

Article

Some High-Order Iterative Methods for Nonlinear Models Originating from Real Life Problems

Malik Zaka Ullah ^{1,*} , Ramandeep Behl ¹ and Ioannis K. Argyros ²¹ Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; ramanbehl87@yahoo.in² Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu

* Correspondence: zmalek@kau.edu.sa

Received: 13 July 2020; Accepted: 28 July 2020; Published: 31 July 2020



Abstract: We develop a sixth order Steffensen-type method with one parameter in order to solve systems of equations. Our study's novelty lies in the fact that two types of local convergence are established under weak conditions, including computable error bounds and uniqueness of the results. The performance of our methods is discussed and compared to other schemes using similar information. Finally, very large systems of equations (100×100 and 200×200) are solved in order to test the theoretical results and compare them favorably to earlier works.

Keywords: local convergence; Steffensen's method; Banach space; system of equations

MSC: 65H05; 65G99

1. Introduction

A plenty of problems from Biology, Chemistry, Economics, Engineering, Mathematics, and Physics are converted to a mathematical expression of the following form

$$F(u) = 0. \quad (1)$$

Here, $F : \Omega \subset \mathbb{B} \rightarrow \mathbb{B}$, is differentiable, $\mathbb{B}$ is a Banach space and Ω is nonempty and open. Closed form solutions are rarely found, so iterative methods [1–16] are used converging to the solution u_* .

In particular, we propose the following new scheme

$$\begin{aligned} y_p &= u_p - [u_p + F(u_p), u_p; F]^{-1} F(u_p) \\ z_p &= u_p - \lambda [u_p + F(u_p), u_p; F]^{-1} (F(u_p) + F(y_p)) - (1 - \lambda) [u_p, y_p; F]^{-1} F(u_p) \\ u_{p+1} &= z_p - [z_p + F(z_p), z_p; F]^{-1} F(z_p), \end{aligned} \quad (2)$$

$u_0 \in \Omega$ is an initial point and $\lambda \in \mathbb{R}$ is a free parameter. In addition to this, $[\cdot, \cdot; F] : \Omega \times \Omega \rightarrow \ell(\mathbb{B}, \mathbb{B})$ is a divided difference of order one.

We shall present two convergence analyses. Later, we present the advantages over other methods using similar information.

2. Local Convergence Analysis I

We assume that $\mathbb{B} = \mathbb{R}$. We use method (2) with standard Taylor expansions [9] for studying local convergence.

Theorem 1. Suppose that mapping F is s sufficient differentiable on Ω , with $u_* \in \Omega$, a simple zero of F . We also consider that the inverse of F , $F'(u_*)^{-1} \in \ell(\mathbb{B}, \mathbb{B})$. Then, $\lim_{p \rightarrow \infty} u_p = u_*$ provided that u_0 is close enough to u_* . Moreover, the convergence order is six.

Proof. Set $\epsilon_p = u_p - u_*$ and $Q_p = \frac{F'(u_*)}{p!}$, where $(\epsilon_p)^\gamma = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)^\gamma$, $\epsilon_p \in \mathbb{R}^p$. We shall use some Taylor series expansions, first for $F(u_p)$ and $F(u_p + F(u_p))$:

$$F(u_p) = Q_1 \epsilon_p + Q_2 \epsilon_p^2 + O(\epsilon_p^3) \quad (3)$$

and

$$F(u_p + F(u_p)) = (Q_1 + Q_1^2) \epsilon_p + (3Q_1 Q_2 + Q_2 + Q_1^2 Q_2) \epsilon_p^2 + O(\epsilon_p^3), \quad (4)$$

respectively.

By using the expressions (3) and (4) in the first substep of scheme (2), we have

$$\hat{\epsilon}_p = y_p - u_* = b_1 \epsilon_p^2 + b_2 \epsilon_p^3 + b_3 \epsilon_p^4 + O(\epsilon_p^5), \quad (5)$$

where

$$b_1 = \frac{Q_2}{Q_1} + Q_2,$$

$$b_2 = \frac{2Q_3}{Q_1} - \frac{2Q_2^2}{Q_1} - \frac{2Q_2^3}{Q_1^2} + Q_1 Q_3 + 3Q_3 - Q_2^2$$

and

$$b_3 = \frac{3Q_2^3}{Q_1} - 2Q_1 Q_2 Q_3 + Q_2^3 + Q_1^2 Q_4 + 4Q_1 Q_4 + 6Q_4 - 7Q_2 Q_3 + \frac{3Q_4}{Q_1} + \frac{4Q_2^3}{Q_1^2} + \frac{5Q_3^3}{Q_1^2} - \frac{10Q_2 Q_3}{Q_1}.$$

Secondly, we expand $F(y_p)$

$$F(y_p) = Q_1 \hat{\epsilon}_p + Q_2 \hat{\epsilon}_p^2 + O(\hat{\epsilon}_p^3). \quad (6)$$

In view of (3)–(6), we get in the second substep of scheme (2)

$$\bar{\epsilon}_p = u_{p+1} - u_* = z_p - u_* = b_4 \epsilon_p^3 + O(\epsilon_p^4), \quad (7)$$

where

$$b_4 = \frac{3Q_2^2}{Q_1} + \frac{2Q_2^2}{Q_1^2} + Q_2^2 - \lambda \left(\frac{4Q_2^2}{Q_1} + Q_2^2 + \frac{3Q_2^2}{Q_1^2} \right).$$

Thirdly, we need the expansions for $F(z_p)$ and $F(z_p + F(z_p))$

$$F(z_p) = Q_1 \bar{\epsilon}_p + Q_2 \bar{\epsilon}_p^2 + O(\bar{\epsilon}_p^3), \quad (8)$$

Hence, by (5) and (8), we get

$$F(z_p + F(z_p)) = b_5 \bar{\epsilon}_p + b_6 \bar{\epsilon}_p^2 + O(\bar{\epsilon}_p^3), \quad (9)$$

leading together with the third substep of method (2) to

$$e_{p+1} = u_{p+1} - u_* = b_7 e_p^6 + O(e_p^7), \quad (10)$$

where

$$\begin{aligned}
b_5 &= Q_1 + Q_1^2, \\
b_6 &= 3Q_1Q_2 + Q_2 + Q_1^2Q_2 \\
\text{and} \\
b_7 &= \left(Q_2 + \frac{Q_2}{Q_1}\right) \left[\frac{3Q_2^2}{Q_1} + \frac{2Q_2^2}{Q_1} + Q_2^2 - \lambda \left(\frac{4Q_2^2}{Q_1} + \frac{3Q_2^2}{Q_1} + Q_2^2\right)\right].
\end{aligned}$$

□

According to Theorem 1, the applicability of method (2) is limited to mappings F with derivatives up to the seventh order.

