \Psi\Psi^T A\|_{2,2} \|e^n\| \leq \sqrt{\chi_{d+1}}, \quad n = 1, 2, \dots, L, \quad (12)$$

in which e^n ($n = 1, 2, \dots, L$) are the unit vectors.

Remark 2. Because $M \gg L$ and the positive eigenvalues χ_j ($1 \leq j \leq r$) of the matrix AA^T are the same as those of the matrix $A^T A$, we may firstly calculate the foremost d eigenvalues χ_j ($1 \leq j \leq d$) of $A^T A$ and the relative eigenvectors ϕ_j ($1 \leq j \leq d$), we then could easily get the foremost eigenvectors $\phi_j = A\phi_j / \sqrt{\chi_j}$ ($1 \leq j \leq d$) and make up a group of POD basis $\Psi = (\phi_1, \phi_2, \dots, \phi_d)$ ($d \leq r$).

3.2. Formulation of Matrix-Form ROECNFE Model

If we set $C_d^n = (c_{d1}^n, c_{d2}^n, \dots, c_{dM}^n)^T = \Psi\alpha_d^n = \Psi\Psi^T C^n$, $\alpha_d^n = (\alpha_1^n, \alpha_2^n, \dots, \alpha_d^n)^T$ and $u_d^n = C_d^n \cdot \xi$, then we obtain the first L ROECNFE solutions $u_d^n = \xi \cdot C_d^n$ ($1 \leq n \leq L$) from Section 3.1. Replacing the solutions C^n in Problem 4 with $C_d^n = \Psi\alpha_d^n$ ($L+1 \leq n \leq N$) and multiplying on left by Ψ^T , by the orthonormality of the POD basic vectors we obtain the following elegant matrix-form ROECNFE model.

Problem 5. Find $\alpha_d^n \in \mathbb{R}^d$ and $u_d^n \in \mathbb{V}_h$ ($n = 1, 2, \dots, N$) such that

$$\alpha_d^n = \Psi^T C^n, \quad 1 \leq n \leq L; \quad (13)$$

$$\alpha_d^n = \Psi^T \left(I + \frac{\Delta t}{2} B \right)^{-1} \left(I - \frac{\Delta t}{2} B \right) \Psi \alpha_d^{n-1} + \Delta t \Psi^T \left(I + \frac{\Delta t}{2} B \right)^{-1} F^{n-\frac{1}{2}}, \quad L+1 \leq n \leq N, \quad (14)$$

$$u_d^n = \xi \cdot (\Psi \alpha_d^n), \quad 1 \leq n \leq N, \quad (15)$$

in which C^n ($n = 1, 2, \dots, L$) are the first L coefficient vectors in Problem 4, and the matrix B and the vectors $F^{n-\frac{1}{2}}$ are given in Problem 4.

Remark 3. It is evident that Problem 5 has a unique series of ROECNFE solutions $\{u_d^n\}_{n=1}^N \subset \mathbb{V}_h$ since B and $\left(I + \frac{\Delta t}{2} B \right)$ are positive definite. It is worth noting that Problem 5 at every time node contains d unknowns ($d \ll M$), whereas Problem 4 includes M unknowns at the same time node. Therefore, Problem 5 is distinctly superior over Problem 4, which means that Problem 5 can immensely lessen unknowns, resulting in that it can greatly save CPU elapsed time, decrease the accumulation

of rounded-off errors and improve the accuracy of numerical solutions in the practical computations (see the numerical tests in Section 4). Especially, Problem 5 has the same basic functions as Problem 4 such that they both hold same accuracy.

3.3. The Stability and Error Estimates to the ROECNFE Solutions

To discuss the stability and errors for the ROECNFE solutions, we require the following matrix properties (see [38], Lemmas 1.4.1 and 1.4.2).

Lemma 1. *The positive definite matrix $\mathbf{B}$ in Problem 4 satisfies the following two inequalities:*

$$\|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}(\mathbf{I} - 0.5\Delta t\mathbf{B})\|_{2,2} \leq 1; \quad \|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}\|_{2,2} < 1. \quad (16)$$

There is the following consequence about the stability and error estimates to the ROECNFE solutions in Problem 5.