Now, we choose $\mathbb{B} = \mathbb{R}$, $\Omega = [-\frac{3}{2}, \frac{1}{2}]$ and define a function f , as follows:

$$f(\xi) = \begin{cases} \xi^3 \ln \xi^2 + \xi^5 - \xi^4, & \xi \neq 0 \\ 0, & \xi = 0 \end{cases}. \quad (11)$$

We have the following derivatives of function f

$$\begin{aligned}
f'(\xi) &= 3\xi^2 \ln \xi^2 + 5\xi^4 - 4\xi^3 + 2\xi^2, \\
f''(\xi) &= 12\xi \ln \xi^2 + 20\xi^3 - 12\xi^2 + 10\xi, \\
f'''(\xi) &= 12 \ln \xi^2 + 60\xi^2 - 12\xi + 22.
\end{aligned}$$

However, $f'''(\xi)$ is not bounded on Ω , so Section 2, cannot be used. In this case, we have a more general alternative given in the up coming section.

3. Local Convergence Analysis II

Consider $a \geq 0$ and $b > 0$. Let $w_0 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be a increasingly continuous map with $w_0(0, 0) = 0$.

Suppose equation

$$w_0(at, t) = 1 \quad (12)$$

has ρ_1 as the smallest positive zero. In addition, we assume that $w : [0, \rho_1) \times [0, \rho_1) \rightarrow [0, \infty)$ is a increasingly continuous map with $w(0, 0) = 0$.

Consider functions g_1 and h_1 defined on semi open interval $[0, \rho_1)$ as follow:

$$g_1(t) = \frac{w(bt, t)}{1 - w_0(at, t)},$$

and

$$h_1(t) = g_1(t) - 1.$$

By these definitions, we have $h_1(0) = -1$ and $h_1(t) \rightarrow \infty$ as $t \rightarrow \rho_1^-$. Subsequently, the intermediate value theorem assures that function h_1 has minimum one solution in $(0, \rho_1)$. Let r_1 be the minimal such zero.

The expression

$$w_0(t, g_1(t)t) = 1 \quad (13)$$

has the smallest positive zero ρ_2 . Set $\rho_3 = \min\{\rho_1, \rho_2\}$.

We construe the functions g_2 and h_2 on interval $[0, \rho_3)$ in the following way

$$g_2(t) = g_1(t) + \frac{b|1 - \lambda|w\left(bt, \left(1 + g_1(t)\right)t\right)g_1(t)}{\left(1 - w_0(at, t)\right)\left(1 - w_0\left(t, g_1(t)t\right)\right)},$$

and

$$h_2(t) = g_2(t) - 1.$$

We yield $h_2(0) = -1$ and $h_2(t) \rightarrow \infty$ since $t \rightarrow \rho_3^-$. The r_2 stand for the minimal such zero of function h_2 on $(0, \rho_3)$.

The equation

$$w_0\left(ag_2(t)t, g_2(t)t\right) = 1 \quad (14)$$

has ρ_4 as the smallest positive solution. Set $\rho = \min\{\rho_3, \rho_4\}$. Define functions g_3 and h_3 on $[0, \rho]$ as

$$g_3(t) = \frac{w\left(bg_2(t)t, g_2(t)t\right)g_2(t)}{1 - w_0\left(ag_2(t)t, g_2(t)t\right)},$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain $h_3(0) = -1$ and $h_3(t) \rightarrow \infty$ as $t \rightarrow \rho^-$. The r_3 imply the minimal zero of h_3 on $(0, \rho)$. Moreover, define

$$r = \min\{r_i\}, \text{ for } i = 1, 2, 3. \quad (15)$$

Accordingly, we have

$$0 \leq w_0(at, t) < 1, \quad (16)$$

$$0 \leq w_0\left(t, g_1(t)t\right) < 1, \quad (17)$$

$$0 \leq w_0\left(ag_2(t)t, g_2(t)t\right) < 1, \quad (18)$$

and

$$0 \leq g_i(t) < 1, \quad (19)$$

for all $t \in [0, r)$.

$S(v, c)$ denotes the open ball centered at $v \in \mathbb{B}$ and of radius $c > 0$. By $\bar{S}(v, c)$, we denote the closure of $S(v, c)$

We use the following conditions (A) in order to study the local convergence:

- (a₁) $F : \Omega \rightarrow \mathbb{B}$ is a differentiable operator in the Fréchet sense, $[\cdot, \cdot; F] : \Omega \times \Omega \rightarrow \ell(\mathbb{B}, \mathbb{B})$ is a divided difference of order one. In addition to this, we assume that $u_* \in \Omega$ is a simple zero of F . At last, the inverse of operator F , $F'(u_*)^{-1} \in \ell(\mathbb{B}, \mathbb{B})$.
- (a₂) Let $w_0 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be a increasingly continuous function with $w_0(0, 0) = 0$, parameters $a \geq 0$ and $b > 0$, such that for each $u, y \in \Omega$

$$\left\|F'(u_*)^{-1}([u, y; F] - F'(u_*))\right\| \leq w_0(\|u - u_*\|, \|y - u_*\|),$$

$$\|I + [u, u_*; F]\| \leq a,$$

and

$$\|[u, u_*; F]\| \leq b.$$

Set $\Omega_0 = \Omega \cap S(u_*, \rho_1)$, where ρ_1 exists and is given by (12).

(a₃) We assume that $w : [0, \rho_1) \times [0, \rho_1) \rightarrow [0, \infty)$ is a increasingly continuous $\forall x, y, \zeta, \eta \in \Omega_0$

$$\left\| F'(u_*)^{-1} \left([u, y; F] - [\zeta, \eta; F] \right) \right\| \leq w(\|u - \zeta\|, \|y - \eta\|).$$

(a₄) $\bar{S}(u_*, R) \subseteq \Omega$, where $R = \max\{r, ar, br\}$, r is defined in (15) and ρ_2, ρ_4 exist and are given by (13) and (13), respectively.

(a₅) There exists $\bar{r} \geq r$, such that

$$w_0(0, \bar{r}) < 1 \text{ or } w_0(\bar{r}, 0) < 1.$$

$$\text{Set } \Omega_1 = \Omega \cap \bar{S}(u_*, \bar{r}).$$

Theorem 2. Under the hypotheses (A) further consider that $u_0 \in S(u_*, r) - \{u_*\}$. Accordingly, the proceeding assertions hold

$$\{u_p\} \subseteq S(u_*, r), \quad (20)$$

$$\lim_{p \rightarrow \infty} \{u_p\} = u_*, \quad (21)$$

$$\|y_p - u_*\| \leq g_1(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\| < r, \quad (22)$$

$$\|z_p - u_*\| \leq g_2(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\|, \quad (23)$$

and

$$\|u_{p+1} - u_*\| \leq g_3(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\|. \quad (24)$$

In addition, the u_* is the unique solution of $F(u) = 0$ in the set Ω_1 mentioned in hypothesis (a₅).