Theorem 2. *Under the hypotheses in Theorem 1, the set of ROECNFE solutions $\{u_d^n\}_{n=1}^N$ in Problem 5 is unconditionally stable ([6], Definition 1.1.2) and satisfies the following error estimates*

$$\|u(t_n) - u_d^n\|_0 \leq \sigma \left(\Delta t^2 + h^{l+1} + \sqrt{\chi_{d+1}} \right), \quad n = 1, 2, \dots, N, \quad (17)$$

in which $u(t_n)$ are the function values for solution $u(\mathbf{x}, t)$ to Problem 2 at the time nodes $t_n = n\Delta t$ ($1 \leq n \leq N$).

Proof. (1) The unconditionally stability of the solutions for Problem 5.

When $1 \leq n \leq L$, owing to the orthonormality of vectors in Ψ , we obtain

$$\|u_d^n\|_0 = \|\mathbf{C}_d^n \cdot \boldsymbol{\xi}\|_0 = \|\Psi\Psi^T \mathbf{C}_d^n \cdot \boldsymbol{\xi}\|_0 \leq \sigma \|u_h^n\|_0. \quad (18)$$

Owing to the unconditionally stability of $\{u_h^n\}_{n=1}^N$ obtained by Theorem 1, we immediately conclude that $\{u_d^n\}_{n=1}^L$ is unconditionally stable.

Suppose $L + 1 \leq n \leq N$. Using $\mathbf{C}_d^n = \Psi\boldsymbol{\alpha}_d^n$, we could revert (14) into

$$\mathbf{C}_d^n = (\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}(\mathbf{I} - 0.5\Delta t\mathbf{B})\mathbf{C}_d^{n-1} + \Delta t(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}\mathbf{F}^{n-\frac{1}{2}}. \quad (19)$$

Using Lemma 1, from (19) we have

$$\begin{aligned} \|\mathbf{C}_d^n\| &\leq \|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}(\mathbf{I} - 0.5\Delta t\mathbf{B})\|_{2,2} \|\mathbf{C}_d^{n-1}\| + \Delta t \|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}\|_{2,2} \|\mathbf{F}^{n-\frac{1}{2}}\| \\ &\leq \|\mathbf{C}_d^{n-1}\| + \Delta t \|\mathbf{F}^{n-\frac{1}{2}}\|. \end{aligned} \quad (20)$$

Note that $\|\mathbf{C}_d^L\| = \|\Psi\Psi^T \mathbf{C}^L\| \leq \sigma \|\mathbf{C}^L\|$. From Problem 4 and Lemma 1, we have

$$\begin{aligned} \|\mathbf{C}^n\| &\leq \|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}(\mathbf{I} - 0.5\Delta t\mathbf{B})\mathbf{C}^{n-1}\| + \Delta t \|(\mathbf{I} + 0.5\Delta t\mathbf{B})^{-1}\mathbf{F}^{n-\frac{1}{2}}\| \\ &\leq \|\mathbf{C}^{n-1}\| + \Delta t \|\mathbf{F}^{n-\frac{1}{2}}\| \leq \|\mathbf{C}^0\| + \Delta t \sum_{i=1}^n \|\mathbf{F}^{i-\frac{1}{2}}\| \leq \sigma, \quad n = 1, 2, \dots, N. \end{aligned} \quad (21)$$

Summating from $L + 1$ to n for (20) and using (21), we obtain

$$\|\mathbf{C}_d^n\| \leq \|\mathbf{C}_d^L\| + \Delta t \sum_{i=L+1}^n \|\mathbf{F}^{i-\frac{1}{2}}\| \leq \sigma \|\mathbf{C}^L\| + \Delta t \sum_{i=L+1}^n \|\mathbf{F}^{i-\frac{1}{2}}\| \leq \sigma, \quad L + 1 \leq n \leq N. \quad (22)$$

Thus, noting that $u_d^n = \xi \cdot C_d^n$ and $\|\xi\|_0 \leq \sigma$, we have

$$\|u_d^n\|_0 \leq \|C_d^n \cdot \xi\|_0 \leq \|C_d^L\| \|\xi\|_0 \leq \sigma, \quad L+1 \leq n \leq N, \quad (23)$$

which means that $\{u_d^n\}_{n=L+1}^N$ are also unconditionally stable. Therefore, $\{u_d^n\}_{n=1}^N$ in Problem 5 are unconditionally stable.