Proof. We first show items (20)–(24) by adopting mathematical induction. Because $p \in S(u_*, r) - \{u_*\}$ hold and by condition (a₂), we have

$$\begin{aligned} \|p + F(p) - u_*\| &= \|(I + [p, u_*; F])(p - u_*)\| \\ &\leq \|I + [p, u_*; F]\| \|p - u_*\| \\ &\leq a \|p - u_*\| \end{aligned}$$

and

$$\begin{aligned} \|F(p)\| &= \|F(p) - F(u_*)\| \\ &\leq \|[p, u_*; F](p - u_*)\| \\ &\leq \|[p, u_*; F]\| \|p - u_*\| \\ &\leq b \|p - u_*\| \end{aligned}$$

so $p + F(p) - u_*$ and $F(p)$ belong in $\bar{S}(u_*, R)$. Afterwards, for $u, y, \in S(u_*, r) - \{u_*\}$, and

$$\begin{aligned} \left\| F'(u_*)^{-1} ([u, y; F] - F'(u_*)) \right\| &\leq w_0(\|u - u_*\|, \|y - u_*\|) \\ &\leq w_0(r, r) < 1, \end{aligned}$$

so the Banach lemma on invertible operators [3–5,12] gives $[u, y; F]^{-1} \subset \ell(\mathbb{B}, \mathbb{B})$, and

$$\left\| [u, y; F]^{-1} F'(u_*) \right\| \leq \frac{1}{1 - w_0(\|u_0 - u_*\|, \|y - u_*\|)}. \quad (25)$$

It also follows that y_0 is defined.

Adopting (15), (16), (19) (for $i = 1$), (a₂), (a₃), (25) and y_0 , we get

$$\begin{aligned}
\|y_0 - u_*\| &= \left\| u_0 - u_* - [u_0 + F(u_0), u_0; F]^{-1} F(u_0) \right\| \\
&= \left\| [u_0 + F(u_0), u_0; F]^{-1} \left([u_0 + F(u_0), u_0; F] - [u_0, u_*; F](u_0 - u_*) \right) \right\| \\
&\leq \left\| [u_0 + F(u_0), u_0; F]^{-1} F'(u_*) \right\| \left\| F'(u_*)^{-1} \left([u_0 + F(u_0), u_0; F] - [u_0, u_*; F] \right) \right\| \|u_0 - u_*\| \\
&\leq \frac{w \left(\|F(u_*)\|, \|u_0 - u_*\| \right) \|u_0 - u_*\|}{1 - w_0(a\|u_0 - u_*\|, \|u_0 - u_*\|)} \\
&\leq \frac{w \left(b\|u_0 - u_*\|, \|u_0 - u_*\| \right) \|u_0 - u_*\|}{1 - w_0(a\|u_0 - u_*\|, \|u_0 - u_*\|)} \\
&= g_1(\|u_0 - u_*\|) \|u_0 - u_*\| < \|u_0 - u_*\| < r,
\end{aligned} \tag{26}$$

so $y_0 \in S(u_*, r)$ (for $y_0 \neq u_*$) and (22) holds for $n = 0$.

$$\begin{aligned}
\left\| F'(u_*) \left([u_0, y_0; F] - F'(u_*) \right) \right\| &\leq w_0 \left(\|u_0 - u_*\|, \|y_0 - u_*\| \right) \\
&\leq w_0 \left(\|u_0 - u_*\|, g_1(\|u_0 - u_*\|) \|u_0 - u_*\| \right) \\
&\leq w_0 \left(r, g_1(r)r \right) < 1,
\end{aligned}$$

so $[u_0, y_0; F]^{-1} \in \ell(\mathbb{B}, \mathbb{B})$ and

$$\left\| [u_0, y_0; F]^{-1} F'(u_*) \right\| \leq \frac{1}{1 - w_0 \left(\|u_0 - u_*\|, g_1(\|u_0 - u_*\|) \|u_0 - u_*\| \right)}. \tag{27}$$

It also follows that z_0 is well defined by the second substep of method (2) for $n = 0$. In particular, we have

$$\begin{aligned}
z_0 &= u_0 - \lambda [u_0 + F(u_0), u_0; F]^{-1} (F(u_0) + F(y_0)) - (1 - \lambda) [u_0 + F(u_0), u_0; F]^{-1} F(u_0) \\
&= \left(u_0 - [u_0 + F(u_0), u_0; F]^{-1} F(u_0) \right) + [u_0 + F(u_0), u_0; F]^{-1} F(u_0) - \lambda [u_0 + F(u_0), u_0; F]^{-1} F(u_0) \\
&\quad - (1 - \lambda) [u_0, y_0; F]^{-1} F(u_0) - \lambda [u_0 + F(u_0), u_0; F]^{-1} F(y_0) \\
&= y_0 + (1 - \lambda) \left([u_0 + F(u_0), u_0; F]^{-1} - [u_0, y_0; F]^{-1} \right) F(u_0) - \lambda [u_0 + F(u_0), u_0; F]^{-1} F(y_0)
\end{aligned} \tag{28}$$

Next, by (15), (19) (for $i = 2$) and (25)–(28), we get, in turn, that

$$\begin{aligned}
\|z_0 - u_0\| &\leq \|y_0 - u_*\| + |1 - \lambda| \left\| [u_0 + F(u_0), u_0; F]^{-1} F'(u_*) \right\| \left\| F'(u_*)^{-1} \left([u_0, y_0; F] - [u_0 + F(u_0), u_0; F] \right) \right\| \\
&\quad \times \frac{\left\| [u_0, y_0; F]^{-1} F'(u_*) \right\| \left\| F'(u_*)^{-1} [u_0, u_*; F] \right\| \|u_0 - u_*\|}{\left(1 - w_0(a\|u_0 - u_*\|, \|u_0 - u_*\|) \right) \left(1 - w_0(\|u_0 - u_*\|, g_1(\|u_0 - u_*\|) \|u_0 - u_*\|) \right)} \\
&\quad + |\lambda| \left\| [u_0 + F(u_0), u_0; F]^{-1} F'(u_*) \right\| \left\| F'(u_*)^{-1} [y_0, u_*; F] \right\| \|y_0 - u_*\| \\
&\leq \left[g_1(\|u_0 - u_*\|) + \frac{b|1 - \lambda|w \left(b\|u_0 - u_*\|, (1 + g_1(\|u_0 - u_*\|)) \|u_0 - u_*\| \right)}{\left(1 - w_0(a\|u_0 - u_*\|, \|u_0 - u_*\|) \right) \left(1 - w_0(\|u_0 - u_*\|, g_1(\|u_0 - u_*\|) \|u_0 - u_*\|) \right)} \right] \\
&\quad + |\lambda| \frac{bg_1(\|u_0 - u_*\|)}{1 - w_0(a\|u_0 - u_*\|, \|u_0 - u_*\|)} \|u_0 - u_*\| \\
&= g_2(\|u_0 - u_*\|) \|u_0 - u_*\| \leq \|u_0 - u_*\|,
\end{aligned} \tag{29}$$

so $z_0 \in S(u_*, r)$ (for $z_0 \neq u_*$) and (23) holds for $p = 0$.

We have by (15), (18) and (29)

$$\begin{aligned} \left\| F'(u_*)^{-1}([z_0 + F(u_0), z_0; F] - F'(u_*)) \right\| &\leq w_0(b\|z_0 - u_*\|, \|z_0 - u_*\|) \\ &\leq w_0(bg_2(\|u_0 - u_*\|)\|u_0 - u_*\|, g_2(\|u_0 - u_*\|)\|u_0 - u_*\|) \\ &\leq w_0(bg_2(r)r, g_2(r)r) < 1 \end{aligned}$$

Accordingly, $[z_0 + F(z_0), z_0; F]^{-1} \in \ell(\mathbb{B}, \mathbb{B})$ and

$$\left\| [z_0 + F(z_0), z_0; F]^{-1} F'(u_*) \right\| \leq \frac{1}{1 - w_0(bg_2(\|u_0 - u_*\|)\|u_0 - u_*\|, g_2(\|u_0 - u_*\|)\|u_0 - u_*\|)}. \quad (30)$$

It also follows that u_1 is well defined by (30) and the last substep of method (2) for $n = 0$. Then, as in (25) and (26) (for $z = 3$) and (30), we obtain in turn

$$\begin{aligned} \|u_1 - u_*\| &\leq \frac{w(b\|z_0 - u_*\|, \|z_0 - u_*\|)\|z_0 - u_*\|}{1 - w_0(a\|z_0 - u_*\|, \|z_0 - u_*\|)} \\ &\leq \frac{w(bg_2(\|u_0 - u_*\|)\|u_0 - u_*\|, g_2(\|u_0 - u_*\|)\|u_0 - u_*\|)g_2(\|u_0 - u_*\|)\|u_0 - u_*\|}{1 - w_0(ag_2(\|u_0 - u_*\|)\|u_0 - u_*\|, g_2(\|u_0 - u_*\|)\|u_0 - u_*\|)} \quad (31) \\ &= g_3(\|u_0 - u_*\|)\|u_0 - u_*\| \leq \|u_0 - u_*\|, \end{aligned}$$

so, $u_1 \in S(u_*, r)$ (for $u_1 \neq u_*$) and (24) holds for $n = 0$. Subsequently, substituting u_0, y_0, z_0, u_1 by u_m, y_m, z_m, u_{m+1} , respectively. Hence, the induction for (30) and (22)–(24) is complete. Using the estimation

$$\|u_{m+1} - u_*\| < \alpha\|u_m - u_*\| < r, \quad (32)$$

where $\alpha = g_3(\|u_0 - u_*\|) \in [0, 1]$, we deduce that $\lim_{m \rightarrow \infty} u_m = u_*$ and $u_{m+1} \in S(u_*, r)$.

Finally, we want to illustrate that the required solution is unique. Therefore, let $T = [u_*, y_*; F]$ for $y_* \in \Omega_1$, so that $F(y_*) = 0$. Then, by (a_2) and (a_5) , we get

$$\begin{aligned} \|F'(u_*)^{-1}(T - F'(u_*))\| &\leq w_0(0, \|u_* - y_*\|) \\ &\leq w_0(0, \bar{r}) < 1, \end{aligned}$$

so $T^{-1} \in \ell(\mathbb{B}, \mathbb{B})$. Finally, $u_* = y_*$ is deduced from $0 = F(u_*) - F(y_*) = T(u_* - y_*)$. $\square$

Remark 1. Another way of defining functions g_i, h_i and radii r_i is as follows:

Let $\alpha = \max\{\beta, a\}$, $i = 1, 2, 3$. Subsequently, as in (12)–(18), we shall have instead:

Suppose that equation

$$w_0(\alpha t, t) = 1 \quad (33)$$

has a smallest positive solution $\bar{\rho}_1$. Let $\bar{w} : [0, \bar{\rho}_1] \times [0, \bar{\rho}_1]$ be a increasingly continuous function with $\bar{w}(0, 0) = 0$.

Let functions $\bar{g}_1$ and $\bar{h}_1$ be defined in the interval $[0, \bar{\rho}_1]$ by

$$\bar{g}_1(t) = \frac{\bar{w}(bt, t)}{1 - w_0(\alpha t, t)} \text{ and } \bar{h}_1(t) = \bar{g}_1(t) - 1.$$

The $\bar{r}_1$ stands for the smallest positive root of $\bar{h}_1(t) = 0$ in $(0, \bar{\rho}_1)$. Moreover, define functions $\bar{g}_2, \bar{g}_3, \bar{h}_2$ and $\bar{h}_3$ on the closed interval $[0, \bar{\rho}_1]$, as follows:

$$\begin{aligned}\bar{g}_2(t) &= \bar{g}_1(t) + \frac{b|1 - \lambda|w\left(bt, \left(1 + \bar{g}_1(t)\right)t\right)\bar{g}_1(t)}{\left(1 - w_0(\alpha t, t)\right)^2}, \\ \bar{g}_3(t) &= \frac{w\left(b\bar{g}_2(t)t, \bar{g}_2(t)t\right)\bar{g}_2(t)}{1 - w_0(\alpha t, t)}, \\ \bar{h}_2(t) &= \bar{g}_2(t) - 1 \\ \text{and} \\ \bar{h}_3(t) &= \bar{g}_3(t) - 1.\end{aligned}$$

The $\bar{r}_2$ and $\bar{r}_3$ serve as the minimal positive roots of $\bar{h}_2(t) = 0$ and $\bar{h}_3(t) = 0$ on closed interval $[0, \bar{\rho}_1]$, respectively. Subsequently, Theorem 2 can be written by using the “bar” conditions and functions, with $\bar{r} = \min\{\bar{r}_i\}$.

Remark 2. The convergence of method (2) to u_* is established under the conditions of Theorem 1. However, the order convergence under the conditions of Theorem 2 can be established by using (COC) and (ACOC) (for the details, please see Section 5).

4. Numerical Examples

Here, we monitor the convergence conditions on three problems (1)–(3). We choose $[u, y; F] = \int_0^1 F'(y + \theta(u - y))d\theta$ in the examples. We can confirm the verification the hypotheses of Theorem 2 for the given choices of the “ w ” functions and parameters a and b .