(2) The error estimates of the ROECNFE solutions.

Let $\|\cdot\|_\infty$ stand for the max-sum norm of a vector ([38], p. 28). When $n = 1, 2, \dots, L$, noting that $u_h^n = C^n \cdot \xi$, $\|\cdot\|_\infty \leq \|\cdot\|$, $C_d^n = \Psi \Psi^T C^n$ and $\|\xi\|_0 \leq \sigma$, by (12), we gain

$$\|u_h^n - u_d^n\|_0 \leq \|\xi\|_0 \|C^n - C_d^n\|_\infty \leq \sigma \|C^n - \Psi \Psi^T C^n\| \leq \sigma \sqrt{\chi_{d+1}}. \quad (24)$$

When $L+1 \leq n \leq N$, from (9) and (19), using Lemma 1, we get

$$\begin{aligned} \|C^n - C_d^n\| &= \|(I + 0.5\Delta t B)^{-1} (I - 0.5\Delta t B) (C^{n-1} - C_d^{n-1})\| \\ &\leq \|(I + 0.5\Delta t B)^{-1} (I - 0.5\Delta t B)\|_{2,2} \|C^{n-1} - C_d^{n-1}\| \\ &\leq \|C^{n-1} - C_d^{n-1}\|. \end{aligned} \quad (25)$$

Thus, from (25) and (12), we obtain

$$\|C^n - C_d^n\| \leq \|C^L - C_d^L\| \leq \|C^L - \Psi \Psi^T C^L\| \leq \sqrt{\chi_{d+1}}, \quad L+1 \leq n \leq N. \quad (26)$$

Further, we obtain

$$\begin{aligned} \|u_h^n - u_d^n\|_0 &= \|\xi \cdot (C^n - C_d^n)\|_0 \leq \|\xi\|_0 \|C^n - C_d^n\| \leq \sigma \|C^n - C_d^n\| \\ &\leq \sigma \sqrt{\chi_{d+1}}, \quad L+1 \leq n \leq N. \end{aligned} \quad (27)$$

Combining (24) and (27) with Theorem 1 yields (17). This completes the proof of Theorem 2. $\square$

Remark 4. The error term $\sqrt{\chi_{d+1}}$ in Theorem 2 is caused by the reduced-order for the CNFE model, but it may serve as the suggestion how to choose the POD bases, namely it is only necessary to choose the foremost d POD bases such that $\sqrt{\chi_{d+1}} \leq \Delta t^2 + h^{l+1}$. A large number of tests have verified that the eigenvalue $\sqrt{\chi_j}$ would rapidly fall off to 0. Usually when $d = 5$ or 6 , $\sqrt{\chi_{d+1}}$ is already extremely small and satisfies $\sqrt{\chi_{d+1}} \leq \Delta t^2 + h^{l+1}$. Especially, if the ROECNFE solution $u_d^{n_0+1}$ obtained by Problem 5 at some of time node t_{n_0+1} cannot satisfy the desired accuracy, but $u_d^{n_0}$ at t_{n_0} can still satisfies the accuracy, then we can choose a new snapshot matrix $A = (C^{n_0+1-L}, C^{n_0+2-L}, \dots, C^{n_0-1}, C^{n_0})$ to generate new a set of POD basis and establish the new matrix-form ROECNFE model and to find the ROECNFE solutions satisfied accuracy. So goes on, we can calculate out the ROECNFE solutions at arbitrary time node. This is something that the classical CNFE model cannot do.

4. Some Numerical Tests

In order to expediently compare the errors between the CNFE solutions and the analytical solution with the errors between the ROECNFE solutions and the analytical solution, and to reveal the superiority of the matrix-form ROECNFE model, we here adopt the numerical tests that there exists the analytical solution in Problem 1 (generally has no the analytical solution).

Taking $\Omega = (0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$, the initial function $u^0(x) = \sin x_1 \sin x_2$, and the source function $g(x, t) = 0$, we may obtain an analytical solution $u(x, t) = e^{-2t} \sin x_1 \sin x_2$ for Problem 1 satisfying $u(x, t) = 0$ on $\partial\Omega$.

Cutting $\Omega = (0, 2\pi) \times (0, 2\pi)$ into 2000×2000 squares with side-length $1/(1000\pi)$ and linking the diagonal in the square to cut every square into two triangles in the uniform direction, we can form an uniformly regular triangulation $\mathfrak{S}_h$ with $h = \sqrt{2}/(1000\pi)$. If $\Delta t = 1/1000$, $\sqrt{\chi_{d+1}} = O(10^{-6})$,

and $\mathbb{P}_1(K)$ is formed by 1st degree polynomials (i.e., $l = 1$) in Problem 4, then the theoretical errors reach $O(10^{-6})$ according on Theorems 1 and 2.

Firstly, based on previous experiences ([6]), we calculate out the 20 initial CNFE solutions C^n ($n = 1, 2, \dots, 20$) by Problem 4, making up the snapshot matrix $A = (C^1, C^2, \dots, C^{20})$ and calculate out the eigenvalues χ_i (arranged in decreasing order) of the matrix $A^T A$. Then we achieve by reckoning that $\sqrt{\chi_7} \leq 2.1 \times 10^{-6}$, which means that we only need to take the initial six eigenvectors ϕ_i ($i = 1, 2, \dots, 6$) of the matrix $A^T A$ to make up the POD basis $\Psi = (\phi_1, \phi_2, \dots, \phi_6)$ by means of $\phi_i = A\phi_i / \sqrt{\chi_i}$ ($1 \leq i \leq 6$). Finally we gain the FOECNFE solutions at $t = 0.5, 1.0$, and 1.5 (i.e., $n = 500, 1000$, and 1500) by solving Problem 5, exhibited in Figures 1a, 2a, and 3a, respectively.

In order to reveal the superiority for the ROECNFE matrix-model (Problem 5), we also calculate out the CNFE solutions at $t = 0.5, 1.0$, and 1.5 by the CNFE model (Problem 4), shown in Figures 1b, 2b, and 3b, respectively. Moreover, the analytical solutions at $t = 0.5, 1.0$, and 1.5 are shown in Figures 1c, 2c, and 3c, respectively.

Figure 1. (a) The ROECNFE solution at $t = 0.5$. (b) The CNFE solution at $t = 0.5$. (c) The analytical solution at $t = 0.5$.

Figure 2. (a) The ROECNFE solution at $t = 1.0$. (b) The CNFE solution at $t = 1.0$. (c) The analytical solution at $t = 1.0$.

Figure 3. (a) The ROECNFE solution at $t = 1.5$. (b) The CNFE solution at $t = 1.5$. (c) The analytical solution at $t = 1.5$.

Comparing each three diagrams in Figures 1–3, we can view that the diagrams corresponding to the ROECNFE, CNFE, and analytical solutions at $t = 0.5$ are roughly the same to each other. The diagrams corresponding to the ROECNFE and analytical solutions at $t = 1.0$ are roughly the same to each other, but the diagram corresponding to the CNFE solution at $t = 1.0$ reveals dispersion. Specially, the diagrams corresponding to the analytical and ROECNFE solutions at $t = 1.5$ are basically agreement with each other, whereas the diagram corresponding to the CNFE solution at $t = 1.5$ has appeared very serious dispersion (where all three diagrams in Figure 3 for a better look are revolved by 90°). We cannot go on calculating the desired CNFE solutions at time nodes $t_n > 1.5$ by the CNFE model, but still can go on to find the ROECNFE solutions at any time nodes $t_n > 1.5$ by the ROECNFE model. This shows that the ROECNFE model is much superior over the CNFE model.