Example 1. Here, we investigate the application of our results on Hammerstein integral equations (see [9], pp. 19–20) for $\mathbb{B} = C[0, 1]$ as follows:

$$F(u(s_1)) = u(s_1) - \frac{1}{5} \int_0^1 S(s_1, s_2) u(s_2)^3 ds_2 = 0, \quad u \in C[0, 1], \quad s_1, s_2 \in [0, 1], \quad (34)$$

where

$$S(s_1, s_2) = \begin{cases} s(1 - s_2), & s \leq s_2, \\ (1 - s)s_2, & s_2 \leq s. \end{cases}$$

We use $\int_0^1 \phi(t)dt \simeq \sum_{k=1}^8 w_k \phi(t_k)$ in (34), where t_k and w_k are the abscissas and weights, respectively. Using $u(t_j)$ for u_j ($j = 1, 2, 3, \dots, 8$), leads to

$$\begin{aligned}5u_i - 5 - \sum_{k=1}^8 a_{jk} u_k^3 &= 0, \quad j = 1, 2, 3, \dots, 8, \\ a_{jk} &= \begin{cases} w_k t_k (1 - t_j), & k \leq j, \\ w_k t_j (1 - t_k), & j < k. \end{cases}\end{aligned}$$

The values of t_k and w_k when $k = 8$, are illustrated in Table 1. Subsequently, we have

$$u_* = (1.002096 \dots, 1.009900 \dots, 1.019727 \dots, 1.026436 \dots, 1.026436 \dots, 1.019727 \dots, 1.009900 \dots, 1.002096 \dots)^T.$$

Accordingly, we set $w_0(s_1, s_2) = w(s_1, s_2) = \frac{3}{80}(s_1 + s_2)$, $a = \frac{163}{80}$ and $b = \frac{83}{80}$. The radii for Example 1 are listed in Tables 2 and 3:

Table 1. Abscissas and weights for $k = 8$.

j	t_j	w_j
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

Table 2. Convergence radii for Example 1.

λ	r_1	r_2	r_3	r
0	5.25452	3.87208	4.09301	3.87208
0.5	5.25452	4.26006	4.42602	4.26006
1	5.25452	5.25452	5.25452	5.25452

Table 3. Convergence radii for Example 1 with bar functions.

λ	r_1	r_2	r_3	r
0	5.25452	3.67748	3.87626	3.67748
0.5	5.25452	4.07351	4.17413	4.07351
1	5.25452	5.25452	4.89162	4.89162

Example 2. Here, we choose as integral equation [17,18], for $\mathbb{B} = C[0, 1]$ as

$$[F(\mu)](\gamma_1) = \mu(\gamma_1) - \int_0^1 G(\gamma_1, \gamma_2) \left(\mu(\gamma_2)^{\frac{3}{2}} + \frac{\mu(\gamma_2)^2}{2} \right) d\gamma_2 = 0, \quad (35)$$

where

$$G(\gamma_1, \gamma_2) = \begin{cases} (1 - \gamma_2)\gamma_2, & \gamma_2 \leq \gamma_1, \\ \gamma_1(1 - \gamma_2), & \gamma_1 \leq \gamma_2. \end{cases} \quad (36)$$

Because $\mathbb{B} = C[0, 1]$ so, $F : C[0, 1] \rightarrow C[0, 1]$ is given as

$$[F(\mu)](\gamma_1) = \mu(\gamma_1) - \int_0^{\gamma_1} G(\gamma_1, \gamma_2) \left(\mu(\gamma_2)^{\frac{3}{2}} + \frac{\mu(\gamma_2)^2}{2} \right) d\gamma_2. \quad (37)$$

We get

$$\left\| \int_0^{\gamma_1} G(\gamma_1, \gamma_2) d\gamma_2 \right\| \leq \frac{1}{8}. \quad (38)$$

Moreover,

$$[F'(\mu)\eta](\gamma_1) = \eta(\gamma_1) - \int_0^{\gamma_1} G(\gamma_1, \gamma_2) \left(\frac{3}{2}\mu(\gamma_2)^{\frac{1}{2}} + \mu(\gamma_2) \right) \eta(\gamma_2) d\gamma_2,$$

so $\mu_*(\gamma_1) = 0$, since $F'(\mu_*(\gamma_1)) = I$,

$$\left\| F'(\mu_*)^{-1}(F'(\mu) - F'(\eta)) \right\| \leq \frac{1}{8} \left(\frac{3}{2} \|\mu - \eta\|^{\frac{1}{2}} + \|\mu - \eta\| \right). \quad (39)$$

Hence, we have

$$w_0(s, t) = w(s, t) = \frac{1}{16} \left[\frac{3}{2}(\sqrt{s} + \sqrt{t}) + s + t \right], \quad a = \frac{53}{16}, \text{ and } b = \frac{37}{16}.$$

Therefore, our results can be utilized even though F' is not bounded on Ω . The radii for Example 2 are given in Table 4.

Table 4. Convergence radii for Example 2 with bar functions.

λ	r_1	r_2	r_3	r
0	1.03137	0.502403	0.61211	0.502403
0.5	1.03137	0.61199	0.70738	0.61199
1	1.03137	1.03137	1.03137	1.03137

Example 3. We assume the following differential equations

$$\begin{aligned} q_1'(\mu) - q_1(\mu) - 1 &= 0 \\ q_2'(\eta) - (e-1)\eta - 1 &= 0 \\ q_3'(\theta) - 1 &= 0 \end{aligned} \quad (40)$$

characterizes the progress/movement of a molecule in 3D with $(\mu, \eta, \theta) \in \Omega$ for $q_1(0) = q_2(0) = q_3(0) = 0$. The required solution $v = (\mu, \eta, \theta)^T$ describes to $K := (q_1, q_2, q_3) : \Omega \rightarrow \mathbb{R}^3$ given as

$$K(v) = \left(e^\mu - 1, \frac{e-1}{2}\eta^2 + \eta, \theta \right)^T = 0. \quad (41)$$

It follows from (41) that

$$K'(v) = \begin{bmatrix} e^\mu & 0 & 0 \\ 0 & (e-1)\eta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which yields

$$w_0(s, t) = \frac{1}{2}(e-1)(s+t), \quad w(s, t) = \frac{1}{2}e(s+t), \quad a = \frac{1}{2}(e+3), \quad \text{and} \quad b = \frac{1}{2}(e+1).$$

We depicted the radii of Example 3 in Tables 5 and 6.

Table 5. Convergence radii for Example 3.

λ	r_1	r_2	r_3	r
0	0.1388596	0.921375	0.083356	0.083356
0.5	0.1388596	0.921375	0.086297	0.086297
1	0.1388596	0.1388596	0.1388596	0.1388596

Table 6. Convergence radii for Example 3 with bar functions.

λ	r_1	r_2	r_3	r
0	0.1388596	0.0487471	0.1229551	0.0487471
0.5	0.1388596	0.0487471	0.1377815	0.0487471
1	0.1388596	0.1388596	0.1380780	0.1380780

Example 4. By the example of Section 2, for $\Omega = \mathbb{B} = \mathbb{R}$, $f(\xi) = 0$, we get

$$w_0(s, t) = w(s, t) = \frac{96.66297}{2}(s+t), \quad a = \frac{5}{2}, \quad \text{and} \quad b = \frac{3}{2}.$$

The radii of method (2) for Example 4 are listed in Tables 7 and 8.

Table 7. Convergence radii for Example 4.

λ	r_1	r_2	r_3	r
0	0.00344841	0.00239612	0.00256623	0.00239612
0.5	0.00344841	0.00267769	0.00280807	0.00267769
1	0.00344841	0.00344841	0.00344841	0.00344841

Table 8. Convergence radii for Example 4 with bar functions.