In order to reveal further superiority of the ROECNFE model, we recorded the L^2 errors $\|u(t_n) - u_h^n\|_0$ between the analytical solutions $u(t_n)$ and the CNFE solutions u_h^n , the L^2 errors $\|u(t_n) - u_d^n\|_0$ between the analytical solutions $u(t_n)$ and the ROECNFE solutions u_d^n , and the CPU run time for the CNFE model and the ROECNFE model when $n = 250, 500, 750, 1000, 1250, 1500, 1750$, and 2000 (i.e., at $t = 0.25, 0.50, 0.75, 1.0, 1.25, 1.50, 1.75, 2.00$), listed in the following Table 1.

Table 1. CPU run time and errors between the analytical and the CNFE as well as ROECNFE solutions.

t	n	CNFE Model		ROECNFE Matrix-Model	
		$\ u(t_n) - u_h^n\ _0$	CPU Run Time	$\ u(t_n) - u_d^n\ _0$	CPU Run Time
0.25	250	1.010356×10^{-6}	43.568 s	4.050523×10^{-6}	1.623 s
0.50	500	2.012083×10^{-6}	86.865 s	4.350732×10^{-6}	2.265 s
0.75	750	3.125338×10^{-6}	129.914 s	4.671732×10^{-6}	2.873 s
1.00	1000	1.015376×10^{-5}	172.931 s	4.952762×10^{-6}	3.492 s
1.25	1250	1.414376×10^{-5}	215.982 s	5.251718×10^{-6}	4.013 s
1.50	1500	2.534283×10^{-4}	259.173 s	5.552123×10^{-6}	4.621 s
1.75	1750	3.241232×10^{-4}	302.842 s	5.856431×10^{-6}	5.276 s
2.00	2000	2.562183×10^{-3}	345.874 s	6.172762×10^{-6}	5.813 s

It has been shown in the Table 1 that as time nodes go on, the CPU run time for the CNFE model (having 4×10^6 unknowns at every time node) increases rapidly, but the CPU run time for the ROECNFE model (only having six unknowns at every time node) increases very slowly. When $t = 1.5$, the CPU run time for the CNFE model is more 56th times greater than the ROECNFE matrix-model. The ROECNFE model can greatly save the CPU run time even though the errors for the ROECNFE solutions at some initial time nodes are slightly larger than those for the CNFE ones, which is caused by using the POD method to reduce the order for the CNFE model. Owing to the accumulation of round-off errors, the CNFE solutions gradually appear divergence, resulting in that the errors for the CNFE solutions gradually increase and cannot reach the theoretical errors $O(10^{-6})$. The errors for the ROECNFE solutions increase very slowly, always staying within the range of the theoretical error $O(10^{-6})$. It is further shown that the CNFE model is incomparable to the ROECNFE model and that the ROECNFE model is very effective for settling the parabolic type PDE.

5. Conclusions and Discussions

This paper has dealt with the reduced-order of the CNFE method for the parabolic type PDE by means of the reduced-order of the unknown coefficient vectors in the CNFE solutions. The matrix-form ROECNFE model for the parabolic type PDE has tactfully been set up by means of the POD basic vectors constructed with the initial few known CNFE solution coefficient vectors. The stability and error estimates of the ROECNFE solutions have been proven faultlessly by means of the matrix analysis, and some numerical tests have verified adequately the superiority of the ROECNFE method. Its unknowns are far fewer than those in the CNFE model, resulting that it not only greatly lessens the calculated burden and retard the accumulation of round-off errors, but also spares the CPU

run time in the calculations on computers. In particular, the order-reduction about the solution coefficient vectors in the CNFE model for the parabolic type PDE is proposed for the first time, so the matrix-form ROECNFE model is totally new and different from the reduced-order FE models based on the continuous POD basic functions in [32,33]. Therefore, the work in this paper is a new development in study of the parabolic type PDE.

Though we have developed the ROECNFE method for the linear parabolic type PDE by means of the matrix analysis, this approach can be generalized to the nonlinear problems (such as the Allen-Cahn equation [39]) and to the more complex real-world engineering problems. Therefore, the ROECNFE method here has very broad applications.

Funding: This work was supported by the National Natural Science Foundation of China (11671106) and Qian Science Cooperation Platform Talent of Guizhou Province in China ([2017]5726-41).

Conflicts of Interest: The author declares no conflict of interest.

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