λ	r_1	r_2	r_3	r
0	0.00344841	0.00225955	0.00246765	0.00225955
0.5	0.00344841	0.00225955	0.00246765	0.00225955
1	0.00344841	0.00344841	0.00334891	0.00344841

5. Applications with Large Systems

We choose $\lambda = 0$, $\lambda = 0.5$ and $\lambda = 1$ in our scheme (2), called by (PS1), (PS2) and (PS3), respectively. Now, we compare our schemes with a 6th-order iterative methods suggested by Abbasbandy et al. [19] and Hueso et al. [20], among them we picked the methods (8) and (14–15) (for $t_1 = -\frac{9}{4}$ and $s_2 = \frac{9}{8}$), respectively, known as (AS) and (HS). Moreover, a comparison of them has been done with the 6th-order iterative methods given by Wang and Li [21], among their method we chose expression (6), denoted by and (WS). At the last, we contrast (2) with sixth-order scheme given by Sharma and Arora [22], we pick expression (13), known as (SM). The details of all the iterative expressions are given, as follows:

method AS:

$$\begin{aligned}
 y_j &= u_j - \frac{2}{3}F'(u_j)^{-1}F(u_j), \\
 z_j &= u_j - \left[I + \frac{21}{8}F'(u_j)^{-1}F'(y_j) - \frac{9}{2}(F'(u_j)^{-1}F'(y_j))^2 + \frac{15}{8}(F'(u_j)^{-1}F'(y_j))^3 \right] F'(u_j)^{-1}F(u_j), \\
 u_{j+1} &= z_j - \left[3I - \frac{5}{2}F'(u_j)^{-1}F'(y_j) + \frac{1}{2}(F'(u_j)^{-1}F'(y_j))^2 \right] F'(u_j)^{-1}F(z_j).
 \end{aligned} \quad (42)$$

scheme HS:

$$\begin{aligned}
 y_j &= u_j - F'(u_j)^{-1}F(u_j), \\
 H(u_j, y_j) &= F'(u_j)^{-1}F(y_j), \quad H(y_j, u_j) = F'(y_j)^{-1}F(u_j), \\
 G_s(u_j, y_j) &= s_1I + s_2H(y_j, u_j) + s_3H(u_j, y_j) + s_4H(y_j, u_j)^2, \\
 z_j &= u_j - G_s(u_j, y_j)F'(u_j)^{-1}F(u_j), \\
 u_{j+1} &= z_j.
 \end{aligned} \quad (43)$$

where s_1, s_2, s_3 and s_4 are real numbers.

iterative method WS:

$$\begin{aligned}
 y_j &= u_j - F'(u_j)^{-1}F(u_j), \\
 z_j &= y_j - \left[2I - F'(u_j)^{-1}F'(y_j) \right] F'(u_j)^{-1}F(y_j), \\
 u_{j+1} &= z_j - \left[2I - F'(u_j)^{-1}F'(y_j) \right] F'(u_j)^{-1}F(z_j).
 \end{aligned} \quad (44)$$

scheme SM:

$$\begin{aligned}
 y_j &= u_j - \frac{2}{3}F'(u_j)^{-1}F(u_j), \\
 z_j &= u_j - \left[pI + F'(u_j)^{-1}F'(y_j) \left(qI + rF'(u_j)^{-1}F'(y_j) \right) \right] F'(u_j)^{-1}F(u_j), \\
 u_{j+1} &= z_j - \left[\frac{5}{2}I - \frac{3}{2}F'(u_j)^{-1}F'(y_j) \right] F'(u_j)^{-1}F(z_j),
 \end{aligned} \quad (45)$$

where $p = \frac{23}{8}$, $q = -3$ and $r = \frac{9}{8}$.

The (j) , $(\|F(u_j)\|)$, $\|u_{j+1} - u_j\|$, and $\rho^* \approx \frac{\log [\|u_{j+1} - u_j\| / \|u_j - u_{j-1}\|]}{\log [\|u_j - u_{j-1}\| / \|u_{j-1} - u_{j-2}\|]}$ stands for index of iteration, absolute residual errors in the function F , error between two successive iterations and computational convergence order, receptively. There values are listed in Tables 9–11. Moreover, the quantity η is the final obtained value of $\frac{\|u_{j+1} - u_j\|}{\|u_j - u_{j-1}\|^6}$.

The estimation of all the above parameters have been calculated by Mathematica-9. For minimizing the round-off errors, we have chosen multiple precision arithmetic with 1000 digits of mantissa. The term $b_1 (\pm b_2)$ symbolizes the $b_1 \times 10^{(\pm b_2)}$ in all mentioned tables. We adopted the command "AbsoluteTiming[]" in order to calculate the CPU time. We run our programs three times and depicted the average CPU time in Table 12, also one can observe the times used for each iterative method, where we want to point out that for big size problems the method PS1 uses the minimum time, so it is being very competitive. The configuration of the used computer is given below:

Processor: Intel(R) Core(TM) i7-4790 CPU @ 3.60 GHz

Made: HP

RAM: 8:00 GB

System type: 64-bit-Operating System, x64-based processor.

Example 5. Here, we deal with a boundary value problem from Ortega and Rheinboldt [9], given by

$$y'' = \frac{y^3 + 6y' + 1}{2} - \frac{3}{2-x}, \quad y(0) = 0, \quad y(1) = 1. \quad (46)$$

We assume

$$u_0 = 0 < u_1 < u_2 < u_3 < \dots < u_p, \text{ where } u_{p+1} = u_p + h, \quad h = \frac{1}{p}, \quad (47)$$

partition of the interval $[0, 1]$ and $y_0 = y(u_0) = 0$, $y_1 = y(u_1)$, $\dots$, $y_{n-1} = y(u_{n-1})$, $y_p = y(u_p) = 1$.

Now, we discretize expression (46) by adopting following numerical formula for derivatives

$$y'_j = \frac{y_{j+1} - y_{j-1}}{2h}, \quad y''_j = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2}, \quad j = 1, 2, \dots, p-1,$$

which leads to

$$y_{j+1} - 2y_j + y_{j-1} - \frac{h^2}{2}y_j^3 - \frac{3}{2}h(y_{k+1} - y_{k-1}) - \frac{3}{2-u_j}h^2 - \frac{1}{h^2} = 0, \quad j = 1, 2, \dots, p-1,$$

$(p-1) \times (p-1)$ system of nonlinear equations.

For specific value of $p = 7$, we have a 6×6 system and the required solution is

$$u_* = (0.07654393 \dots, 0.1658739 \dots, 0.2715210 \dots, 0.3984540 \dots, 0.5538864 \dots, 0.7486878 \dots)^T.$$

The computational estimations are listed in Table 9 on the basis of initial approximation $y_j^{(0)} = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2})^T$.

Table 9. Comparisons of different methods on a Boundary value problem in Example 5.

Methods	j	$\ F(u_j)\ $	$\ u_{j+1} - u_j\ $	ρ^*	$\frac{\ u_{j+1} - u_j\ }{\ u_j - u_{j-1}\ }$
AS	1	1.9(−4)	6.1(−4)	5.9985	5.133234733(−7)
	2	8.7(−27)	2.8(−26)		
	3	8.1(−161)	2.6(−160)		
HS	1	1.3(−4)	5.7(−4)	4.9954	8.252588019(−3)
	2	8.1(−23)	2.8(−22)		
	3	3.5(−114)	1.0(−113)		
WS	1	2.6(−4)	1.1(−3)	5.9957	1.977528884(−7)
	2	1.1(−25)	3.1(−25)		
	3	8.6(−155)	2.4(−154)		
SM	1	7.3(−5)	2.7(−3)	5.9973	7.804847473(−7)
	2	8.2(−29)	2.7(−28)		
	3	1.1(−172)	3.6(−172)		
PS1	1	4.9(−6)	1.6(−5)	6.0010	2.474537279(−9)
	2	1.6(−38)	4.8(−38)		
	3	9.3(−234)	2.7(−233)		
PS2	1	1.1(−5)	3.7(−5)	6.0013	4.513404180(−9)
	2	3.7(−36)	1.1(−35)		
	3	2.4(−219)	7.2(−219)		
PS3	1	1.9(−5)	6.5(−5)	6.0016	7.168046437(−9)
	2	1.9(−34)	5.6(−34)		
	3	6.6(−209)	1.9(−208)		

Example 6. The classical 2D Bratu problem [23,24] is given by

$$u_{\mu\mu} + u_{\theta\theta} + Ce^u = 0, \quad \Omega = \{(\mu, \theta) \in 0 \leq \mu \leq 1, 0 \leq \theta \leq 1\}, \text{ with boundary hypotheses } u = 0 \text{ on } \Omega. \quad (48)$$

By adopting finite difference discretization, we can deduced the above PDE (48) to a nonlinear system. For this purpose, we denote $\Delta_{i,j} = u(\mu_i, \theta_j)$ as numerical solution at the grid points of the mesh. In addition to this, M_1 and M_2 stand for the number of steps in the directions of μ and θ , respectively. The h and k called as the respective step sizes in the directions of μ and θ . Adopt the following central difference formula to $u_{\mu\mu}$ and $u_{\theta\theta}$

$$u_{\mu\mu}(u_i, \theta_j) = \frac{\Delta_{i+1,j} - 2\Delta_{i,j} + \Delta_{i-1,j}}{h^2}, \quad C = 0.1, \quad \theta \in [0, 1], \quad (49)$$

leads to us

$$\Delta_{i,j+1} + \Delta_{i,j-1} - \Delta_{i,j} + \Delta_{i+1,j} + \Delta_{i-1,j} + h^2 C \exp(\Delta_{i,j}) \quad i = 1, 2, 3, \dots, M_1, j = 1, 2, 3, \dots, M_2 \quad (50)$$

For obtaining a large system of 100×100 , we choose $M_1 = M_2 = 11$, $C = 0.1$ and $h = \frac{1}{11}$. The numerical results are listed in Table 10 based on the initial guess $u_0 = 0.1 \left(\sin(\pi h i) \sin(\pi h j) \right)^T$, $i = j = 10$.

Table 10. Comparisons of different methods on two-dimensional (2D) Bratu problem in Example 6.

Methods	j	$\ F(u_j)\ $	$\ u_{j+1} - u_j\ $	ρ_*	$\frac{\ u_{j+1} - u_j\ }{\ u_j - u_{j-1}\ ^6}$
AS	1	4.4(−15)	2.4(−14)	5.9994	1.428095547(−12)
	2	6.9(−95)	3.5(−94)		
	3	7.9(−574)	3.9(−573)		
HS	1	2.1(−13)	1.2(−12)	4.9997	7.368055345(−11)
	2	2.1(−71)	1.2(−70)		
	3	1.7(−361)	9.3(−361)		
WS	1	5.0(−19)	2.9(−18)	1.0(−121)	1.754949400(−16)
	2		1.7(−122)		
	3	3.1(−743)	1.8(−742)		
SM	1	4.4(−15)	2.4(−14)	5.9994	1.433541371(−12)
	2	7.1(−95)	3.6(−94)		
	3	9.2(−574)	4.5(−573)		
PS1	1	9.1(−21)	5.3(−20)	6.0000	3.060974255(−18)
	2	1.2(−134)	7.1(−134)		
	3	6.9(−818)	4.0(−817)		
PS2	1	1.9(−20)	1.1(−19)	6.0000	6.095821945(−18)
	2	1.7(−132)	1.0(−131)		
	3	1.1(−804)	6.7(−804)		
PS3	1	3.1(−20)	1.8(−19)	6.0000	1.016575545(−17)
	2	6.7(−131)	3.9(−130)		
	3	6.3(−795)	3.7(−794)		

Example 7. Let us consider the following nonlinear system

$$F(x) = \begin{cases} u_j^2 u_{j+1} - 1 = 0, & 1 \leq j \leq p-1, \\ x_p^2 u_1 - 1 = 0. \end{cases} \quad (51)$$

For specific value $p = 200$, we have 200×200 system, and chose the following starting point

$$x^{(0)} = (1.25, 1.25, 1.25, \dots, 1.25)^T.$$

The $u_* = (1, 1, 1, \dots, 1)^T$ is the required solution of system 7. Table 11 provides the numerical results.

Remark 3. On the basis of Tables 9–11, we conclude that our methods namely PS1, PS2 and PS3 perform better in the contrast of existing schemes AS, HS, SM and SM on the basis of residual errors, errors between two consecutive iterations, and asymptotic error constant. In addition, our methods also demonstrate the stable computational order of convergence. Finally, we concluded that our methods not only perform better than existing methods in numerical results, but also take half of the CPU time in contrast to other existing methods (results can be easily found in Table 12).

Table 11. Comparisons of different methods on Example 7.

Methods	j	$\ F(u_j)\ $	$\ u_{j+1} - u_j\ $	ρ_*	$\frac{\ u_{j+1} - u_j\ }{\ u_j - u_{j-1}\ ^6}$
AS	1	5.2(−3)	1.7(−3)	6.0000	7.686036043(−5)
	2	6.2(−21)	2.1(−21)		
	3	1.7(−128)	5.8(−129)		
HS	1	2.3(−3)	7.7(−4)	5.0000	8.659247536(−2)
	2	5.4(−20)	1.8(−20)		
	3	3.9(−103)	1.3(−103)		
WS	1	3.5(−3)	1.2(−3)	6.0000	5.299207889(−5)
	2	4.0(−22)	1.3(−22)		
	3	8.4(−136)	2.8(−136)		
SM	1	3.0(−3)	1.0(−3)	6.0000	4.671758076(−5)
	2	1.4(−22)	4.6(−23)		
	3	1.3(−138)	4.3(−139)		
PS1	1	5.1(−3)	1.7(−3)	6.000	1.130483172(−4)
	2	8.4(−21)	2.8(−21)		
	3	1.6(−127)	5.5(−128)		
PS2	1	1.0(−1)	3.3(−2)	5.9989	9.906447117(−4)
	2	4.0(−12)	1.3(−12)		
	3	1.7(−74)	5.8(−75)		
PS3	1	3.3(−1)	1.1(−1)	5.9943	2.565472254(−3)
	2	1.3(−8)	4.3(−9)		
	3	5.6(−53)	1.9(−53)		

Table 12. CPU time of different methods on Examples 5–7.

Methods	Example 5	Example 6	Example 7	Total Time	Average Time
AS	0.465330	210.079553	356.906591	567.451474	189.1504913
HS	0.583412	189.541919	366.511753	556.637084	185.5456947
WS	0.274193	128.377322	182.956711	311.608226	103.8694087
SM	1.130812	126.641140	401.627979	529.399931	176.4666437
PS1	0.101071	120.094370	52.204957	172.400398	57.46679933
PS2	0.100071	117.901198	52.146903	170.148172	56.71605733
PS3	0.100083	117.923227	51.972773	169.996083	56.665361

According to the CPU time, method PS3 is taking the lowest time for executing the results. All of the other schemes AS, HS, SM; and, SM consuming at least double CPU timing as compare to our methods namely PS1, PS2 and PS3. So, we conclude that our methods provide results faster than the other existing methods.

6. Conclusions

We presented a new family of Steffensen-type methods with one parameter. The local convergence is studied in Section 2 while using Taylor expansion and derivative up to the order seven, when $\mathbb{B} = \mathbb{R}^j$. To extend the suitability of these iterative methods, we only use hypotheses on the first derivative in Section 3 and Banach space valued operators. This way, we also find computable error bounds on $\|u_p - u_*\|$ as well as uniqueness results based on generalized Lipschitz-type real functions. Numerical examples of equations, favorable comparisons to other methods can be found in Section 4.

Author Contributions: M.Z.U.: Validation; Review & Editing, R.B. and I.K.A.: Conceptualization; Methodology; Validation; Writing—Original Draft Preparation; Writing—Review & Editing. All authors have read and agreed to the published version of the manuscript.

Funding: Research and development office (RDO) at the ministry of Education, Kingdom of Saudi Arabia. Grant no (HIQI-22-2019).

Acknowledgments: This project was funded by the research and development office (RDO) at the ministry of Education, Kingdom of Saudi Arabia. Grant No. (HIQI-22-2019). The authors also, acknowledge with thanks research and development office (RDO-KAU) at King Abdulaziz University for technical support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Amat, S.; Bermudez, C.; Hernández-Verón, M.A.; Martínez, E. On an efficient k-step iterative method for nonlinear equations. *J. Comput. Appl. Math.* **2016**, *302*, 258–271. [\[CrossRef\]](#)
2. Argyros, I.K. *Convergence and Applications of Newton-Type Iterations*; Springer: New York, NY, USA, 2008.
3. Argyros, I.K.; George, S. *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications*; Nova Publishers: New York, NY, USA, 2019; Volume III.
4. Argyros, I.K.; Hillout, S. Weaker conditions for the convergence of Newton’s method. *J. Complex.* **2012**, *28*, 364–387. [\[CrossRef\]](#)
5. Argyros, I.K.; Magrenan, A.A. *A Contemporary Study of Iterative Methods*; Academy Press: Cambridge, MA, USA; Elsevier: Amsterdam, The Netherlands, 2018.
6. Cordero, A.; Torregrosa, J.R. Low-complexity root finding iteration functions with no derivatives of any order of convergence. *J. Comput. Appl. Math.* **2015**, *275*, 502–515. [\[CrossRef\]](#)
7. Ezquerro, J.A.; Hernández, M.A. How to improve the domain of starting points for Steffensen’s method. *Stud. Appl. Math.* **2014**, *132*, 354–380. [\[CrossRef\]](#)
8. Potra, F.A.; Pták, V. *Nondiscrete Induction and Iterative Processes*; Pitman Advanced Publishing Program: Boston, MA, USA, 1984; Volume 103.
9. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
10. Rheinboldt, W.C. An adaptive continuation process for solving systems of equations. *Pol. Acad. Sci. Banach Cent. Publ.* **1978**, *3*, 129–142. [\[CrossRef\]](#)
11. Sharma, J.R.; Ghua, R.K.; Sharma, R. An efficient fourth-order weighted Newton method for systems of nonlinear equations. *Numer. Algorithms* **2013**, *62*, 307–325. [\[CrossRef\]](#)
12. Traub, J.F. *Iterative Methods for the Solutions of Equations*; American Mathematical Society: Providence, RI, USA, 1982.
13. Zúñic, J.D.; Petković, M.S. A cubically convergent Steffensenlike method for solving nonlinear equations. *Appl. Math. Lett.* **2012**, *25*, 1881–1886.
14. Alarcón, V.; Amat, S.; Busquier, S.; López, D.J. A Steffensen type method in Banach spaces with applications on boundary-value problems. *J. Comput. Appl. Math.* **2008**, *216*, 243–250. [\[CrossRef\]](#)
15. Behl, R.; Argyros, I.K.; Machado, J.A.T. Ball comparison between three sixth order methods for Banach space valued operators. *Mathematics* **2020**, *8*, 667. [\[CrossRef\]](#)
16. Iliev, A.; Kyurkchiev, N. *Nontrivial Methods in Numerical Analysis: Selected Topics in Numerical Analysis*; LAP LAMBERT Academic Publishing: Saarbrücken, Germany, 2010; ISBN 978-3-8433-6793-6.
17. Ezquerro, J.A.; Hernández, M.A. New iterations of R-order four with reduced computational cost. *BIT Numer. Math.* **2009**, *49*, 325–342. [\[CrossRef\]](#)
18. Hernández, M.A.; Martínez, E. On the semilocal convergence of a three steps Newton-type process under mild convergence conditions. *Numer. Algor.* **2015**, *70*, 377–392. [\[CrossRef\]](#)
19. Abbasbandy, S.; Bakhtiari, P.; Cordero, A.; Torregrosa, J.R.; Lotfi, T. New efficient methods for solving nonlinear systems of equations with arbitrary even order. *Appl. Math. Comput.* **2016**, *287*, 287–288. [\[CrossRef\]](#)
20. Hueso, J.L.; Martínez, E.; Teruel, C. Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems. *J. Comput. Appl. Math.* **2015**, *275*, 412–420. [\[CrossRef\]](#)
21. Wang, X.; Li, Y. An Efficient Sixth Order Newton Type Method for Solving Nonlinear Systems. *Algorithms* **2017**, *10*, 45. [\[CrossRef\]](#)
22. Sharma, J.R.; Arora, H. Efficient Jarratt-like methods for solving systems of nonlinear equations. *Calcolo* **2014**, *51*, 193–210. [\[CrossRef\]](#)

23. Kapania, R.K. A pseudo-spectral solution of 2-parameter Bratu's equation. *Comput. Mech.* **1990**, *6*, 55–63. [[CrossRef](#)]
24. Simpson, R.B. A method for the numerical determination of bifurcation states of nonlinear systems of equations. *SIAM J. Numer. Anal.* **1975**, *12*, 439–451. [[CrossRef](#)]



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).